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SEPTIC FIELDS WITH DISCRIMINANT $\pm 2^a 3^b$

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Abstract. We classify septic number fields which are unramified outside of $\{\infty, 2, 3\}$ by a targeted Hunter search; there are exactly 10 such fields, all with Galois group S_7. We also describe separate computations which strongly suggest that none of these fields come from specializing septic genus zero three-point covers.

1. Introduction

Let $S \subset \{\infty, 2, 3, 5, \dots\}$ be a finite set of primes and let n be a positive integer. The set of isomorphism classes of extensions of $\mathbf{Q}$ of degree n which are unramified outside of S has long been known to be finite by a classical theorem of Hermite [Her], [Lan], yet little else is known about these sets in general. We are particularly interested in the case where S is a small set of small primes. Number fields with such sharply restricted ramification are extremely rare in lists of number fields with absolute discriminant less than a given bound. On the other hand, they appear in a variety of places in mathematics. For discussions of why fields ramified at only a few small primes are interesting, see [JR1], [Har].

Here we consider the case $S = \{\infty, 2, 3\}$, i.e., fields with discriminant $\pm 2^a 3^b$. In [JR1], we presented the previously known cases of degrees $n \leq 5$ and worked out the computationally much harder case $n = 6$. In this paper, we settle the computationally yet harder case $n = 7$, thereby completing the following table, giving the number of fields unramified away from ∞, 2 and 3 for small n:

n	1	2	3	4	5	6	7
Total number of fields	1	7	9	62	6	398	10
Number of nonsolvable fields					5	62	10.

It is fair to say that nonsolvable fields just barely exist for $S = \{\infty, 2, 3\}$ in these degrees. Certainly for $S = \{\infty, 2\}$, $\{\infty, 3\}$, or $\{2, 3\}$ in degrees ≤ 7, all fields are solvable; indeed the nonsolvable fields in our case $S = \{\infty, 2, 3\}$ are all wildly ramified at 2 and 3. On the other hand, for $S = \{\infty, 2, 3, 5\}$ there are 1,415 nonsolvable quintic fields [Jon], and surely many nonsolvable sextic and septic fields too.

Section 2 discusses how our search technique is an improvement on [JR1]. Section 3 presents a defining polynomial for each of the ten fields. Also it discusses quantitatively how the ten fields were found early in the search, and analyzes the wild ramification at 2 and 3. It is natural to ask if our septic fields have conceptual sources, like many of the fields in [JR1] do. Section 4 reports on our attempts to

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produce some of the ten fields by specializing 3-point covers of $\mathbf{P}^1$. We are able to produce one field by specializing a nonic cover, and we present evidence that none of the fields can be produced by specializing septic covers.

2. The computer search

The computer search was based on Hunter's theorem, a method used extensively for determining all number fields of a given degree with absolute discriminant less than a prescribed bound. See [Coh, §9.3] for a full discussion of this approach.

For septic fields, a Hunter search with bound B enumerates a long but finite list of polynomials guaranteed to contain the characteristic polynomial of at least one primitive element for each septic field with absolute discriminant $\leq B$. The number of polynomials examined grows roughly in proportion to $B^{9/4}$ for septic fields.

For this paper, we need to consider absolute discriminants up to $2^{14}3^{11}$. A Hunter search with this value for B would be prohibitively long. On the other hand, we are only interested in a few isolated discriminant values, and for each candidate discriminant value, the defining polynomials will satisfy p-adic conditions for $p = 2$ and 3. We refer to a Hunter search which incorporates these p-adic conditions as *targeted*. The primary difference in methodology for this search as compared with [JR1] is how we implemented the targeting.

Here, as in [JR1], we divide the search into smaller searches, each indexed by a 2-adic ramification structure and a 3-adic ramification structure. A p-adic ramification structure (see [JR1, §2]) contains the information of ramification exponents for the factorization of the ideal (p) in the ring of integers $\mathcal{O}_K$ of a number field K, along with corresponding contributions to the discriminant.

In each smaller search of [JR1], we searched for polynomials $f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in \mathbf{Z}[x]$ which could arise as characteristic polynomials f_η of elements $\eta \in I$, where I is a particular ideal of $\mathcal{O}_K$. The field K is unknown, but satisfies the prescribed ramification properties which translate to congruences on the coefficients a_i of f. The attraction of working in I rather than $\mathcal{O}_K$ is that one has strong congruences having the very simple form $p^e \mid a_i$. The method of [JR1] uses only linear congruences on individual coefficients of this sort. It therefore oversearches substantially, as is explained via an example in [JR1, §2.5].

The improvement suggested in [JR1, §2.5] is to use all congruences on the vector $(a_1, \ldots, a_n)$ of coefficients, not just linear congruences on individual coefficients a_i. Here we indeed use all such congruences. Having done this, there is no longer any point in restricting η to lie in the search ideal I. Rather we let η be anywhere in $\mathcal{O}_K$. Note that the timing analysis in [JR2] of each of these smaller searches applies to our new method.

To illustrate how ramification properties of the fields sought translate to congruences on coefficients, consider looking for fields K whose local form at 2 is $K \otimes \mathbf{Q}_2 \cong K_t \times K_w$, where K_t is a cubic field of discriminant 2^2 while K_w is a quartic field of discriminant 2^{10}. These invariants force the two fields to be totally ramified, which simplifies our discussion. The cubic field K_t is necessarily tamely ramified while the quartic field K_w must be wildly ramified, which explains our notation. Let $\mathcal{O}_t \supset I_t$ and $\mathcal{O}_w \supset I_w$ be the corresponding local rings of integers and maximal ideals.

Consider first K_t. An element of I_t has characteristic polynomial congruent to x^3 modulo 2. Every element of $\mathcal{O}_t$ either is in I_t, or is a translate by 1 from an element of the I_t. Hence every element of $\mathcal{O}_t$ has characteristic polynomial congruent modulo 2 to $(x+a)^3$, where $a \in \{0,1\}$.

Consider next K_w. Elements of I_w have characteristic polynomials of the form

$$x^4 + 4Ax^3 + 4Bx^2 + 8Cx + 2D,$$

where $A, B, C, D \in \mathbf{Z}_2$. The numerical coefficients 4, 4, and 8 correspond to what we called Ore congruences in [JR1]; they come from the fact that totally ramified quartics can have discriminant as low as 2^4 while here our quartic discriminant is 2^{10}. To get all elements of $\mathcal{O}_w$, we again have to allow for a shift by 1. We get polynomials of the form

$$(x+E)^4 + 4A(x+E)^3 + 4B(x+E)^2 + 8C(x+E) + 2D, \tag{2.1}$$

where $A, B, C, D \in \mathbf{Z}_2$ and $E \in \{0,1\}$. Looking modulo 8, we have

$$(x+E)^4 + 4A(x+E)^3 + 4B(x+E)^2 + 2D,$$

where $A, B, E \in \{0,1\}$ and $D \in \{0,1,2,3\}$.

So, septic polynomials for fields with the given ramification structure at 2 are congruent modulo 8 to a polynomial of the form

$$\left((x+a)^3 + 2bx^2 + 2cx + 2d\right)\left((x+E)^4 + 4A(x+E)^3 + 4B(x+E)^2 + 2D\right),$$

where $a, A, B, E \in \{0,1\}$ and $b, c, d, D \in \{0,1,2,3\}$. Enumerating all such polynomials modulo 8, we find 2560 possibilities for the reduction of f modulo 8. However, the coefficient a_1 of x^6 is constrained to be simply 0, 1, 2, or 3 for a Hunter search. This cuts the list down to 1280 possibilities.

This number can be reduced a bit further by handling the wildly ramified extension more carefully. If D is odd and A is even in equation (2.1), then the polynomial necessarily defines a quartic extension of $\mathbf{Q}_2$ with discriminant 2^{11}, not 2^{10}. Eliminating those cases and carrying through as above brings the list to 1024 possibilities, 256 for each allowed value of a_1. This data is stored in one of our many 2-adic input files. Similarly, we created a 3-adic input file for each possible 3-adic ramification structure.

When conducting the actual search for a given combination of ramification structures at 2 and 3, the computer program starts by reading the appropriate file of 2-adic bounds and the appropriate file of 3-adic bounds. We then conduct Hunter searches using the method of [Poh] to deduce the archimedean bounds on coefficients. Note that there is still a remaining issue of how to combine the p-adic bounds with the archimedean ones.

Looking over all ramification structures, we have a total of $22,782$ congruence vectors modulo powers of 2 and $101,699$ congruence vectors modulo powers of 3. If we were to look directly at each possible combination and conduct a Hunter search for f congruent to the resultant $\bar{f}$ modulo $2^j 3^k$, we would have $2,316,906,618$ such searches. The run time would then be dominated by setting up many searches only to find that strong congruences on the coefficients of f were incompatible with the archimedean inequalities. In other words, we would have spent most of the computer time initializing searches which would end up looking at no polynomials. To avoid this, we adopted the following strategy.

A Hunter search program is built around nested loops, one for each a_i. For the outermost 3 loops, we simply loop through all values for (a_1, a_2, a_7) of a Hunter search for the targeted discriminant, skipping only those triples which are incompatible with all of our congruence vectors for 2 or for 3. Then, we find the congruence vectors modulo 2 and 3 which are consistent with the current values of a_1, a_2, and a_7. This takes very little time, since the input files of p-adic bounds have been appropriately presorted. Then, we step through the relatively few consistent pairs of congruence vectors, running for each pair a search for the remaining coefficients a_3, a_4, a_5, a_6.

The main program was written in C, making use of the *Pari* library. The searches were conducted on a Sun workstation, inspected approximately 700 million polynomials, and had a run time of approximately 13 hours. For comparison, we had our new programs rerun the sextic searches of [JR1]. While the original sextic searches took several months, the rerun searches took approximately 2 hours.

3. The ten fields

The ten fields are $\mathbf{Q}[x]/f_i(x)$ with defining polynomials $f_i(x)$ as in Table 3.1. We have chosen the defining polynomials to minimize the quantity r appearing in equation (3.1) below.

The Galois group corresponding to each of the ten fields is the full symmetric group S_7. Another common feature shared by the ten fields is that they all have the maximal possible number of complex places, namely three. Fields with no complex places are unramified at ∞, so in this sense all ten fields are maximally ramified at ∞. Table 3.2 presents more numeric invariants of the fields.

The column D of Table 3.2 gives the discriminant of each field. Because the fields all have an odd number of complex places, all the discriminants are negative. Somewhat surprisingly, the maximum locally allowed exponent at 2, namely 14, does not arise.

All 10 fields were found near the very beginning of the search. Also, the coefficients of our defining polynomials are quite small. The next block of three columns gives numerical quantities related to these two observations as follows. Given a degree n polynomial, $f(x) = x^n + a_1x^{n-1} + \cdots \in \mathbf{Z}[x]$, associate to it the traceless polynomial $g(x) = f(x - a_1/n) = x^n + 0x^{n-1} + \cdots \in \mathbf{Q}[x]$. Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ be the roots of f and let $\beta_1, \ldots, \beta_n \in \mathbf{C}$ be the roots of g. Define the root-length $\sqrt{T_2}$

Table 3.1. Defining polynomials

$$
\begin{aligned}
f_1(x) &= x^7 + x^6 + 3x^5 + 5x^4 + 2x^3 + 12x^2 + x + 7\\
f_2(x) &= x^7 - x^6 - x^4 - 2x^3 + 4x + 2\\
f_3(x) &= x^7 - 2x^6 + 6x^5 - 10x^4 + 14x^3 - 12x^2 + 4x + 4\\
f_4(x) &= x^7 + 2x^6 + 27x^3 + 54x^2 + 42x + 12\\
f_5(x) &= x^7 - 3x^6 + 9x^5 - 3x^4 + 12x - 36\\
f_6(x) &= x^7 - 3x^5 + x^4 + 3x^3 - 6x^2 - x + 9\\
f_7(x) &= x^7 - 2x^6 + 3x^5 + x^4 - 5x^3 + 12x^2 - 7x + 5\\
f_8(x) &= x^7 - 3x^6 + 3x^5 + 3x^4 + 3x^3 - 9x^2 - 11x - 3\\
f_9(x) &= x^7 - 3x^6 + 3x^5 + x^4 - 3x^3 - 3x^2 - 5x - 3\\
f_{10}(x) &= x^7 + 2x^6 + 3x^5 + 4x^4 + 5x^3 + 6x^2 + 7x + 8
\end{aligned}
$$

TABLE 3.2. Some invariants of the 10 fields

		root-length information			p-adic factors		p-adic slope content		
#	D	r	R	Earliness	2	3	2	3	GRD
1	$-2^9\ 3^{11}$	3.70	5.05	4,580	3_9^2	1_{11}^6	$322\mathit{0}$	$\frac{5}{2}2\mathit{10}$	51.41
2	$-2^9\ 3^{11}$	3.16	5.05	321,000	$1_9^4\ 3_0^1$	1_{11}^6	$\frac{7}{2}32\mathit{0}$	$\frac{5}{2}2\mathit{10}$	72.71
3	$-2^{13}3^9$	3.71	5.30	15,100	$1_{11}^4 1_2^3$	1_9^6	$43\mathit{1}0$	$2\frac{3}{2}\mathit{1}$	50.09
4	$-2^{13}3^{11}$	4.58	6.37	7,420	$1_{10}^4 1_3^2$	1_{11}^6	$\frac{7}{2}320$	$\frac{5}{2}2\mathit{10}$	72.71
5	$-2^{12}3^{11}$	4.89	6.01	318	$1_{10}^4 1_2^2$	1_{11}^6	$\frac{7}{2}320$	$\frac{5}{2}2\mathit{1}$	72.71
6	$-2^{11}3^8$	3.63	4.31	105	$1_{11}^4 3_0^1$	2_8^3	$430\mathit{0}$	$22\mathit{10}$	47.43
7	$-2^{10}3^{10}$	3.49	4.88	9,010	$1_{10}^4 3_0^1$	2_{10}^3	$\frac{7}{2}320$	$\frac{5}{2}\frac{3}{2}\mathit{10}$	64.35
8	$-2^{10}3^{10}$	3.99	4.88	251	1_{10}^6	2_{10}^3	$\frac{8}{3}\frac{8}{3}2\mathit{1}0$	$\frac{5}{2}\frac{3}{2}\mathit{10}$	48.21
9	$-2^{12}3^{10}$	3.48	5.48	206,000	$1_{10}^4 1_2^3$	1_{10}^6	$\frac{7}{2}32\mathit{1}0$	$\frac{9}{4}\frac{9}{4}\mathit{110}$	70.29
10	$-2^{12}3^{10}$	3.56	5.48	115,000	$1_9^4\ 1_3^2$	1_{10}^6	$\frac{7}{2}32$	$\frac{9}{4}\frac{9}{4}\mathit{110}$	66.35

and the reduced root-length r of f by

$$\sqrt{T_2} = \sqrt{\sum_{j=1}^{n} |\alpha_j|^2} \quad \text{and} \quad r = \sqrt{\sum_{j=1}^{n} |\beta_j|^2}. \tag{3.1}$$

These two quantities are related by

$$T_2 = \frac{a_1^2}{n} + r^2.$$

In Hunter searches one normally restricts attention to polynomials with $0 \leq a_1 \leq n/2$. Thus, the difference between the quantities $\sqrt{T_2}$ and r is rather small. It is standard in the literature to work with T_2. We will work instead with r, as it figures directly into (3.2) and (3.3) below.

Hunter's theorem, which is the theoretical foundation for archimedean bounds we use, asserts that any septic field with discriminant D has a defining polynomial $f(x) \in \mathbf{Z}[x]$ satisfying

$$r \leq R := \left(\frac{64|D|}{7}\right)^{1/12}. \tag{3.2}$$

Let V_r be the volume in traceless coefficient space of the image of the ball in traceless root space of radius r, notions being as in [JR2]. Then V_r grows as a constant times r^{27}. The exponent 27 comes from multiplying the $2n-2$ appearing in the first displayed equation of [JR2, §4] by the $(n+2)/4$ in [JR2, Prop 5.1] and specializing to the case $n = 7$; see also [Coh, p. 448] for a similar statement in terms of T_2. The table prints r and R to two decimal places, and

$$\text{Earliness} := (R/r)^{27} \tag{3.3}$$

to three significant figures. Note that if we had we decreased our cutoffs so as to decrease search volumes by a factor of say 100, we would have still found all 10 fields, as $100 < 105$. The discussion of this paragraph emphasizes that the main mathematical result of this paper is not that we have found 10 septic fields with discriminant $\pm 2^a 3^b$, but rather that we have proved there are no more. Also our discussion suggests that in other situations where complete searches would be prohibitively long, very much shorter searches have a good chance of finding most of the fields sought.

For exactly two of the ten fields, the polynomial in Table 3.1 which minimizes r^2 does not minimize T_2. In these cases, polynomials minimizing T_2 are

$$\begin{aligned} f_5^*(x) &= x^7 - x^6 + 18x^3 - 18x^2 - 24x - 48, \\ f_8^*(x) &= x^7 - 3x^5 + 2x^4 + 3x^3 - 12x^2 + 19x - 6. \end{aligned}$$

Numerical quantities compare as follows.

	T_2	$\lvert a_1\rvert$	r^2	Earliness
f_5^*	24.10	1	23.96	253
f_5	24.84	3	23.56	318
f_8^*	15.89	0	15.89	241
f_8	17.13	3	15.84	251

Thus the true earliness of fields 5 and 8 is not captured by what would be traditional defining polynomials, f_5^* and f_8^*.

Returning to our description of Table 3.2, we consider next the two columns under "p-adic factors," which give some basic 2-adic and 3-adic information about each field K. The p-adic algebra $K \otimes \mathbf{Q}_p$ factors as a product of p-adic fields. Each of the factor fields has an inertial degree f, a ramification degree e, and a discriminantal exponent c. We represent such a factor field by f_c^e. For example, a 1_0^1 would represent a factor isomorphic to $\mathbf{Q}_p$ itself, but we omit printing these degree one factors. Note that, rather remarkably, all ten fields factor 3-adically as a sextic field times $\mathbf{Q}_3$.

The next two columns give the slope content of the local algebras K_p, following the conventions of [JR1]. For example, the decomposition group of field 8 at 2 has the form $D_2 = S_4 \times S_2$ while the decomposition group D_3 has the form $C_3^2.C_4$. The slope-filtrations of these groups are

$$\begin{array}{lcccccccccl} \{e\} = I^\infty & \overset{2^2}{\subset} & I^{8/3} & \overset{2}{\subset} & I^2 & \overset{3}{\subset} & I^1 & \overset{2}{\subset} & I^0 = D_2, \\ \{e\} = I^\infty & \overset{3^2}{\subset} & I^{9/4} & & & \overset{2}{\subset} & I^1 & \overset{2}{\subset} & I^0 = D_3. \end{array} \tag{3.4}$$

A minimal subquotient I^s/I^t is associated with the slope s, and its order is given above the corresponding inclusion in (3.4). For all ten fields, the orders $|D_2|$ and $|D_3|$ are a power of 2 times a power of 3. If the slope s contributes $2^A 3^B$ to $|D_2|$, we print it A times in roman and B times in italics. If s contributes $2^A 3^B$ to $|D_3|$, then we print it A times in italics and B times in roman. So $s > 1$, which corresponds to wild ramification, can be printed only in roman; $s = 1$, which corresponds to tame ramification, can be printed only in italics; $s = 0$, corresponding to unramified extensions of $\mathbf{Q}_p$, can be printed in either.

Finally, the entry under GRD is the Galois root discriminant, i.e., $\Delta^{1/7!}$, where Δ is the discriminant of a Galois closure of the given septic field, the 7! arising as the order of the Galois group S_7. The GRD is a numerical invariant of ramification which lets one meaningfully compare fields with different Galois groups. It is of the form $2^\alpha 3^\beta$ with α and β weighted averages of the slopes just examined. Slope s

appears with weight

$$\frac{1}{|I^t|} - \frac{1}{|I^s|} = \frac{|I^s| - |I^t|}{|I^s||I^t|},$$

with t being the next greater appearing slope. In the case of field 8,

$$\begin{aligned} \alpha &= \frac{3}{4} \cdot \frac{8}{3} + \frac{1}{8} \cdot 2 + \frac{1}{24} \cdot 1 = 2\frac{1}{3}, \\ \beta &= \frac{8}{9} \cdot \frac{9}{4} + \frac{1}{18} \cdot 1 = 2\frac{1}{18}. \end{aligned}$$

So for field 8, the GRD is $2^\alpha 3^\beta \approx 48.21$. For a more detailed discussion of slope content and Galois root discriminants, see [JR1, §1.2]. Some more p-adic information on our ten fields is available in the format of [JR1] at [Jon].

Our main result can be compared with the main result of [Bru], which says that there are no nonsolvable septic fields with discriminant of the form $\pm 7^a$. The maximal locally-allowed GRD in our situation is $2^{37/12} 3^{13/6} \approx 91.61$, while the maximal locally-allowed GRD in the situation of [Bru] is $7^{83/42} \approx 46.78$. In the light of the fields found by our search and their GRD's, the fact that [Bru] found no nonsolvable fields is not too surprising. In fact, the lowest GRD found here is 47.43 from field 6, and the smallest GRD we are aware of for any S_7 field is 45.39.

4. Comparison with three-point covers

We investigated the question of whether any of the 10 septic fields with discriminant $\pm 2^a 3^b$ can be obtained from a family of fields arising from a three-point cover of $\mathbf{P}^1$. Interestingly, the only field which we able to produce in this way came from a degeneration of a degree 9 cover. After discussing this example, we consider degree 7 covers and explain in some detail why the most promising cover did not produce any of our 10 fields.

As a technical note, all the covers considered here have genus zero, and in fact correspond to maps from a projective line $\mathbf{P}^1_x$ with coordinate x to a projective line $\mathbf{P}^1_t$ with coordinate t, with ramification only above $t = 0,\ 1,\ \infty$, everything being defined over $\mathbf{Q}$. We restricted to genus zero because one cannot yet systematically compute covers in higher genus. However, moduli algebras as in [Mal2] tend to have large degrees in higher genus cases, and no higher genus cover looks particularly promising as a source for septic fields with discriminant $\pm 2^a 3^b$.

Consider the two-variable polynomial

$$f_t(x) = x^9 - 9tx + 8t. \tag{4.1}$$

Its discriminant with respect to x is

$$D_t = -2^{24} 3^{18} t^8 (t-1). \tag{4.2}$$

Since $D_1 = 0$, the polynomial $f_1(x)$ has a repeated root; in fact, its factorization into irreducibles is

$$f_1(x) = (x-1)^2 (x^7 + 2x^6 + 3x^5 + 4x^4 + 5x^3 + 6x^2 + 7x + 8).$$

The septic factor has polynomial discriminant $-2^{16} 3^{12}$, and the septic field that it defines is field 10 from the previous section. One couldn't ask for more simply-behaved coefficients, and this septic polynomial is an example of an abc-polynomial, in the sense of [Bor].

The polynomial $f_t(x)$ in (4.1) can be regarded as a family of separable nonic polynomials in $\mathbf{Q}[x]$, indexed by $t \in \mathbf{Q} - \{0, 1\}$. One sees from (4.2) that for some of these t, the discriminant of $f_t(x)$ has the form $\pm 2^a 3^b$. For other t, the polynomial discriminant will have an extra square factor but the algebra discriminant of $\mathbf{Q}[x]/f_t(x)$ will be of the form $\pm 2^a 3^b$. We report on this example in [Rob]: from 21 specialization points of the first type and 14 specialization points of the second type, one gets 25 S_9 fields and 10 A_9 fields with discriminant $\pm 2^a 3^b$, all distinct.

Similarly, many of the fields of degree ≤ 6 in [JR1] come from specializing families degenerating only at $t = 0$, 1, and ∞. For example, $x^4 - 4tx + 3t$ has discriminant $-2^8 3^3 t^3 (t-1)$. The 22 S_4 fields listed in [JR1], except the totally real one, come from specializing this family. For degree six examples with more complicated defining equations, see [Rob].

We expected to similarly find some of the 10 fields by septic covers. Using the computations in [Mal2] to eliminate many cases, we found that there are ten septic genus zero covers such that any septic genus zero cover which could possibly lead to a field of the sort we seek is a base change [Mal1] of one of these ten covers. We will describe the details of our investigation for only one of these covers, the one that came the closest to producing a septic field of discriminant $\pm 2^a 3^b$.

This cover is defined by the partition data

$$\begin{aligned} \lambda_0 &= 331, \\ \lambda_1 &= 2221, \\ \lambda_\infty &= 43. \end{aligned} \tag{4.3}$$

To find a corresponding polynomial, one looks for relatively prime polynomials $g_0(x)$, $g_1(x)$, $g_\infty(x)$ satisfying

$$g_0(x) + g_1(x) + g_\infty(x) = 0,$$

with the multiplicities of the roots of g_c being given by λ_c. The unique solution, up to a fractional linear change in the x-variable and a constant scale factor, is

$$\begin{aligned} g_0(x) &= (256x^2 - 448x + 189)^3 x, \\ g_1(x) &= -(4096x^3 - 8704x^2 + 5400x - 729)^2 (x-1), \\ g_\infty(x) &= 27(28x - 27)^3. \end{aligned}$$

The cover $F : \mathbf{P}^1_x \to \mathbf{P}^1_t$ is given by $F(x) = -g_0(x)/g_\infty(x)$. Our x-variable is normalized by the requirement that $F(c) = c$ for $c = 0$, 1, ∞. The corresponding family of septic polynomials is

$$f_t(x) = g_0(x) + t g_\infty(x), \tag{4.4}$$

which has discriminant

$$D_t = -2^{118} 3^{60} 7^7 t^4 (t-1)^3. \tag{4.5}$$

The problem is that one has to choose $t \in \mathbf{Q} - \{0, 1\}$ such that 7 is not a factor of the discriminant of $\mathbf{Q}[x]/f_t(x)$ and no extraneous primes are introduced. This problem with 7 is present in all nine other covers as well. For seven of the other nine covers, one has to get rid of 5 too.

Table 4.1 summarizes our p-adic analysis of the family (4.4). Consider first $t \in \mathbf{Q}_2 - \{0, 1\}$, defining a 2-adic algebra $\mathbf{Q}_2[x]/f_t(x)$. The specialization point t is 2-adically closest to one of three cusps $c = 0$, 1, ∞. Suppose it is m-close to 0, meaning it has the form $2^m u$ with u a unit and $m \geq 1$. Then the first data line of Table 4.1 gives d with $\mathrm{disc}(\mathbf{Q}_2[x]/f_t(x)) = 2^d$. The parentheses indicate repetition, e.g., $m = 9$, 10, 11 gives $d = 8$, 8, 6, etc. Suppose next that t is m-close to ∞,

TABLE 4.1. p-adic variation of the discriminantal exponents of $\mathbf{Q}_p[x]/f_t(x)$ with $f_t(x) = (256x^2 - 448x + 189)^3 x + 27t(28x - 27)^3$

gen	c	1	2	3	4	5	6	7	8	9	10
$p = 2$	0	12	6	10	8	4	(8	8	6)		
	1	13	(8	9, 11)							
	∞	13	(0	2	2)						
			(8, 4, 2	11	6, 4	11)					
$p = 3$	0	(10	10	6, 4)							
	1	7	6	(3	4)						
6	∞	(3, 1	5	5)							
		(0	3	2	3)						
$p = 7$	0	11	9	7	(5	3	5)				
	1	10	7	(4	5)						
5, 3	∞	12	11	10	9	8	7	(0	2	2)	
								(0	3	2	3)
$p \neq 2, 3, 7$	0	(4	4	0)							
	1	(3	0)								
0	∞	(2	2	0)							
		(3	2	3	0)						

meaning $t = u2^{-m}$, with $m \geq 1$. If $m = 1$, the table says $\mathrm{disc}(\mathbf{Q}_2[x]/f_t(x)) = 2^{13}$. If $m > 1$, then $\mathbf{Q}_2[x]/f_t(x)$ splits naturally into a cubic and a quartic. This splitting is reflected in the table by the use of two lines for ∞. The presence of several numbers in a slot on Table 4.1 indicates an ambiguity. For example, if $m \equiv 2\ (4)$, then the quartic algebra has discriminant 2^d with $d = 8$, 4 or 2. Finally, t may be a 2-adic unit, in which case $t = 1 + 2^m u$ for some $m \geq 1$. For the remaining primes, conventions are similar; however here $t \in \mathbf{Z}_p$ may reduce to an element of $\mathbf{F}_p - \{0, 1\}$, in which case we say that t is generic. Ramification at primes $p \neq 2, 3, 7$ is always tame, and the last block on Table 4.1 follows from general considerations of tame ramification. The blocks for primes $p = 2$, 3, and 7 reflect wild ramification, and needed to be computed directly from (4.4). We should add that we have not written out a proof of the correctness of these blocks. However, we are convinced that the information given is correct because of a number of theoretical consistencies, and because the data agrees with numerical computation of discriminants for a great many $t \in \mathbf{Q} - \{0, 1\}$.

Table 4.1 says that 7 does not divide the discriminant of $\mathbf{Q}[x]/f_t(x)$ iff t is m-close to ∞ for $m \equiv 7\ (12)$. We actually do not need the 2- and 3-blocks of Table 4.1 at the moment, because we are not trying to get rid of 2 or 3. Note, however, that they say that 2 and 3 cannot be eliminated. Similar analysis of the other nine covers says that one cannot eliminate 7 in one of them and cannot eliminate 5 in two others. So, besides the cover we are treating in detail, there are now only six more to consider.

Next, there is a global component to our analysis. Let $t \neq 0, 1$ be a rational number, and write it in lowest terms as $-au^3/cw^{12}$ with a, u, c, and w being integers with a cube-free and c twelfth-power-free. Define integers b and v by

$$au^3 + bv^2 + cw^{12} = 0 \tag{4.6}$$

and the requirement that b be square-free. Impose the normalization condition that u, v, and w are all positive and exactly one of a, b, and c is negative. Then the integers a, b, c, u, v, and w are all uniquely determined by t. The previous local analysis using the last block of Table 4.1 says that $\mathbf{Q}[x]/f_t(x)$ is ramified within $\{2,3,7\}$ iff

$$\text{all primes dividing } abc \text{ are in } \{2,3,7\}. \tag{4.7}$$

The local analysis using the 7-block says that 7 does not divide $\mathrm{disc}(\mathbf{Q}[x]/f_t(x))$ iff

$$7^7 \text{ exactly divides } c \text{ in (4.6)}. \tag{4.8}$$

We carried out a computer search which found four t-values corresponding to the following solutions of (4.6) satisfying (4.7) and (4.8):

$$\begin{array}{rlrl} 3\ \ 139^3 & -\ 2^4 & 745^2 & +\ \ 7^7 = 0, \\ 197^3 & -\ 2 & 2249^2 & +\ \ 3\ 7^7 = 0, \\ 2\ \ 113^3 & - & 3209^2 & +\ \ 3^2 7^7 = 0, \\ 2351^3 & + & 46745^2 & -\ 2^{11}3^2 7^7 = 0. \end{array} \tag{4.9}$$

The six other covers have, their analogs of (4.6)–(4.8). However, the form of these analogs makes global solutions seem much less likely than solutions to (4.6)–(4.8). For while 7 behaves similarly, in four cases one has to kill 5 also. Also the smaller two exponents in the analogs of (4.6) are always larger than the 2 and 3 in (4.6). The importance of having small exponents is clear from the solutions (4.9) found for our cover. At any rate, computer searches did not find any global solutions for these other six systems.

We know that for the t-values corresponding to (4.9), the algebra $\mathbf{Q}[x]/f_t(x)$ is ramified exactly at 2 and 3. However, unfortunately from the point of view of constructing septic fields, all four $f_t(x)$ are reducible over $\mathbf{Q}$, factoring into a linear factor and a sextic factor! Our experience with specializing three-point covers tells us these factorizations constitute extremely unusual behavior. Much more typical is the nonic case (4.1), where the 35 specialization points behave generically, as we discussed at the beginning of this section.

Note that the analysis just completed did not make use of our list of ten fields. Nor did it make use of of the 2- and 3-blocks on Table 4.1. But now let's compare Table 3.2 with Table 4.1. We will consider only the 3-adic information, because the situation is remarkably simple here. Note that fields 1–5 on Table 3.2 have discriminant of the form -2^a3^b with b either 9 or 11. But, by Table 4.1, specializations of our cover can have 3-adic discriminantal exponent only 1, 3, 4, 5, 6, 7, 8, or 10. More subtly, field 6 has $b = 8$, but 3-adically factors into a sextic field and $\mathbf{Q}_3$. But by Table 4.1, specializations of our cover with 3-adic discriminantal exponent 8 factor into a wild cubic field and a tame quartic field. Finally, fields 7–10 all have 3-adic discriminantal exponent $b = 10$ and factor into a sextic field and $\mathbf{Q}_3$. But one can check by a 3-adic Newton polygon calculation that any specialization of our cover with $b = 10$ factors 3-adically as a cubic field times a cubic field times $\mathbf{Q}_3$. In short, no specialization of our cover matches one of our 10 fields even 3-adically. So, while we were unable to produce any of the 10 fields by septic 3-point covers, the local analysis of the cover examined above combined with our classification of septic fields unramified outside $\{\infty, 2, 3\}$ explains why this cover unexpectedly split at our four specialization points.

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