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COMPUTATION OF STARK-TAMAGAWA UNITS

W. BLEY

Abstract. Let K be a totally real number field and let l denote an odd prime number. We design an algorithm which computes strong numerical evidence for the validity of the "Equivariant Tamagawa Number Conjecture" for the $\mathbb{Q}[G]$-equivariant motive $h^0(\mathrm{Spec}(L))$, where L/K is a cyclic extension of degree l and group G. This conjecture is a very deep refinement of the classical analytic class number formula. In the course of the algorithm, we compute a set of special units which must be considered as a generalization of the (conjecturally existing) Stark units associated to first order vanishing Dirichlet L-functions.

1. Introduction

Let L/K denote a finite Galois extension of number fields of group G. In this paper we provide numerical evidence for the so-called "Equivariant Tamagawa Number Conjecture" for the $\mathbb{Q}[G]$-equivariant motive $h^0(\mathrm{Spec}(L))$ formulated in [3] and [6]. Our approach is based on the results of [2], where for a large class of abelian extensions L/K the conjectural vanishing of the Tamagawa number $T\Omega(L/K)$ of $h^0(\mathrm{Spec}(L))$ is interpreted in terms of the existence of S-units satisfying a variety of explicit conditions.

These conditions are in the same spirit as the conditions studied by Rubin in [16] and Popescu in [14], but are in general much finer. Indeed, recent work of Burns [5] shows that in the context of this paper the "Equivariant Tamagawa Number Conjecture" implies a certain natural refinement of the conjectures of Rubin and Popescu. In turn, their conjectures are generalizations of the well-known refined Stark conjecture "over $\mathbb{Z}$" for first order vanishing Dirichlet L-functions (cf. [18, Chap. IV]). Therefore our examples also provide new evidence in favour of these Stark-type conjectures.

The article is organized in the following way: in Section 2 we recall the main result of [2]. In Section 3 we describe an algorithm which verifies the above mentioned conjecture (up to the precision of the computation) for cyclic extensions L of odd prime degree of a totally real number field K, and Section 4 contains a worked-out example.

2. The equivariant Tamagawa number conjecture

We fix a Galois extension L/K of number fields and set $G = \mathrm{Gal}(L/K)$. Let $K_0(\mathbb{Z}[G], \mathbb{R})$ denote the Grothendieck group of the fibre category of the functor

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$\_ \otimes_{\mathbb{Z}} \mathbb{R}$ from the category of finitely generated projective left $\mathbb{Z}[G]$-modules to the category of finitely generated left $\mathbb{R}[G]$-modules. If G is abelian (which is the case we will be concerned with), this group is canonically isomorphic to the group of invertible $\mathbb{Z}[G]$-sublattices of $\mathbb{R}[G]$ (cf. [1, Lem. 2.6(i)]).

In [4] Burns used complexes arising from étale cohomology to define a canonical element $T\Omega(L/K) \in K_0(\mathbb{Z}[G], \mathbb{R})$. For a precise definition in the general case the reader is referred to loc.cit.; an easier accessible version in the abelian case is contained in [1]. In [4] it is shown that the Stark conjecture [18, Ch.I, 5.1] is equivalent to asserting $T\Omega(L/K) \in K_0(\mathbb{Z}[G], \mathbb{Q})$, the Strong Stark conjecture [8, Conj. 2.2] is equivalent to the containement $T\Omega(L/K) \in K_0(\mathbb{Z}[G], \mathbb{Q})_{tor}$, and finally, the "equivariant Tamagawa number conjecture" (for short, ETNC) of [4], [6] for the special motive $h^0(\mathrm{Spec}(L))$ is equivalent to the equality $T\Omega(L/K) = 0$.

Very recently, Burns and Greither [7] have proven ETNCfor all abelian extensions $L/\mathbb{Q}$ of odd conductor, but beyond this, very little concerning ETNC is known so far.

From now on we assume that G is abelian and fix the following notation. For any finite set S of places of K which contains the set S_∞ of archimedian places, we write $S(L)$ for the set of places of L lying above places in S, $\mathcal{O}_S$ for the $S(L)$-integers of L, cl_S for the $S(L)$-class group, and we set $h_S := |cl_S|$. We let U_S denote the $S(L)$-units of L, μ_L the torsion subgroup of U_S, and we set $E_S := U_S/\mu_L$. Analogously we write $U_{K,S}$ and $E_{K,S}$ for the corresponding groups on the base field level. We let Y_S denote the free abelian group on the set $S(L)$ and write X_S for the kernel of the homomorphism $Y_S \to \mathbb{Z}$ which sends each element of $S(L)$ to 1. For later reference we recall that if $(h_S, |G|) = 1$, then there exists an exact sequence of finitely generated G-modules

$$0 \longrightarrow U_S \longrightarrow A \longrightarrow B \longrightarrow X_S \longrightarrow 0 \tag{1}$$

with A, B of finite projective dimension. (This is shown in [17] under the assumption $h_S = 1$, but the same argument works with our slightly weaker hypothesis.)

For each place w of L we let $|\cdot|_w$ denote the absolute value of w, which is normalised as in [18, Chap. 0, 0.2]. We let $R_S : U_S \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow X_S \otimes_{\mathbb{Z}} \mathbb{R}$ denote the $\mathbb{R}[G]$-equivariant isomorphism given by

$$R_S(u) = -\sum_{w \in S(L)} \log |\, u \,|_w \cdot w$$

for each $u \in U_S$.

We let G^* denote the group of abelian characters of G. For each $\chi \in G^*$ we write $L_S(s, \chi)$ for the associated S-truncated Dirichlet L-function and e_χ for the primitive idempotent $|G|^{-1} \sum_{g \in G} \chi(g) g^{-1}$. In this way we obtain a $\mathbb{C}[G]$-valued function of the complex variable s by setting

$$L_S(s) := \sum_{\chi \in G^*} L_S(s, \chi^{-1}) e_\chi.$$

We let $L_S^*(0, \chi)$ denote the leading coefficient in the Taylor expansion of $L_S(s, \chi)$ at $s = 0$, and set $L_S^*(0) := \sum_{\chi \in G^*} L_S^*(0, \chi^{-1}) e_\chi$. Roughly speaking, ETNC predicts a conjectural formula for the $\mathbb{Z}[G]$-sublattice of $\mathbb{R}[G]$ which is generated by $L_S^*(0)$ (note that $L_S(s, \bar{\chi}) = \overline{L_S(s, \chi)}$ for $s \in \mathbb{R}$ implies $L_S^*(0) \in \mathbb{R}[G]$).

Let E be the field generated over $\mathbb{Q}$ by the values of elements of G^*, and write $\mathcal{O}$ for its ring of algebraic integers. For any commutative ring R and each R-module

M we write $\mathrm{Fitt}_R(M)$ for the (first) Fitting ideal of M. (We refer the reader to [11, App.] or [13, Sec. 1.4] for the basic properties of Fitting ideals.) If M is any finite G-module with $(|M|, |G|) = 1$, then each e_χ acts naturally on $\mathcal{O} \otimes_{\mathbb{Z}} M$, and we set

$$\mathrm{Fitt}_\chi(M) := \mathrm{Fitt}_{\mathcal{O}}(e_\chi(\mathcal{O} \otimes_{\mathbb{Z}} M)).$$

Note that $\mathrm{Fitt}_\chi(M)$ coincides with the usual $\mathcal{O}$-order ideal of the finitely generated $\mathcal{O}$-torsion module $e_\chi(\mathcal{O} \otimes_{\mathbb{Z}} M)$.

We now introduce a natural simplifying hypothesis on the extension L/K under which ETNCcan be interpreted as asserting the existence of special units in L. For each place v of K we write G_v for the decomposition subgroup in G.

Hypothesis (S). *There exists a finite set S of places of K which satisfies all of the following conditions: S contains all archimedean places and all places which ramify in L/K; $(|G|, h_S) = 1$; there exists a place $v_0 \in S$ for which $G_{v_0} = G$; and for each place $v \in S_0 := S \setminus \{v_0\}$, the group G_v is cyclic.*

Assuming that S is as described in Hypothesis (S), we fix a generator g_v of G_v for each $v \in S_0$. We let w_0 denote the (unique) place of L lying above v_0, and for each $v \in S_0$ we choose a place w_v of L lying above v. For each $v \in S_0$, and each place $w \in S(L) \setminus \{w_0\}$, we define $\delta_{v,w}$ to be 1 if $w = w_v$ and to be 0 otherwise. If $w \in S(L)$, we write $v(w)$ for the unique place in S defined by w. For each $\chi \in G^*$ we let S_χ denote the set $\{v \in S_0 : \chi(g_v) = 1\}$, and we define $S^\chi := S_0 \setminus S_\chi$.

If N/M is a finite abelian extension of p-adic fields, we write $(-, N/M)$ for the associated Artin map $M^\times \longrightarrow \mathrm{Gal}(N/M)$. If U is a subgroup of $\mathrm{Gal}(N/M)$, then N^U denotes the subfield of N fixed by U.

The next two theorems are just reformulations of [2, Th. 3.2].

Theorem 2.1. *Assume that $(|\mu_L|, |G|) = 1$ and in addition that S is as described in Hypothesis (S). Then there exist elements $\epsilon_v \in U_{L^{G_v},S}$ for each $v \in S_0$ such that*

(i) *the index of $\mathcal{E}_S := \langle \epsilon_v : v \in S_0 \rangle_{\mathbb{Z}}[G]$ in U_S is finite and coprime to $|G|$, and*
(ii) *for each place $v \in S_0$ and each place w of $S(L) \setminus \{w_0\}$ one has*

$$(\epsilon_v, L_w/L_w^{G_v \cap G_{v(w)}}) = g_v^{\delta_{v,w}}.$$

Proof. This is immediate from [2, Prop. 2.1] and the first part of the proof of [2, Th. 3.2]. □

Theorem 2.2. *Assume the notation of Theorem 2.1. Then $T\Omega(L/K) = 0$ if and only if for each $\chi \in G^*$ there exists an element a_χ of $E^\times$ such that*

(i) $\prod_{v \in S^\chi} (\chi(g_v) - 1) \cdot \bigwedge_{v \in S_\chi} \frac{1}{|G_v|} R_S(e_\chi(\epsilon_v)) = a_\chi L_S^*(0, \chi^{-1}) \cdot \bigwedge_{v \in S_\chi} e_\chi(w_v - w_0)$,
(ii) $a_\chi \mathcal{O} = \mathrm{Fitt}_\chi(U_S/\mathcal{E}_S)\mathrm{Fitt}_\chi(cl_S)^{-1}$, *and*
(iii) $h_S a \in \mathbb{Z}[G]$, *where* $a := \sum_{\chi \in G^*} a_\chi e_\chi$.

Proof. This is proved in the second part of the proof of [2, Th. 3.2]. □

Remarks 2.3. a) Equality (i) in Theorem 2.2 is an equality in the one-dimensional $\mathbb{C}$-vector space $\bigwedge_{v \in S_\chi} e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} X_S)$ and therefore uniquely determines complex numbers a_χ for each $\chi \in G^*$. Stark's conjecture for all characters $\chi \in G^*$ is equivalent to the assertion $a \in \mathbb{Q}[G]$. If we assume Stark's conjecture, then equality (ii) holds if and only if the Strong Stark conjecture is true. Finally, (ii) and (iii) together imply

$$a\mathbb{Z}[G] = \mathrm{Fitt}_{\mathbb{Z}}[G](U_S/\mathcal{E}_S)\mathrm{Fitt}_{\mathbb{Z}}[G](cl_S)^{-1}.$$

This statement in turn is equivalent to the equality $T\Omega(L/K) = 0$.

Very recently, Burns and Greither have proven ETNC for any abelian extension $L/\mathbb{Q}$ of odd conductor. Furthermore, ETNC is known to be valid for a natural family of non-abelian extensions $L/\mathbb{Q}$ for which G is isomorphic to the quaternion group of order 8 (cf. [12]). Apart from these extensions and their subextensions, we are not aware of any other abelian extensions L/K for which ETNC is known to be true. In fact, the extensions we will consider in the next section are not even known to validate Stark's conjecture or the Strong Stark conjecture.

b) If we assume in addition to Hypothesis (S) that S contains precisely r places that split completely in L/K, then the $(r-1)$-st derivative $L_S^{(r-1)}(0,\chi)$ equals 0 for all $\chi \in G^*$ (cf. [18, Chap. I, Prop. 3.4]). Therefore $L_S^{(r)}(0,\chi)$ is of particular interest, and Rubin in [16], and subsequently Popescu in [14], conjectured certain natural integrality properties for the r-th derivative $L_S^{(r)}(0)$ of $L_S(s)$. Their conjectures generalize the refined conjecture "over $\mathbb{Z}$" formulated by Stark in the case $r = 1$ (cf. [18, Chap. IV]).

If μ_L is cohomologically trivial, then Rubin's and Popescu's conjecture are equivalent as a consequence of [14, Th. 5.5.1]. In addition, the main result of [5, Sec. 3.2] shows that under our hypothesis ETNC implies a strong refinement of Popescu's conjecture (cf. [5, Rem. 3.3(iv)]).

For a brief discussion about what is known about these Stark-type conjectures of Rubin and Popescu, we recommend the interested reader to consult [14, §6].

3. An algorithm

Let K denote a totally real number field of degree n. We fix an odd prime number l and let L/K denote a cyclic extension of number fields of degree l. Note that for each such extension there exists a set S such that the assumptions of Theorems 2.1 and 2.2 are satisfied. Our aim is to use these results to develop an algorithm to check the validity of ETNCfor L/K up to the precision of the computation.

We assume that L/K is given by class field theoretic data as described in [9, Chap. 3 and 4]. In particular, we let $\mathfrak{f} = \mathfrak{f}_{L/K}$ denote the conductor of L/K and write $cl_{\mathfrak{f}}(K)$ for the ray class group modulo $\mathfrak{f}$. Let $\mathcal{H} \le cl_{\mathfrak{f}}(K)$ denote the subgroup of index l corresponding to the given extension L. Then recently developed algorithms due to Cohen and Roblot (cf. [9, Chap. 6] or [15]) allow us to compute defining polynomials for L. Based on this, we further assume that we are able to compute all basic invariants of L, such as the ring of algebraic integers, the ideal class group and a system of fundamental units (and also the S-versions of these objects).

In the following we write $S_\infty = \{\infty_1, \dots, \infty_n\}$, $S_{ram} = \{\mathfrak{p}_0, \dots, \mathfrak{p}_r\} = \{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{f}\}$ and choose a set $S' = \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$ of primes which split completely in L/K and such that for $S = S_\infty \cup S_{ram} \cup S'$ one has $l \nmid h_S$. We will use $\mathfrak{p}_0$ as the distinguished place v_0 of Hypothesis (S). Then

$$X_S = \bigoplus_{i=1}^{n} \mathbb{Z}[G](\hat{\infty}_i - \hat{\mathfrak{p}}_0) \oplus \bigoplus_{i=1}^{s} \mathbb{Z}[G](\hat{\mathfrak{q}}_i - \hat{\mathfrak{p}}_0) \oplus \bigoplus_{i=1}^{r} \mathbb{Z}(\hat{\mathfrak{p}}_i - \hat{\mathfrak{p}}_0), \tag{2}$$

where for each place $\mathfrak{p} \in S$ we choose a place $\hat{\mathfrak{p}}$ of L above $\mathfrak{p}$.

The $\mathbb{Z}$-rank of U_S is then given by $m := l(n+s)+r$, and we let $\delta_1, \dots, \delta_m$ denote a system of fundamental $S(L)$-units. Finally we assume that the representation of G induced by its action on E_S and this choice of fundamental units is explicitly known: for $g \in G$ we denote by $D(g)$ the corresponding matrices. Each element $u = \prod_{i=1}^m \delta_i^{x_i}$ of E_S is then represented by the vector $x := (x_1, \dots, x_m)^t$, and the action of $g \in G$ on u is translated into the matrix multiplication $D(g)x$. We will perform all our computations in $U_{S,\mathbb{Q}} := U_S \otimes_{\mathbb{Z}} \mathbb{Q}$ and thus represent each $u \in E_S$ by a vector $x \in \mathbb{Q}^m$.

We will simply view this data as input for our algorithm. Its actual computation is, of course, a very hard problem on its own. It is remarkable that the PARI system provides almost all routines to compute this input, at least for small degrees l and small conductors.

Let ζ denote a primitive l-th root of unity and set $E = \mathbb{Q}(\zeta)$. We fix a generator g_0 of G and define a character $\chi \in G^*$ by $\chi(g_0) = \zeta$. Thus $G^* = \langle \chi \rangle$. We also write χ_0 for the trivial character of G, $e_0 = \frac{1}{l}\sum_{g \in G} g$ and $e_1 = 1 - e_0$ for the primitive idempotents of $\mathbb{Q}[G]$, and note that the map $\lambda \mapsto (\chi_0(\lambda), \chi(\lambda))$, $\lambda \in \mathbb{Q}[G]$, defines a natural identification of $\mathbb{Q}[G]$ and $\mathbb{Q} \oplus E$. Without further mention we will henceforth identify $\mathbb{Q}[G]$ and $\mathbb{Q} \oplus E$. The lattice $e_1 E_S \subseteq U_{S,\mathbb{Q}}$ is then naturally endowed with the structure of an $\mathcal{O}$-module.

3.1. **Computation of $\mathcal{E}_S$.** In this subsection we explain how to compute a $\mathbb{Z}[G]$-sublattice $\mathcal{E}_S$ of U_S such that the assertions of Theorem 2.1 are satisfied. Concretely, we have to exhibit S-units $\epsilon_{\infty_1}, \dots, \epsilon_{\infty_n}, \epsilon_{\mathfrak{q}_1}, \dots, \epsilon_{\mathfrak{q}_s} \in U_S$ and $\epsilon_{\mathfrak{p}_1}, \dots, \epsilon_{\mathfrak{p}_r} \in U_{K,S}$. To simplify our notation we write $\epsilon_1, \dots, \epsilon_{n+s}$, resp. $\eta_1, \dots, \eta_r$, for $\epsilon_{\infty_1}, \dots, \epsilon_{\infty_n}$, $\epsilon_{\mathfrak{q}_1}, \dots, \epsilon_{\mathfrak{q}_s}$, resp. $\epsilon_{\mathfrak{p}_1}, \dots, \epsilon_{\mathfrak{p}_r}$.

Lemma 3.1. *Suppose that $\mathcal{A} = \langle \epsilon_1, \dots, \epsilon_{n+s} \rangle_{\mathbb{Z}}[G] \subseteq U_S$ satisfies $l \nmid [e_1 E_S : e_1 \mathcal{A}]$. Then there exist S-units $\eta_1, \dots, \eta_r \in U_{K,S}$ such that $l \nmid [U_{K,S} : \langle l e_0 \mathcal{A}, \eta_1, \dots, \eta_r \rangle_{\mathbb{Z}}]$. In addition, if one sets $\mathcal{E}_S := \langle \epsilon_1, \dots, \epsilon_{n+s}, \eta_1, \dots, \eta_r \rangle_{\mathbb{Z}}[G]$, then $l \nmid [U_S : \mathcal{E}_S]$.*

Proof. For any finitely generated $\mathbb{Z}$-module M we write $M_l := M \otimes_{\mathbb{Z}} \mathbb{Z}_l$ for its l-completion. By [2, Prop. 2.1] we know there exist $\omega_1, \dots \omega_{n+s} \in U_S, \nu_1, \dots, \nu_r \in U_{K,S}$ such that $U_{S,l} = \langle \omega_1, \dots, \omega_{n+s}, \nu_1, \dots, \nu_r \rangle_{\mathbb{Z}_l[G]}$. This will be the key observation in the proof of Lemma 3.1.

The short exact sequence $0 \to E_{K,S} \to E_S \to e_1 E_S \to 0$ together with our assumption $l \nmid [e_1 E_S : e_1 \mathcal{A}]$ implies

$$U_{S,l} = \langle \epsilon_1, \dots, \epsilon_{n+s}, U_{K,S} \rangle_{\mathbb{Z}_l[G]} = \langle \omega_1, \dots, \omega_{n+s}, U_{K,S} \rangle_{\mathbb{Z}_l[G]}. \tag{3}$$

We let $\mathcal{O}_l$ denote the localization of $\mathcal{O}$ with respect to the unique prime of E above l, and recall that the maximal $\mathbb{Z}_l$-order in $\mathbb{Q}_l[G]$ is isomorphic to $\mathbb{Z}_l \oplus \mathcal{O}_l$. Via this identification one has $\mathbb{Z}_l[G] = \{(a, \alpha) \in \mathbb{Z}_l \oplus \mathcal{O}_l \mid a \equiv \alpha (\mathrm{mod}\ (1-\zeta))\}$.

Let $A \in \mathrm{Gl}_{n+s}(\mathcal{O}_l)$ denote the matrix such that

$$(e_1\epsilon_1, \dots, e_1\epsilon_{n+s}) = (e_1\omega_1, \dots, e_1\omega_{n+s})A,$$

and choose a matrix $A_1 \in \mathrm{Mat}_{n+s,n+s}(\mathbb{Z}_l)$ such that $A \equiv A_1 (\mathrm{mod}\ (1-\zeta))$, where here and in the following congruences of matrices are meant componentwise. Then $A + A_1 \in \mathrm{Mat}_{n+s,n+s}(\mathbb{Z}_l[G])$, $\det(A_1) \in \mathbb{Z}_l^\times$ and

$$(e_0\epsilon_1, \dots, e_0\epsilon_{n+s}) \equiv (e_0\omega_1, \dots, e_0\omega_{n+s})A_1 (\mathrm{mod}\ U_{K,S,l}). \tag{4}$$

We write H^i for the Tate cohomology groups and let N_G denote the norm operator. From (1) and (2) we obtain $U_{K,S}/N_G U_S = H^0(G, U_S) = H^0(G, X_S) = \bigoplus_{i=1}^r \mathbb{Z}/l\mathbb{Z}$. Since $rk_{\mathbb{Z}}(U_{K,S}) = n + s + r$ we conclude that $e_0\omega_i \notin U_{K,S}$ for $i = 1, \ldots, n+s$, and $U_{K,S,l} = \langle le_0\omega_1, \ldots, le_0\omega_{n+s}, \nu_1, \ldots, \nu_r\rangle_{\mathbb{Z}_l}$.

If we let $u_1, \ldots, u_{n+r+s}$ denote a system of fundamental units of $U_{K,S}$, then there exists an invertible matrix $B \in \mathrm{Gl}_{n+r+s}(\mathbb{Z}_l)$ such that

$$(le_0\omega_1, \ldots, le_0\omega_{n+s}, \nu_1, \ldots, \nu_r) = (u_1, \ldots, u_{n+r+s})B.$$

Writing $B = (B_1 \mid B_2)$ with $B_1 \in \mathrm{Mat}_{n+r+s,n+s}(\mathbb{Z}_l), B_2 \in \mathrm{Mat}_{n+r+s,r}(\mathbb{Z}_l)$, it follows that B_1 has an invertible $(n+s) \times (n+s)$ minor.

From (4) we deduce

$$(e_0\epsilon_1, \ldots, e_0\epsilon_{n+s}) = (e_0\omega_1, \ldots, e_0\omega_{n+s})A_1 + (v_1, \ldots, v_{n+s}) \tag{5}$$

with $v_1, \ldots, v_{n+s} \in U_{K,S,l}$, and furthermore

$$\begin{aligned}(le_0\epsilon_1, \ldots, le_0\epsilon_{n+s}) &= (u_1, \ldots, u_{n+r+s})B_1A_1 + l(v_1, \ldots, v_{n+s}) \\ &= (u_1, \ldots, u_{n+r+s})H\end{aligned}$$

with a matrix $H \in \mathrm{Mat}_{n+r+s,n+s}(\mathbb{Z}_l)$. Since $B_1A_1 \equiv H(\mathrm{mod}\ l)$ and $\det(A_1) \in \mathbb{Z}_l^\times$, the matrix H also contains an invertible $(n+s) \times (n+s)$ minor. Therefore we can complement $le_0\epsilon_1, \ldots, le_0\epsilon_{n+s}$ with $\eta_1, \ldots, \eta_r$ to obtain a basis of $U_{K,S,l}$, and it is now obvious from (3) that $U_{S,l} = \langle \epsilon_1, \ldots, \epsilon_{n+s}, \eta_1, \ldots, \eta_r\rangle_{\mathbb{Z}_l[G]}$. □

Given the result of Lemma 3.1, we achieve the computation of $\mathcal{E}_S$ in three steps.

Step 1: Computation of $\mathcal{A} := \langle \epsilon_1, \ldots, \epsilon_{n+s}\rangle_{\mathbb{Z}}[G] \subseteq U_S$ such that $l \nmid [e_1E_S : e_1\mathcal{A}]$.

Step 2: Computation of $\eta_1', \ldots, \eta_r' \in U_{K,S}$ such that

$$l \nmid [E_{K,S} : \langle le_0\mathcal{A}, \eta_1', \ldots, \eta_r'\rangle_{\mathbb{Z}}].$$

Step 3: Adaption of $\eta_1', \ldots, \eta_r'$ such that part (ii) of Theorem 2.1 is satisfied.

We begin with the description of Step 1. Recall that we always identify $e_1\mathbb{Q}[G]$ and E. Inductively we construct an E-basis $\xi_1, \ldots, \xi_{n+s}$ of $e_1U_{S,\mathbb{Q}}$: suppose that $\xi_1, \ldots, \xi_{i-1}$ are already computed. Then we choose $\delta \in \{\delta_1, \ldots, \delta_m\}$ such that $e_1\delta \in e_1U_{S,\mathbb{Q}} \setminus \langle \xi_1, \ldots, \xi_{i-1}\rangle_{e_1\mathbb{Q}[G]}$ (a condition which is easily checked by solving a system of linear equations) and set $\xi_i = e_1\delta$.

Next we compute the $(n+s) \times m$ matrix A with coefficients in E such that

$$(e_1\delta_1, \ldots, e_1\delta_m) = (\xi_1, \ldots, \xi_{n+s})A.$$

We will use the Hermite normal form algorithm in Dedekind domains (cf. [9, Th. 1.4.6 and Alg. 1.4.7]) and also stick to the notation of loc.cit. In this way we obtain a matrix $U \in \mathrm{Gl}_m(E)$ and nonzero fractional $\mathcal{O}$-ideals $\mathfrak{c}_1, \ldots, \mathfrak{c}_{n+s}$ such that

$$e_1U_S = \mathfrak{c}_1\omega_1 \oplus \ldots \oplus \mathfrak{c}_{n+s}\omega_{n+s}$$

with

$$\omega_i = (e_1\delta_1, \ldots, e_1\delta_m)v_i, \quad i = 1, \ldots, n+s,$$

where v_i is the $(m-n-s+i)$-th column of U. For $i = 1, \ldots, n+s$ we choose integral ideals $\mathfrak{c}_i'$ such that $(l, \mathfrak{c}_i') = 1$ and $\mathfrak{c}_i\mathfrak{c}_i' = (c_i), c_i \in E$. We then define the e_1-component of ϵ_i by

$$e_1\epsilon_i := c_i\omega_i, \quad i = 1, \ldots, n+s.$$

By [9, Th. 1.4.6(1)] $c_i v_i$ has coefficients in $\mathfrak{c}_i' \subseteq \mathcal{O}$. Hence we may choose a vector $w_i \in \mathbb{Z}^m$ such that $c_i v_i \equiv w_i (\mathrm{mod}\ (1-\zeta))$. Then the element

$$\epsilon_i := (e_1\delta_1, \ldots, e_1\delta_m)c_i v_i + (e_0\delta_1, \ldots, e_0\delta_m)w_i$$

is actually an S-unit with e_1-component $c_i\omega_i$. By construction we have

$$[e_1\mathcal{E}_S : e_1\mathcal{A}]_{\mathcal{O}} = \prod_{i=1}^{n+s} \frac{c_i}{\mathfrak{c}_i} = \prod_{i=1}^{n+s} \mathfrak{c}_i' =: \mathfrak{c}',$$

which is prime to l, as desired.

To accomplish Step 2 we compute the matrix $H \in \mathrm{Mat}_{n+r+s,n+s}(\mathbb{Z})$ such that $(le_0\epsilon_1, \ldots, le_0\epsilon_{n+s}) = (u_1, \ldots, u_{n+r+s})H$, where $u_1, \ldots, u_{n+r+s}$ denotes a system of fundamental units for $U_{K,S}$. As shown at the end of the proof of Lemma 3.1, there exists an $(n+s)\times(n+s)$ minor H_1 of H with $(\det(H_1), l) = 1$. Without loss of generality we may assume $H = \begin{pmatrix} * \\ H_1 \end{pmatrix}$, so that we can take $\eta_i' = u_i$ for $i = 1, \ldots, r$.

Note that by now we have constructed a sublattice $\mathcal{E}_S'$ of U_S generated over $\mathbb{Z}[G]$ by $\epsilon_1, \ldots, \epsilon_{n+s}, \eta_1', \ldots, \eta_r'$ with the following property:

$$[e_1U_S : e_1\mathcal{E}_S'] = N_{E/\mathbb{Q}}(\mathfrak{c}'), \quad [e_0U_S : e_0\mathcal{E}_S'] = 2|\det(H_1)|.$$

For $j = 0, \ldots, r$ we let L_j/K_j denote the completion of L/K at the totally ramified prime $\mathfrak{p}_j$. In the final Step 3 we will compute a matrix $C \in \mathrm{Mat}_{r,r}(\mathbb{Z})$ and set $(\eta_1, \ldots, \eta_r) = (\eta_1', \ldots, \eta_r')C$. The matrix C will be chosen such that $l \nmid \det(C)$ and $(\eta_i, L_j/K_j) = g_0^{\delta_{ij}}$ Kronecker delta purpose we consider the matrix $D = (d_{ij}) \in \mathrm{Mat}_{r,r}(\mathbb{Z}/l\mathbb{Z})$ defined by

$$(\eta_i', L_j/K_j) = g_0^{d_{ij}}.$$

Lemma 3.2. $D \in \mathrm{Gl}_r(\mathbb{Z}/l\mathbb{Z})$.

Proof. By [2, Prop. 2.1] we know that there exist $\omega_1, \ldots, \omega_{n+s} \in U_S$ and $\nu_1, \ldots, \nu_r \in U_{K,S}$ such that

$$U_{S,l} = \langle \omega_1, \ldots, \omega_{n+s}, \nu_1, \ldots, \nu_r \rangle_{\mathbb{Z}_l[G]} \text{ and } (\nu_i, L_j/K_j) = g_0^{\delta_{ij}}, \quad 1 \le i, j \le r.$$

The proof of Lemma 3.1 shows that there exists a matrix $X = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathrm{Gl}_{n+r+s}(\mathbb{Z}_l)$ such that

$$(le_0\epsilon_1, \ldots, le_0\epsilon_{n+s}, \eta_1', \ldots, \eta_r') = (le_0\omega_1, \ldots, le_0\omega_{n+s}, \nu_1, \ldots, \nu_r)X.$$

Moreover, it follows from (5) that $R \equiv 0 (\mathrm{mod}\ l)$ and therefore $S \in \mathrm{Gl}_r(\mathbb{Z}_l)$. We now conclude that

$$(\eta_i', L_j/K_j) = \prod_{k=1}^{n+s}(le_0\omega_k, L_j/K_j)^{q_{ki}} \prod_{l=1}^{r}(\nu_l, L_j/K_j)^{s_{li}} = \prod_{l=1}^{r}(\nu_l, L_j/K_j)^{s_{li}} = g_0^{s_{ji}},$$

where $Q = (q_{kl}), S = (s_{kl})$ and all exponents are read in $\mathbb{Z}/l\mathbb{Z}$. Hence $D \equiv S^t (\mathrm{mod}\ l)$. $\square$

Now let $C \in \mathrm{Mat}_{r,r}(\mathbb{Z})$ such that $C \equiv D^{-1} (\mathrm{mod}\ l)$, and define $\eta_1, \ldots, \eta_r$ by the equality $(\eta_1, \ldots, \eta_r) = (\eta_1', \ldots, \eta_r')C$.

Remark 3.3. Lemmas 3.1 and 3.2 should be considered as very explicit versions of the main results of Holland (cf. [10, Th. 3.1]) applied to our very special situation.

For the computation of D it remains to show how to determine the local Artin symbols $(\eta_i', L_j/K_j)$. We let $v_j = v_{\mathfrak{p}_j}$ denote the $\mathfrak{p}_j$-valuation of K for $j = 0, \ldots, r$. Set $e_{ij} = v_j(\eta_i')$ and choose an element $\pi_j \in \mathcal{O}_K$ such that $v_i(\pi_j) = \delta_{ij}$ (Kronecker delta). Let $\mathfrak{f} = \prod_{k=0}^r \mathfrak{p}_k^{s_k}$ be the prime ideal factorization of the conductor $\mathfrak{f}$. If we let $\xi_{ij} \in \mathcal{O}_K$ denote a solution of the simultanous congruences

$$\begin{aligned} \xi_{ij} &\equiv \pi_j^{e_{ij}} (\mathrm{mod}\ \mathfrak{p}_k^{s_k}), \quad k = 0, \ldots, r, k \neq j, \\ \xi_{ij} &\equiv \pi_j^{e_{ij}}/\eta_i' (\mathrm{mod}\ \mathfrak{p}_j^{s_j}), \end{aligned}$$

then class field theory shows that $(\eta_i', L_{\mathfrak{p}_j}/K_j) = (\mathfrak{c}_{ij}, L/K)$, where

$$\mathfrak{c}_{ij} = \xi_{ij} \prod_{\mathfrak{p} | \pi_j, \mathfrak{p} \neq \mathfrak{p}_j} \mathfrak{p}^{-e_{ij} v_{\mathfrak{p}}(\pi_j)}.$$

Recall that L/K is given by $cl_{\mathfrak{f}}(K)$ and a subgroup $\mathcal{H}$ of index l. Assume that the integral ideal $\mathfrak{c}_0$ corresponds to g_0 via the global reciprocity isomorphism. Using [9, Alg. 4.3.2], it is then easy to compute $d_{ij} \in \mathbb{Z}/l\mathbb{Z}$ such that $\mathfrak{c}_{ij} = \mathfrak{c}_0^{d_{ij}}$ in $cl_{\mathfrak{f}}(K)/\mathcal{H}$.

3.2. **Computation of a_χ.** We assume that we have computed a sublattice $\mathcal{E}_S = \langle \epsilon_v : v \in S_0 \rangle_{\mathbb{Z}}[G]$ of U_S satisfying the assumptions of Theorem 2.1. In this subsection we show how to compute complex approximations to the elements $a_\chi, \chi \in G^*$, which are uniquely determined by part (i) of Theorem 2.2. Writing $S_1 = \{\infty_1, \ldots, \infty_n, \mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$ and $S_2 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$, we have

$$R_S(\epsilon) = -\sum_{v \in S_1} \sum_{g \in G} \log |\epsilon|_{g w_v} g(w_v - w_0) - \sum_{v \in S_2} \log |\epsilon|_{w_v} (w_v - w_0) \tag{6}$$

for any $\epsilon \in U_S$. Considering the trivial character χ_0, we have $S_{\chi_0} = S_0, S^{\chi_0} = \emptyset$, and from (6) we conclude that

$$R_S(e_{\chi_0}\epsilon) = -\sum_{v \in S_1} \sum_{g \in G} \log |g^{-1}\epsilon|_{w_v} e_{\chi_0}(w_v - w_0) - \sum_{v \in S_2} \log |\epsilon|_{w_v} e_{\chi_0}(w_v - w_0).$$

Hence a_{χ_0} is given by

$$\frac{1}{l^r} \cdot \frac{\det(R_{\chi_0}(\mathcal{E}_S))}{L_S^*(0, \chi_0)}, \tag{7}$$

where $R_{\chi_0}(\mathcal{E}_S)$ denotes the matrix $(r_{st})_{s,t \in S_0}$ with

$$r_{st} = \begin{cases} -\sum_{g \in G} \log |g^{-1}\epsilon_s|_{w_t}, & \text{if } t \in S_1, \\ -\log |\epsilon_s|_{w_t}, & \text{if } t \in S_2. \end{cases}$$

If χ is non-trivial, then $S_\chi = S_1, S^\chi = S_2$ and

$$R_S(e_\chi \epsilon) = -\sum_{v \in S_1} \sum_{g \in G} \log |g^{-1}\epsilon|_{w_v} e_\chi(w_v - w_0).$$

Thus a_χ is given by

$$a_\chi = (\chi(g_0) - 1)^r \cdot \frac{\det(R_\chi(\mathcal{E}_S))}{L_S^*(0, \chi^{-1})}, \tag{8}$$

where $R_\chi(\mathcal{E}_S)$ is given by

$$R_\chi(\mathcal{E}_S) = \left(-\sum_{g \in G} \log |g^{-1}\epsilon_s|_{w_t} \chi(g) \right)_{s,t \in S_1}.$$

In conclusion, the complex approximations to $R_\chi(\mathcal{E}_S), \chi \in G^*$, can easily be computed provided that we know how to evaluate $|\alpha|_w$ for $\alpha \in L$ and a place w of L. This is straightforward and left to the reader. An algorithm for the computation of complex approximations of the L-values is already implemented in the PARI system and explained in [9, Ch. 6].

3.3. **Numerical verification of ETNC.** This subsection is devoted to the verification (up to the precision of the computation) of the conjectural assertions

$$h_s a \in \mathbb{Z}[G] \quad \text{and} \quad a_\chi \mathcal{O} = \mathrm{Fitt}_\chi(U_S/\mathcal{E}_S)/\mathrm{Fitt}_\chi(cl_S).$$

We first describe how to compute the relevant Fitting ideals. Since $(l, |U_S/\mathcal{E}_S|) = 1$, one easily shows that $e_\chi(\mathcal{O} \otimes_\mathbb{Z} U_S)/e_\chi(\mathcal{O} \otimes_\mathbb{Z} \mathcal{E}_S) \simeq e_\chi(\mathcal{O} \otimes_\mathbb{Z} U_S/\mathcal{E}_S)$ for each $\chi \in G^*$. By construction we have

$$\mathrm{Fitt}_{\chi_0}(U_S/\mathcal{E}_S) = \mathrm{Fitt}_\mathbb{Z}(e_0 U_S/e_0\mathcal{E}_S)\mathcal{O} = 2|\det(H_1)\det(C)|\mathcal{O}.$$

If, in addition, $h_E = 1$, then our algorithm produces a sublattice $\mathcal{E}_S$ such that $e_1 U_S = e_1 \mathcal{E}_S$, and hence $\mathrm{Fitt}_\chi(U_S/\mathcal{E}_S) = \mathcal{O}$ for each $\chi \neq \chi_0$. In general, we observe that $e_\chi\epsilon_1, \ldots, e_\chi\epsilon_{n+s}$ constitutes an E-basis of $e_\chi U_S$. We compute the matrix $A \in \mathrm{Mat}_{m,n+s}(E)$ such that

$$(e_\chi\delta_1, \ldots, e_\chi\delta_m) = (e_\chi\epsilon_1, \ldots, e_\chi\epsilon_{n+s})A.$$

Applying the HNF algorithm of [9, Alg. 1.4.7] to A, we obtain fractional $\mathcal{O}$-ideals $\mathfrak{c}_1, \ldots, \mathfrak{c}_{n+s}$ such that $\mathrm{Fitt}_\chi(U_S/\mathcal{E}_S) = \left(\prod_{i=1}^{n+s} \mathfrak{c}_i\right)^{-1}$.

For the computation of $\mathrm{Fitt}_\chi(cl_S)$ we assume that the S-class group cl_S is given by a direct product of cyclic subgroups $\langle[\mathfrak{g}_1]\rangle \times \ldots \times \langle[\mathfrak{g}_k]\rangle$, where for an integral $\mathcal{O}_L$-ideal $\mathfrak{g}$ we write $[\mathfrak{g}]$ for its class in cl_S. Let n_i denote the order of $[\mathfrak{g}_i]$. We further assume that the action of G on cl_S is known and given by $g([\mathfrak{g}_1], \ldots, [\mathfrak{g}_k]) = ([\mathfrak{g}_1], \ldots, [\mathfrak{g}_k])\, T(g)$ with $T(g) \in \mathrm{Mat}_{k,k}(\mathbb{Z})$. Then $e_\chi[\mathfrak{g}_i]$ is represented by a column vector $v_i \in E^k$. We write e_i for the i-th unit vector and let A denote the matrix with columns $v_1, \ldots, v_k, n_1 e_1, \ldots, n_k e_k$. Again applying the HNF algorithm for Dedekind domains, we compute ideals $\mathfrak{c}_1, \ldots, \mathfrak{c}_k$ such that

$$\mathrm{Fitt}_\chi(cl_S) = \frac{h_S}{(\prod_{i=1}^k \mathfrak{c}_i)}.$$

At this stage of the algorithm we already have approximations to the complex numbers a_χ for all $\chi \in G^*$, so that it remains to identify a_χ, $\chi \in G^*$, as elements of $E^\times$. By Stark's conjecture $h_S a_{\chi_0}$ should be a rational integer. If this is confirmed by our computations, we round $h_S a_{\chi_0}$ to the nearest integer and consider a_{χ_0} as a rational number.

The elements $h_S a_\chi$ are conjecturally (again by Stark) conjugated integral numbers of E, so that the polynomial

$$f(x) = \prod_{\chi \neq \chi_0} (x - h_s a_\chi)$$

should have coefficients in $\mathbb{Z}$. If this is true up to the precision of the computation, we again round the coefficients of f and factor f, $f = \prod_{i=1}^s f_i^{a_i}$. Each of the irreducible polynomials f_i is expected to define a subextension of E, and if this is true, we can identify the a_χ as elements of E.

Finally, after checking the equality $a_\chi \mathcal{O} = \mathrm{Fitt}_\chi(U_S/\mathcal{E}_S)/\mathrm{Fitt}_\chi(cl_S)$, we need to test whether $h_S a_\chi \equiv h_S a_{\chi_0} (\mathrm{mod}\ (1-\zeta))$ for all $\chi \neq \chi_0$. If these congruences hold, then $h_s a \in \mathbb{Z}[G]$, and the numerical confirmation of ETNCis complete.

4. An example

The algorithm described in Section 3 was implemented under PARI-GP, Version 2.0.20. We describe an explicit example. All computations were done with a real precision of 28 significant digits. Let $K = \mathbb{Q}(\sqrt{5})$ and set $\omega = (1+\sqrt{5})/2$. We let $\mathfrak{f} = \mathfrak{p}_0\mathfrak{p}_1\mathfrak{p}_2$ with

$$\mathfrak{p}_0 = (19, -10+2\omega), \quad \mathfrak{p}_1 = (31, -7+2\omega), \quad \mathfrak{p}_2 = (61, -27+2\omega).$$

The PARI function `bnrinit` computes the the ray class group $cl_\mathfrak{f}(K)$, which is of order 90, generated by two elements $[\mathfrak{g}_1], [\mathfrak{g}_2]$, where $\mathfrak{g}_1 = (15191), \mathfrak{g}_2 = (-15+42\omega)$ and $\mathrm{ord}([\mathfrak{g}_1]) = 30, \mathrm{ord}([\mathfrak{g}_1]) = 3$. There is precisely one subgroup $\mathcal{H}$ of $cl_\mathfrak{f}(K)$ of index $l = 3$ such that the corresponding class field has conductor $\mathfrak{f}$. Explicitly, $\mathcal{H}$ is generated by $3[\mathfrak{g}_1]$ and $2[\mathfrak{g}_1]+[\mathfrak{g}_2]$. We let L denote the extension corresponding to $\mathcal{H}$ und use the PARI-routine `bnrstark` to compute the defining polynomial

$$h(x) = x^6 - x^5 - 127x^4 + 182x^3 + 4192x^2 - 8472x - 17776.$$

Let α denote a root of h so that $L = \mathbb{Q}(\alpha)$.

By applying `bnfinit` we obtain the ring of integers, the ideal class group and a system of fundamental units for L. The class number of L is 3, and it is easily checked that the ideal class group is generated by the ramified primes. Therefore we may use $\{\infty_1, \infty_2, \mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2\}$ as our set S.

We let $\mathfrak{c}_0 = \mathfrak{g}_0$ be a fixed representative of $cl_\mathfrak{f}(K)/\mathcal{H}$ and use `nfgaloisconj` to compute $G = \mathrm{Gal}(L/K)$. It is absolutely essential for the subsequent computations that we choose $g_0 \in G$ such that $(\mathfrak{c}_0, L/K) = g_0$. In this specific example g_0 is given by the substitution

$$x \leftarrow \frac{19}{7616}x^5 + \frac{223}{7616}x^4 - \frac{2579}{7616}x^3 - \frac{69}{28}x^2 + \frac{2353}{238}x + \frac{23999}{952}.$$

If we carry out the algorithm of Section 3 we obtain the special units

$$\begin{aligned}
\epsilon_1 &= \frac{-53}{1904}\alpha^5 - \frac{869}{1904}\alpha^4 + \frac{8665}{1904}\alpha^3 + \frac{1011}{28}\alpha^2 - \frac{20142}{119}\alpha - \frac{66125}{238}, \\
\epsilon_2 &= \frac{47}{7616}\alpha^5 - \frac{565}{7616}\alpha^4 - \frac{2543}{7616}\alpha^3 + \frac{139}{28}\alpha^2 + \frac{571}{119}\alpha - \frac{77493}{952}, \\
\eta_1 &= \frac{7}{2}\omega + \frac{123}{2}, \\
\eta_2 &= \frac{11}{2}\omega + \frac{19}{2}.
\end{aligned}$$

We have $\mathrm{Fitt}_{\chi_0}(U_S/\mathcal{E}_S) = 16\mathcal{O}$ and $\mathrm{Fitt}_\chi(U_S/\mathcal{E}_S) = \mathcal{O}, \chi \neq \chi_0$. We let χ be the character determined by $\chi(g_0) = \exp(2\pi i/l)$. For the extensions $\hat{\infty}_1$ and $\hat{\infty}_2$ we choose the embeddings uniquely determined by

$$\begin{aligned}
\hat{\infty}_1 &: \alpha \mapsto -8.179796812075983731456745607, \\
\hat{\infty}_2 &: \alpha \mapsto -7.947029995151366546855650171.
\end{aligned}$$

Then the computation of the regulator matrices and its determinants leads to

$$\begin{aligned}
\det(R_{\chi_0}(\mathcal{E}_S)) &= -1440.1381979031601504340229 76, \\
\det(R_{\chi}(\mathcal{E}_S)) &= -27.507688225132416320744470 34 \\
&\quad +25.6433241228474488848227768 5i, \\
\det(R_{\bar\chi}(\mathcal{E}_S)) &= -27.507688225132416320744470 34 \\
&\quad -25.6433241228474488848227768 5i.
\end{aligned}$$

We use the routine `bnrL1` to compute approximations to the L-values, and obtain

$$\begin{aligned}
L_S^*(0,\chi_0) &= -10.000959707660834378014048 38, \\
L_S^*(0,\chi) &= 25.361778045893969223234896 22 \\
&\quad -109.932056591311421225607521 7i, \\
L_S^*(0,\bar\chi) &= 25.361778045893969223234896 11 \\
&\quad +109.932056591311421225607522 0i.
\end{aligned}$$

Putting everything together, approximations of the complex numbers a_χ (defined in (7) and (8)) are given by

$$\begin{aligned}
a_{\chi_0} &\approx 16.000000000000000000000000 10, \\
a_{\chi} &\approx 0.9999999999999999999999999992 58 \\
&\quad -0.0000000000000000000000000010 6i, \\
a_{\bar\chi} &\approx 0.9999999999999999999999999992 37 \\
&\quad +0.0000000000000000000000000010 8i,
\end{aligned}$$

This suggests that $a_{\chi_0} = 16, a_\chi = a_{\bar\chi} = 1$ and hence $a = 6 + 5g_0 + 5g_0^2$. Altogether these numerical results confirm the validity of ETNC in this example.

The algorithm has been applied to a lot more examples, each time establishing the validity of ETNC. These numerical results can be found under

`http://www.math.uni-augsburg.de/~ bley`.

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Institut für Mathematik, Universität Augsburg, Universitätsstrasse 8, D-86159 Augsburg, Germany

E-mail address: `bley@math.uni-augsburg.de`