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A NONCONFORMING COMBINATION OF THE FINITE ELEMENT AND VOLUME METHODS WITH AN ANISOTROPIC MESH REFINEMENT FOR A SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATION

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ABSTRACT. In this paper we formulate and analyze a discretization method for a 2D linear singularly perturbed convection-diffusion problem with a singular perturbation parameter ε. The method is based on a nonconforming combination of the conventional Galerkin piecewise linear triangular finite element method and an exponentially fitted finite volume method, and on a mixture of triangular and rectangular elements. It is shown that the method is stable with respect to a semi-discrete energy norm and the approximation error in the semi-discrete energy norm is bounded by $Ch\sqrt{\left|\frac{\ln \varepsilon}{\ln h}\right|}$ with C independent of the mesh parameter h, the diffusion coefficient ε and the exact solution of the problem.

1. INTRODUCTION

Many phenomena in engineering, physics and finance are governed by a convection-diffusion equation with a diffusion coefficient $1 > \varepsilon > 0$ which is much smaller than the average magnitude of the convection coefficient function. These equations are convection-dominated or singularly perturbed, and solutions to them normally have sharp boundary or interior layers so that applications of conventional numerical methods to these problems often yield solutions with nonphysical, spurious oscillations. To overcome this stability problem, many methods have been proposed. These include upwind methods (cf., for example, [5, 11, 12, 4, 2]), streamline diffusion methods (cf., for example, [13]) and exponentially fitted methods (cf., for example, [14, 15, 18, 23, 24]). However, no method guarantees, in general, that a numerical solution converges to the exact one uniformly in ε on an unstructured triangular partition. Several uniformly convergent schemes (cf., for example, [18, 22]) have been proposed and analyzed based on the Shishkin mesh technique (cf. [16]) and on piecewise uniform, structured partitions. Thus, these schemes can hardly be used for problems with nonrectangular geometries. Work on a least-squares finite element using a Gartland-type mesh is reported in [21]. Recently, Feistauer et al. ([7, 8, 9]) proposed and analyzed some semi-implicit and explicit schemes based on

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combinations of the finite element and finite volume methods for nonlinear time-dependent convection-diffusion problems. These methods have been successfully used in solving practical problems. A combination of the finite element and volume methods for a linear, stationary singularly perturbed convection-diffusion problem was also proposed and analyzed in [19]. All these combined methods are based on the idea that the convection and diffusion terms are discretized by a finite volume method and a finite element method, respectively. Although these methods are useful in practice, theoretically they do not guarantee that the errors in the approximate solutions do not deteriorate significantly as the diffusion coefficient goes to zero.

Intuitively, a solution to a singularly perturbed problem is smooth outside the boundary layers, as it is almost equal to the solution of the reduced problem (with $\varepsilon = 0$). Therefore, any classic method should be able to resolve this part of the solution satisfactorily. The solution shows sharp layers in a (small) subregion near part of the boundary of the solution domain, and so a sophisticated method, such as anisotropic mesh refinement, the Shishkin mesh technique, the exponential fitting technique, or a combination of these, needs to be used so as to accurately resolve that part of the solution. This suggests that a combination of classic and other methods can solve a singularly perturbed problem efficiently.

In this paper, we investigate a combination of the conventional piecewise linear triangular finite element method and the exponentially fitted finite element (or volume) method (cf. [14, 15]) with anisotropic meshes for a linear, time-dependent singularly perturbed convection-diffusion problem. In this approach, a solution region is divided into two subregions, of which one does not contain layers and the other does. The equation is then discretized by conventional piecewise linear finite elements on a regular triangulation of the first subregion and by exponentially fitted finite elements on an anisotropic mesh for the second subregion. These two discretizations are then coupled to each other by imposing continuity at the mesh nodes along the intersection of the boundaries of the two subregions. Since the finite element basis functions may not be continuous across the boundaries between the two subregions, the resulting finite element space is nonconforming. The stability of the method is proved, and an $O(h\sqrt{|\ln\varepsilon/\ln h|})$ upper bound for the error in the approximate solution in a semi-discrete energy norm is established. It is shown that the error bound is almost independent of ε. We comment that our approach is completely different from the coupled method in [19]. The latter is based on the discretization of the diffusion term by the finite element method and the convection term by a finite volume method. Also, [19] does not take into consideration the uniformity in ε, and thus the resulting error estimate depends strongly on ε. Although the problem considered in this paper is linear, the idea can be used for solving nonlinear convection-diffusion problems. This will be one of our future research topics. Theoretically the method needs a priori knowledge of the layer locations. When this information is not available a priori, a mesh adaption technique can be used in conjunction with the method so that layers' locations can be identified approximately from the numerical solution of the previous step.

The rest of the paper is organized as follows.

The continuous problem and some preliminaries are described in the next section. The finite element formulation is discussed in Section 3. This formulation is a combination of those of the conventional Galerkin piecewise linear finite element and the Petrov-Galerkin exponentially fitted finite element proposed in [14]. The finite element method is globally nonconforming. In Section 4, we will first show that the finite element is uniformly stable by proving that the corresponding bilinear form is coercive. Then, we will show that the approximation error in a semi-discrete energy norm is bounded by $O(h)$ almost uniformly in ε.

2. Preliminaries

Consider stationary, linear, convection-diffusion problems of the form

$$-\nabla \cdot \boldsymbol{f}_u + Gu = F \quad \text{in} \quad \Omega := (0,1)^2, \tag{1}$$

$$\boldsymbol{f}_u = \varepsilon \nabla u - \boldsymbol{a} u, \tag{2}$$

$$u|_{\partial\Omega} = 0, \tag{3}$$

where $\partial\Omega$ denotes the boundary of Ω, $\varepsilon > 0$ is a positive parameter, $\boldsymbol{a} = (a_1, a_2)$ is a known vector-valued function and F is a given function.

In what follows $L^p(S)$ denotes the space of p-integrable functions on an open and measurable set S with norm $\|\cdot\|_{0,p,S}$, and $W^{m,p}(\Omega)$ is the usual Sobolev space with norm $\|\cdot\|_{m,p,S}$ and the kth order seminorm $|\cdot|_{k,p,S}$ for any $1 \leq p < \infty$, nonnegative integer m and $0 \leq k \leq m$. Obviously $W^{0,p}(S) = L^p(S)$. When $S = \Omega$ we omit the subscript in the above notation. Furthermore, we let $H^m(\Omega) := W^{m,2}(\Omega)$, $||\cdot||_m := ||\cdot||_{m,2,\Omega}$ and $|\cdot|_k := |\cdot|_{k,2,\Omega}$. The inner product on $L^2(\Omega)$ or on $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^2$ is denoted by $(\cdot,\cdot)$. We put $H^1_0(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, and the set of functions which together with their derivatives of order $\leq m$ are continuous on Ω (or $\bar\Omega$) is denoted by $C^m(\Omega)$ (or $C^m(\bar\Omega)$).

For the coefficient functions we assume that $\boldsymbol{a} \in (C^1(\bar\Omega))^2$, $G \in C(\Omega) \cap H^1(\Omega)$ and $F \in L^\infty(\Omega)$. We also assume that $\boldsymbol{a}$ satisfies

$$\frac{1}{2}\nabla \cdot \boldsymbol{a} + G \geq \alpha > 0 \qquad \text{in} \quad \Omega \tag{4}$$

for some positive number α. This condition has been used in many papers and books on uniform convergence analysis, such as [2, 17, 18, 21]. The existence and uniqueness of the solutions to both the continuous and the finite element problem do not need this condition, but it will be used in the proof of the error estimates in Section 5. For simplicity, we assume that each of the components of $\boldsymbol{a}$ is bounded below by a positive constant, i.e.,

$$a_1 \geq \alpha_1 > 0, \qquad a_2 \geq \alpha_2 > 0 \qquad \text{in} \quad \Omega. \tag{5}$$

In this case, the solution to (1)–(3) has two exponential boundary layers of width $O(\varepsilon)$ at $x = 1$ and $y = 1$. The variational problem corresponding to (1), (2) and (3) is

Problem 2.1. Find $u \in H^1_0(\Omega)$ such that for all $v \in H^1_0(\Omega)$

$$A(u,v) = (F,v), \tag{6}$$

where $A(\cdot,\cdot)$ is a bilinear form on $(H^1_0(\Omega))^2$ defined by

$$A(u,v) = (\varepsilon\nabla u - \boldsymbol{a}u, \nabla v). \tag{7}$$

Let $||\cdot||_\varepsilon$ be a functional on $H^1_0(\Omega)$ defined by $||v||_\varepsilon = \big(A(v,v)\big)^{1/2}$. Then it is easy to show that (cf., for example, [14])

$$||v||^2_\varepsilon = (\varepsilon\nabla v, \nabla v) + ((\frac{1}{2}\nabla \cdot \boldsymbol{a} + G)v, v).$$

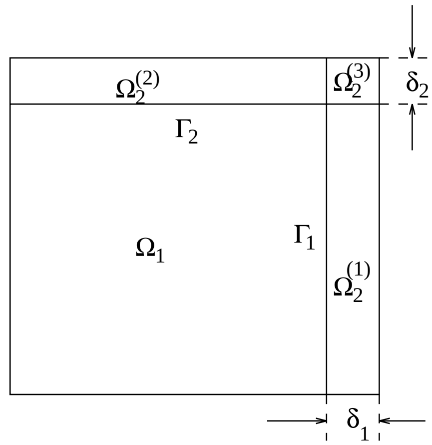

FIGURE 1. Subdomains Ω_1 and $\Omega_2 = \Omega_2^{(1)} \cup \Omega_2^{(2)} \cup \Omega_2^{(3)}$, $\Gamma = \Gamma_1 \cup \Gamma_2$.

Thus, $||\cdot||_\varepsilon$ is a norm on $H_0^1(\Omega)$, because $\frac{1}{2}\nabla\cdot\boldsymbol{a} + G \geq 0$ by (4) and $(\nabla u, \nabla v)$ is a norm on $H_0^1(\Omega)$ by the well-known Poincaré-Friedrichs inequality. Now, from the definition of the norm we have

$$A(u,u) = ||u||_\varepsilon^2, \qquad \forall u \in H_0^1(\Omega). \tag{8}$$

This implies that $A(\cdot,\cdot)$ is coercive on $H_0^1(\Omega)$, and thus, by the well-known Lax-Milgram lemma, Problem 2.1 has a unique solution in $H_0^1(\Omega)$.

Because of (5), the solution to Problem 2.1 has two boundary layers of width $O(\varepsilon)$ at $x = 1$ and $y = 1$ respectively. Thus we divide the solution region Ω into two parts Ω_1 and Ω_2, given respectively by

$$\Omega_1 = (0, 1-\delta_1) \times (0, 1-\delta_2), \qquad \Omega_2 = (1-\delta_1, 1) \times (0,1) \cup (0, 1-\delta_1) \times (1-\delta_2, 1),$$

with $\delta_1, \delta_2 \in (0,1)$ (cf. Figure 1). Obviously $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$. The region Ω_2 consists of the three subregions

$$\begin{aligned}
\Omega_2^{(1)} &= (1-\delta_1, 1) \times (0, 1-\delta_2),\\
\Omega_2^{(2)} &= (0, 1-\delta_1) \times (1-\delta_2, 1),\\
\Omega_2^{(3)} &= (1-\delta_1, 1) \times (1-\delta_2, 1).
\end{aligned}$$

The choice of the transition parameters δ_1 and δ_2 is rather arbitrary, but it is required that Ω_2 contains the boundary layers and $\delta_1, \delta_2 = O(\varepsilon)$. One choice is

$$\delta_1 = \frac{\beta}{\alpha_1}\varepsilon \ln(1/\varepsilon) \quad \text{and} \quad \delta_2 = \frac{\beta}{\alpha_2}\varepsilon \ln(1/\varepsilon), \tag{9}$$

where $\beta \geq 1$ is a positive constant (cf., for example, [18]). We let $\Gamma_1 = \bar{\Omega}_1 \cap \bar{\Omega}_2^{(1)}$ and $\Gamma_2 = \bar{\Omega}_1 \cap \bar{\Omega}_2^{(2)}$, and put $\Gamma = \Gamma_1 \cup \Gamma_2$. In the rest of this paper Γ is sometimes regarded as an oriented curve. If Γ is oriented counterclockwise, then it is denoted as Γ^+. Otherwise, we use Γ^- to denote it.

Now we need to make the following assumption on the solution to Problem 2.1:

Assumption 2.2. The solution u to Problem 2.1 has the representation

$$u = U_1 + U_2 + U_3 + U_4, \tag{10}$$

where U_1 satisfies

$$||U_1||_{k,\infty,\Omega} \leq C \qquad \text{for} \quad k = 0, 1, 2, \tag{11}$$

and U_2, U_3 and U_4 satisfy

$$\left|\frac{\partial^{i+j} U_2}{\partial x^i \partial y^j}\right| \leq C\varepsilon^{-i} \exp\left(-\frac{\alpha_1(1-x)}{\varepsilon}\right), \tag{12}$$

$$\left|\frac{\partial^{i+j} U_3}{\partial x^i \partial y^j}\right| \leq C\varepsilon^{-j} \exp\left(-\frac{\alpha_2(1-y)}{\varepsilon}\right), \tag{13}$$

$$\left|\frac{\partial^{i+j} U_4}{\partial x^i \partial y^j}\right| \leq C\varepsilon^{-(i+j)} \exp\left(-\frac{\alpha_1(1-x)}{\varepsilon}\right) \exp\left(-\frac{\alpha_2(1-y)}{\varepsilon}\right), \tag{14}$$

for $0 \leq i + j \leq 2$ and some positive constant C.

This assumption shows that the solution u is decomposed into 4 parts, U_i ($i = 1, 2, 3, 4$). The part U_1 is globally smooth and uniformly bounded, while U_2, U_3 and U_4 contain layers in $\Omega_2^{(1)}, \Omega_2^{(2)}$ and $\Omega_2^{(3)}$ respectively. Sufficient conditions for the existence of this decomposition have been discussed in various papers and books such as [6], [16] and [22], but necessary and sufficient conditions are unknown. The following theorem shows that u and all its first and second partial derivatives are uniformly bounded in Ω_1.

Theorem 2.3. *If $\beta \geq 3/2$, then the solution u to Problem 2.1 satisfies*

$$||u||_{i,\Omega_1} \leq C, \qquad i = 0, 1, 2, \tag{15}$$

for some positive constants C, independent of u. Furthermore, if $\beta \geq 2$, then

$$||u||_{i,\infty,\Omega_1} \leq C, \qquad i = 0, 1, 2. \tag{16}$$

Proof. Let C be a generic positive constant independent of u and ε. We first prove (15). We only show that $||u||_{2,\Omega_1} \leq C$. The other cases are similar to this one. From (10) we see that u is decomposed into the sum of U_i ($i = 1, 2, 3, 4$). Thus it suffices to show that $||U_i||_{2,\Omega_1} \leq C$ for $i = 1, 2, 3, 4$. We now prove that $||\partial^2 U_2/\partial x^2||_{0,\Omega_1} \leq C$.

From the definition of Ω_1 and (12) we have

$$\begin{aligned}
||\partial^2 U_2/\partial x^2||_{0,\Omega_1}^2 &\leq C\varepsilon^{-4} \int_{\Omega_1} \exp\left(-\frac{2\alpha_1}{\varepsilon}(1-x)\right) dx \\
&= \frac{C(1-\delta_2)}{\varepsilon^4} e^{-2\alpha_1/\varepsilon} \frac{e^{2\alpha_1 x/\varepsilon}}{2\alpha_1/\varepsilon}\Big|_0^{1-\delta_1} \\
&= \frac{C(1-\delta_2)}{2\alpha_1 \varepsilon^3} \left(e^{-2\alpha_1\delta_1/\varepsilon} - e^{-2\alpha_1/\varepsilon}\right) \\
&= C\frac{1-\delta_2}{2\alpha_1\varepsilon^3} \left(e^{-2\beta \ln \frac{1}{\varepsilon}} - e^{-2\alpha_1/\varepsilon}\right) \\
&= \frac{C(1-\delta_2)}{2\alpha_1} \left(\varepsilon^{2\beta-3} - \varepsilon^{-3} e^{-2\alpha_1/\varepsilon}\right).
\end{aligned}$$

Note that $\varepsilon^{-3}e^{-2\alpha_1/\varepsilon}$ is uniformly bounded above for all $0 < \varepsilon \leq O(1)$, since $\alpha_1 > 0$. Thus if $\beta \geq 3/2$, we have from the above inequality

$$||\partial^2 U_2/\partial x^2||^2_{0,\Omega_1} \leq C.$$

The proof of boundedness for the first and second seminorms of all other terms are analogous. Therefore we have shown (15).

We now show (16). Again we take $\partial^2 U_2/\partial x^2$ for example. Proofs for other terms are similar. For any fixed ε, the function in the right side of (12) is increasing in x. Thus we have

$$||\partial^2 U_2/\partial x^2||_{0,\infty,\Omega_1} \leq C\varepsilon^{-2}e^{-\alpha_1\delta_1/\varepsilon} = C\varepsilon^{\beta-2}.$$

The last term in the above is uniformly bounded for all $\beta \geq 2$. This completes the proof for (16). □

3. The finite element formulation

To formulate the finite element method, we first define a mesh for the solution region Ω which is a combination of regular triangles in Ω_1 and rectangles in Ω_2. Let $T_h^{(1)}$ denote a triangular mesh on Ω_1 with each triangle t having diameter $h_t \leq h$. For all $0 < h < \dim(\Omega_1)$, $\{T_h^{(1)}\}$ forms a family of triangular meshes on Ω_1. For any $t \in T_h^{(1)}$, let ρ_t denote the diameter of the incircle of t. Then, we assume that this family satisfies

$$\max_{t\in T_h^{(1)}} \frac{h_t}{\rho_h} \leq \gamma \qquad \forall h \in (0, \dim(\Omega_1))$$

for some positive constant γ. In this case, $\{T_h^{(1)}\}$ is said to be a regular family of triangular meshes. Now, the set of vertices of $T_h^{(1)}$ not on $\partial\Omega$ is denoted $\{x_i\}_1^{N_1}$. We assume that the number of vertices on Γ not on $\partial\Omega$ is N_Γ. Note that Ω_2 contains two thin overlapped stripes $\Omega_2^{(1)}\cup\Omega_2^{(3)}$ and $\Omega_2^{(2)}\cup\Omega_2^{(3)}$ with widths δ_1 and δ_2 respectively. Thus we divide these two strips into rectangles so that the resulting mesh is uniform along the x-axis with M_1 subintervals in the former subregion and along the y-axis with M_2 subintervals in the latter (cf. Figure 2). We also require that the mesh points on Γ match those from $T_h^{(1)}$. This mesh is denoted by $T_h^{(2)}$. Without loss of generality, we assume that the vertices in $T_h^{(2)}$ not on $\partial\Omega$ are numbered from N_1+1 to N_1+N_2. The set of edges of $T_h^{(2)}$ not on $\partial\Omega$ is denoted by $E_h^{(2)}$. Obviously all rectangles in $\Omega_2^{(1)}$ and $\Omega_2^{(2)}$ have lengths $O(h)$ and widths either δ_1/M_1 or δ_2/M_2, and rectangles in $\Omega_2^{(3)}$ have length δ_1/M_1 and width δ_2/M_2. The meshes $T_h^{(1)}$ and $T_h^{(2)}$ form a conforming mesh on Ω, and we denote it by T_h. A typical case is depicted in Figure 2.

Along with $T_h^{(2)}$, we define two meshes dual to it. The first dual mesh, denoted by $D_h^{(2)}$, is the Dirichlet tessellation associated with the mesh nodes in $T_h^{(2)}$, i.e., the element $d_i \in D_h^{(2)}$ associated with the node x_i of $T_h^{(2)}$ is given by

$$d_i = \{x \in \Omega_2 : |x - x_i| < |x - x_j|, i \neq j\} \tag{17}$$

for any other node x_j of the mesh $T_h^{(2)}$. For each edge in $T_h^{(2)}$ not on $\partial\Omega$, we construct a quadrilateral element by connecting the two end-points of the edge and the mid-points of the rectangles (or rectangle, if the edge is on Γ) sharing the edge (see Figure 2). All these quadrilaterals form the second dual mesh, denoted by

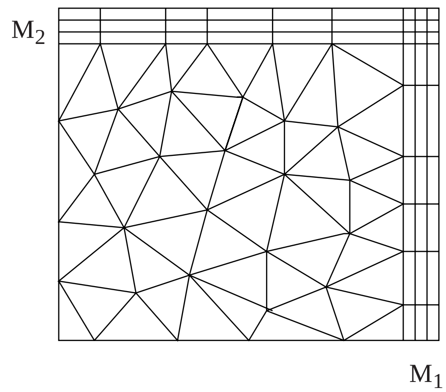

FIGURE 2. A typical hybrid mesh for Ω.

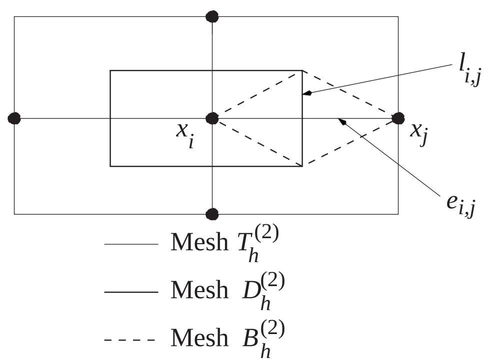

FIGURE 3. Elements and edges associated with the node x_i.

$B_h^{(2)}$. An example of these nested meshes is depicted in Figure 3. For a detailed description of the constructions of the dual meshes, we refer to [14], Section 3.

Using the meshes defined above, we now construct finite element trial and test spaces. Let $U_h^{(1)}$ denote the conventional piecewise linear finite element space of dimension N_1 constructed on the partition $T_h^{(1)}$. In our formulation below, we will use $U_h^{(1)}$ as both trial and test spaces in Ω_1. Corresponding to $T_h^{(2)}$, we construct a test space and a trial space in the same way as in Section 3 of [14]. The test space is chosen to be $V_h^{(2)} = \text{span}\{\xi_i\}$, where ξ_i is piecewise constant, given by

$$\xi_i = \begin{cases} 1 & \text{on } d_i, \\ 0 & \text{otherwise.} \end{cases}$$

To construct the trial space $U_h^{(2)}$, we follow the discussion in [14] based on the idea of exponential fitting proposed independently in [1] and [20]. For each $e_{i,j} \in E_h^{(2)}$ connecting the two neighbouring nodes x_i and x_j, we define an exponential function

$\phi_{i,j}$ on $e_{i,j}$ by

$$\frac{d}{d\boldsymbol{e}_{i,j}}(\varepsilon \frac{d\phi_{i,j}}{d\boldsymbol{e}_{i,j}} - \bar{a}_{i,j}\phi_{i,j}) = 0, \qquad \phi_{i,j}(x_i) = 1,\ \phi_{i,j}(x_j) = 0, \tag{18}$$

where $\boldsymbol{e}_{i,j}$ denotes the unit vector from x_i to x_j and $\bar{a}_{i,j}$ is a constant approximation to $\boldsymbol{a}\cdot\boldsymbol{e}_{i,j}$ on $e_{i,j}$ such that the mapping $\boldsymbol{a}\cdot\boldsymbol{e}_{i,j} \mapsto \bar{a}_{i,j}$ from $C(e_{i,j}) \mapsto P_0(e_{i,j})$ preserves constants (e.g., $\bar{a}_{i,j} = (\boldsymbol{a}(x_i)+\boldsymbol{a}(x_j))\cdot\boldsymbol{e}_{i,j}/2$), where $C(e_{i,j})$ and $P_0(e_{i,j})$ denote respectively the spaces of all continuous functions and all 0th order polynomials on $e_{i,j}$. The above linear, constant coefficient two-point boundary value problem can be solved exactly, yielding the local 1D basis function $\phi_{i,j}$ on the edge $e_{i,j}$. We then extend $\phi_{i,j}$ to $b_{i,j}$ by defining it to be constant along perpendiculars to $e_{i,j}$. Using this exponential function, we define a global basis function for $U_h^{(2)}$ on Ω as follows:

$$\phi_i = \begin{cases} \phi_{i,j} & \text{on } b_{i,j} \text{ if } j \in I_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $b_{i,j}$ denotes the element of B_h containing $e_{i,j}$ and

$$I_i = \{j : e_{i,j} \in E_h\} \tag{19}$$

denotes the index set of all neighbour nodes of x_i. The support of ϕ_i is star-shaped. We put $U_h^{(2)} = \text{span}\{\phi_i\}_1^{N'}$. Obviously we have $U_h^{(2)} \subset L^2(\Omega)$, and thus the trial space is nonconforming. This finite element space has the property that for any sufficiently smooth function u, the projection of the flux of the $U_h^{(2)}$-interpolant u_I of u on $\boldsymbol{e}_{i,j}$ satisfies

$$f_{i,j} := \varepsilon\frac{du_I}{d\boldsymbol{e}_{i,j}} - \bar{a}_{i,j}u_I = \frac{\varepsilon}{|e_{i,j}|}\big(B(\frac{\bar{a}_{i,j}|e_{i,j}|}{\varepsilon})u_j - B(-\frac{\bar{a}_{i,j}|e_{i,j}|}{\varepsilon})u_i\big) \tag{20}$$

on the edge $e_{i,j}$, where B denotes the Bernoulli function defined by

$$B(x) = \begin{cases} x/(e^x-1), & x \neq 0, \\ 1, & x = 1. \end{cases} \tag{21}$$

Also, the approximation error in $f_{i,j}$ satisfies

$$||\boldsymbol{f}_u\cdot\boldsymbol{e}_{i,j} - f_{i,j}||_{\infty,e_{i,j}} \leq C\big(|\boldsymbol{f}_u|_{1,\infty,e_{i,j}} + |\boldsymbol{a}|_{1,\infty,e_{i,j}}||u||_{\infty,e_{i,j}}\big), \tag{22}$$

where C is a positive constant independent of h, u and ε. For a detailed discussion, we refer to [14].

Now we choose the global trial and test spaces to be $U_h = U_h^{(1)} \oplus U_h^{(2)}$ and $V_h = U_h^{(1)} \oplus V_h^{(2)}$ respectively. Obviously both U_h and V_h are nonconforming, as they are not continuous across Γ. To define the discrete problem using U_h and V_h, it is convenient to introduce some notation. Let $l_{i,j} = \partial d_i \cap \partial d_j$ denote the intersection of the boundaries of d_i and d_j (cf. Figure 3). Clearly $|l_{i,j}| = \frac{2|b_{i,j}|}{|e_{i,j}|}$ if $j \in I_i$, and $|l_{i,j}| = 0$ otherwise. Corresponding to each $l_{i,j}$, we introduce a unit vector $\boldsymbol{l}_{i,j}$ directed so that $\arg(\boldsymbol{l}_{i,j}) = \arg(\boldsymbol{e}_{i,j}) + \pi/2$. For convenience we let $\hat{\boldsymbol{a}}$ be the approximation of $\boldsymbol{a}$ defined on $\bar{\Omega}_2$ such that, for all $b_{i,j} \in B_h$,

$$\hat{\boldsymbol{a}}|_{b_{i,j}} = \bar{a}_{i,j}\boldsymbol{e}_{i,j} + \bar{a}_{i,j}^{\perp}\boldsymbol{l}_{i,j}, \tag{23}$$

where $\bar{a}_{i,j}$ is the constant used in (18) and $\bar{a}_{i,j}^{\perp} = \sup_{x\in e_{i,j}} \boldsymbol{a}(x)\cdot\boldsymbol{l}_{i,j}$. Obviously, $\hat{\boldsymbol{a}}$ is a piecewise constant approximation to $\boldsymbol{a}$ on Ω_2. Note that the component $\bar{a}_{i,j}^{\perp}$ will make a contribution to the finite element formulation only on the integrals along

Γ^- that appear in the bilinear form $a_2(\cdot,\cdot)$ defined below. This contribution will be used in the proof of coercivity of the bilinear form in the next section. Before defining the finite element problem, we first introduce the mass lumping operator $P : C(\bar{\Omega}_2) \mapsto V_h^{(2)}$ such that

$$P(w)(x) = \sum_{i=N_1+1}^{N_2} w(x_i)\xi_i(x), \qquad x \in \bar{\Omega}_2, \tag{24}$$

for all $w \in C(\bar{\Omega}_2)$. When restricted to $U_h^{(2)}$, it is easy to show that the mapping P is one-to-one from $U_h^{(2)}$ to $V_h^{(2)}$. Using this mapping, we define the following finite element problem:

Problem 3.1. Find $u_h \in U_h$ such that

$$a(u_h, v_h) := a_1(u_h, v_h) + a_2(u_h, v_h) = (F, v_h), \qquad \forall v_h \in V_h, \tag{25}$$

where $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$ are bilinear forms on Ω_1 and Ω_2 defined by

$$a_1(u_h, v_h) = \int_{\Omega_1} (\varepsilon \nabla u_h - \boldsymbol{a} u_h) \cdot \nabla v_h dx + (G u_h, v_h)_{\Omega_1}, \tag{26}$$

$$a_2(u_h, v_h) = -\sum_{d \in D_h^{(2)}} \int_{\partial d \setminus \partial\Omega_2} (\varepsilon \nabla u_h - \hat{\boldsymbol{a}} u_h) \cdot \boldsymbol{n} v_h|_d ds + (P(G u_h), v_h)_{\Omega_2}. \tag{27}$$

Here $v_h|_d$ denotes the restriction of v_h to d.

It is clear that the bilinear form $a(\cdot,\cdot)$ is a combination of two terms. The former is the standard bilinear form on Ω_1, and the latter is a nonstandard Petrov-Galerkin form on Ω_2 originally defined in [14]. It has been shown in [14] that, using the mass lumping operator P defined in (24), the second bilinear form $a_2(u_h, v_h)$ can be transformed into a Bubnov-Galerkin formulation, yielding the following Bubnov-Galerkin problem corresponding to Problem 3.1:

Problem 3.2. Find $u_h \in U_h$ such that

$$b(u_h, v_h) = (F, v_h)_{\Omega_1} + (F, P(v_h))_{\Omega_2}, \qquad \forall v_h \in U_h, \tag{28}$$

where $(\cdot,\cdot)_{\Omega_k}$ denotes the inner product on $L^2(\Omega_k)$ for $k = 1, 2$ and $b(\cdot,\cdot)$ is a bilinear form on $U_h \times U_h$ defined by

$$b(u_h, v_h) := a_1(u_h, v_h) + a_2(u_h, P(v_h)). \tag{29}$$

4. Coercivity of the bilinear form $b(\cdot,\cdot)$

In this section we show that the bilinear form on the right side of (29) is coercive. This implies that Problem 3.2 is uniquely solvable and the finite element formulation is stable, irrespectively of ε. The coercivity result is also the foundation for the proof of convergence of the solutions to Problem 3.2. We now consider the two parts of $a(\cdot,\cdot)$ separately.

Before further discussion, we make the following assumption:

Assumption 4.1. Let the mesh $T_h^{(2)}$ be sufficiently fine such that the inequality

$$\frac{1}{2}\int_{\partial d_i} \hat{\boldsymbol{a}} \cdot \boldsymbol{n} ds + G(x_i)|d_i| \geq \alpha_0 > 0 \tag{30}$$

holds for all $d_i \in D_h^{(2)}$, where $\hat{\boldsymbol{a}}$ is the approximation of $\boldsymbol{a}$ defined in (23) and x_i denotes the mesh node contained in d_i.

We comment that (30) is essentially a discrete analogue of (4). In fact, it can be obtained by integrating (4) over $d_i \in D_h^{(2)}$, integrating the first term by parts, and then approximating $\boldsymbol{a}$ and G by $\hat{\boldsymbol{a}}$ and $G(x_i)$. Since the errors in the quadrature approximations are of order $O(h)$ and α in (4) is a positive constant, (30) will always be satisfied when h is sufficiently small.

Furthermore, since all the mesh lines in $T_h^{(2)}$ are parallel to one of the axes and $\boldsymbol{a}$ satisfies (5), it is obvious that

$$\min_{e_{i,j} \in E_h^{(2)}} |\hat{\boldsymbol{a}} \cdot \boldsymbol{e}_{i,j}| = \min_{e_{i,j} \in E_h^{(2)}} |a_{i,j}| \geq \min\{\alpha_1, \alpha_2\}. \tag{31}$$

Note that, when restricted to Ω_1, the test and trial spaces are equal to each other, and thus from (26) we have

$$\begin{aligned} a_1(u,u) &= \int_{\Omega_1} (\varepsilon \nabla u - \boldsymbol{a} u) \cdot \nabla u ds + \int_{\Omega_1} G u^2 dx \\ &= \varepsilon ||\nabla u||^2_{0,\Omega_1} - \int_{\Omega_1} \boldsymbol{a} u \cdot \nabla u ds + (Gu, u)_{\Omega_1} \end{aligned} \tag{32}$$

for any $u \in U_h$. Observing that $u\nabla u = \frac{1}{2}\nabla(u^2)$, a direct application of the Gauss-Green-Stokes formula then yields

$$-\frac{1}{2}\int_{\Omega_1} \boldsymbol{a} \cdot \nabla(u^2) dx = -\frac{1}{2}\int_{\partial\Omega_1} u^2 \boldsymbol{a} \cdot \boldsymbol{n} ds + \frac{1}{2} u^2 \nabla \cdot \boldsymbol{a} dx.$$

Substituting the above into (32), we obtain, using (4),

$$\begin{aligned} a_1(u,u) &= \varepsilon ||\nabla u||^2_{0,\Omega_1} + \frac{1}{2}(\nabla \cdot \boldsymbol{a} u, u)_{\Omega_1} - \frac{1}{2}\int_{\partial\Omega_1} u^2 \boldsymbol{a} \cdot \boldsymbol{n} ds \\ &= \varepsilon ||\nabla u||^2_{0,\Omega_1} + ((\frac{1}{2}\nabla \cdot \boldsymbol{a} + G) u, u)_{\Omega_1} - \frac{1}{2}\int_{\Gamma^+} u^2 \boldsymbol{a} \cdot \boldsymbol{n} ds \\ &\geq \varepsilon ||\nabla u||^2_{0,\Omega_1} + \alpha ||u||^2_{0,\Omega_1} - \frac{1}{2}\int_{\Gamma^+} u^2 \boldsymbol{a} \cdot \boldsymbol{n} ds, \end{aligned} \tag{33}$$

since $u = 0$ on $\partial\Omega_1 \setminus \Gamma^+$.

Now, for any $u \in U_h$ it is shown in Section 4 of [14] that

$$\begin{aligned} a_2(u, P(u)) = &- \sum_{d_i \in D_h^{(2)}} \int_{\partial d_i \setminus \partial\Omega_N} (\varepsilon \nabla u_h - \hat{\boldsymbol{a}} u_h) \cdot \boldsymbol{n} P(u_h) ds + \sum_{d_i \in D_h^{(2)}} G(x_i) u_i^2 |d_i| \\ = &\sum_{e_{i,j} \in E_h^{(2)}} \sigma_{i,j} B(\rho_{i,j})(1 + e^{\rho_{i,j}})(u_j - u_i)^2 \frac{|b_{i,j}|}{|e_{i,j}|} \\ &+ \sum_{e_{i,j} \in E_h^{(2)}} \frac{\bar{a}_{i,j}}{2}(u_i^2 - u_j^2)|l_{i,j}| + \sum_{d_i \in D_h^{(2)}} G(x_i) u_i^2 |d_i|, \end{aligned} \tag{34}$$

where $l_{i,j} = \partial d_i \cap \partial d_j$, $\sigma_{i,j} = \varepsilon/|e_{i,j}|$, $\rho_{i,j} = \bar{a}_{i,j}/\sigma_{i,j}$ and $B(\cdot)$ is the Bernoulli function defined in (21). Transforming from a summation over the edges to a summation over the nodes of $T_h^{(2)}$, the last two terms in the right-hand side of (34) can be written as

$$
\begin{aligned}
&\sum_{e_{i,j}\in E_h^{(2)}} \frac{\bar{a}_{i,j}}{2}(u_i^2-u_j^2)|l_{i,j}| + \sum_{d_i\in D_h^{(2)}} G(x_i)u_i^2|d_i| \\
&\quad = \frac{1}{2}\sum_{d_i\in D_h^{(2)}} u_i^2 \sum_{j\in I_i} \bar{a}_{i,j}|l_{i,j}| + \sum_{d_i\in D_h^{(2)}} G(x_i)u_i^2|d_i| \\
&\quad = \frac{1}{2}\sum_{d_i\in D_h^{(2)}} u_i^2 \int_{\partial d_i\setminus\Gamma} \hat{\boldsymbol{a}}\cdot\boldsymbol{n}ds + \sum_{d_i\in D_h^{(2)}} G(x_i)u_i^2|d_i| \\
&\quad = \sum_{d_i\in D_h^{(2)}} u_i^2\left(\frac{1}{2}\int_{\partial d_i}\hat{\boldsymbol{a}}\cdot\boldsymbol{n}ds + G(x_i)|d_i|\right) \\
&\qquad - \frac{1}{2}\sum_{d_i\in D_h^{(2)}} u_i^2\int_{\partial d_i\cap\Gamma^-}\hat{\boldsymbol{a}}\cdot\boldsymbol{n}ds \\
&\quad \geq \alpha_0\sum_{d_i\in D_h^{(2)}} u_i^2 - \frac{1}{2}\sum_{d_i\in D_h^{(2)}} u_i^2\int_{\partial d_i\cap\Gamma^-}\hat{\boldsymbol{a}}\cdot\boldsymbol{n}ds,
\end{aligned} \tag{35}
$$

because of (30).

Using the definition of the Bernoulli function $B(\cdot)$ in (21), we have

$$\sigma_{i,j}B(\rho_{i,j})(1+e^{\rho_{i,j}}) = \bar{a}_{i,j}\frac{e^{\rho_{i,j}}+1}{e^{\rho_{i,j}}-1} \geq |\bar{a}_{i,j}|.$$

Using this inequality, (35), and (34), we obtain

$$
\begin{aligned}
a_2(u,P(u)) \geq C&\left(\sum_{e_{i,j}\in E_h^{(2)}} |e_{i,j}|\left(\frac{u_j-u_i}{|e_{i,j}|}\right)^2|b_{i,j}| + \sum_{d_i\in D_h^{(2)}} u_i^2\right) \\
&- \frac{1}{2}\sum_{d_i\in D_h^{(2)}} u_i^2\int_{\partial d_i\cap\Gamma^-}\boldsymbol{a}\cdot\boldsymbol{n}ds.
\end{aligned} \tag{36}
$$

Let $||\cdot||$ be a functional on the finite element space U_h defined by

$$||v|| = \left(||v||^2_{\varepsilon,\Omega_1} + ||v||^2_{h,\Omega_2}\right)^{1/2}, \tag{37}$$

where

$$||v||^2_{\varepsilon,\Omega_1} = \varepsilon||\nabla v||^2_{0,\Omega_1} + ||v||^2_{0,\Omega_1}, \tag{38}$$

$$||v||^2_{h,\Omega_2} = \sum_{e_{i,j}\in E_h^{(2)}} |e_{i,j}|\left(\frac{v_j-v_i}{|e_{i,j}|}\right)^2|b_{i,j}| + \sum_{d_i\in D_h^{(2)}} v_i^2. \tag{39}$$

It is easy to see that $||\cdot||$ is a semi-discrete energy norm on U_h consisting of a continuous H^1-norm on Ω_1 and a discrete H^1-norm on Ω_2. Using this norm, we have the following theorem:

Theorem 4.2. *Let Assumption 4.1 be fulfilled. Then, for all $u\in U_h$,*

$$b(u,u) \geq C||u||^2, \tag{40}$$

where C denotes a generic positive constant independent of ε, h and u.

Proof. For any $u \in U_h$, using (33) and (36) we have

$$
\begin{aligned}
b(u,u) &= a_1(u,u) + a_2(u,P(u)) \\
&\geq \varepsilon||\nabla u||^2_{0,\Omega_1} + ||u||^2_{0,\Omega_1} - \frac{1}{2}\int_{\Gamma^+} u^2 \boldsymbol{a}\cdot\boldsymbol{n}ds \\
&\quad + C\left(\sum_{e_{i,j}\in E_h^{(2)}} |e_{i,j}| \left(\frac{u_j - u_i}{|e_{i,j}|}\right)^2 |b_{i,j}| + \sum_{d_i \in D_h^{(2)}} u_i^2\right) \\
&\quad - \frac{1}{2}\sum_{d_i\in D_h^{(2)}} u_i^2 \int_{\partial d_i \cap \Gamma^-} \hat{\boldsymbol{a}}\cdot\boldsymbol{n}ds \\
&\geq C||u||^2 - \frac{1}{2}\int_{\Gamma^+} u^2\boldsymbol{a}\cdot\boldsymbol{n}ds - \frac{1}{2}\sum_{d_i\in D_h^{(2)}} u_i^2 \int_{\partial d_i\cap\Gamma^-} \hat{\boldsymbol{a}}\cdot\boldsymbol{n}ds \\
&=: C||u||^2 + \frac{1}{2}J,
\end{aligned}
$$

where J denotes the sum of the two line integrals. It remains to show that J is nonnegative. Let $\bar{u}(x)$ denote the piecewise constant function on Γ defined by $\bar{u}(x) = u_i$ if $x \in \partial d_i \cap \Gamma$, i.e., $\bar{u}$ is the restriction of $P(u|_{\Omega_2})$ to Γ. Using $\bar{u}$ and transforming the second term of J from the summation over the boundaries of Dirichlet tiles to a summation over the edges of triangles, we have

$$
J = \sum_{k=1}^{N_\Gamma} \int_{\Gamma_k^+} (\hat{\boldsymbol{a}}\bar{u}^2 - \boldsymbol{a}u^2)\cdot\boldsymbol{n}ds,
$$

where Γ_k^+ denotes an edge of the mesh on Γ^+ and N_Γ is the number of edges on Γ defined in the previous section. Note that Γ^+ is the positive side of Γ, and its normal direction $\boldsymbol{n}$ is the same as the outward normal direction of Ω_1. Thus, $\boldsymbol{a}\cdot\boldsymbol{n} \geq \alpha_1$ or α_2 on Γ. Let us now consider the integral along a typical edge $\Gamma_k := e_{i,j}$ on Γ^+ connecting the two mesh vertices x_i and x_j. On Γ_k we have $\hat{\boldsymbol{a}}\cdot\boldsymbol{n} = \bar{a}_{i,j}^{\perp} = \sup_{x\in e_{i,j}} \boldsymbol{a}\cdot\boldsymbol{n}$ by (23), and u and $\bar{u}$ are defined by

$$
u = u_i + \frac{|x - x_i|}{|x_j - x_i|}(u_j - u_i) \tag{41}
$$

and

$$
\bar{u} = \begin{cases} u_i, & x_i \leq x < (x_i + x_j)/2, \\ u_j, & (x_i+x_j)/2 < x \leq x_j. \end{cases} \tag{42}
$$

Parametrizing $e_{i,j}$ as $\{s : 0 \leq s \leq |e_{i,j}|\}$, we have

$$
\begin{aligned}
J_{i,j} &:= \int_0^{|e_{i,j}|} (\bar{a}_{i,j}^{\perp}\bar{u}^2 - \boldsymbol{a}u^2\cdot\boldsymbol{n})ds \\
&= \bar{a}_{i,j}^{\perp}\int_0^{|e_{i,j}|}(\bar{u}^2 - u^2)ds + \int_0^{|e_{i,j}|}(a_{i,j}^{\perp} - \boldsymbol{a}\cdot\boldsymbol{n})u^2 ds \\
&\geq \bar{a}_{i,j}^{\perp}\int_0^{|e_{i,j}|}(\bar{u}^2 - u^2)ds,
\end{aligned} \tag{43}
$$

since $a_{i,j}^{\perp} \geq \boldsymbol{a} \cdot \boldsymbol{n} > 0$ on $e_{i,j}$. Using (41) and (42), it is easily seen that $\int_0^{|e_{i,j}|} \bar{u}^2 = (u_i^2 + u_j^2)\frac{|e_{i,j}|}{2}$ is equal to the numerical integral of $\int_0^{|e_{i,j}|} u^2 ds$ by the trapezoidal quadrature rule. Therefore, by a standard argument for the trapezoidal rule there exists $\xi \in (0, |e_{i,j}|)$ such that

$$\int_0^{|e_{i,j}|} (\bar{u}^2 - u^2) ds = \frac{1}{12} \frac{d^2(u^2)}{ds^2}\Big|_{s=\xi} |e_{i,j}|^3 = \frac{|e_{i,j}|}{6}(u_j - u_j)^2 \geq 0,$$

since $d^2(u^2)/ds^2 = 2(u_j - u_i)^2/|e_{i,j}|^2$ by (41). Combining the above bound with (43), we obtain $J_{i,j} \geq 0$, and so

$$J = \sum_{e_{i,j} \subset \Gamma^+} J_{i,j} \geq 0.$$

This completes the proof. $\square$

5. Convergence

In this section we establish an upper bound for $||u_I - u_h||$, where u_I and u_h denote respectively the U_h-interpolant of the solution u to Problem 2.1 and the solution to Problem 3.1. We start this discussion by proving the following lemma, which will be used in the proof of the main result of this section.

Lemma 5.1. *Let Assumptions 2.2 be fulfilled. If $\beta \geq 3$ in* (9) *and $M_1 = M_2 = M$, a positive integer, then, for any element edge $e_{i,j} \in E_h^{(2)}$, there exists a positive integer C, independent of h, u and ε, such that*

$$\int_{l_{i,j}} |\boldsymbol{f}_u \cdot \boldsymbol{e}_{i,j} - f_{i,j}| ds \quad \leq \quad \begin{cases} C|l_{i,j}| h K_1, & e_{i,j} \subset \bar{\Omega}_2^{(1)} \cup \bar{\Omega}_2^{(2)}, \\ C|l_{i,j}| M^{-1} \ln(1/\varepsilon), & e_{i,j} \subset \Omega_2^{(3)}, \end{cases} \tag{44}$$

where $\boldsymbol{f}_u$ and $f_{i,j}$ are defined in (2) *and* (20) *respectively and*

$$K_1 = \max\{1, h^{-1}\varepsilon^{\beta/2M}, h^{-1}\varepsilon \ln \frac{1}{\varepsilon}\}. \tag{45}$$

Proof. From the construction of the mesh on Ω_2 we see that the mesh lines are parallel to either the x-axis or the y-axis. We now discuss these two cases separately.

Case 1: $\boldsymbol{e}_{i,j} = (1, 0)$. In this case $e_{i,j}$ and $l_{i,j}$ are parallel to the x-axis and y-axis respectively, and have lengths either $M^{-1}\varepsilon \ln \frac{1}{\varepsilon}$ and $O(h)$ respectively (if $e_{i,j} \subset \Omega_2^{(1)} \cup \Omega_2^{(3)}$), or $O(h)$ and $M^{-1}\varepsilon \ln \frac{1}{\varepsilon}$ respectively (if $e_{i,j} \subset \Omega_2^{(2)}$).

$$\begin{aligned} &\int_{l_{i,j}} |\boldsymbol{f}_u \cdot \boldsymbol{e}_{i,j} - f_{i,j}| ds \\ &\leq \int_{l_{i,j}} [|(\boldsymbol{f}_u - \boldsymbol{f}_u(x_{i,j})) \cdot \boldsymbol{e}_{i,j}| + |\boldsymbol{f}_u(x_{i,j}) \cdot \boldsymbol{e}_{i,j} - f_{i,j}|]\, ds \\ &\leq C\left(|l_{i,j}| \int_{l_{i,j}} |\frac{\partial \boldsymbol{f}_u \cdot \boldsymbol{e}_{i,j}}{\partial y}| ds + |l_{i,j}||e_{i,j}| \cdot ||\frac{\partial \boldsymbol{f}_u \cdot \boldsymbol{e}_{i,j}}{\partial x}||_{0,b_{i,j},\infty}\right) \\ &\leq C|l_{i,j}|\left[\int_{l_{i,j}} |\varepsilon \frac{\partial^2 u}{\partial x \partial y} - a_1 \frac{\partial u}{\partial y}| ds + |e_{i,j}| \cdot ||\varepsilon \frac{\partial^2 u}{\partial x^2} - a_1 \frac{\partial u}{\partial x}||_{0,b_{i,j},\infty}\right] \\ &=: C|l_{i,j}|(I_1 + I_2), \end{aligned} \tag{46}$$

where $x_{i,j} := (x_i + x_j)/2$. We discuss the two terms I_1 and I_2 in (46). From (1) and (2) we have

$$-\varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + a_1\frac{\partial u}{\partial x} + a_2\frac{\partial u}{\partial y} + (\nabla\cdot\boldsymbol{a} + G)u = F.$$

Using this, Assumption 2.2 and the regularity assumptions on $\boldsymbol{a}, G$ and F, we get

$$\begin{aligned}&||\varepsilon\frac{\partial^2 u}{\partial x^2} - a_1\frac{\partial u}{\partial x}||_{0,b_{i,j},\infty}\\ &\qquad\le ||\varepsilon\frac{\partial^2 u}{\partial y^2} - a_2\frac{\partial u}{\partial y}||_{0,b_{i,j},\infty} + ||(\nabla\cdot\boldsymbol{a} + G)u||_{0,b_{i,j},\infty} + ||F||_{0,b_{i,j},\infty} \le C,\end{aligned}$$

if $e_{i,j} \subset \Omega_2^{(1)}$. By Theorem 2.3, $||\varepsilon\frac{\partial^2 u}{\partial x^2} - a_1\frac{\partial u}{\partial x}||_{0,b_{i,j},\infty}$ is also bounded above by C when $e_{i,j} \subset \Omega_2^{(2)}$, because $b_{i,j}$ is away from the layer in which the partial derivatives of u with respect to x are bounded. When $e_{i,j} \subset \Omega_2^{(3)}$, from Assumption 2.2 we have

$$||\varepsilon\frac{\partial^2 u}{\partial x^2} - a_1\frac{\partial u}{\partial x}||_{0,b_{i,j},\infty} \le C(\varepsilon\varepsilon^{-2} + \varepsilon^{-1}) = C\varepsilon^{-1}.$$

Therefore, combining all the above estimates for $||\varepsilon\frac{\partial^2 u}{\partial x^2} - a_1\frac{\partial u}{\partial x}||_{0,b_{i,j},\infty}$ on $\Omega_2^{(k)}, k = 1,2,3$, and noticing that $|e_{i,j}| = O(M^{-1}\varepsilon\ln(1/\varepsilon))$ in $\Omega_2^{(1)}$ and $\Omega_2^{(3)}$ and $O(h)$ in $\Omega_2^{(2)}$, we obtain

$$I_2 = |e_{i,j}|\cdot||\varepsilon\frac{\partial^2 u}{\partial x^2} - a_1\frac{\partial u}{\partial x}||_{0,b_{i,j},\infty} \le \begin{cases} CM^{-1}\varepsilon\ln(1/\varepsilon), & e_{i,j} \subset \Omega_2^{(1)},\\ Ch, & e_{i,j} \subset \Omega_2^{(2)},\\ CM^{-1}\ln(1/\varepsilon), & e_{i,j} \subset \Omega_2^{(3)}.\end{cases} \tag{47}$$

We now consider I_1. When $e_{i,j} \subset \Omega_2^{(1)}$ we have $|l_{i,j}| = O(h)$ and so, from Assumption 2.2,

$$I_1 \le C|l_{i,j}|(\varepsilon\varepsilon^{-1} + 1) \le Ch.$$

If $e_{i,j} \subset \Omega_2^{(2)}$, then $l_{i,j}$ is perpendicular to the boundary segment $y = 1$ and has length $\frac{\beta}{\alpha_2}M^{-1}\varepsilon\ln(1/\varepsilon)$. So after a proper parametrization and using Assumption 2.2 we have

$$\begin{aligned} I_1 &\le C\varepsilon^{-1}\int_{y_*}^{y_*+|l_{i,j}|} e^{-\alpha_2(1-y)/\varepsilon}dy\\ &= C\varepsilon^{-1}e^{-\alpha_2/\varepsilon}\frac{e^{\alpha_2 y/\varepsilon}}{\alpha_2/\varepsilon}\Big|_{y_*}^{y_*+|l_{i,j}|}\\ &\le Ce^{-\alpha(1-y_*)/\varepsilon}\left(e^{\alpha|l_{i,j}|/\varepsilon} - 1\right),\end{aligned} \tag{48}$$

where y_* denotes the lower end point of $l_{i,j}$. From the construction of the mesh we see that $1 - y_* \ge 3|l_{i,j}|/2$, i.e., the y-coordinate of the segment $l_{i,j}$ closest to $y = 1$ ranges from $1 - 3|l_{i,j}|/2$ to $1 - |l_{i,j}|/2$, where $|l_{i,j}| = \delta_2/M$. Therefore, from (48) we have

$$\begin{aligned} I_1 &\leq Ce^{-3\alpha|l_{i,j}|/(2\varepsilon)}\left(e^{\alpha|l_{i,j}|/\varepsilon}-1\right)\\ &\leq C\left(e^{-\alpha|l_{i,j}|/(2\varepsilon)}-e^{-3\alpha|l_{i,j}|/(2\varepsilon)}\right)\\ &\leq C\left(\varepsilon^{\beta/2M}-\varepsilon^{3\beta/2M}\right)\\ &\leq C\varepsilon^{\beta/2M}. \end{aligned}$$

When $e_{i,j}\subset\Omega_2^{(3)}$, from Assumption 2.2 we have

$$I_1 \leq C(\varepsilon\varepsilon^{-2}+\varepsilon^{-1})|l_{i,j}| \leq C\varepsilon^{-1}|l_{i,j}| \leq CM^{-1}\ln(1/\varepsilon).$$

Combining the three cases, we obtain

$$I_1 \leq \begin{cases} Ch, & e_{i,j}\subset\Omega_2^{(1)},\\ C\varepsilon^{\beta/2M}, & e_{i,j}\subset\Omega_2^{(2)},\\ CM^{-1}\ln(1/\varepsilon), & e_{i,j}\subset\Omega_2^{(3)}. \end{cases} \tag{49}$$

Finally, substituting (49) and (47) into (46), we obtain

$$\begin{aligned} \int_{l_{i,j}}|\boldsymbol{f}_u\cdot\boldsymbol{e}_{i,j}-f_{i,j}|ds &\leq \begin{cases} C|l_{i,j}|(h+M^{-1}\varepsilon\ln(1/\varepsilon)), & e_{i,j}\subset\bar\Omega_2^{(1)},\\ C|l_{i,j}|(\varepsilon^{\beta/2M}+h), & e_{i,j}\subset\bar\Omega_2^{(2)},\\ C|l_{i,j}|M^{-1}\ln(1/\varepsilon), & e_{i,j}\subset\Omega_2^{(3)}, \end{cases}\\ &\leq \begin{cases} C|l_{i,j}|hK_1, & e_{i,j}\subset\bar\Omega_2^{(1)}\cup\bar\Omega_2^{(2)},\\ C|l_{i,j}|M^{-1}\ln(1/\varepsilon), & e_{i,j}\subset\Omega_2^{(3)}, \end{cases} \end{aligned} \tag{50}$$

where K_1 is defined in (45).

Case 2: $\boldsymbol{e}_{i,j}=(0,1)$. From the symmetry of the problem it is easily seen that, by the same argument as in Case 1, the estimate (44) is also satisfied. This completes the proof. □

The error estimate for the method is established in the following theorem.

Theorem 5.2. *Let Assumptions 2.2 and 4.1 be fulfilled. If $\beta\geq 3$ in (9) and $M_1=M_2=M$, a positive integer, then there exists a positive integer C, independent of h, u and ε, such that*

$$||u_I-u_h|| \leq Ch\left(K_1^{1/2}+K_2\right), \tag{51}$$

where u_I and u_h denote respectively the U_h-interpolation of the solution u to Problem 2.1 and the solution to Problem 3.1, K_1 is defined in (45) and K_2 is defined as

$$K_2=\max\{M^{1/2}K_1, M^{-1/2}\sqrt{\varepsilon}\ln\frac{1}{\varepsilon}\}. \tag{52}$$

Proof. Note that Problems 3.1 and 3.2 are equivalent to each other. For any $v_h \in V_h$, subtracting $a(u_I, v_h)$ from both sides of (25), we have

$$a(u_h - u_I, v_h) = (F, v_h) - a(u_I, v_h). \tag{53}$$

For $k = 1, 2$, let $v_h^{(k)} = v_h|_{\Omega_k}$. Clearly, $v_h^{(1)}$ and $v_h^{(2)}$ are in $V_h^{(1)}$ and $V_h^{(2)}$ respectively. Multiplying (1) by v_h and integrating by parts, we get

$$\begin{aligned}
(F, v_h) &= (\boldsymbol{f}_u, \nabla v_h^{(1)})_{\Omega_1} + (Gu, v_h^{(1)})_{\Omega_1} - \sum_{d \in D_h^{(2)}} \int_{\partial d \setminus \Gamma^-} v_h^{(2)} \boldsymbol{f}_u \cdot \boldsymbol{n} ds \\
&\quad + (Gu, v_h^{(2)})_{\Omega_2} - \int_{\Gamma^-} \boldsymbol{f}_u \cdot \boldsymbol{n}(v_h^{(2)} - v_h^{(1)}) ds \\
&= a(u, v_h) + (Gu - P(Gu), v_h^{(2)})_{\Omega_2} - \int_{\Gamma^-} \boldsymbol{f}_u \cdot \boldsymbol{n}(v_h^{(2)} - v_h^{(1)}) ds,
\end{aligned}$$

where P is the mapping defined in (24). Substituting the above into (53) gives

$$\begin{aligned}
& a(u_h - u_I, v_h) \\
&\quad = a(u - u_I, v_h) + (Gu - P(Gu), v_h^{(2)})_{\Omega_2} - \int_{\Gamma^-} \boldsymbol{f}_u \cdot \boldsymbol{n}(v_h^{(2)} - v_h^{(1)}) ds \\
&\quad \le \left| \int_{\Omega_1} (\boldsymbol{f}_u - \boldsymbol{f}_{u_I}) \cdot \nabla v_h^{(1)} dx + (Gu - Gu_I, v_h^{(1)})_{\Omega_1} \right| \\
&\qquad + \left| \int_{\Gamma^-} \boldsymbol{f}_u \cdot \boldsymbol{n}(v_h^{(2)} - v_h^{(1)}) ds \right| \\
&\qquad + \left| \sum_{d \in D_h^{(2)}} v_h^{(2)} (\boldsymbol{f}_u - \boldsymbol{f}_{u_I}) \cdot \boldsymbol{n} ds \right| + |(Gu - P(Gu), v_h^{(2)})_{\Omega_2}| \\
&\quad =: R_1 + R_2 + R_3 + R_4.
\end{aligned} \tag{54}$$

We now consider the error terms R_1, R_2, R_3 and R_4 separately. Notice that $u_I|_{\Omega_1}$ is the piecewise linear interpolant of u on the triangular mesh $T_h^{(1)}$. Using the standard argument, we have

$$\begin{aligned}
R_1 &= \left| \int_{\Omega_1} (\varepsilon \nabla (u - u_I) - \boldsymbol{a}(u - u_I)) \cdot \nabla v_h^{(1)} dx + (Gu - Gu_I, v_h^{(1)})_{\Omega_1} \right| \\
&\le C \left(h\varepsilon + h^2 \right) ||u||_{2,\Omega_1} ||\nabla v_h^{(1)}||_{0,\Omega_1} + Ch^2 ||u||_{2,\Omega_1} ||v_h^{(1)}||_{0,\Omega_1} \\
&\le Ch \left(\varepsilon ||\nabla v_h^{(1)}||_{0,\Omega_1} + ||v_h^{(1)}||_{0,\Omega_1} \right) ||u||_{2,\Omega_1} \\
&\le Ch \left(\sqrt{\varepsilon} ||\nabla v_h^{(1)}||_{0,\Omega_1} + ||v_h^{(1)}||_{0,\Omega_1} \right) \\
&= Ch ||v_h^{(1)}||_{\varepsilon,\Omega_1}.
\end{aligned} \tag{55}$$

In the above we used the inequality $h||\nabla v_h^{(1)}||_{0,\Omega_1} \le C||v_h^{(1)}||_{0,\Omega_1}$ and (15).

Let us consider R_2. This term is due to the discontinuity or nonconformity of the test space across Γ. For this term we have

$$
\begin{aligned}
R_2 &= \left| \int_{\Gamma^-} \boldsymbol{f}_u \cdot \boldsymbol{n} (v_h^{(2)} - v_h^{(1)}) ds \right| \\
&= \left| \sum_{e_{i,j} \subset \Gamma} \int_{e_{i,j}} \boldsymbol{f}_u \cdot \boldsymbol{n} (v_h^{(2)} - v_h^{(1)}) ds \right|,
\end{aligned} \tag{56}
$$

where $e_{i,j} \in E_h$ denotes the edge connecting x_i and x_j. On $e_{i,j}$, $v_h^{(1)}$ is linear and $v_h^{(2)}$ is piecewise constant. They are in the form of (41) and (42) respectively. For clarity, we let $g(x) = \boldsymbol{f}_u \cdot \boldsymbol{n}$ and $G(x) = \int g dx$. We also put $v_i := v_h^{(1)}(x_i) = v_h^{(2)}(x_i)$ and $v_j := v_h^{(1)}(x_j) = v_h^{(2)}(x_j)$. Parametrizing $e_{i,j}$ as $\{s : 0 \le s \le |e_{i,j}|\}$ and integrating by parts, we have

$$
\begin{aligned}
\int_{e_{i,j}} \boldsymbol{f}_u \cdot \boldsymbol{n} (v_h^{(2)} - v_h^{(1)}) ds &= \int_0^{\frac{|e_{i,j}|}{2}} g (v_h^{(2)} - v_h^{(1)}) ds + \int_{\frac{|e_{i,j}|}{2}}^{|e_{i,j}|} g (v_h^{(2)} - v_h^{(1)}) ds \\
&= G(\frac{|e_{i,j}|}{2})(\frac{v_i + v_j}{2} - v_i) - \frac{v_j - v_i}{|e_{i,j}|} \int_0^{\frac{|e_{i,j}|}{2}} G ds \\
&\quad - G(\frac{|e_{i,j}|}{2})(\frac{v_i + v_j}{2} - v_j) - \frac{v_j - v_i}{|e_{i,j}|} \int_{\frac{|e_{i,j}|}{2}}^{|e_{i,j}|} G ds \\
&= \frac{v_j - v_i}{|e_{i,j}|} \left[\int_0^{|e_{i,j}|} G(s) ds - |e_{i,j}| G(\frac{|e_{i,j}|}{2}) \right].
\end{aligned} \tag{57}
$$

The last term of (57) is essentially the error in the numerical integration of G by the mid-point quadrature rule. Using Taylor expansion, it is easy to show that

$$
\left| \int_0^{|e_{i,j}|} G(s) ds - |e_{i,j}| G(\frac{|e_{i,j}|}{2}) \right| \le C ||G||_{2,\infty} |e_{i,j}|^3 \le C ||\boldsymbol{f}_u||_{1,\infty} |e_{i,j}|^3.
$$

Substituting the above into (57) and then the result into (56), we obtain

$$
R_2 \le C \sum_{e_{i,j} \subset \Gamma} |v_j - v_i| |e_{i,j}|^3 ||\boldsymbol{f}_u||_{1,\infty} \le C h^2 ||v_h||_\infty, \tag{58}
$$

since $||\boldsymbol{f}_u||_{1,\infty} \le C$ on Γ.

We now consider R_3. Analogously to the derivation of (35), transforming from a summation over the edges to a summation over the nodes of $T_h^{(2)}$ gives

$$
\begin{aligned}
R_3 &= \left| \sum_{e_{i,j} \in E_h^{(2)}} (v_i - v_j) \int_{l_{i,j}} (\boldsymbol{f}_u - \boldsymbol{f}_{u_I}) \cdot \boldsymbol{n} ds \right| \\
&\le \sum_{e_{i,j} \in E_h^{(2)}} |v_i - v_j| \int_{l_{i,j}} |\boldsymbol{f}_u \cdot \boldsymbol{n} - f_{i,j}| ds.
\end{aligned} \tag{59}
$$

Combining (44) and (59) and noting that $|l_{i,j}| = 2|b_{i,j}|/|e_{i,j}|$, we have

$$
\begin{aligned}
R_3 &\le \sum_{e_{i,j}\in E_h^{(2)}} |v_i - v_j| \int_{l_{i,j}} |\boldsymbol{f}_u \cdot \boldsymbol{e}_{i,j} - f_{i,j}| ds \\
&\le ChK_1 \sum_{e_{i,j}\subset\bar\Omega_2^{(1)}\cup\bar\Omega_2^{(2)}} |v_i - v_j| \frac{|b_{i,j}|}{|e_{i,j}|} + \frac{C}{M}\ln\frac{1}{\varepsilon} \sum_{e_{i,j}\subset\bar\Omega_2^{(3)}} |v_i - v_j| \frac{|b_{i,j}|}{|e_{i,j}|} \\
&\le ChK_1 \left(\sum_{e_{i,j}\in\bar\Omega_2^{(1)}\cup\bar\Omega_2^{(2)}} \left(\frac{v_i - v_j}{|e_{i,j}|}\right)^2 |b_{i,j}| \right)^{1/2} \left(\sum_{e_{i,j}\subset\bar\Omega_2^{(1)}\cup\bar\Omega_2^{(2)}} |b_{i,j}| \right)^{1/2} \\
&\quad + \frac{C}{M}\ln\frac{1}{\varepsilon} \left(\sum_{e_{i,j}\subset\bar\Omega_2^{(3)}} \left(\frac{v_i - v_j}{|e_{i,j}|}\right)^2 |b_{i,j}| \right)^{1/2} \left(\sum_{e_{i,j}\subset\bar\Omega_2^{(3)}} |b_{i,j}| \right)^{1/2} \\
&\le ChK_1\sqrt{\varepsilon}\ln^{1/2}\frac{1}{\varepsilon} \left(\sum_{e_{i,j}\subset\bar\Omega_2^{(1)}\cup\bar\Omega_2^{(2)}} \left(\frac{v_i - v_j}{|e_{i,j}|}\right)^2 |b_{i,j}| \right)^{1/2} \\
&\quad + \frac{C}{M}\varepsilon\ln^{3/2}\frac{1}{\varepsilon} \left(\sum_{e_{i,j}\subset\Omega_2^{(3)}} \left(\frac{v_i - v_j}{|e_{i,j}|}\right)^2 |b_{i,j}| \right)^{1/2} \\
&\le ChK_2 \left(\sum_{e_{i,j}\in E_h^{(2)}} |e_{i,j}| \left(\frac{v_i - v_j}{|e_{i,j}|}\right)^2 |b_{i,j}| \right)^{1/2} \\
&= ChK_2 ||v||_{h,\Omega_2},
\end{aligned}
\tag{60}
$$

where $||\cdot||_{h,\Omega_2}$ is the discrete norm defined by (39) and K_2 is defined in (52). In (60) we used the facts that $|e_{i,j}| \ge \frac{C}{M}\varepsilon\ln\frac{1}{\varepsilon}$, $(\sum_{e_{i,j}\subset\bar\Omega_2^{(1)}\cup\bar\Omega_2^{(2)}} |b_{i,j}|)^{1/2} = O(\sqrt{\varepsilon\ln\frac{1}{\varepsilon}})$ and $(\sum_{e_{i,j}\subset\Omega_2^{(3)}} |b_{i,j}|)^{1/2} = O(\varepsilon\ln\frac{1}{\varepsilon})$.

Now, we consider R_4 in (46). Let $G_i = G(x_i)$. Since $v_h = 0$ on $\partial\Omega_2 \cap \partial\Omega$, the term R_4 can be estimated as

$$
\begin{aligned}
R_4 &= \sum_{d_i\in D_h^{(2)}} \int_{d_i} |(Gu - P(Gu))v_h| dx \\
&\le \sum_{d_i\in D_h^{(2)}} |G_i v_i| \int_{d_i} |u - u_i||v_h| dx \\
&\quad + \sum_{d_i\in D_h^{(2)}} |v_i| ||u||_{0,\infty,d_i} \int_{d_i} |G - G_i| dx \\
&\le ||G||_\infty ||v_h||_\infty \sum_{d_i\in D_h^{(2)}} \int_{d_i} |u - u_i| dx + Ch|G|_{1,\infty}||v_h||_\infty \varepsilon\ln\frac{1}{\varepsilon},
\end{aligned}
\tag{61}
$$

since $\sum_{d_i \in D_h^{(2)}} |d_i| = O(\varepsilon \ln \frac{1}{\varepsilon})$. Similarly to the discussion of R_3, using Assumption 2.2, the sum of the integrals in (61) can be estimated as

$$\begin{aligned}\sum_{d_i \in D_h^{(2)}} \int_{d_i} |u - u_i| dx &\le Ch \sum_{d_i \in D_h^{(2)}} \int_{d_i} \left(\left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right| \right) dxdy \\ &\le Ch \int_0^{1-\delta_2/M} \int_{1-\delta_1}^{1-\delta_1/2M} \left(\left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right| \right) dxdy \\ &\quad + Ch \int_0^{1-\delta_1} \int_{1-\delta_2}^{1-\delta_2/2M} \left(\left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right| \right) dydx \\ &\le Ch(\varepsilon \ln \frac{1}{\varepsilon} + \varepsilon^{\beta/2M}),\end{aligned}$$

where δ_1 and δ_2 are defined in (9). Substituting the above estimate into (61), we obtain

$$\begin{aligned}R_4 &\le C||G||_\infty ||v_h||_\infty (\varepsilon \ln \frac{1}{\varepsilon} + \varepsilon^{\beta/2M}) + Ch|G|_{1,\infty} ||v_h||_{0,\infty} \varepsilon \ln \frac{1}{\varepsilon} \\ &\le Ch||v_h||_\infty (\varepsilon \ln \frac{1}{\varepsilon} + \varepsilon^{\beta/2M}).\end{aligned} \tag{62}$$

Finally, substituting (55), (58), (60) and (62) into (54), we have

$$\begin{aligned}a(u_h - u_I, v_h) \quad &\le \quad Ch||v||_{\varepsilon,\Omega_1} + ChK_2||v||_{h,\Omega_2} \\ &\quad + C\big[h^2 + h(\varepsilon \ln \frac{1}{\varepsilon} + e^{\beta/2M})\big]||v_h||_\infty \\ &\le \quad C\big[hK_2||v|| + h^2 K_1 ||v_h||_\infty\big],\end{aligned}$$

where K_1 and K_2 are defined in (45) and (52). Choosing $v_h = P(u_h - u_I)$ and using (40), we obtain

$$||u_h - u_I||^2 \le C\big[hK_2||u_h - u_I|| + h^2 K_1\big].$$

This is of the form

$$y^2 \le CK_2 hy + CK_1 h^2 \quad \text{or} \quad (y - \frac{1}{2} CK_2 h)^2 \le CK_1 h^2 + \frac{(CK_2)^2}{4} h^2,$$

which reduces to

$$y \le h\sqrt{CK_1 + \frac{(CK_2)^2}{4}} + \frac{CK_2}{2} h \le Ch(K_2 + K_1^{1/2}).$$

Replacing y with $||u_h - u_I||$, we obtain

$$||u_h - u_I|| \le Ch(K_1^{1/2} + K_2).$$

This completes the proof of the theorem. $\square$

Theorem 5.2 yields the following corollary.

Corollary 5.3. *Let the assumptions in Theorem 5.2 be fulfilled. If ε, h, M and β are such that $h^{-1}\varepsilon^{\beta/2M} \le O(1)$ and $h^{-1}\varepsilon \ln(1/\varepsilon) \le O(1)$, then*

$$||u_I - u_h|| \le ChM^{1/2}. \tag{63}$$

Furthermore, if we choose $h^{-1}\varepsilon^{\beta/2M} = O(1)$, then

$$||u_I - u_h|| \le Ch\sqrt{\left| \frac{\ln \varepsilon}{\ln h} \right|}. \tag{64}$$

Proof. Since $h^{-1}\varepsilon^{\beta/2M} \leq O(1)$ and $h^{-1}\varepsilon \ln(1/\varepsilon) \leq O(1)$, it follows that $K_1 = 1$ and $K_2 \leq M^{1/2}$. Substituting these estimates into (51) yields (63).

We now prove (64). Consider $h^{-1}\varepsilon^{\beta/2M} = 1$. Solving this equation for M, we get $M = \frac{\beta}{2}|\ln\varepsilon/\ln h|$, from which we see that if $h^{-1}\varepsilon^{\beta/2M} = O(1)$, then $M = O(|\ln\varepsilon/\ln h|)$. Substituting this into (63), we obtain (64). $\square$

We comment that obviously the above corollary implies that if $\varepsilon << h$, then $||u_I - u_h||$ converges to zero at the rate of h almost uniformly in ε. This improves the results $O(h\ln^{1/2}\frac{1}{\varepsilon})$ in [3] and [10]. Unlike most of the previous results, the present method uses a mixture of triangular and rectangular elements. Thus, the results can be easily extended to singularly perturbed problems with nonrectangular, polygonal regions, if the a priori estimates corresponding to those in Assumption 2.2 can be established on those regions. Moreover, unlike most of the previous cases, the discrete part of the energy norm $||\cdot||$ is weighted by the local mesh size $|e_{i,j}|$, though some of them are of length $O(\varepsilon\ln\frac{1}{\varepsilon})$.

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