MATHEMATICS OF COMPUTATION
Volume 73, Number 245, Pages 493–513
S 0025-5718(03)01513-8
Article electronically published on June 19, 2003

GREEDY SUMS OF DISTINCT SQUARES

HUGH L. MONTGOMERY AND ULRIKE M. A. VORHAUER

ABSTRACT. When a positive integer is expressed as a sum of squares, with each successive summand as large as possible, the summands decrease rapidly in size until the very end, where one may find two 4's, or several 1's. We find that the set of integers for which the summands are distinct does not have a natural density but that the counting function oscillates in a predictable way.

1. INTRODUCTION

Let n be a positive integer. Suppose that s_1^2 is the largest square not exceeding n, that s_2^2 is the largest square not exceeding $n - s_1^2$, and so on, so that

$$n = s_1^2 + s_2^2 + \cdots + s_r^2$$

for some r. Clearly the s_i are weakly decreasing, but if they are strictly decreasing, $s_1 > s_2 > \cdots > s_r$, then we say that n is a *greedy sum of distinct squares.* We study the asymptotic and local distribution of these numbers.

If $s^2 \le n < (s+1)^2$, so that s^2 is the largest square not exceeding n, then $n - s^2 \le 2s$, which is $< s^2$ if $s \ge 3$. Thus the question of whether the summands are distinct depends entirely on the first remainder less than 9. If this remainder is one of the numbers 0, 1, 4, or 5, then the summands are distinct, but if it is one of the numbers 2, 3, 6, 7 or 8, then they are not. Since there are four good cases and five bad, it might be expected that the set of greedy sums of distinct squares would have natural density 4/9, but the initial numerical data suggest otherwise (see Table 1 on the next page). Mike Sheppard has suggested that perhaps the natural density is 1/2. We determine the asymptotics of this set and find that it has no density although it has a very distinctive limiting behavior.

Put $a(n) = 1$ if n is a greedy sum of distinct squares and $a(n) = 0$ otherwise. It is convenient to set $a(0) = 1$. Let

$$A(v) = \sum_{0 \le n < v} a(n)\,. \tag{1}$$

We find that $A(v)/v$ has persistent wobbles on a log log-scale, in the following sense:

Theorem 1. *There is a continuous nonconstant function f with period 1 such that*

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{Z}}} \frac{A\big(4\exp(2^{k+x})\big)}{4\exp(2^{k+x})} = f(x)\,.$$

Received by the editor May 18, 2001.
2000 *Mathematics Subject Classification.* Primary 11B83, 11A63, 11Y70, 34J10.
Key words and phrases. Greedy algorithm, differential-difference equations.
The first author was supported in part by NSF Grant DMS 0070720.

TABLE 1. Values of $a(n)$ and $A(n)$ for $0 \le n \le 51$.

n	$a(n)$	$A(n)$	n	$a(n)$	$A(n)$	n	$a(n)$	$A(n)$	n	$a(n)$	$A(n)$
0	1	0	13	1	6	26	1	13	39	0	20
1	1	1	14	1	7	27	0	14	40	1	20
2	0	2	15	0	8	28	0	14	41	1	21
3	0	2	16	1	8	29	1	14	42	0	22
4	1	2	17	1	9	30	1	15	43	0	22
5	1	3	18	0	10	31	0	16	44	0	22
6	0	4	19	0	10	32	0	16	45	1	22
7	0	4	20	1	10	33	0	16	46	1	23
8	0	4	21	1	11	34	1	16	47	0	24
9	1	4	22	0	12	35	1	17	48	0	24
10	1	5	23	0	12	36	1	18	49	1	24
11	0	6	24	0	12	37	1	19	50	1	25
12	0	6	25	1	12	38	0	20	51	0	26

The situation when the squares are to be distinct but are not chosen greedily has been dealt with by Pall [7], Wright [8], Halter-Koch [6], and Bateman, Hildebrand and Purdy [2]. It may be observed that the customary radix expansion of an integer is also computed by the greedy algorithm and that averages of digital sums also display an oscillation that is periodic after an exponential change of variable is made; see Delange [4] and Flajolet et al. [5].

It is clear that $a(n)$ can be calculated by computing $O(\log\log n)$ integer square roots. Hence the most immediate way to compute $A(v)$ involves somewhat more than v operations. In Section 2 we derive recurrences that permit the exact calculation of $A(v)$ in something closer to $v^{1/4}$ operations. Thus, for example, we are able to compute the quotient $A(4\exp(2^x))/(4\exp(2^x))$ for $2 \le x \le 5$, as depicted in Figure 1.

The exact recurrence for $A(v)$ does not seem to offer much scope for analyzing the asymptotic behavior, so in Section 3 we pass to a continuous function $B(v)$ determined by an integral equation, and we show that $\alpha(v) = A(v)/v$ is uniformly approximated by $\beta(v) = B(v)/v$. Consideration of $B(v)$ leads us to the linear differential-difference equation

$$\delta(x-1) - \delta(x) \;=\; \frac{\delta'(x)}{2^x \log 2}\,.$$

The limiting distribution is shown to exist, and it seems likely that

$$\frac{A\big(4\exp(2^x)\big)}{4\exp(2^x)} \;=\; f(x) + O(2^{-x})\,.$$

The one detail that remains to be addressed in order to complete the proof of Theorem 1 is to show that f is nonconstant. Later we find that $\min f = 0.50307$ and $\max f = 0.50964$, a difference of less than 10^{-2}. Hence we need to calculate f with an error less than 10^{-3}, say. In principle this could be accomplished by using the relation above, but this would involve calculating $A(v)$ with v of the order of $\exp(10^3)$, which does not seem to be feasible. In Section 4 we introduce a more efficient method for recovering the limiting distribution from the early values of A, and thus we are able to calculate f to within ε in not much more than ε^{-2} operations. The proof of Theorem 1 is then completed in Section 5.

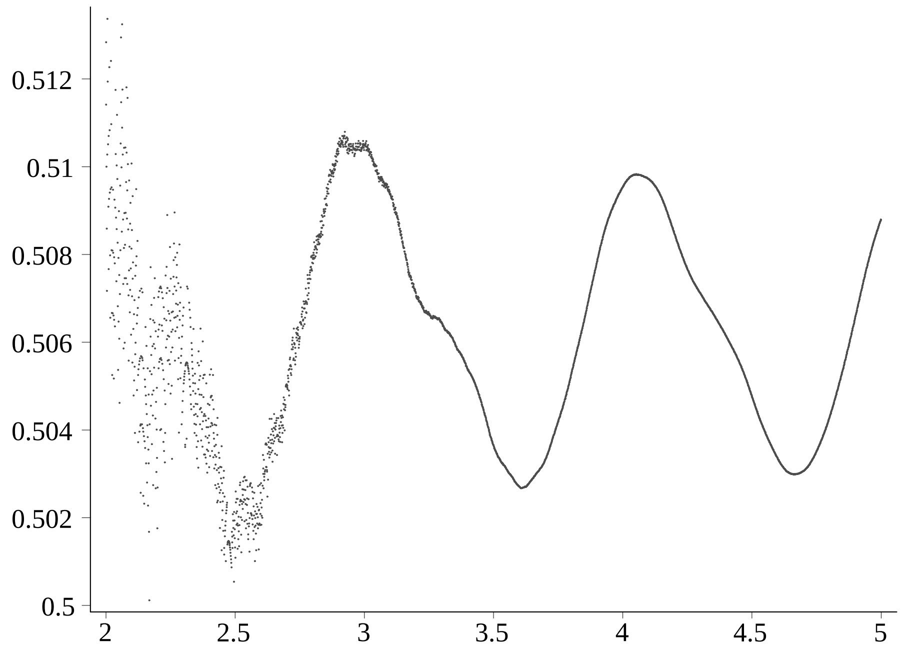

FIGURE 1. Graph of $A(4\exp(2^x))/(4\exp(2^x))$ for $2 \le x \le 5$.

In Section 6 we consider the patterns of 0's and 1's that can occur among the numbers $a(n)$. In view of the highly repetitive nature of the $a(n)$, it is not surprising that the strings that arise are rather restricted. By analyzing how strings of length h give rise to strings of length $h+1$, we are able to count the number of strings occurring, as follows.

Theorem 2. *Of the 2^h possible patterns of 0's and 1's of length h, let $S(h)$ denote the number of such patterns that arise as $a(n+1), \ldots, a(n+h)$ for some n. Then $S(h)$ is a linear recurrence sequence of order 9, specified by the initial values $S(1) = 2$, $S(2) = 4$, $S(3) = 7$, $S(4) = 11$, $S(5) = 18$, $S(6) = 30$, $S(7) = 49$, $S(8) = 79$, and the recurrence*

$$S(h+1) = S(h) + S(h-2) + S(h-3) + S(h-8) \qquad (h \ge 8).$$

In Section 7 we describe computer programs that underly our work.

2. Exact recurrences

We show first that if s is an integer, $s \ge 3$, and v is real, $s^2 \le v \le (s+1)^2$, then

$$A(v) = A(v - s^2) + A(s^2)\,. \tag{2}$$

To see this, suppose that $s^2 \le n < v$, so that $a(n)$ is counted in $A(v)$ but not in $A(s^2)$. Then we write $n = s^2 + (n - s^2)$. Here s^2 is the first greedy square. Since $s \ge 3$, it follows that $n - s^2 < v - s^2 \le (s+1)^2 - s^2 = 2s + 1 < s^2$. Hence any squares used to represent $n - s^2$ will be strictly smaller than s^2. Thus n is a sum of distinct greedy squares if and only if $n - s^2$ is. That is, $a(n) = a(n - s^2)$. Hence the sum of $a(n)$ for $s^2 \le n < v$ is $A(v - s^2)$, and we have (2).

By taking $v = (s+1)^2$ in (2), we see that $A((s+1)^2) - A(s^2) = A(2s+1)$ for $s \ge 3$. On summing this, it follows that

$$A(s^2) - A(9) \;=\; \sum_{i=3}^{s-1} A(2i+1)\,.$$

By appealing again to (2) and reindexing, we see that if s is an integer, $s \ge 3$, and v is real, $s^2 \le v \le (s+1)^2$, then

$$A(v) \;=\; A(v - s^2) + A(9) + \sum_{i=4}^{s} A(2i-1)\,. \tag{3}$$

This recurrence allows the computation of $A(v)$ in $O\big(v^{1/2}\big)$ arithmetic operations. Further application of (2) allows the calculation of $A(v)$ still more efficiently. For example, suppose that $j \ge 2$ and that $(2j-1)^2 \le 2i-1 < (2j)^2$, which is to say that $2j^2 - 2j + 1 \le i \le 2j^2$, then by (2), $A(2i-1) = A((2j-1)^2) + A(2i-1-(2j-1)^2)$. Hence

$$\sum_{i=2j^2-2j+1}^{2j^2} A(2i-1) \;=\; 2j\,A\big((2j-1)^2\big) \;+\; \sum_{i=0}^{2j-1} A(2i)\,.$$

Similarly, if $j \ge 2$ and $(2j)^2 \le 2i-1 < (2j+1)^2$, which is to say that $2j^2+1 \le i \le 2j^2+2j$, then $A(2i-1) = A((2j)^2) + A(2i-1-(2j)^2)$. Hence

$$\sum_{i=2j^2+1}^{2j^2+2j} A(2i-1) \;=\; 2j\,A\big((2j)^2\big) \;+\; \sum_{i=1}^{2j} A(2i-1)\,.$$

On summing these identities, we find that

$$\sum_{\substack{i \\ (2j-1)^2 \le 2i-1 < (2j+1)^2}} A(2i-1) \;=\; 2j\,\Big(A\big((2j-1)^2\big) + A\big((2j)^2\big)\Big) \;+\; \sum_{i=1}^{4j-1} A(i)\,.$$

On summing this over j, it follows that

$$\sum_{\substack{i \\ 9 \le 2i-1 < (2J+1)^2}} A(2i-1) \;=\; \sum_{j=2}^{J} \bigg(2j\,\Big(A\big((2j-1)^2\big) + A\big((2j)^2\big)\Big) + \sum_{i=0}^{4j-1} A(i)\bigg)\,. \tag{4}$$

Alternatively, if $j \ge 3$ and $(2j-2)^2 + 1 \le 2i-1 \le (2j-1)^2$, which is to say that $2j^2 - 4j + 3 \le i \le 2j^2 - 2j + 1$, then $A(2i-1) = A((2j-2)^2) + A(2i-1-(2j-2)^2)$. Hence

$$\sum_{i=2j^2-4j+3}^{2j^2-2j+1} A(2i-1) \;=\; (2j-1)\,A\big((2j-2)^2\big) \;+\; \sum_{i=1}^{2j-1} A(2i-1)\,.$$

Similarly, if $j \ge 3$ and $(2j-1)^2 < 2i-1 \le (2j)^2 - 1$, which is to say that $2j^2 - 2j + 2 \le i \le 2j^2$, then $A(2i-1) = A((2j-1)^2) + A(2i-1-(2j-1)^2)$. Hence

$$\sum_{i=2j^2-2j+2}^{2j^2} A(2i-1) \;=\; (2j-1)\,A\big((2j-1)^2\big) \;+\; \sum_{i=1}^{2j-1} A(2i)\,.$$

On summing these last two identities, we find that

$$\sum_{\substack{i \\ (2j-2)^2+1\le 2i-1<(2j)^2}} A(2i-1) \;=\; (2j-1)\Big(A\big((2j-2)^2\big)+A\big((2j-1)^2\big)\Big) \;+\; \sum_{i=1}^{4j-2} A(i)\,.$$

On summing this over j, it follows that

$$(5)\qquad \sum_{\substack{i \\ 17\le 2i-1<(2J)^2}} A(2i-1) \;=\sum_{j=3}^{J}\bigg((2j-1)\Big(A\big((2j-2)^2\big)+A\big((2j-1)^2\big)\Big)+\sum_{i=1}^{4j-2}A(i)\bigg)\,.$$

To calculate $A(v)$ more efficiently, we first use (3) and then determine the largest K so that $K^2 \le 2s-1 < (K+1)^2$. Depending on the parity of K, we use (4) or (5). The terms on the right-hand side are easily calculated recursively, and the values arising in that calculation facilitate the calculation of the remaining sum

$$\sum_{\substack{i \\ K^2\le 2i-1\le 2s-1}} A(2i-1)\,,$$

in which the number of terms is $\ll s^{1/2} \ll v^{1/4}$. In this way we can calculate $A(v)$ in $\ll v^{1/4}\log\log v$ operations. In principle it should be possible to calculate $A(v)$ in $\ll_\varepsilon v^\varepsilon$ operations, but the details appear to be complicated.

For $v>0$ put $\alpha(v)=A(v)/v$. Since

$$v \;=\; (v-s^2)+9+\sum_{i=4}^{s}(2i-1)\,,$$

we see from (3) that if $v\ge 9$, then $\alpha(v)$ is a weighted average of α at smaller arguments. Thus, in particular, $4/9=\alpha(9)\le\alpha(v)\le 1$ for all $v>0$. We now elaborate on this idea.

Lemma 3. *Let s_0 be an integer, $s_0\ge 4$. If*

$$(6)\qquad av \le A(v) \le bv$$

for all real v in the interval $2s_0-1\le v\le s_0^2$, then (6) holds for all $v\ge 2s_0-1$.

Proof. Let $P(N)$ be the proposition that (6) is true for $2s_0-1\le v\le N$. We prove that $P(N)$ is true for all $N\ge s_0^2$, by induction on N. For $N=s_0^2$ there is nothing to prove. Suppose that $N>s_0^2$, that $P(N-1)$ is true, and that $N-1<v\le N$. Choose s so that $s^2\le v<(s+1)^2$. We consider two cases.

Case 1. $s^2+2s-1\le v<(s+1)^2$. Take $v_1=v-s^2$ and $v_2=s^2$. Then $A(v)=A(v_1)+A(v_2)$ by (2). Since

$$2s_0-1\le 2s-1\le v_1<v-1\le N-1\,,$$

the induction hypothesis $P(N-1)$ implies that

$$(7)\qquad av_1\le A(v_1)\le bv_1\,.$$

Similarly, since

$$2s_0-1\le s_0^2\le v_2\le v-2s+1<v-1\le N-1\,,$$

by $P(N-1)$ we know that

$$(8)\qquad av_2\le A(v_2)\le bv_2\,.$$

Since $v_1+v_2=v$, by summing (7) and (8) we see that (6) holds for v in this first range.

Case 2. $s^2 \le v < s^2+2s-1$. Put $v_1 = 2s-1$ and $v_2 = v-2s+1$. Then

$$2s_0-1 \le v_1 \le s^2-1 \le v-1 \le N-1$$

so that $av_1 \le A(v_1) \le bv_1$ by $P(N-1)$. Similarly,

$$2s_0-1 \le 2s-1 \le (s-1)^2 \le v_2 < v-1 \le N-1$$

so that $av_2 \le A(v_2) \le bv_2$. Furthermore, $(s-1)^2 \le v_2 < s^2$ so that by (3)

$$A(v_2) = A(v_2-(s-1)^2) + A(9) + \sum_{i=4}^{s-1} A(2i-1)\,.$$

But $v_2-(s-1)^2 = v-s^2$, so by a second application of (3) to $A(v)$ we see that $A(v) = A(v_1)+A(v_2)$. Since $v = v_1+v_2$, by summing we see that (6) holds for v in this second range. Thus $P(N)$ is true and the induction is complete. □

The function $A(v)/v$ is continuous and strictly decreasing in intervals where $A(v)$ is constant, and it has upward jump discontinuities otherwise. It is also continuous from the left, so that its minimum over any interval $I=[a,b]$ will be assumed at an integer. The supremum over an interval is not necessarily attained, but it can be calculated as the maximum of $A(n+1)/n$ for $a \le n < b$, $n \in \mathbb{Z}$.

Take $s_0 = 10^5/2$. Then

$$\sup_{2s_0-1\le v\le s_0^2} \frac{A(v)}{v} \;=\; 0.50983,$$

attained at $v = 64784549^+$, and

$$\min_{2s_0-1\le v\le s_0^2} \frac{A(v)}{v} \;=\; 0.50267,$$

attained at $v = 779010$.

We also note that $A(1086) = 543 = 1086/2$, and by Lemma 3 it is easy to verify that $A(v) > v/2$ for $v > 1086$.

3. The continuous analogue

Since $A(v)$ is weakly increasing, it follows that

$$\tfrac{1}{2}\int_5^{2s-1} A(t)\,dt \;\le\; \sum_{i=4}^{s} A(2i-1) \;\le\; \tfrac{1}{2}\int_7^{2s+1} A(t)\,dt$$

and hence that

$$A(v) \;=\; \tfrac{1}{2}\int_0^{2\sqrt{v}} A(t)\,dt \;+\; O\big(\sqrt{v}\,\big)\,. \tag{9}$$

We first eliminate the error term from consideration. Let $v_0 \ge 9$. Suppose that

$$B(v) \;=\; \begin{cases} A(v) & \text{for } 0 \le v \le v_0, \\ \tfrac{1}{2}\displaystyle\int_0^{2\sqrt{v}} B(t)\,dt & \text{for } v > v_0\,. \end{cases} \tag{10}$$

We show that $B(v)$ is a good approximation to $A(v)$ if v_0 is large.

Lemma 4. *Let v_0 and $B(v)$ be defined as above and let $\{v_k\}_{k=0}^{\infty}$ be a sequence defined by the recurrence $v_{k+1} = (v_k/2)^2$. Suppose that the implicit constant in (9) is bounded by c_1 uniformly for $v \ge v_0$. Then*

$$|A(v) - B(v)| \;\le\; c_2 v \tag{11}$$

for all $v \ge 0$, where $c_2 = c_1 \sum_{k=0}^{\infty} 1/\sqrt{v_k}$.

Proof. Let $\varepsilon_k = \sup |A(v) - B(v)|/v$ for $v_0 \le v \le v_k$. If $v_k \le v \le v_{k+1}$, then

$$A(v) - B(v) \;=\; \tfrac{1}{2}\int_0^{2\sqrt{v}} \Big(A(t) - B(t)\Big)dt \;+\; O\big(\sqrt{v}\,\big)\,.$$

Since $2\sqrt{v} \le v_k$, it follows that

$$|A(v) - B(v)| \;\le\; \frac{\varepsilon_k}{2}\int_0^{2\sqrt{v}} t\,dt \;+\; c_1\sqrt{v}\,.$$

Thus we see that

$$\varepsilon_{k+1} \le \varepsilon_k + \frac{c_1}{\sqrt{v_k}}\,.$$

Hence we have the stated result. □

The transition from $2\sqrt{v}$ to v can be made less drastic by means of an exponential change of variable $v = 4e^w$. We put $C(w) = B(4e^w)$ with the result that $C(w) = A(4e^w)$ for $-\infty < w \le w_0 := \log(v_0/4)$, and

$$C(w) \;=\; 2\int_{-\infty}^{w/2} C(t)e^t\,dt \tag{12}$$

for $w > w_0$. Thus $C(w)$ is continuous for $w > w_0$, and we have

$$C'(w) \;=\; C(w/2)\,e^{w/2} \qquad \text{for } w > 2w_0\,. \tag{13}$$

We know that $C(w)$ is of the order of magnitude e^w, so we put $\gamma(w) = C(w)/(4e^w)$. We write $\alpha(v) = A(v)/v$ and $\beta(v) = B(v)/v$, so that $\gamma(w) = \beta(4e^w)$ and

$$\gamma'(w) \;=\; \gamma(w/2) - \gamma(w) \qquad \text{for } w > 2w_0\,. \tag{14}$$

Next we show that $\gamma'(w)$ becomes small as w tends to infinity.

Lemma 5. *Suppose that w_0 is a given real number, that $C(w)$ is a function such that $C(w) = O(e^w)$ for $-\infty < w \le w_0$, and such that (12) holds for $w > w_0$. Put $\gamma(w) = C(w)/(4e^w)$. Then $\gamma'(w) = O(1/w)$ as $w \to \infty$.*

Proof. Let $d_k = \sup w\,|\gamma'(w)|$ for $2w_0 < w \le 2^k w_0$. We show that

$$d_k \;\le\; d_{k-1}\big(1 + O(2^{-k})\big) \tag{15}$$

for $k \ge 4$. Suppose that $2^{k-1}w_0 < w \le 2^k w_0$. Then by (14) and (12) we see that

$$\begin{aligned}\gamma'(w) &= \gamma(w/2) - \gamma(w)\\ &= \gamma(w/2) - \frac{1}{2e^w}\int_{-\infty}^{w/2} C(t)\,e^t\,dt\\ &= \gamma(w/2) - \frac{1}{4e^w}\,C(w/2) - \frac{2}{e^w}\int_{w/4}^{w/2}\gamma(t)\,e^{2t}\,dt\\ &= \frac{2}{e^w}\int_{w/4}^{w/2}\big(\gamma(w/2)-\gamma(t)\big)e^{2t}\,dt\,.\end{aligned}$$

We integrate by parts and again use (14) to see that the above is

$$= \gamma'(w/2)e^{-w/2} + e^{-w}\int_{w/4}^{w/2}\gamma'(t)\,e^{2t}\,dt\,.$$

Here γ' occurs with arguments in the interval $(2^{k-3}w_0, 2^{k-1}w_0] \subseteq (2w_0, 2^{k-1}w_0]$, and thus by the inductive hypothesis

$$|w\gamma'(w)| \le d_{k-1}\Big(2e^{-w/2} + we^{-w}\int_{w/4}^{w/2}\frac{e^{2t}}{t}\,dt\Big)\,.$$

The integral here is $e^w/w + O(e^w/w^2)$, so we obtain (15). The sequence of numbers d_k is bounded; hence we have the stated result. □

We can now show that $A(v)$ has a kind of limiting behavior. Suppose that $1 \le w \le 2$. By (14) and Lemma 5 the telescoping series

$$\sum_{k=1}^{\infty}\Big(\gamma(2^k w) - \gamma(2^{k-1}w)\Big)$$

is uniformly convergent. Hence the sequence $\gamma(2^k w)$ converges uniformly to a continuous function $g_{w_0}(w)$. With $\alpha(v)$ and $\beta(v)$ as above, by taking $w = 2^x$ we see that $\beta(4e^{2^{x+k}})$ tends to a limit $f_{w_0}(x)$ as k tends to infinity. Likewise, by Lemma 4, $\alpha(4e^{2^{x+k}})$ is uniformly close to $f_{w_0}(x)$ when k is large and hence has a continuous limit $f(x)$. The proof of the theorem is now complete, apart from the need to show that $f(x)$ is nonconstant.

4. Iterated recurrences and an invariant functional

Suppose that $C(w)$ is given by (12) for $w > w_0$. Note that $C(w) = ce^w$ is such a function. If $w > 2^k w_0$, then (12) can be used $k+1$ times to express $C(w)$ in terms of $C(t)$ for $t \le w/2^{k+1}$. The formulas that arise in this way (see Lemma 7) involve a family of polynomials, which we now describe. As is familiar in the theory of q-series (see p. 487 of [1]), we write $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$. We also write $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$. For nonnegative integers k let

$$P_k(x) = \sum_{j=0}^{k}\frac{(-1)^j 2^{-j(j-1)/2}}{(1/2;1/2)_j\,(1/2;1/2)_{k-j}}\;x^{2^j-1}\,. \tag{16}$$

Lemma 6. *Let the polynomials P_k be defined as above. Then for any positive integer k we have*

$$P_k'(x) \;=\; -2P_{k-1}(x^2)\,, \tag{17}$$

P_k is strictly monotonically decreasing for $0 \le x < 1$, and $P_k(1) = 0$. Furthermore for any nonnegative integer k we have $\int_0^1 P_k(x)\,dx = 1$, $P_k(x) > 0$ for $0 \le x < 1$ and

$$P_k(x) \;\sim\; \frac{2^{k(k+1)/2}}{k!}\,(1-x)^k \tag{18}$$

for x near 1.

Proof. The first statement is easy to verify by direct calculation. Gauß's q-Binomial Theorem (see (10.0.9) on p. 484 of [1]) asserts that

$$\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{j(j-1)/2} x^j \;=\; \prod_{j=0}^{k-1}\bigl(1 + xq^j\bigr) \;=\; (-x;q)_k, \tag{19}$$

where $\left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right]_q = (q;q)_k/\bigl((q;q)_j(q;q)_{k-j}\bigr)$. On taking $q = 1/2$ and $x = -1$, it follows that

$$P_k(1) \;=\; \frac{(1;1/2)_k}{(1/2;1/2)_k} \;=\; \frac{\prod_{i=0}^{k-1}(1-2^{-i})}{\prod_{i=1}^{k}(1-2^{-i})} \;=\; 0 \tag{20}$$

for $k \ge 1$. Similarly for $k \ge 0$, by (19) with $q = 1/2$, $x = -1/2$, we see

$$\int_0^1 P_k(x)\,dx \;=\; \sum_{j=0}^{k} \frac{2^{-j(j-1)/2}(-1/2)^j}{(1/2;1/2)_j\,(1/2;1/2)_{k-j}} \;=\; \frac{(1/2;1/2)_k}{(1/2;1/2)_k} \;=\; 1\,.$$

From (17) and (20) we see that

$$P_k(x) \;=\; 2\int_x^1 P_{k-1}(u^2)\,du\,. \tag{21}$$

Hence by induction on k it is clear that $P_k(x)$ is positive and strictly monotonically decreasing for $0 \le x < 1$. Suppose that $P_{k-1}(x) \sim a_{k-1}(1-x)^{k-1}$ when x is near 1. Then by (21) it follows that $P_k(x) \sim a_{k-1}2^k(1-x)^k/k$. Thus we obtain (18) by induction on k. $\square$

Lemma 7. *Suppose that $w_0 > 0$ and that $C(w)$ satisfies* (12) *for all $w > w_0$. If k is a nonnegative integer and $w > 2^k w_0$, then*

$$\begin{aligned} C(w) \;&=\; \sum_{j=0}^{k} C\bigl(2^{k-j}w_0\bigr)\,e^{(1-2^{-j})w}\,P_j\bigl(e^{(2^k w_0 - w)/2^j}\bigr) \\ &\quad+\; 2\,e^{(1-2^{-k})w}\int_{w_0/2}^{w/2^{k+1}} C(u)\,e^u\,P_k\bigl(e^{2u - w/2^k}\bigr)\,du, \end{aligned} \tag{22}$$

where the polynomials P_j are given by (16).

By taking $w = 2^{k+1}w_0$ in the above lemma, we obtain an expression for $C(2^{k+1}w_0)$ in terms of $C(2^j w_0)$ for $j = 0, \ldots, k$ and $C(u)$ for $w_0/2 \le u \le w_0$. Thus the numbers $C(2^k w_0)$ can be determined iteratively. To calculate $C(w)$ in terms of $C(u)$ for $w_0/2 \le u \le w_0$, we choose k such that $2^k w_0 < w \le 2^{k+1} w_0$.

Proof. We proceed by induction on k. For $k=0$ it suffices to note that by (12)

$$C(w) \;=\; 2\int_{-\infty}^{w/2} C(u)\, e^u\, du \;=\; C(w_0) \,+\, 2\int_{w_0/2}^{w/2} C(u)\, e^u\, du \tag{23}$$

for $w > w_0$. Suppose the identity (22) holds for k. Since (12) holds for all $w > w_0$, it holds for all $w > w_1$, for any $w_1 \ge w_0$. Therefore the identity holds with w_0 replaced by w_1; in particular, it holds for $w_1 = 2w_0$. Hence if $w > 2^{k+1}w_0$, then

$$\begin{aligned} C(w) \;&=\; \sum_{j=0}^{k} C\big(2^{k+1-j}w_0\big)\, e^{(1-2^{-j})w} P_j\big(e^{(2^{k+1}w_0-w)/2^j}\big) \\ &\quad+\; 2\, e^{(1-2^{-k})w} \int_{w_0}^{w/2^{k+1}} C(v)\, e^v P_k\big(e^{2v-w/2^k}\big)\, dv\,. \end{aligned} \tag{24}$$

In the above integral we use (23). Thus we get for this

$$C(w_0)\int_{w_0}^{w/2^{k+1}} e^v P_k\big(e^{2v-w/2^k}\big)\, dv \;+\; 2\int_{w_0}^{w/2^{k+1}}\int_{w_0/2}^{v/2} C(u)\, e^u\, e^v P_k\big(e^{2v-w/2^k}\big)\, du\, dv\,.$$

From (17) we see that $\frac{d}{dv}(-1/2)e^{w/2^{k+1}}P_{k+1}(e^{v-w/2^{k+1}}) = e^v P_k(e^{2v-w/2^k})$. Since $P_{k+1}(1) = 0$ by Lemma 6, the first term is

$$\frac{1}{2}\, C(w_0)\, e^{w/2^{k+1}}\, P_{k+1}\big(e^{w_0-w/2^{k+1}}\big)\,. \tag{25}$$

In the second term we exchange the order of integration and obtain similarly

$$\begin{aligned} &2\int_{w_0}^{w/2^{k+2}} C(u)\, e^u \int_{2u}^{w/2^{k+1}} e^v P_k\big(e^{2v-w/2^k}\big)\, dv\, du \\ &\qquad = e^{w/2^{k+1}}\int_{w_0}^{w/2^{k+2}} C(u)\, e^u P_{k+1}\big(e^{2u-w/2^{k+1}}\big)\, du\,. \end{aligned}$$

Inserting this together with (25) for the integral into (24), we get the identity (22) for $k+1$. $\square$

For $|x| \le 1$ the polynomials P_k tend uniformly to a limiting power series, which thus inherits the stated properties and is extremely useful to us. Therefore for $w \ge 0$ let

$$K(w) \;=\; c\sum_{j=0}^{\infty} (-2)^j\, \frac{\exp(-2^{j+1}w)}{\prod_{i=1}^{j}(2^i-1)}, \tag{26}$$

where $c = 2/(1/2;1/2)_\infty = 6.925493\ldots$.

Lemma 8. *Let $K(w)$ be defined as above. Then $K(w)$ is continuous and has the following properties:*

$$\big(e^w K(w/2)\big)' \;=\; 2e^w K(w) \tag{27}$$

for $w > 0$, $K(0) = 0$, $K(w) \ge 0$ for all $w \ge 0$, $\int_0^\infty K(w)dw = 1$ and $K(w) = O\big(e^{-2w}\big)$ for $w \ge 0$.

Apart from some differences in notation and some changes of variables, the differential-delay equation (27) is the adjoint of the equation (14), in the sense of Bellman [3], pp. 304–305.

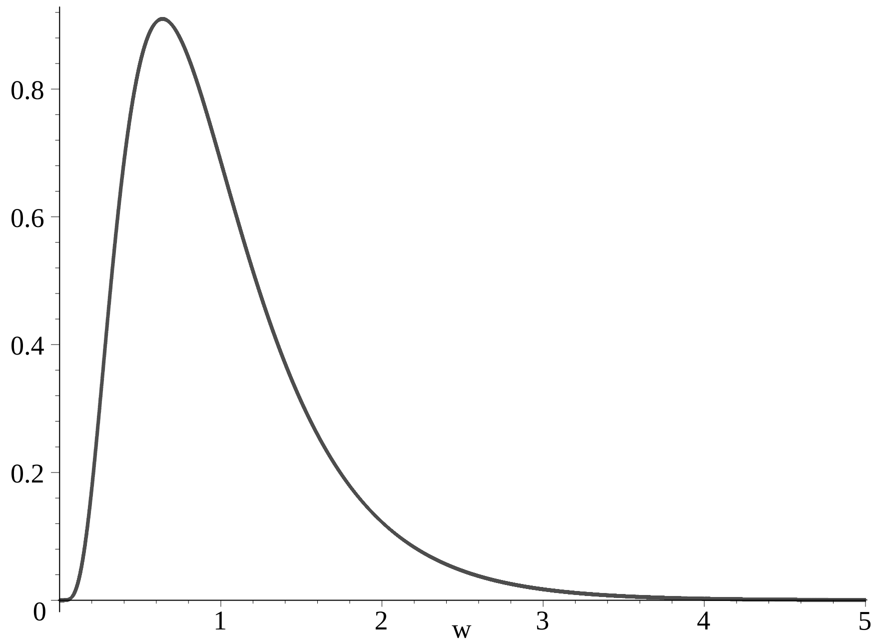

FIGURE 2. Graph of $K(w)$ for $0 \le w \le 5$.

By using the above properties of $K(w)$, it is not hard to show that the zero of $K(w)$ at $w = 0$ is of infinite order.

Proof. Let

$$P(x) \;=\; \frac{c}{2} \sum_{j=0}^{\infty} \frac{(-2)^j}{\prod_{i=1}^{j} \left(2^i - 1\right)} \; x^{2^j - 1}$$

for $|x| \le 1$. By direct calculation we find that $P'(x) = -2P(x^2)$ for $|x| < 1$. Since $\lim_{k\to\infty} P_k(x) = P(x)$ uniformly for $|x| \le 1$, from Lemma 6 it is immediate that $P(1) = 0$, $\int_0^1 P(x)\,dx = 1$, and that $P(x) \ge 0$ for $0 \le x \le 1$. Since $K(w) = 2e^{-2w} P\big(e^{-2w}\big)$, the asserted properties of $K(w)$ now follow. □

Lemma 9. *Let $\gamma(w) = C(w)/(4e^w)$ where $C(w)$ is defined as in Lemma 5, let $K(w)$ be defined by (26), and put*

$$I(w) \;=\; \int_{w_0/2}^{w} \gamma(u)\, K(w-u)\, du\,.$$

If $w \ge w_0/2$, then

$$I(2w) \;=\; I(w) \;+\; \tfrac{1}{2}\gamma(w_0) K\big(w - \tfrac{w_0}{2}\big) \;+\; \int_{w_0/2}^{w_0} \gamma(u)\, K(2w-u)\, du\,. \tag{28}$$

If the integral equation (12) held for all w, then we would have $I(2w) = I(w)$ with no secondary terms. As things stand, the linear functional $I(w)$ is only approximately invariant, but the additional terms are easily computed numerically.

Proof. The above identity is clear for $w = w_0/2$. Therefore assume $w > w_0/2$. We write

$$I(2w) = \tfrac{1}{4}\int_{w_0}^{2w} C(u)e^{-u}K(2w-u)\,du + \int_{w_0/2}^{w_0} \gamma(u)K(2w-u)\,du\,. \tag{29}$$

The second integral on the right is found in (28). In the first one we use (12) in the form of (23) since here $u > w_0$ and we obtain

$$= \tfrac{1}{4}C(w_0)\int_{w_0}^{2w} e^{-u}K(2w-u)\,du + \tfrac{1}{2}\int_{w_0}^{2w}\int_{w_0/2}^{u/2} C(v)e^{v-u}K(2w-u)\,dv\,du\,.$$

From (27) we see that $(-1/2)e^{-u}K(w-u/2)$ is an antiderivative of $e^{-u}K(2w-u)$. Furthermore, $K(0) = 0$ by Lemma 8. Hence, by exchanging the order of integration in the second integral, the above is

$$\begin{aligned} &\tfrac{1}{8}C(w_0)e^{-w_0}K(w-w_0/2) + \tfrac{1}{2}\int_{w_0/2}^{w} C(v)e^{v}\int_{2v}^{2w} e^{-u}K(2w-u)\,du\,dv \\ &\quad = \tfrac{1}{2}\gamma(w_0)K\big(w-w_0/2\big) + \tfrac{1}{4}\int_{w_0/2}^{w} C(v)e^{-v}K(w-v)\,dv\,. \end{aligned}$$

Here the second term is exactly $I(w)$ and therefore the proof is complete. $\square$

Lemma 10. *Let $w_0/2 \le w \le w_0$, let the functions I, K, γ be defined as in Lemma 9, and set $K(w) = 0$ when $w < 0$. Then*

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{Z}}} \gamma(2^k w) = \tfrac{1}{2}\gamma(w_0)\sum_{j=0}^{\infty} K\big(2^j w - \tfrac{w_0}{2}\big) + \int_{w_0/2}^{w_0} \gamma(u)\sum_{j=0}^{\infty} K\big(2^j w - u\big)\,du\,.$$

Proof. By Lemma 8 we know that $\int_0^\infty K(t)\,dt = 1$ and $K(w) = O(e^{-2w})$. Furthermore, $\gamma(u)$ is bounded and $\gamma'(u) \to 0$ as $u \to \infty$ by Lemma 5. Hence

$$\lim_{w\to\infty} \big(\gamma(w) - I(w)\big) = 0\,.$$

In particular, the limit to be calculated is $\lim I(2^k w)$ as k tends to infinity through integer values. Let $j \ge 0$. Then for $w \ge w_0/2$ we have by Lemma 9

$$I(2^{j+1}w) = I(2^j w) + \tfrac{1}{2}\gamma(w_0)K\big(2^j w - \tfrac{w_0}{2}\big) + \int_{w_0/2}^{w_0} \gamma(u)\,K\big(2^{j+1}w - u\big)\,du\,.$$

We sum over j, $0 \le j \le k$, and let $k \to \infty$. This gives

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{Z}}} I(2^k w) = I(w) + \tfrac{1}{2}\gamma(w_0)\sum_{j=0}^{\infty} K\big(2^j w - \tfrac{w_0}{2}\big) + \int_{w_0/2}^{w_0} \gamma(u)\sum_{j=1}^{\infty} K\big(2^j w - u\big)\,du,$$

and hence by

$$I(w) = \int_{w_0/2}^{w_0} \gamma(u)\,K(w-u)\,du$$

we have the stated identity. $\square$

Let $w_0/2 \le w \le w_0$. Since $\gamma(u) = C(u)/(4e^u) = A(4e^u)/(4e^u)$ for $u \le w_0$, we see that

$$\begin{aligned}
4\int_{w_0/2}^{w_0} &\gamma(u)\, K\big(2^j w - u\big)\, du \\
&= \int_{w_0/2}^{w_0} A\big(4e^u\big)\, e^{-u} K\big(2^j w - u\big)\, du \\
&= A\big(4e^{w_0/2}\big) \int_{w_0/2}^{w_0} e^{-u} K\big(2^j w - u\big)\, du \\
&\quad + \int_{w_0/2}^{w_0} \sum_{4e^{w_0/2} \le n < 4e^u} a(n)\, e^{-u} K\big(2^j w - u\big)\, du.
\end{aligned}$$

Keeping in mind that $(-1/2)e^{-u}K(w-u/2)$ is an antiderivative of $e^{-u}K(2w-u)$, we see that the first integral gives

$$-\tfrac{1}{2}\, A\big(4e^{w_0/2}\big)\, e^{-w_0} K\big(2^{j-1}w - \tfrac{w_0}{2}\big) \;+\; \tfrac{1}{2}\, A\big(4e^{w_0/2}\big)\, e^{-w_0/2} K\big(2^{j-1}w - \tfrac{w}{4}\big)\,.$$

To evaluate the second integral, we exchange integration and summation and get similarly

$$\begin{aligned}
\sum_{4e^{w_0/2} \le n < 4e^{w_0}} a(n) &\int_{\log(n/4)}^{w_0} e^{-u} K\big(2^j w - u\big)\, du \\
&= \; -\tfrac{1}{2} a(n)\Big(A\big(4e^{w_0}\big) - A\big(4e^{w_0/2}\big)\Big) e^{-w_0} K\big(2^{j-1}w - \tfrac{w_0}{2}\big) \\
&\quad + \; 2 \sum_{4e^{w_0/2} \le n < 4e^{w_0}} \frac{a(n)}{n}\; K\big(2^{j-1}w - \tfrac{1}{2}\log \tfrac{n}{4}\big)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{w_0/2}^{w_0} \gamma(u)\, K\big(2^j w - u\big)\, du \; &= \tfrac{1}{2}\gamma\big(\tfrac{w_0}{2}\big) K\big(2^{j-1}w - \tfrac{w_0}{4}\big) \;-\; \tfrac{1}{2}\gamma(w_0) K\big(2^{j-1}w - \tfrac{w_0}{2}\big) \\
&\quad + \; \tfrac{1}{2} \sum_{4e^{w_0/2} \le n < 4e^{w_0}} \frac{a(n)}{n}\; K\big(2^{j-1}w - \tfrac{1}{2}\log(\tfrac{n}{4})\big)\,.
\end{aligned}$$

Inserting this in Lemma 10 and taking into account that $K\big((w - w_0)/2\big) = 0$, we see that

$$\begin{aligned}
\lim_{\substack{k\to\infty \\ k\in\mathbb{Z}}} \gamma(2^k w) \; &= \tfrac{1}{2}\gamma\big(\tfrac{w_0}{2}\big) \sum_{j=0}^{\infty} K\big(2^{j-1}w - \tfrac{w_0}{4}\big) \\
&\quad + \; \tfrac{1}{2} \sum_{4e^{w_0/2} \le n < 4e^{w_0}} \frac{a(n)}{n}\; \sum_{j=0}^{\infty} K\big(2^{j-1}w - \tfrac{1}{2}\log(\tfrac{n}{4})\big)
\end{aligned} \tag{30}$$

for $w_0/2 \le w \le w_0$. This allows us to compute the limit in terms of $a(n)$ for $0 \le n \le v_0 = 4e^{w_0}$, in a computationally efficient way. We set $\delta(x) = \gamma(2^x)$. Let

$$f_{w_0}(x) = \lim_{\substack{k\to\infty \\ k\in\mathbb{Z}}} \delta(x+k)\,.$$

We use the above formula to evaluate f_{17} at $1,000$ equally spaced points, and thus we obtain the graph in Figure 3.

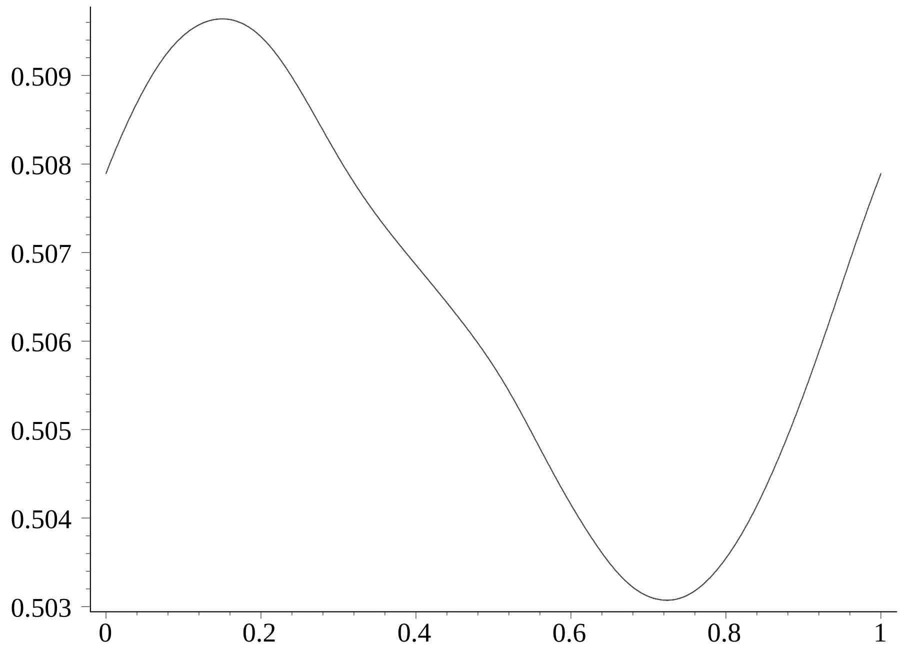

FIGURE 3. Graph of $f_{17}(x)$ for $0 \le x \le 1$.

For isolated values of x we can compute $f_{w_0}(x)$ for even larger choices of w_0. In particular, we find that

$$\begin{aligned} f_{20}(0.149) &= 0.5096380535, \\ f_{20}(0.724) &= 0.503072305\,. \end{aligned} \tag{31}$$

5. Completion of the proof

We now show that in Lemma 4 we can take $c_1 = 2$ provided that $v_0 \ge 16$. When $w_0 = 20$, this gives $v_0 = 4e^{20} > 1.9 \cdot 10^9$, and hence $c_2 < 5 \cdot 10^{-5}$. From (31) it follows that $f(0.149) > 0.50958$ and that $f(0.724) < 0.50312$. Thus f is not constant, and in particular $\lim A(v)/v$ does not exist.

Suppose that $s^2 \le v \le (s+1)^2$ where $s \ge 3$. Since $A(1) + A(3) + A(5) = 1 + 2 + 3 = 6$ while $A(9) = 4$, it follows that

$$\begin{aligned} A(9) + \sum_{i=4}^{s} A(2i-1) &= -2 + \sum_{i=1}^{s} A(2i-1) \\ &= -2 + \sum_{i=1}^{s} \sum_{n=0}^{2i-2} a(n) \\ &= -2 + \sum_{n=0}^{2s-2} a(n) \sum_{\frac{n}{2}+1 \le i \le s} 1 \\ &= -2 + \sum_{n=0}^{2s} a(n) \left(s - \left[\frac{n+1}{2}\right]\right), \end{aligned}$$

where $[x]$ denotes the integral part of x. On the other hand,

$$\begin{aligned}\int_0^{2\sqrt{v}} A(t)\,dt &= \int_0^{2\sqrt{v}} \sum_{0\le n<t} a(n)\,dt \\ &= \sum_{0\le n<2\sqrt{v}} a(n) \int_n^{2\sqrt{v}} 1\,dt \\ &= \sum_{0\le n<2\sqrt{v}} a(n)\left(2\sqrt{v}-n\right).\end{aligned}$$

Hence by (3) we see that

$$\begin{aligned}&A(v) - \frac{1}{2}\int_0^{2\sqrt{v}} A(t)\,dt \\ &\quad = A(v-s^2) - 2 - d - \{\sqrt{v}\}A(2s+1) - \sum_{n=0}^{2s} a(n)\left(\left[\frac{n+1}{2}\right] - \frac{n}{2}\right),\end{aligned} \tag{32}$$

where $\{x\} = x - [x]$ denotes the fractional part of x and

$$d = \begin{cases} 0 & s \le \sqrt{v} < s + 1/2\,, \\ a(2s+1)\big(\sqrt{v} - s - 1/2\big) & s + 1/2 \le \sqrt{v} \le s+1\,. \end{cases}$$

To derive an upper bound, it suffices to note that

$$A(v-s^2) \;\le\; A(2s+1) \;\le\; 2s \;\le\; 2\sqrt{v}\,.$$

For a lower bound it suffices to note that the right-hand side of (32) is

$$\ge -3 - A(2s+1) - \frac{1}{2}\sum_{\substack{n=0\\ n\ \text{odd}}}^{2s} a(n)\,.$$

By taking $s_0 = 5$ in Lemma 3, we see that $A(v) \le 0.6\,v$ for $v \ge 7$. Since $v \ge v_0 \ge 16$, it follows that $s \ge 4$ and hence that $A(2s+1) = A(2s^+) \le 1.2\,s$. Thus the above is

$$\ge -3 - \tfrac{6}{5}\,s - \tfrac{1}{2}\sum_{\substack{n=0\\ n\ \text{odd}}}^{2s} 1 \;=\; -3 - \tfrac{17}{10}\,s \;\ge\; -2s \;\ge\; -2\sqrt{v}\,,$$

and the proof of Theorem 1 is complete.

The reasoning above can be tightened considerably. For example, since $0.5 \le A(v)/v \le 0.51$ for all sufficiently large v, it is possible to show that $\big|A(v-s^2) - \{\sqrt{v}\}\,A(2s+1)\big| \le \sqrt{v}/50$ if v_0 is large. Thus one can derive an estimate for c_1 that is not much larger than $1/4$, which is to say that the error in our approximation is expected to be roughly $1/8$ of what we have proved it to be. Consequently, our computed approximation f_{17} to $f(x)$, proved to be accurate to within $5 \cdot 10^{-5}$, is probably accurate to within 10^{-5}. We find it interesting to compare the exact values displayed in Figure 1 with the limiting behavior.

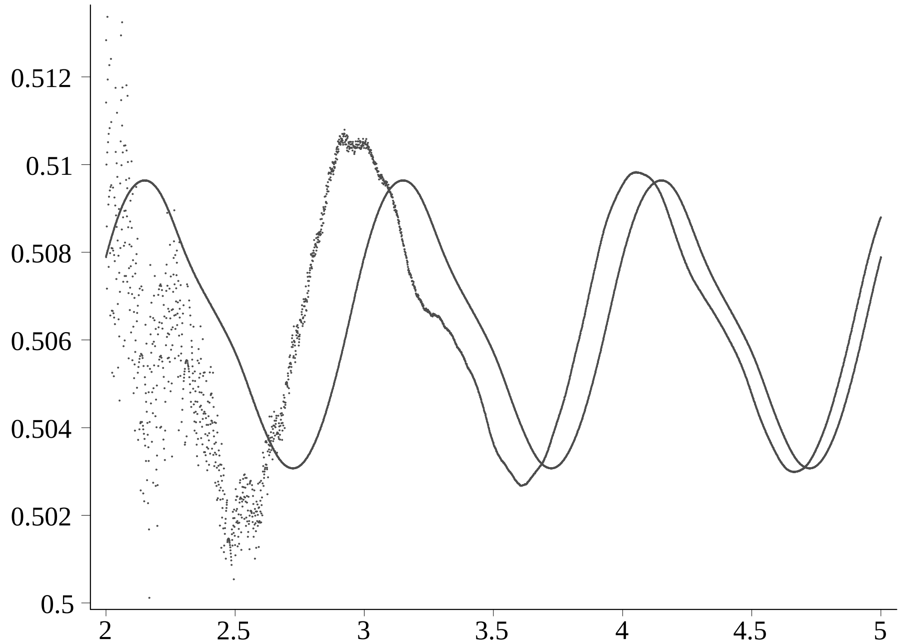

FIGURE 4. Comparison of $A(4\exp(2^x))/(4\exp(2^x))$ with $f_{17}(x)$.

6. LOCAL BEHAVIOR OF THE SEQUENCE $a(n)$

It is interesting to consider what patterns of consecutive 0's and 1's can occur among the $a(n)$. Since the initial segment $a(0), \dots, a(2s)$ is repeated as $a(s^2), \dots, a(s^2+2s)$, for any integer $s \ge 3$, any pattern found is repeated infinitely often. Consider the pattern formed by $a(n+1), \dots, a(n+h)$. If $s^2 \le n+1 \le n+h < (s+1)^2$, $s \ge 3$, then the same pattern occurred earlier. Thus when a pattern $a(n+1), \dots, a(n+h)$ occurs for the first time, at least one of the following must be true: (a) $n+1 \le 8$, or (b) there is an $s \ge 3$ such that $n+1 < s^2 \le n+h$. In the latter case, we may assume that s^2 is the largest such square, which is to say that $n+h < (s+1)^2$, and hence the terms $a(s^2), \dots, a(n+h)$ of our string coincide with the initial segment $a(0), \dots, a(n+h-s^2)$. Conversely, suppose that we have a particular pattern $a(n+1), \dots, a(n+k)$ with $k < h$. If $n+k$ is odd or if $n < h-2k-4$, then we choose an integer $s \ge 3$ such that also $2s > n+k$, and we observe that the same pattern occurs at $a(s^2+n+1), \dots, a(s^2+n+k)$ where s^2+n+k is even if the parity of s is chosen appropriately, and n is replaced by $n+s^2$, which is larger than $h-2k-4$ if s is large. Thus without loss of generality $n+k$ is even, say $n+k=2s$. We may assume that we are not in case (a) and hence that $s \ge 3$. Thus the pattern $a(n+1), \dots, a(n+k)$ is repeated as $a(s^2+n+1), \dots, a(s^2+n+k)$ where it is immediately followed by $a((s+1)^2), \dots, a((s+1)^2+h-k-1)$, which is to say by the initial segment of length $h-k$, $a(0), \dots, a(h-k-1)$. Thus we see that the patterns arising in case (b) are precisely those that arise by concatenating a pattern of length k, where $k < h$, with an initial segment of length $h-k$.

Table 2. Classification of strings and their successors.

Initial String		Append 0		Append 1	
Tag	Description	Tag	Description	Tag	Description
1	... 1 0 0 1 1	2	... 1 0 0 1 1 0	14	... 1 1 1
2	... 1 0 0 1 1 0	3	... 1 0 0 1 1 0 0	15	... 1 0 1
3	... 1 0 0 1 1 0 0	4	... 0 0 0	16	... 1 0 0 1
4	... 0 0 0			5	... 0 0 0 1
5	... 0 0 0 1			6	... 0 0 0 1 1
6	... 0 0 0 1 1	7	... 0 0 0 1 1 0	14	... 1 1 1
7	... 0 0 0 1 1 0	8	... 0 0 0 1 1 0 0	15	... 1 0 1
8	... 0 0 0 1 1 0 0			16	... 1 0 0 1
9	... 1 0 1 1	10	... 1 0 1 1 0	14	... 1 1 1
10	... 1 0 1 1 0	11	... 1 0 1 1 0 0	15	... 1 0 1
11	... 1 0 1 1 0 0			16	... 1 0 0 1
12	... 1 1 1 0	13	... 1 1 1 0 0	15	... 1 0 1
13	... 1 1 1 0 0			16	... 1 0 0 1
14	... 1 1 1	12	... 1 1 1 0	14	... 1 1 1
15	... 1 0 1			9	... 1 0 1 1
16	... 1 0 0 1			1	... 1 0 0 1 1

By means of the above reasoning, we can deduce that the following strings never occur among the $a(n)$:

0 0 0 0
0 0 0 1 1 0 0 0
1 0 1 1 0 0 0
1 1 1 0 0 0
0 0 0 1 0
1 0 0 1 0
1 0 1 0

A string of length h can always have a 1 added at the end to form a string of length $h+1$, since $a(0) = 1$. Whether a 0 can also be added depends on the trailing entries of the string. For example, if a string ends in 01, then a further 0 cannot be added. On the other hand, if a string ends in 11, then a 0 definitely can be added. However, if a string ends in 10, then the situation is indeterminate and can only be resolved by examining additional terms of the string. In this way cases lead to subcases until we reach strings of length 7. At this point and for all $h \geq 7$, the strings fall into 16 classes, according as their trailing entries are as in Table 2. In each of these cases it is clear whether a 0 can be added (since the patterns above do not occur), and by adding a 1 or 0 (when it is allowed), one is taken again to one of the 16 cases, as indicated in Table 2.

For $1 \leq i \leq 16$ let $S_i(h)$ denote the number of strings of length h that occur and that end in the i-th pattern, and let $\boldsymbol{S}(h) \in \mathbb{R}^{16}$ be the column vector with coordinates $S_i(h)$. The sequence $\boldsymbol{S}(h)$ is determined by $\boldsymbol{S}(7)$ and the recurrence

$\boldsymbol{S}(h+1) = T\boldsymbol{S}(h)$ where

$$T = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&1&0&0&1&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&1&0&0&1&0&1&0&0&0 \end{bmatrix}, \quad \boldsymbol{S}(7) = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 4 \\ 2 \\ 1 \\ 7 \\ 4 \\ 11 \\ 7 \\ 4 \end{bmatrix}.$$

We find that

$$\operatorname{charpoly}(T) \;=\; (x^9 - x^8 - x^6 - x^5 - 1)\, x^7\,.$$

The matrix T is not diagonalizable since 0 is an eigenvalue of algebraic multiplicity 7 and geometric multiplicity only 2. However, if we let $r(h)$ be the linear recurrence sequence of order 9 defined by the initial conditions $r(h) = 1$ for $-6 \le h \le 0$, $r(1) = 2$, $r(2) = 4$ together with the recurrence

$$r(h+1) = r(h) + r(h-2) + r(h-3) + r(h-8), \tag{33}$$

then by an easy induction on h we find that

$$\begin{aligned} S_1(h) &= r(h-6)\,, & S_9(h) &= r(h-5)\,, \\ S_2(h) &= r(h-7)\,, & S_{10}(h) &= r(h-6)\,, \\ S_3(h) &= r(h-8)\,, & S_{11}(h) &= r(h-7)\,, \\ S_4(h) &= r(h-9), & S_{12}(h) &= r(h-4)\,, \\ S_5(h) &= r(h-10)\,, & S_{13}(h) &= r(h-5)\,, \\ S_6(h) &= r(h-11)\,, & S_{14}(h) &= r(h-3)\,, \\ S_7(h) &= r(h-12)\,, & S_{15}(h) &= r(h-4)\,, \\ S_8(h) &= r(h-13)\,, & S_{16}(h) &= r(h-5) \end{aligned}$$

for $h \ge 7$. From the above we discover that $S_{12}(h) = S_{15}(h)$, that $S_9(h) = S_{13}(h) = S_{16}(h)$, that $S_1(h) = S_{10}(h)$, and that $S_2(h) = S_{11}(h)$ for all $h \ge 7$. If we let $S(h) = \sum_{i=1}^{16} S_i(h)$ be the total number of strings of length h that occur, then $S(h) = r(h)$ for all $h \ge 7$. By direct computation we find that indeed this also holds for $1 \le h \le 6$. This completes the proof of Theorem 2.

The polynomial $x^9 - x^8 - x^6 - x^5 - 1$ is irreducible over $\mathbb{Q}$ and has roots

$$\begin{aligned}
\alpha_1 &= 1.62866794\,, \\
\alpha_2 &= 0.13830350 + 1.05660492\,i\,, \quad \alpha_3 = \overline{\alpha_2}\,, \\
\alpha_4 &= -0.33184251 + 0.88329102\,i\,, \quad \alpha_5 = \overline{\alpha_4}\,, \\
\alpha_6 &= 0.72811073 + 0.46035377\,i\,, \quad \alpha_7 = \overline{\alpha_6}\,, \\
\alpha_8 &= -0.84890569 + 0.31268265\,i\,, \quad \alpha_9 = \overline{\alpha_8}\,.
\end{aligned}$$

It is easy to verify that if α is one of these roots, then the associated eigenvector of the transition matrix T has coordinates α^{-6}, α^{-7}, α^{-8}, α^{-9}, α^{-10}, α^{-11}, α^{-12}, α^{-13}, α^{-5}, α^{-6}, α^{-7}, α^{-4}, α^{-5}, α^{-3}, α^{-4}, α^{-5}. The eigenvalue of T of largest absolute value is positive real, and the coordinates of its associated eigenvector are positive, in accordance with the Perron–Frobenius theory of positive matrices. We find that

$$S(h) = \sum_{i=1}^{16} p(\alpha_i)\alpha_i^h \qquad (h \ge 1),$$

where $p(x)$ is the polynomial

$$\begin{aligned}
p(x) \;=& -\frac{22515926}{289348113}\,x^8 + \frac{4019460}{96449371}\,x^7 + \frac{7630461}{96449371}\,x^6 \\
&+ \frac{9432697}{96449371}\,x^5 + \frac{43618717}{289348113}\,x^4 + \frac{29446171}{289348113}\,x^3 \\
&+ \frac{13853450}{289348113}\,x^2 + \frac{2790374}{289348113}\,x - \frac{17955847}{289348113}\,.
\end{aligned}$$

Hence in particular $S(h) \sim c\alpha_1^h$ as $h \to \infty$, where $c = p(\alpha_1) = 1.592655\ldots$.

7. Programs

In Section 6, Maple was used to find the characteristic polynomial of the transition matrix T, to locate its roots in the complex plane, to determine its factorization over $\mathbb{Q}$, and to compute the polynomial p. All other computations were accomplished by means of more specialized programs written in TurboPascal. These are described individually below. These programs, including source code, are available at the URL `http://trident.mcs.kent.edu/~vorhauer/programs/`. Times indicated below refer to runs on a 500MHz Pentium III PC. For the logical integrity of the main arguments only the program gs6 is needed, to calculate $f_{w_0}(x)$ at two different values of x, with w_0 sufficiently large to guarantee that the limiting function $f(x)$ is nonconstant. To this end it would suffice to take $w_0 = 14$, but in (31) we took $w_0 = 20$, which gives greater accuracy and still in a reasonable time. The other programs were useful for discovery, or for providing numerical confirmation of various phenomena. For example, the data provided by gs9 led to the surmise that each of the $S_i(h)$ is given by the same linear recurrence sequence with suitably retarded arguments. Once this is suspected, it is easily confirmed by induction.

gs1. For any given n, $0 \le n \le 10^{18}$, the values $a(n)$, $r(n)$, $A(n)$, and $A(n)/n$ are returned. Here $r(n)$ is the first remainder < 9. The computation of $A(n)$ uses the identities (4) and (5) and thus runs in time $\ll n^{1/4} \log\log n$. The value of n can be entered on the command line.

gs2. The same data is provided as in gs1 but now in the form of a table with a primitive GUI.

gs3. The user is prompted to give a value of w_0, and then $A\big(4\exp(2^x)\big)$ is calculated for $0 \le x \le 5$ at increments of 10^{-3}. The points calculated are exported to a file, in a form suitable for plotting by Maple. The time of the calculation is also reported. The computation takes 11 seconds, and the resulting data is used in Figures 1 and 4.

gs4. The user is prompted for a value of N and then reports the number $R_r(N)$ of integers n, $0 \le n < N$, for which $r(n) = r$, where $0 \le r \le 8$. The calculation proceeds by calculating $r(n)$ for each individual n, and thus the running time is $\approx N \log\log N$. Since each of the R_r satisfy the same recurrences as $A(v)$, the values could be calculated in time $\ll N^{1/4}\log\log N$. The program checks that $\sum_{r=0}^{8} R_r(N) = N$ and also that $R_0(N) + R_1(N) + R_4(N) + R_5(N) = A(N)$. The value of $A(N)$ is computed using (3), in time $\ll N^{1/2}\log\log N$.

gs5. The user is prompted to enter an integer s_0, $4 \le s_0 \le 10^9$, and then the sup and min of $A(v)/v$ is computed for $2s_0 \le v \le s_0^2$. This is useful in conjunction with Lemma 3. The running time is $\approx s_0^2$. With $s_0 = 10^4$, the computation took 2 minutes 11 seconds. With $s_0 = 10^5/2$, the computation took 48 minutes. For each n in the range considered, the successive summands, remainders and maximum allowable remainders are held in arrays. In this way the calculation of $a(n)$ is reduced from $\ll \log\log n$ to an average of $\ll 1$ operations. For the s_0 used, the running time is less than half of what it would have been.

gs6. The user is prompted to choose a value of w_0, $1 \le w_0 \le 40$. The value of $f_{w_0}(x)$ is calculated for $0 \le x < 1$, in increments of 10^{-3}, using formula (30). The points found are saved to a file, for use in Figures 3 and 4. With $w_0 = 17$, this ran for 178.5 hours. Since each $a(n)$ is computed 1,000 times, some time could be saved by rearranging the computation so that $a(n)$ is computed only once, the associated multipliers are computed, and the partial results are saved in an array of 1,000 elements. This might reduce the running time by as much as 25%.

gs7. The user is prompted to enter a real number w_0, $1 \le w_0 \le 40$, and then a real number x, $0 \le x \le 1$. The program uses formula (30) to calculate $f_{w_0}(x)$. The time of the calculation is also reported. It took 3 hours 42 minutes to calculate $f_{20}(x)$.

gs8. The user is prompted to provide a string of 0's and 1's, and then this string is searched for among the $a(n)$, up to 10^7. This upper limit is a constant in the source code and can be adjusted.

gs9. Constructs a table with 18 columns labeled h, $S_i(h)$ $(1 \le i \le 16)$, $S(h)$, and 13 rows, with h running from 7 to 20. The coordinates $S_i(7)$ are fixed at the outset, and then subsequent vectors $\boldsymbol{S}(h)$ are computed by matrix multiplication by means of the recurrence $\boldsymbol{S}(h+1) = T\,\boldsymbol{S}(h)$.

gs10. Searches for a string of consecutive 1's, either by looking for the first string of a given length or else by listing all strings of length at least $m - c$ where m is the length of the longest such string found so far and c is a parameter provided by the user.

Acknowledgments

The authors would like to thank George Andrews, George Askey, Harold Diamond, Don Lutz, Jeff Rauch, Imre Ruzsa and Mike Sheppard for their comments.

References

1. G. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, Cambridge, 1999. MR **2000g:**33001
2. P. T. Bateman, A. J. Hildebrand and G. B. Purdy, *Sums of distinct squares*, Acta Arith. **67** (1994), 349–380. MR **95j:**11092
3. R. Bellman and K. L. Cooke, *Differential-difference equations*, Mathematics in Science and Engineering, vol. 6, Academic Press, New York, 1963. MR **26:**5259
4. H. Delange, *Sur la fonction sommatoire de la fonction "somme de chiffres"*, Enseignement Math. (2) **21** (1975), 31–47. MR **52:**319
5. P. Flajolet, P. Grabner, P. Kirschenhofer, H. Prodinger and R. Tichy, *Mellin transforms and asymptotics: digital sums*, Theoret. Comput. Sci. **123** (1994), 291–314. MR **94m:**11090
6. F. Halter-Koch, *Darstellung natürlicher Zahlen als Summe von Quadraten*, Acta Arith. **42** (1982), 11–20. MR **84b:**10025
7. G. Pall, *On sums of squares*, Amer. Math. Monthly **40** (1933), 10–18.
8. E. M. Wright, *The representation of a number as a sum of five or more squares*, Quart. J. Math. Oxford Ser. **4** (1933), 37–51.

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109-1109
E-mail address: `hlm@umich.edu`

Department of Mathematics, Kent State University, Kent, Ohio 44242
E-mail address: `vorhauer@math.kent.edu`
URL: `http://trident.mcs.kent.edu/~vorhauer/`