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ERROR BOUNDS FOR GAUSS-TURÁN QUADRATURE FORMULAE OF ANALYTIC FUNCTIONS

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This paper is dedicated to Professor Walter Gautschi on the occasion of his 75th birthday

ABSTRACT. We study the kernels of the remainder term $R_{n,s}(f)$ of Gauss-Turán quadrature formulas

$$\int_{-1}^{1} f(t)w(t)\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,s}(f) \qquad (n \in \mathbb{N};\ s \in \mathbb{N}_0)$$

for classes of analytic functions on elliptical contours with foci at ± 1, when the weight w is one of the special Jacobi weights $w^{(\alpha,\beta)}(t) = (1-t)^\alpha(1+t)^\beta$ ($\alpha = \beta = -1/2$; $\alpha = \beta = 1/2 + s$; $\alpha = -1/2$, $\beta = 1/2 + s$; $\alpha = 1/2 + s$, $\beta = -1/2$). We investigate the location on the contour where the modulus of the kernel attains its maximum value. Some numerical examples are included.

1. Introduction

Quadrature formulae with multiple nodes appeared more than 100 years after the famous Gaussian quadratures. Starting from the Hermite interpolation formula and taking any system of n distinct nodes $\{\tau_1, \ldots, \tau_n\}$ with arbitrary multiplicities m_ν ($\nu = 1, \ldots, n$), in 1948 Chakalov [2] obtained such a general quadrature, which is exact for all algebraic polynomials of degree at most $m_1 + \cdots + m_n - 1$. Taking all the multiplicities to be equal, Turán [44] was the first who introduced the corresponding quadrature formula of Gaussian type.

Let w be an integrable weight function on the interval $(-1, 1)$. In this paper we consider the Gauss-Turán quadrature formula with multiple nodes,

$$\int_{-1}^{1} f(t)w(t)\,dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,s}(f), \tag{1.1}$$

where $A_{i,\nu} = A_{i,\nu}^{(n,s)}$, $\tau_\nu = \tau_\nu^{(n,s)}$ ($i = 0, 1, \ldots, 2s$; $\nu = 1, \ldots, n$), which is exact for all algebraic polynomials of degree at most $2(s+1)n - 1$. The nodes τ_ν in (1.1)

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must be the zeros of the (monic) polynomial $\pi_{n,s}(t)$ which minimizes the integral

$$\Phi(a_0, a_1, \ldots, a_{n-1}) = \int_{\mathbb{R}} \pi_n(t)^{2s+2} w(t)\, dt,$$

where

$$\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

In order to minimize Φ, we must have the following orthogonality conditions

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k w(t)\, dt = 0, \quad k = 0, 1, \ldots, n-1. \tag{1.2}$$

Polynomials $\pi_n = \pi_{n,s}$ which satisfy this type of orthogonality (1.2) (so-called "*power orthogonality*") are known as s-orthogonal (or s-self-associated) polynomials with respect to the measure $d\lambda(t) = w(t)\, dt$. For $s = 0$ this reduces to the standard case of orthogonal polynomials. Several classes of s-orthogonal polynomials, as well as their generalizations known as σ-orthogonal polynomials, were investigated mainly by Italian mathematicians, e.g., Ossicini [26], [27], Ghizzetti and Ossicini [12], [13], Ossicini and Rosati [29], [30], [31], Gori Nicolò-Amati [15] (see the survey paper [22] for details and references).

A generalization of the Gauss-Turán quadrature formula (1.1) to rules having nodes with arbitrary multiplicities was derived independently by Chakalov [3], [4] and Popoviciu [36]. Important theoretical progress on this subject was made by Stancu [41], [42] (see also [43]).

Methods for constructing the nodes τ_ν and/or coefficients $A_{i,\nu}$ in the Gauss-Turán quadratures, as well as in the generalized Chakalov-Popoviciu-Stancu formulas, can be found in [8], [14], [21], [23], [24], [25], [39], [40], [43], [45].

The remainder term in formulas with multiple nodes was studied by Chakalov [3], Ionescu [19], Ossicini [27], Pavel [32], [33],[34], and Milovanović and Spalević [25]. The case of holomorphic functions f in the Gauss-Turán quadrature (1.1) was considered by Ossicini and Rosati [29].

In this paper we consider the remainder term $R_{n,s}(f)$ of Gauss-Turán quadrature formulas for classes of analytic functions on elliptical contours, when the weight function w in (1.1) is one of the special Jacobi weights $w^{(\alpha,\beta)}(t) = (1-t)^\alpha(1+t)^\beta$, with parameters

$$\alpha = \beta = -\frac{1}{2}\,; \quad \alpha = \beta = \frac{1}{2} + s; \quad \alpha = -\frac{1}{2}\,,\ \beta = \frac{1}{2} + s; \quad \alpha = \frac{1}{2} + s,\ \beta = -\frac{1}{2}\,,$$

where $s \in \mathbb{N}_0$. The reason for these choices is explained near the end of Section 2.

The paper is organized as follows. The remainder term of Gauss-Turán formulas for analytic functions and some properties of the kernels in the contour representations of the remainder terms are given in Section 2. The cases of elliptic contours with foci at the points ± 1, when w is any one of the four Jacobi weight functions, are studied in Section 3. More precisely, the location on the contour where the modulus of the kernel attains its maximum value is investigated. Some numerical examples are included.

2. The remainder term in Gauss-Turán quadrature formulae

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and let $\mathcal{D}$ be its interior. If the integrand f is analytic in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$, then the remainder term $R_{n,s}(f)$ in (1.1) admits the contour integral representation (cf. [29])

$$R_{n,s}(f) = \frac{1}{2\pi i} \oint_\Gamma K_{n,s}(z) f(z)\, dz. \tag{2.1}$$

The kernel is given by

$$K_{n,s}(z) = \frac{\varrho_{n,s}(z)}{[\pi_{n,s}(z)]^{2s+1}}, \qquad z \notin [-1,1], \tag{2.2}$$

where

$$\varrho_{n,s}(z) = \int_{-1}^{1} w(t) \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t}\, dt, \qquad n \in \mathbb{N}, \tag{2.3}$$

and $\pi_{n,s}(t)$ is the monic s-orthogonal polynomial with respect to the measure $d\lambda(t) = w(t)\,dt$ on $(-1,1)$. For $s=0$, the formulas (2.1) and (2.2) reduce to the corresponding formulas for Gaussian quadratures.

An alternative representation for $K_{n,s}(z)$ is

$$K_{n,s}(z) = R_{n,s}\Big(\frac{1}{z-\,\cdot}\Big) = \int_{-1}^{1} \frac{w(t)}{z-t}\, dt - \sum_{\nu=1}^{n} \sum_{i=0}^{2s} \frac{i!A_{i,\nu}}{(z-\tau_\nu)^{i+1}}\, .$$

Let $N=(s+1)n$ and

$$t^{2N} = \prod_{\nu=1}^{n} (t-\tau_\nu)^{2s+2} + q(t) = \pi_{n,s}(t)^{2s+2} + q(t) \quad (q \in \mathcal{P}_{2N-1}),$$

where $\mathcal{P}_m$ denotes the set of all algebraic polynomials of degree at most m. Expanding the integrand of (2.3) in descending powers of z, and using (1.2) and

$$R_{n,s}(t^{2N}) = R_{n,s}(\pi_{n,s}^{2s+2}) = \int_{-1}^{1} \pi_{n,s}(t)^2\, d\mu(t) = \|\pi_{n,s}\|_{d\mu}^2,$$

where $d\mu(t) = \pi_{n,s}(t)^{2s} w(t)\,dt$, we conclude that

$$K_{n,s}(z) = \frac{R_{n,s}(t^{2N})}{z^{2N+1}} + \cdots = \frac{\|\pi_{n,s}\|_{d\mu}^2}{z^{2N+1}} \Big(1 + \frac{C_1}{z} + \frac{C_2}{z^2} + \cdots\Big),$$

where $C_1, C_2, \ldots$ are constants.

The integral representation (2.1) leads to the error estimate

$$|R_{n,s}(f)| \le \frac{\ell(\Gamma)}{2\pi} \Big(\max_{z\in\Gamma} |K_{n,s}(z)|\Big) \Big(\max_{z\in\Gamma} |f(z)|\Big), \tag{2.4}$$

where $\ell(\Gamma)$ is the length of the contour Γ. We thus have to study the magnitude of $|K_{n,s}(z)|$ on Γ.

It seems that the first unified approach described above was taken by Donaldson and Elliott [5]. They applied it to several kinds of interpolatory and non-interpolatory quadrature rules. Error bounds for Gaussian quadratures of analytic functions were studied by Gautschi and Varga [10] (see also [11]). In particular, they investigated some cases with special Jacobi weights with parameters $\pm 1/2$ (Chebyshev weights). The cases of Gaussian rules with Bernstein-Szegő weight functions and with some symmetric weights including especially the Gegenbauer weight were studied by Peherstorfer [35] and Schira [38], respectively. Some of the results have been extended to Gauss-Radau and Gauss-Lobatto formulas (cf. Gautschi [6], Gautschi and Li [7], Schira [37], Hunter and Nikolov [18]).

In the sequel we give some properties of the kernel (2.2).

Lemma 2.1. *Let the kernel $K_{n,s}(z)$ be given by (2.2) and (2.3). Then, for each $z \in \mathbb{C} \setminus [-1,1]$,*

$$|K_{n,s}(\overline{z})| = |K_{n,s}(z)|. \tag{2.5}$$

Moreover, if the weight function in (1.1) is even, i.e., $w(-t) = w(t)$, then

$$|K_{n,s}(-\overline{z})| = |K_{n,s}(z)|. \tag{2.6}$$

Proof. According to (2.2) it is clear that

$$K_{n,s}(\overline{z}) = \frac{\varrho_{n,s}(\overline{z})}{[\pi_{n,s}(\overline{z})]^{2s+1}} = \overline{K_{n,s}(z)},$$

implying (2.5).

If w is an even function, i.e., $w(-t) = w(t)$, we have $\pi_{n,s}(-\overline{z}) = (-1)^n \pi_{n,s}(\overline{z})$ and

$$\varrho_{n,s}(z) = \int_{-1}^{1} w(t) \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t}\, dt = \int_{-1}^{1} w(-t) \frac{[\pi_{n,s}(-t)]^{2s+1}}{z+t}\, dt,$$

i.e.,

$$\varrho_{n,s}(z) = (-1)^{n(2s+1)} \int_{-1}^{1} w(t) \frac{[\pi_{n,s}(t)]^{2s+1}}{z+t}\, dt = -(-1)^{n(2s+1)} \varrho_{n,s}(-z),$$

so that

$$K_{n,s}(-\overline{z}) = \frac{\varrho_{n,s}(-\overline{z})}{[\pi_{n,s}(-\overline{z})]^{2s+1}} = \frac{-(-1)^{n(2s+1)} \varrho_{n,s}(\overline{z})}{(-1)^{n(2s+1)} [\pi_{n,s}(\overline{z})]^{2s+1}} = -K_{n,s}(\overline{z}) = -\overline{K_{n,s}(z)}.$$

Thus, in this case we get (2.6). □

A particularly interesting case is the Chebyshev measure

$$d\lambda_1(t) = (1-t^2)^{-1/2} dt.$$

In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial $\hat{T}_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\, dt \qquad (k \geq 0).$$

This means that the Chebyshev polynomials T_n are s-orthogonal on $(-1,1)$ for each $s \geq 0$. Ossicini and Rosati [29] found three other measures $d\lambda_k(t)$ $(k = 2,3,4)$ for which the s-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third, and fourth kind: U_n, V_n, and W_n, which are defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad V_n(\cos\theta) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}, \quad W_n(\cos\theta) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta},$$

respectively (cf. Gautschi and Notaris [9]). However, these measures depend on s,

$$d\lambda_2(t) = (1-t^2)^{1/2+s} dt, \quad d\lambda_3(t) = \frac{(1+t)^{1/2+s}}{(1-t)^{1/2}} dt, \quad d\lambda_4(t) = \frac{(1-t)^{1/2+s}}{(1+t)^{1/2}} dt.$$

It is easy to see that $W_n(-t) = (-1)^n V_n(t)$, so that in the investigation it is sufficient to study only the first three Jacobi measures $d\lambda_k(t)$, $k = 1,2,3$.

Recently, Ossicini, Martinelli, and Rosati [28] have proved the convergence as $n \to +\infty$ (alternatively, as $s \to +\infty$), of the Gauss-Turán quadrature formula (1.1) for the cases $d\lambda_1(t)$ and $d\lambda_2(t)$, on the basis of results from [29], by taking f to be a holomorphic function on $\overline{\operatorname{int}\Gamma}$, where the contour Γ is an ellipse with foci at ± 1 and sum of semiaxes $\varrho > 1$. Using estimates obtained for $R_{n,s}(f)$, they proved the convergence and rate of convergence of the quadrature formulae, $R_{n,s}(f) = O\big(\rho^{-n(2s+1)}\big)$, $n \to +\infty$.

3. The maximum modulus of the kernel on confocal ellipses

In this section we take as the contour Γ an ellipse with foci at the points ± 1 and sum of semiaxes $\varrho > 1$,

$$\mathcal{E}_\varrho = \Big\{ z \in \mathbb{C}\colon z = \frac{1}{2}\big(\varrho e^{i\theta} + \varrho^{-1} e^{-i\theta}\big),\ 0 \le \theta < 2\pi \Big\}. \tag{3.1}$$

When $\varrho \to 1$, then the ellipse shrinks to the interval $[-1, 1]$, while with increasing ϱ it becomes more and more circle-like.

Since the ellipse $\mathcal{E}_\varrho$ has length $\ell(\mathcal{E}_\varrho) = 4\varepsilon^{-1}E(\varepsilon)$, where ε is the eccentricity of $\mathcal{E}_\varrho$, i.e., $\varepsilon = 2/(\varrho + \varrho^{-1})$, and

$$E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2\theta}\, d\theta$$

is the complete elliptic integral of the second kind, the estimate (2.4) reduces to

$$|R_{n,s}(f)| \le \frac{2E(\varepsilon)}{\pi\varepsilon}\Big(\max_{z\in\mathcal{E}_\varrho} |K_{n,s}(z)|\Big)\|f\|_\varrho, \quad \varepsilon = \frac{2}{\varrho + \varrho^{-1}}\,, \tag{3.2}$$

where $\|f\|_\varrho = \max\limits_{z\in\mathcal{E}_\varrho} |f(z)|$. As we can see, the bound on the right in (3.2) is a function of ϱ, so that it can be optimized with respect to $\varrho > 1$.

In this section we study the magnitude of $|K_{n,s}(z)|$ on the contour $\mathcal{E}_\varrho$. More precisely, for the measures $d\lambda_k(t)$ $(k = 1, 2, 3)$ defined at the end of the previous section, we investigate the locations on the confocal ellipses where the modulus of the corresponding kernels attain their maximum values.

Because of (2.5), i.e., symmetry with respect to the real axis, the consideration of $|K_{n,s}(z)|$, when

$$z = \frac{1}{2}\big(\varrho e^{i\theta} + \varrho^{-1}e^{-i\theta}\big) \in \mathcal{E}_\varrho,$$

may be restricted to the interval $0 \le \theta \le \pi$. Moreover, if the weight function is even, as in the cases of $d\lambda_1(t)$ and $d\lambda_2(t)$ (symmetry with respect to both coordinate axes), the consideration may be restricted to the first quarter of $\mathcal{E}_\varrho$, i.e., to the interval $0 \le \theta \le \pi/2$ (see (2.6)).

In the sequel we give explicit representations of the kernels $K_{n,s}^{(\nu)}$ on the ellipse $\mathcal{E}_\varrho$ for the measures $d\lambda_\nu(t)$, $\nu = 1, 2, 3$, and discuss the maximum points on this ellipse in order to get the exact value of $\max_{z\in\mathcal{E}_\varrho} |K_{n,s}^{(\nu)}(z)|$ or some estimate.

3.1. **The measure $d\lambda_1(t) = (1 - t^2)^{-1/2}dt$.** According to (2.3), in this case we have

$$\varrho_{n,s}(z) = \int_{-1}^{1} (1 - t^2)^{-1/2} \frac{[2^{1-n}T_n(t)]^{2s+1}}{z - t}\, dt, \quad n \in \mathbb{N},\ z \notin [-1, 1],$$

where $T_n(t)$ is the Chebyshev polynomial of the first kind of degree n. By substituting $t = \cos\theta$, we obtain

$$\begin{aligned} \varrho_{n,s}(z) &= 2^{(1-n)(2s+1)} \int_0^\pi \frac{[\cos n\theta]^{2s+1}}{z-\cos\theta}\, d\theta \\ &= \frac{2^{(1-n)(2s+1)}}{2^{2s}} \int_0^\pi \frac{1}{z-\cos\theta} \sum_{k=0}^{s} \binom{2s+1}{k} \cos(2s+1-2k)n\theta\, d\theta, \end{aligned}$$

where for the transformation of $[\cos n\theta]^{2s+1}$ we used a formula from [29, p. 232]. Now, the kernel (2.2) has the form

$$K_{n,s}^{(1)}(z) = \frac{2^{-2s} \displaystyle\sum_{k=0}^{s} \binom{2s+1}{k} \int_0^\pi \frac{1}{z-\cos\theta} \cos(2s+1-2k)n\theta\, d\theta}{[T_n(z)]^{2s+1}}.$$

Furthermore, using [17, Eq. 3.613.1], one finds (see also [10, p. 1176])

$$\int_0^\pi \frac{\cos m\theta}{z-\cos\theta}\, d\theta = \frac{\pi}{\sqrt{z^2-1}} \left(z-\sqrt{z^2-1}\right)^m, \quad m \in \mathbb{N}_0, \tag{3.3}$$

and we obtain

$$K_{n,s}^{(1)}(z) = \frac{2^{-2s} \dfrac{\pi}{\sqrt{z^2-1}} \displaystyle\sum_{k=0}^{s} \binom{2s+1}{k} \left(z-\sqrt{z^2-1}\right)^{(2s+1-2k)n}}{[T_n(z)]^{2s+1}}.$$

It is well known that

$$T_n(z) = \frac{1}{2}\left[\left(z+\sqrt{z^2-1}\right)^n + \left(z-\sqrt{z^2-1}\right)^n\right], \quad z \in \mathbb{C}. \tag{3.4}$$

Letting $z = \frac{1}{2}(u+u^{-1})$, we get

$$K_{n,s}^{(1)}(z) = \frac{4\pi}{(u-u^{-1})u^n[u^n+u^{-n}]^{2s+1}} \sum_{k=0}^{s} \binom{2s+1}{k} \frac{1}{u^{2(s-k)n}},$$

i.e.,

$$K_{n,s}^{(1)}(z) = \frac{4\pi Z_{n,s}^{(1)}(u)}{(u-u^{-1})u^n[u^n+u^{-n}]^{2s+1}},$$

where

$$Z_{n,s}^{(1)}(u) = \sum_{k=0}^{s} \binom{2s+1}{s+k+1} u^{-2nk}. \tag{3.5}$$

Introducing

$$a_j = a_j(\varrho) = \frac{1}{2}(\varrho^j + \varrho^{-j}), \qquad j \in \mathbb{N},\ \varrho > 1, \tag{3.6}$$

we have

$$|u-u^{-1}|^2 = 2(a_2 - \cos 2\theta) \quad \text{and} \quad |u^n+u^{-n}|^2 = 2(a_{2n} + \cos 2n\theta),$$

when $u = \varrho e^{i\theta}$, so that

$$|K_{n,s}^{(1)}(z)| = \frac{2^{1-s}\pi}{\varrho^n} \cdot \frac{|Z_{n,s}^{(1)}(\varrho e^{i\theta})|}{(a_2-\cos 2\theta)^{1/2}(a_{2n}+\cos 2n\theta)^{s+1/2}}, \quad z \in \mathcal{E}_\varrho, \tag{3.7}$$

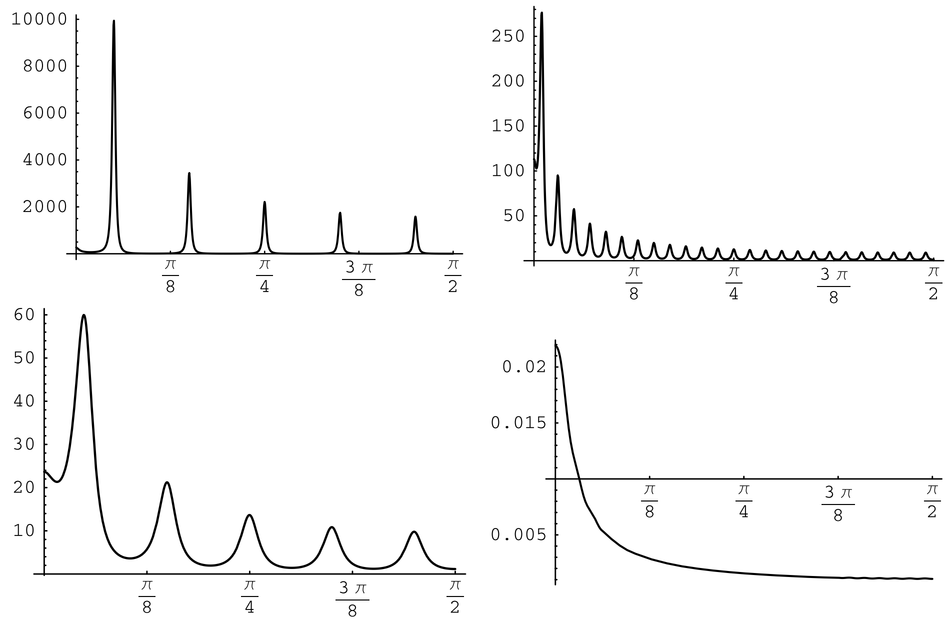

FIGURE 1. The functions $\theta \mapsto |K^{(1)}_{10,1}(z)|$ and $\theta \mapsto |K^{(1)}_{50,1}(z)|$ ($z = \frac{1}{2}(u+u^{-1})$, $u = \varrho e^{i\theta}$) for Chebyshev weight of the first kind and $\varrho = 1.01$ (top) and $\varrho = 1.05$ (bottom)

where $Z^{(1)}_{n,s}(u)$ is given by (3.5) and the ellipse $\mathcal{E}_\varrho$ by (3.1). Note that the case $s = 0$, for which $Z^{(1)}_{n,0}(u) = 1$, was analyzed in [10, Eq. (5.4)].

An analysis of (3.7) shows that the point of the maximum of $|K^{(1)}_{n,s}(z)|$ for a given ϱ depends on n. The graphics $\theta \mapsto |K^{(1)}_{n,1}(z)|$ ($z = (u+u^{-1})/2$, $u = \varrho e^{i\theta}$) for $n = 10$ and $n = 50$ are displayed in Figure 1, when $\varrho = 1.01$ and $\varrho = 1.05$. The cases for $s = 1, 2, 3$, when $n = 10$ and $\varrho = 1.05$, 1.08, 1.10, and 1.12, are presented in Figure 2.

Using the inequality (see [10, Proof of Thm. 5.1]),

$$(a_2 - \cos 2\theta)(a_{2n} + \cos 2n\theta) \geq (a_2 - 1)(a_{2n} + 1), \qquad 0 \leq \theta \leq \pi/2,$$

a simple estimate of (3.7) can be given in the form

$$\begin{aligned} |K^{(1)}_{n,s}(z)| &\leq \frac{4\pi Z^{(1)}_{n,s}(\varrho)}{\varrho^n(\varrho^n - \varrho^{-n})^{2s}(\varrho - \varrho^{-1})(\varrho^n + \varrho^{-n})} \\ &= K^{(1)}_{n,s}\Big(\frac{1}{2}(\varrho + \varrho^{-1})\Big)\left(\frac{\varrho^n + \varrho^{-n}}{\varrho^n - \varrho^{-n}}\right)^{2s}, \end{aligned} \tag{3.8}$$

where $Z^{(1)}_{n,s}(u)$ is defined by (3.5). By the crude inequality $Z^{(1)}_{n,s}(\varrho) < Z^{(1)}_{n,s}(1) = 2^{2s}$ ($\varrho > 1$), the inequality (3.8) can be simplified to

$$|K^{(1)}_{n,s}(z)| \leq \frac{4\pi}{\varrho^n(\varrho^n + \varrho^{-n})(\varrho - \varrho^{-1})}\left(\frac{2}{\varrho^n - \varrho^{-n}}\right)^{2s}. \tag{3.9}$$

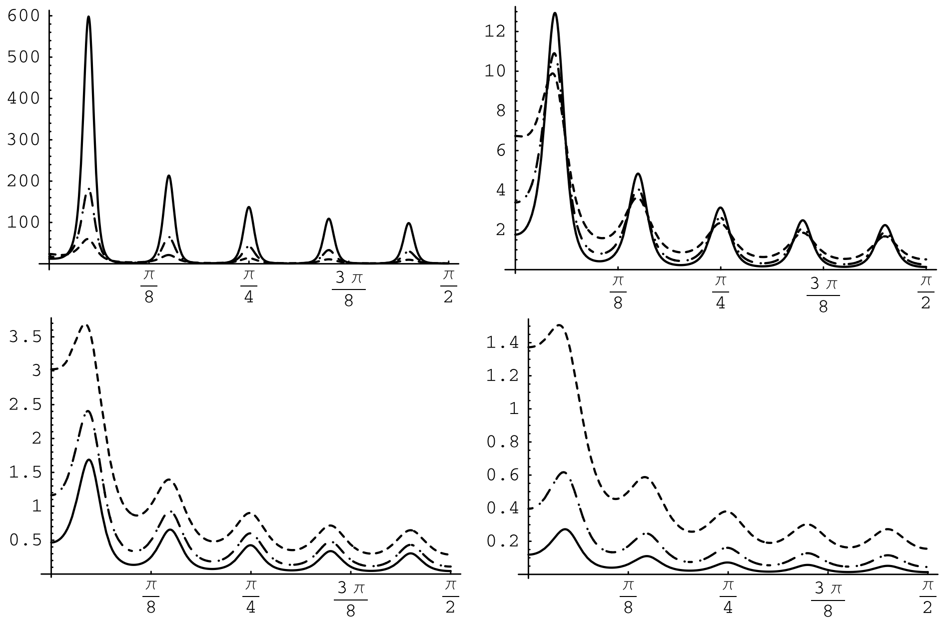

FIGURE 2. The function $\theta \mapsto |K_{10,s}^{(1)}(z)|$, $z = \frac{1}{2}(u + u^{-1})$, $u = \varrho e^{i\theta}$, for $s = 1$ (dashed line), $s = 2$ (dot-dashed line), $s = 3$ (solid line), when $\varrho = 1.05$, $\varrho = 1.08$ (top), and $\varrho = 1.10$, $\varrho = 1.12$ (bottom)

TABLE 1. Maximum value of $|K_{n,s}^{(1)}(z)|$, $z \in \mathcal{E}_\varrho$, and the bound (3.8) for $n = 10, 50, 100$, $s = 1, 2, 3$, and $\varrho = 1.01$, 1.05, and 1.1

		$\varrho = 1.01$		$\varrho = 1.05$		$\varrho = 1.1$	
s	n	max	bound	max	bound	max	bound
1	10	9.996(3)	2.734(4)	5.996(1)	1.155(2)	3.689(0)	5.500(0)
	50	2.779(2)	5.345(2)	2.189(−2)	2.256(−2)	1.040(−6)	1.040(−6)
	100	2.974(1)	4.368(1)	1.291(−6)	1.291(−6)	5.476(−15)	5.476(−15)
2	10	7.583(5)	2.661(6)	1.821(2)	3.993(2)	2.406(0)	3.857(0)
	50	8.106(2)	1.769(3)	5.471(−4)	5.814(−4)	2.514(−10)	2.516(−10)
	100	1.734(1)	2.733(1)	2.489(−10)	2.490(−10)	9.611(−23)	9.611(−23)
3	10	6.369(7)	2.605(8)	5.980(2)	1.417(3)	1.688(0)	2.812(0)
	50	2.554(3)	6.013(3)	1.438(−5)	1.573(−5)	6.385(−14)	6.391(−14)
	100	1.087(1)	1.780(1)	5.037(−14)	5.040(−14)	1.771(−30)	1.771(−30)

Numerical values of the actual maximum of $|K_{n,s}^{(1)}(z)|$, when $z \in \mathcal{E}_\varrho$, and the corresponding bounds (3.8) for some selected values of n, s, and ϱ are presented in Table 1. (Numbers in parenthesis indicate decimal exponents.)

Based on the previous calculation we can state the following conjecture:

Conjecture 3.1. *For each fixed $\varrho > 1$ and $s \in \mathbb{N}_0$ there exists $n_0 = n_0(\varrho, s) \in \mathbb{N}$ such that*

$$\max_{z \in \mathcal{E}_\varrho} |K_{n,s}^{(1)}(z)| = K_{n,s}^{(1)}\Big(\frac{1}{2}(\varrho + \varrho^{-1})\Big)$$

for each $n \geq n_0$.

Using (3.9), the estimate (3.2) becomes

$$|R_{n,s}(f)| \le \frac{M}{\varrho^n(\varrho^n+\varrho^{-n})}\left(\frac{2}{\varrho^n-\varrho^{-n}}\right)^{2s}, \tag{3.10}$$

where $M = 4E(\varepsilon)\|f\|_\varrho(\varrho+\varrho^{-1})/(\varrho-\varrho^{-1})$.

On the basis of (3.10) we conclude that the corresponding Gauss-Turán quadrature formulae converge if s is a fixed integer and $n \to +\infty$, since

$$\lim_{n\to+\infty} R_{n,s}(f) = 0.$$

Moreover, we conclude that $R_{n,s}(f) = O(\varrho^{-2n(s+1)})$, when $n \to +\infty$.

Assuming that $2/(\varrho^n-\varrho^{-n}) < 1$, we also see that $R_{n,s}(f) \to 0$, when $s \to +\infty$ and n is fixed. This condition is satisfied if $\varrho^{2n} - 2\varrho^n - 1 > 0$, i.e., if $\varrho^n > 1+\sqrt{2}$. The same conclusion was obtained in [28, Eq. (5.5)].

Remark 3.2. Recently, Gori and Micchelli [16] have introduced for each n a class of weight functions defined on $[-1,1]$ for which explicit Gauss-Turán quadrature formulas can be found for all s. Indeed, these classes of weight functions have the peculiarity that the corresponding s-orthogonal polynomials, of the same degree, are independent of s. This class includes certain generalized Jacobi weight functions $w_{n,\mu}(t) = |U_{n-1}(t)/n|^{2\mu+1}(1-t^2)^\mu$, where $U_{n-1}(\cos\theta) = \sin n\theta/\sin\theta$ (Chebyshev polynomial of the second kind) and $\mu > -1$. In this case, the Chebyshev polynomials T_n appear as s-orthogonal polynomials. Since

$$|U_{n-1}(\cos\theta)| = \left|\frac{\sin n\theta}{\sin\theta}\right| \le n,$$

i.e., $|U_{n-1}(t)/n|^{2\mu+1}(1-t^2)^{\mu+1/2} \le 1$, by arguing, for example, in an analogous way as in [28], we can obtain in this case that $\lim_{s\to+\infty} R_{n,s}(f) = 0$, under the previous condition $\varrho^n > 1+\sqrt{2}$, where n is a fixed positive integer.

3.2. **The measure** $d\lambda_2(t) = (1-t^2)^{s+1/2}dt,\ s \in \mathbb{N}_0$**.** In this case we have

$$\varrho_{n,s}(z) = \int_{-1}^{1}(1-t^2)^{1/2+s}\frac{[2^{-n}U_n(t)]^{2s+1}}{z-t}\,dt, \quad n \in \mathbb{N},\ z \notin [-1,1],$$

where $U_n(t)$ is the Chebyshev polynomial of the second kind for the weight function $w(x) = \sqrt{1-t^2}$, for which

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

By substituting $t = \cos\theta$, we obtain

$$\begin{aligned}\varrho_{n,s}(z) &= 2^{-n(2s+1)}\int_0^\pi \frac{\sin\theta}{z-\cos\theta}[\sin(n+1)\theta]^{2s+1}\,d\theta\\ &= \frac{2^{-n(2s+1)}}{2^{2s}}\int_0^\pi \frac{\sin\theta}{z-\cos\theta}\sum_{k=0}^{s}(-1)^{s+k}\binom{2s+1}{k}\sin(2s+1-2k)(n+1)\theta\,d\theta,\end{aligned}$$

where for the transformation of $[\sin(n+1)\theta]^{2s+1}$ we used a formula from [29, p. 232]. By using the well-known representation

$$U_n(z) = \frac{1}{2\sqrt{z^2-1}}\left[(z+\sqrt{z^2-1})^{n+1} - (z-\sqrt{z^2-1})^{n+1}\right],$$

the substitution $z = \frac{1}{2}(u + u^{-1})$ and the formula (see [17, Eq. 9.613.3])

$$\int_0^\pi \frac{\sin(m+1)\theta \sin\theta}{z - \cos\theta}\, d\theta = \frac{\pi}{u^{m+1}}, \quad m \in \mathbb{N}_0,$$

yield

$$K_{n,s}^{(2)}(z) = \frac{\pi}{2^{2s}u^{n+1}} \left[\frac{u - u^{-1}}{u^{n+1} - u^{-(n+1)}}\right]^{2s+1} \sum_{k=0}^{s} (-1)^{s+k} \binom{2s+1}{k} \frac{1}{u^{2(s-k)(n+1)}}.$$

We denote the sum on the right-hand side by $Z_{n,s}^{(2)}(u)$ and rewrite it in the form

$$Z_{n,s}^{(2)}(u) = \sum_{k=0}^{s} (-1)^k \binom{2s+1}{s+k+1} u^{-2(n+1)k}, \tag{3.11}$$

so that

$$|K_{n,s}^{(2)}(z)| = \frac{\pi}{4^s \varrho^{n+1}} \left(\frac{a_2 - \cos 2\theta}{a_{2n+2} - \cos(2n+2)\theta}\right)^{s+1/2} |Z_{n,s}^{(2)}(\varrho e^{i\theta})|,$$

i.e.,

$$|K_{n,s}^{(2)}(z)| = \frac{\pi}{4^s \varrho^{n+1}} \cdot \frac{(a_2 - \cos 2\theta)^{s+1/2}}{(a_{2n+2} - \cos(2n+2)\theta)^s} \cdot \frac{|Z_{n,s}^{(2)}(\varrho e^{i\theta})|}{(a_{2n+2} - \cos(2n+2)\theta)^{1/2}}\,, \tag{3.12}$$

where

$$z = \frac{1}{2}(\varrho e^{i\theta} + \varrho^{-1} e^{-i\theta}) \in \mathcal{E}_\varrho$$

and a_j is defined by (3.6).

Now, we consider the last factor in (3.12) when n is odd.

Lemma 3.3. *Let a_j and $Z_{n,s}^{(2)}(u)$ be defined by (3.6) and (3.11), respectively. If n is odd, then*

$$\frac{|Z_{n,s}^{(2)}(\varrho e^{i\theta})|}{(a_{2n+2} - \cos(2n+2)\theta)^{1/2}} \le \frac{Z_{n,s}^{(2)}(i\varrho)}{(a_{2n+2} - 1)^{1/2}}\,, \qquad 0 \le \theta \le \pi/2\,,$$

with equality for $\theta = \pi/2$.

Proof. First we note that

$$Z_{n,s}^{(2)}(u) = \sum_{k=0}^{s} \cdots = \sum_{\nu=0}^{[(s-1)/2]} \left(\sum_{k=2\nu}^{2\nu+1} \cdots\right) + \zeta_{n,s}(u), \tag{3.13}$$

where

$$\zeta_{n,s}(u) := \begin{cases} 0 & \text{if } s \text{ is odd,} \\ u^{-2(n+1)s} & \text{if } s \text{ is even,} \end{cases}$$

as well as $|\zeta_{n,s}(\varrho e^{i\theta})| = \zeta_{n,s}(i\varrho)$. Letting

$$\begin{aligned} S_\nu(u) :\, = \sum_{k=2\nu}^{2\nu+1} \cdots \ &= \binom{2s+1}{s+2\nu+1} u^{-4\nu(n+1)} - \binom{2s+1}{s+2\nu+2} u^{-(4\nu+2)(n+1)} \\ &= \binom{2s+1}{s+2\nu+1} u^{-4\nu(n+1)} \big(1 - \alpha u^{-2(n+1)}\big), \end{aligned}$$

where

$$\alpha = \frac{s-2\nu}{s+2\nu+2} \qquad \text{and} \qquad 0 \le \alpha < 1, \tag{3.14}$$

we see that

$$|S_\nu(\varrho e^{i\theta})| = \binom{2s+1}{s+2\nu+1} \varrho^{-4\nu(n+1)} \sqrt{1 - 2\alpha\varrho^{-2(n+1)}\cos(2n+2)\theta + \alpha^2\varrho^{-4(n+1)}}\,.$$

Now, we consider the quotient

$$F_\nu(\varrho,\theta,n) := \frac{S_\nu(\varrho e^{i\theta})}{(a_{2n+2} - \cos(2n+2)\theta)^{1/2}}\,,$$

when n is odd, and for $0 \le \theta \le \pi/2$ we wish to prove the inequality $|F_\nu(\varrho,\theta,n)| \le F_\nu(\varrho,\pi/2,n)$, i.e.,

$$\frac{|S_\nu(\varrho e^{i\theta})|}{(a_{2n+2} - \cos(2n+2)\theta)^{1/2}} \le \frac{S_\nu(i\varrho)}{(a_{2n+2}-1)^{1/2}}\,, \qquad 0 \le \theta \le \pi/2. \tag{3.15}$$

Using the previous facts, inequality (3.15) reduces to

$$(a_{2n+2}-1)(1-2q\cos(2n+2)\theta + q^2) \le (a_{2n+2} - \cos(2n+2)\theta)(1-q)^2,$$

i.e.,

$$-(1-\cos(2n+2)\theta)(1-2qa_{2n+2}+q^2) \le 0,$$

where $q = \alpha\varrho^{-2(n+1)}$. Using (3.6), we find

$$1 - 2qa_{2n+2} + q^2 = (1-\alpha)(1-\alpha\varrho^{-4(n+1)}),$$

so that the previous inequality becomes

$$-(1-\cos(2n+2)\theta)(1-\alpha)(1-\alpha\varrho^{-4(n+1)}) \le 0. \tag{3.16}$$

Since $1-\cos(2n+2)\theta \ge 0$, $\varrho > 1$, and $0 \le \alpha < 1$ (see (3.14)), we conclude that inequality (3.16) is true. This also proves inequality (3.15).

According to (3.13) and (3.15) we have

$$\begin{aligned}
\frac{|Z_{n,s}^{(2)}(\varrho e^{i\theta})|}{(a_{2n+2} - \cos(2n+2)\theta)^{1/2}} &\le \sum_{\nu=0}^{[(s-1)/2]} |F_\nu(\varrho,\theta,n)| + \frac{|\zeta_{n,s}(\varrho e^{i\theta})|}{(a_{2n+2} - \cos(2n+2)\theta)^{1/2}} \\
&\le \sum_{\nu=0}^{[(s-1)/2]} \frac{S_\nu(i\varrho)}{(a_{2n+2}-1)^{1/2}} + \frac{\zeta_{n,s}(i\varrho)}{(a_{2n+2}-1)^{1/2}} \\
&= \frac{Z_{n,s}^{(2)}(i\varrho)}{(a_{2n+2}-1)^{1/2}}\,,
\end{aligned}$$

with equality holding for $\theta = \pi/2$. $\square$

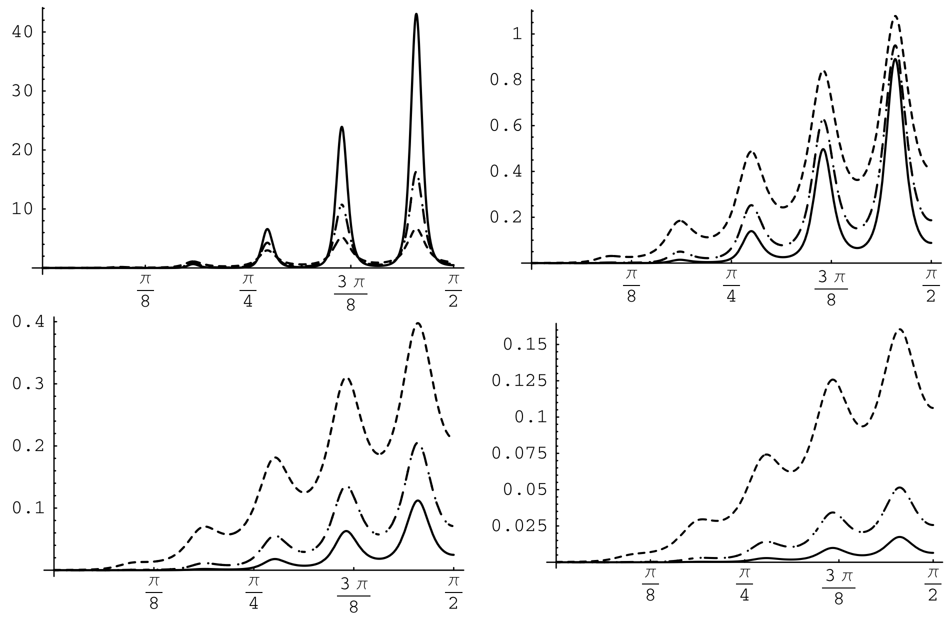

FIGURE 3. The function $\theta \mapsto |K^{(2)}_{10,s}(z)|$, $z = \frac{1}{2}(u+u^{-1})$, $u = \varrho e^{i\theta}$, for $s = 1$ (dashed line), $s = 2$ (dot-dashed line), $s = 3$ (solid line), when $\varrho = 1.05$, $\varrho = 1.08$ (top), and $\varrho = 1.10$, $\varrho = 1.12$ (bottom)

Theorem 3.4. *If $d\lambda(t) = (1-t^2)^{s+1/2}dt$ on $(-1, 1)$, $s \in \mathbb{N}_0$, and n is odd, then*

$$\max_{z\in \mathcal{E}_\varrho} |K^{(2)}_{n,s}(z)| = \Big| \; K^{(2)}_{n,s}\Big(\frac{i}{2}\big(\varrho - \varrho^{-1}\big)\Big) \Big|, \tag{3.17}$$

i.e., the maximum of $|K^{(2)}_{n,s}(z)|$ (n odd) on $\mathcal{E}_\varrho$ is attained on the imaginary axis.

Proof. For the second factor in (3.12), it is obvious that

$$\frac{(a_2 - \cos 2\theta)^{s+1/2}}{(a_{2n+2} - \cos(2n+2)\theta)^s} \le \frac{(a_2+1)^{s+1/2}}{(a_{2n+2}-1)^s}\,, \quad \text{for all } \theta, \text{ all } n,$$

with equality holding when $\theta = \pi/2$ and n is odd. Now, this inequality and Lemma 3.3 give the desired result. □

When n is even in Theorem 3.4, computation shows that the maximum of $|K^{(2)}_{n,s}(z)|$ on the ellipse $\mathcal{E}_\varrho$ is attained slightly off the imaginary axis. The graphics $\theta \mapsto |K^{(2)}_{n,s}(z)|$ ($z = (u+u^{-1})/2$, $u = \varrho e^{i\theta}$) for $n = 10$ and $s = 1, 2, 3$ are displayed in Figure 3, when $\varrho = 1.05$, 1.08, 1.10, and 1.12.

As in the case of the measure $d\lambda_1(t)$ we can get here a simple crude bound of the remainder,

$$|R_{n,s}(f)| \leq \frac{M}{\varrho^{n+1}} \left(\frac{\varrho + \varrho^{-1}}{\varrho^{n+1} - \varrho^{-(n+1)}} \right)^{2s+1}, \qquad M = E(\varepsilon)\|f\|_\varrho(\varrho + \varrho^{-1}), \tag{3.18}$$

which holds for each $n \in \mathbb{N}$.

According to (3.18) we may conclude that the corresponding Gauss-Turán quadrature formulae converge if s is a fixed integer and $n \to +\infty$. Moreover, $R_{n,s}(f) = O(\varrho^{-2n(s+1)})$, when $n \to +\infty$.

Assuming that $(\varrho+\varrho^{-1})/(\varrho^{n+1}-\varrho^{-n-1}) < 1$, we also see that $R_{n,s}(f) \to 0$, when $s \to +\infty$ and n is fixed. This condition is satisfied if $\varrho^{2n+2} - (1+\varrho^2)\varrho^n - 1 > 0$, i.e.,

$$\varrho^n > \frac{1 + \varrho^2 + \sqrt{1 + 6\varrho^2 + \varrho^4}}{2\varrho^2}.$$

3.3. **The measure** $d\lambda_3(t) = (1-t)^{-1/2}(1+t)^{1/2+s}, s \in \mathbb{N}_0$**.** In this case it was shown (see [29], [12]) that the monic s-orthogonal polynomials are the monic Jacobi orthogonal polynomials with parameters $\alpha = -1/2$, $\beta = 1/2$, i.e.,

$$\pi_{n,s}(t) = 2^{-n} V_n(t) = 2^{-n} \frac{\cos \dfrac{(2n+1)\theta}{2}}{\cos \dfrac{\theta}{2}}, \quad t = \cos\theta.$$

Therefore, by (2.3), where $w(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$, substituting $t = \cos\theta$, we have

$$\varrho_{n,s}(z) = 2^{s+1} \cdot 2^{-n(2s+1)} \int_0^\pi \frac{\cos \dfrac{\theta}{2} \left[\cos \dfrac{(2n+1)\theta}{2} \right]^{2s+1}}{z - \cos\theta} \, d\theta, \quad n \in \mathbb{N}, \ z \notin [-1, 1].$$

Using the representation (see [29])

$$V_n(z) = \frac{T_{2n+1}\left(\sqrt{\frac{1}{2}(1+z)}\right)}{\sqrt{\frac{1}{2}(1+z)}},$$

we obtain

$$K^{(3)}_{n,s}(z) = \frac{2^{s+1} \displaystyle\int_0^\pi \frac{\cos \dfrac{\theta}{2}}{z - \cos\theta} \left[\cos \dfrac{(2n+1)\theta}{2} \right]^{2s+1} d\theta}{\left[\dfrac{T_{n+1}\left(\sqrt{\frac{1}{2}(1+z)}\right)}{\sqrt{\frac{1}{2}(1+z)}} \right]^{2s+1}}.$$

The numerator of the last fraction has the form

$$
\begin{aligned}
2^{s+1} & \int_0^\pi \frac{\cos\dfrac{\theta}{2}}{z-\cos\theta}\left[\cos\frac{(2n+1)\theta}{2}\right]^{2s+1} d\theta \\
&= 2^{s+1}\int_0^\pi \frac{\cos\dfrac{\theta}{2}}{z-\cos\theta}\cdot\frac{1}{2^{2s}}\sum_{k=0}^{s}\binom{2s+1}{k}\cos\frac{(2s+1-2k)(2n+1)\theta}{2}\,d\theta \\
&= \frac{1}{2^s}\sum_{k=0}^{s}\binom{2s+1}{k}\int_0^\pi \frac{2\cos\dfrac{\theta}{2}}{z-\cos\theta}\cos\frac{(2m+1)\theta}{2}\,d\theta \\
&= \frac{1}{2^s}\sum_{k=0}^{s}\binom{2s+1}{k}\int_0^\pi\left[\frac{\cos(m+1)\theta}{z-\cos\theta}+\frac{\cos m\theta}{z-\cos\theta}\right]d\theta \\
&= \frac{\pi}{2^s\sqrt{z^2-1}}\sum_{k=0}^{s}\binom{2s+1}{k}\left[\left(z-\sqrt{z^2-1}\right)^{m+1}+\left(z-\sqrt{z^2-1}\right)^{m}\right] \\
&= \frac{2\pi}{2^s(u-u^{-1})}\sum_{k=0}^{s}\binom{2s+1}{k}\frac{u+1}{u^{m+1}},
\end{aligned}
$$

where (3.3) has been used and we have put $m=(2n+1)(s-k)+n$ and $z=\frac{1}{2}(u+u^{-1})$.

On the other hand, according to (3.4) and the fact that

$$\sqrt{\frac{1}{2}(z+1)}=\frac{u+1}{2\sqrt{u}}, \qquad z=\frac{1}{2}(u+u^{-1}),$$

we get

$$\frac{T_{2n+1}\left(\sqrt{\frac{1}{2}(1+z)}\right)}{\sqrt{\frac{1}{2}(z+1)}}=\frac{u^{n+1}+u^{-n}}{u+1}.$$

Therefore,

$$K_{n,s}^{(3)}(z)=\frac{2^{1-s}\pi(u+1)}{u^{n+1}(u-u^{-1})}\left(\frac{u+1}{u^{n+1}+u^{-n}}\right)^{2s+1}\sum_{k=0}^{s}\binom{2s+1}{s+k+1}u^{-(2n+1)k}.$$

Using (3.6) and letting

$$Z_{n,s}^{(3)}(u)=\sum_{k=0}^{s}\binom{2s+1}{s+k+1}u^{-(2n+1)k}, \tag{3.19}$$

we obtain

$$|K_{n,s}^{(3)}(z)|=\frac{2^{1-s}\pi}{\varrho^{n+1/2}}\cdot\frac{(a_1+\cos\theta)^{s+1}|Z_{n,s}^{(3)}(\varrho e^{i\theta})|}{(a_2-\cos 2\theta)^{1/2}(a_{2n+1}+\cos(2n+1)\theta)^{s+1/2}}, \tag{3.20}$$

when

$$z=\frac{1}{2}(\varrho e^{i\theta}+\varrho^{-1}e^{-i\theta})\in\mathcal{E}_\varrho.$$

An analysis of (3.20) shows that the point of the maximum of $|K_{n,s}^{(3)}(z)|$ for a given ϱ depends on n as in the case of the measure $d\lambda_1(t)$. If there exists a sequence of the local maxima, numerical experiments show that it decreases when θ runs over $[0, \pi]$. For this reason and because of better clarity in the following figures, the graphics of the function $\theta \mapsto |K_{n,s}^{(3)}(z)|$ $(z = (u + u^{-1})/2,\ u = \varrho e^{i\theta})$ for some selected n, s, and ϱ are presented only for $\theta \in [0, \pi/2]$. The case $n = 10$, $s = 1$ is given in Figure 4 for $\varrho = 1.01$, 1.05, 1.1, and 1.15. The graphics for $s = 1, 2, 3$, when $n = 10$ and $\varrho = 1.1$ and 1.15, are presented in Figure 5.

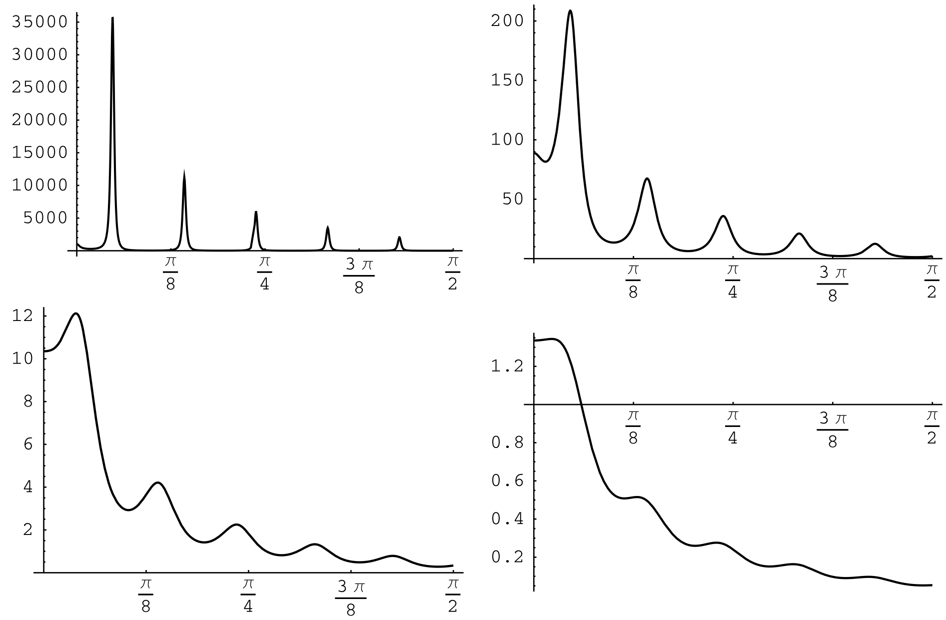

FIGURE 4. The function $\theta \mapsto |K_{10,1}^{(3)}(z)|$ $(z = \frac{1}{2}(u+u^{-1}),\ u = \varrho e^{i\theta})$ for $\varrho = 1.01$, $\varrho = 1.05$ (top) and $\varrho = 1.1$, $\varrho = 1.15$ (bottom)

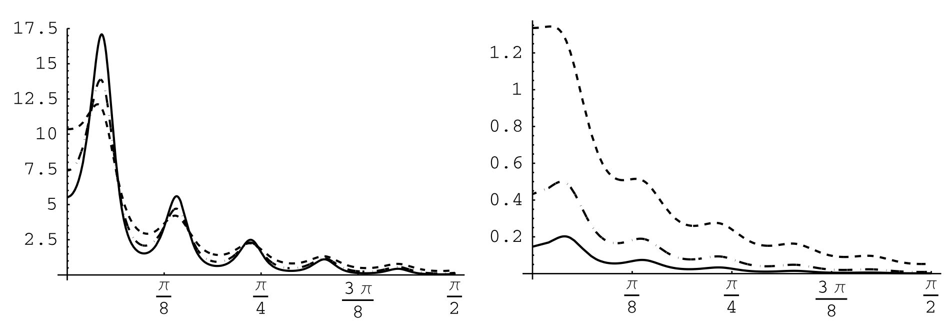

FIGURE 5. The function $\theta \mapsto |K_{10,s}^{(3)}(z)|$, $z = \frac{1}{2}(u+u^{-1})$, $u = \varrho e^{i\theta}$, for $s = 1$ (dashed line), $s = 2$ (dot-dashed line), $s = 3$ (solid line), when $\varrho = 1.1$ (left) and $\varrho = 1.15$ (right)

On the basis of numerical experiments a similar conjecture for $|K_{n,s}^{(3)}(z)|$ on the ellipse $\mathcal{E}_\varrho$ as in Conjecture 3.1 can be stated.

A useful estimate of (3.20) can be given by using the fact that $|Z_{n,s}^{(3)}(\varrho e^{i\theta})| \le Z_{n,s}^{(3)}(\varrho) < Z_{n,s}^{(3)}(1) = 2^{2s}$ and the inequality (see [10, p. 1179])

$$\frac{a_1+\cos\theta}{(a_1-\cos\theta)(a_{2n+1}+\cos(2n+1)\theta)} \le \frac{a_1+1}{(a_1-1)(a_{2n+1}+1)}, \quad 0\le\theta\le\pi,$$

which is equivalent to

$$\frac{a_1+\cos\theta}{\sqrt{(a_2-\cos 2\theta)(a_{2n+1}+\cos(2n+1)\theta)}} \le \frac{a_1+1}{\sqrt{(a_2-1)(a_{2n+1}+1)}}, \quad 0\le\theta\le\pi,$$

because of $a_2 = 2a_1^2-1$ and $a_1-\cos\theta = \frac{1}{2}(a_2-\cos 2\theta)/(a_1+\cos\theta)$. In this way, we get

$$|K_{n,s}^{(3)}(z)| \le \frac{2\pi}{\varrho^{n+1/2}}\cdot\frac{a_1+1}{\sqrt{(a_2-1)(a_{2n+1}+1)}}\left(\frac{2(a_1+\cos\theta)}{a_{2n+1}+\cos(2n+1)\theta}\right)^s,$$

i.e.,

$$|K_{n,s}^{(3)}(z)| \le \frac{2\pi(\varrho+1)}{\varrho^n(\varrho-1)(\varrho^{n+1}+\varrho^{-n})}\left(\frac{\sqrt{2}\,(\varrho^{1/2}+\varrho^{-1/2})}{\varrho^{n+1/2}-\varrho^{-n-1/2}}\right)^{2s}, \tag{3.21}$$

and then, the estimate of the corresponding remainder (3.2) becomes

$$|R_{n,s}(f)| \le \frac{M}{\varrho^n(\varrho^{n+1}+\varrho^{-n})}\left(\frac{\sqrt{2}\,(\varrho+1)}{\varrho^{n+1}-\varrho^{-n}}\right)^{2s}, \qquad M = 2E(\varepsilon)\|f\|_\varrho(\varrho+\varrho^{-1})\frac{\varrho+1}{\varrho-1}.$$

It is clear that $R_{n,s}(f) = O(\varrho^{-2n(s+1)})$ when $n\to+\infty$. Assuming that

$$\sqrt{2}\,\frac{\varrho+1}{\varrho^{n+1}-\varrho^{-n}} < 1,$$

which is equivalent to

$$\varrho^n > \frac{1+\varrho+\sqrt{1+4\varrho+\varrho^2}}{\varrho\sqrt{2}},$$

we conclude that $R_{n,s}(f)\to 0$ when $s\to+\infty$ and n is fixed.

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