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NONCONFORMING ELEMENTS IN LEAST-SQUARES MIXED FINITE ELEMENT METHODS

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ABSTRACT. In this paper we analyze the finite element discretization for the first-order system least squares mixed model for the second-order elliptic problem by means of using nonconforming and conforming elements to approximate displacement and stress, respectively. Moreover, on arbitrary regular quadrilaterals, we propose new variants of both the rotated $\mathcal{Q}_1$ nonconforming element and the lowest-order Raviart-Thomas element.

1. INTRODUCTION

As is well-known, nonconforming elements (e.g., Crouzeix-Raviart (CR) linear elements [11] and the rotated $\mathcal{Q}_1$-element [12], [18], [10]) are very useful to seek numerical solutions of many physical problems (see [11], [12], [13], [15], [16], [17], [18], [27], [10]). A quadrilateral version of the rotated $\mathcal{Q}_1$-element was studied in [18], but it is only suitable for uniform asymptotic rectangles. This is a restrictive condition. In this paper, we propose a new variant which admits arbitrary regular quadrilaterals and allows the finite element equation to be efficiently obtained on the reference element.

In the classical mixed finite element analysis, both triangular and rectangular normal continuous elements [19] are proposed, which are known as Raviart-Thomas-Nédélec (RTN) elements [5], [8], [9] and Brezzi-Douglas-Marini (BDM) elements [7] and Brezzi-Douglas-Fortin-Marini (BDFM) elements [6], and so on. On arbitrary quadrilaterals, Wang and Mathew [25] analyzed variants of these elements, but the very important commuting diagram property does not hold (cf. [35], [19], [37]). In this paper, we propose a new variant of the lowest-order RTN rectangular element. Our variant is the first one which not only admits arbitrary regular quadrilaterals, but also satisfies the commuting diagram property.

These above two new elements will be used for the finite element discretization of the first-order system least-squares mixed model for a second-order elliptic problem with various boundary conditions.

It is well known that one advantage of the least squares mixed method [4] is that coerciveness holds, while the classical mixed method [19], [28] is subject to the Babuška-Brezzi condition. However, it seems that the coerciveness strongly depends on the conformity of the finite dimensional spaces (see [1], [2], [3], [23], [24],

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[26]). Up to now, it is not clear whether the coerciveness still holds if the displacement is approximated by nonconforming elements and the stress by conforming elements. In this paper, on triangular, rectangular and quadrilateral meshes, nonconforming finite element methods are analyzed in a unified way. It is shown that our nonconforming methods are still coercive, and optimal error bounds are derived.

As is known, the so-called inconsistent error is an essential feature of the nonconforming displacement-based finite element method [20], [29], [22]. In this paper, we find that this error does not exist in the first-order system least-squares mixed methods in the case of nonconforming elements. It seems that the theory of the patch test [20], [29], [30] would be lost. Nonetheless, it turns out that the patch test is necessary to obtain coerciveness.

The rest of the paper is arranged as follows. In section 2, the first-order system least-squares mixed model is recalled for the second order elliptic problem. In section 3, nonconforming finite element methods are analyzed for the least-squares mixed model. In section 4, two quadrilateral elements are proposed. In section 5, some comments are made.

2. The least-squares mixed model

Let $\Omega \subset \Re^d$ $(d = 2, 3)$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Given a subdomain $O \subseteq \Omega$ with Lipschitz boundary ∂O, we introduce $L^2(O)$, and $(L^2(O))^d$, with inner product $(\cdot,\cdot)_{0,O}$ and norm $||\cdot||_{0,O}$, and introduce L^2-based Sobolev spaces $H^m(O)$ and $(H^m(O))^d$, with norm $||\cdot||_{m,O}$ and semi-norm $|\cdot|_{m,O}$ ($m \geq 1$ is an integer). In addition, we introduce $H_0^1(O) = \{v \in H^1(O); v_{|\partial O} = 0\}$ with norm $|\cdot|_{1,O}$, and $H(\mathrm{div}; O) = \{\boldsymbol{q} \in (L^2(O))^d; \mathrm{div}\,\boldsymbol{q} \in L^2(O)\}$ with norm $||\cdot||_{H(\mathrm{div};O)}$ (cf. [34]).

In the case $O = \Omega$, we simplify the notation as follows: $|\cdot|_{m,O} \equiv |\cdot|_m$, $||\cdot||_{m,O} \equiv ||\cdot||_m$ $(m \geq 1)$, $(\cdot,\cdot)_{0,O} \equiv (\cdot,\cdot)$, $||\cdot||_{0,O} \equiv ||\cdot||$.

Let $\Gamma = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$, and let $\boldsymbol{n}$ be the unit outward normal vector to Γ. We additionally introduce

$$H_{0,D}^1(\Omega) = \{v \in H^1(\Omega); v = 0 \quad \text{on } \Gamma_D\},$$
$$H_{0,N}(\mathrm{div}; \Omega) = \{\boldsymbol{q} \in H(\mathrm{div}; \Omega); \boldsymbol{q}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_N\},$$
$$H^1(\mathrm{div}; \Omega) = \{\boldsymbol{q} \in (H^1(\Omega))^d; \mathrm{div}\boldsymbol{q} \in H^1(\Omega)\}.$$

Considering the following second-order elliptic problem:

$$\begin{aligned} -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ -\sum_{i,j=1}^{d} n_i\, a_{ij}\frac{\partial u}{\partial x_j} &= 0 \quad \text{on } \Gamma_N, \end{aligned} \tag{2.1}$$

where u is the displacement and $A = (a_{ij}(x)) \in \Re^{d\times d}$ is a sufficiently smooth, symmetric matrix of coefficients, satisfying

$$C\sum_{i=1}^{d} \xi_i^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\,\xi_j \leq C^{-1}\sum_{i=1}^{d} \xi_i^2 \quad \forall(\xi_i) \in \Re^d, \forall x \in \overline{\Omega}. \tag{2.2}$$

Here and below, the letter C (with or without subscripts) is a generic constant which may take different values at different occurrences.

Introducing the stress

$$\boldsymbol{p} = -A \bigtriangledown u \tag{2.3}$$

as an independent variable, we can rewrite (2.1) in the first order system

$$\begin{aligned} \operatorname{div} \boldsymbol{p} = f, \quad \boldsymbol{p} = -A \bigtriangledown u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma_D, \\ \boldsymbol{p} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{2.4}$$

A least-squares variational problem for (2.4) is to find $u \in U = H^1_{0,D}(\Omega)$ and $\boldsymbol{p} \in X = H_{0,N}(\mathrm{div};\Omega)$ such that

$$L(u, \boldsymbol{p}; v, \boldsymbol{q}) := (\boldsymbol{p}+A \bigtriangledown u, \boldsymbol{q}+A \bigtriangledown v) + (\operatorname{div} \boldsymbol{p}, \operatorname{div} \boldsymbol{q}) = (f, \operatorname{div} \boldsymbol{q}) \tag{2.5}$$

for all $(v, \boldsymbol{q}) \in U \times X$.

3. The nonconforming finite element method

3.1. **Method (I).** Let $\mathcal{T}_h$ be the regular triangulation [20], [22] of Ω into triangles or rectangles in $\Re^2$, or tetrahedra or rectangular solids in $\Re^3$. We define

$$U_h \not\subset U, \quad X_h \subset X. \tag{3.1}$$

A finite element method for problem (2.5) is to find $(u_h, \boldsymbol{p}_h) \in U_h \times X_h$ such that

$$\left\{ \begin{array}{l} L_h(u_h, \boldsymbol{p}_h; v_h, \boldsymbol{q}_h) \\ := (\boldsymbol{p}_h + A \bigtriangledown_h u_h, \boldsymbol{q}_h + A \bigtriangledown_h v_h) + (\operatorname{div} \boldsymbol{p}_h, \operatorname{div} \boldsymbol{q}_h) = (f, \operatorname{div} \boldsymbol{q}_h) \\ \forall (v_h, \boldsymbol{q}_h) \in U_h \times X_h, \end{array} \right. \tag{3.2}$$

where

$$\bigtriangledown_h \quad \text{is the gradient operator element-by-element.} \tag{3.3}$$

To investigate both coerciveness and convergence, we define

$$| \cdot |_{1,h} = \sqrt{\sum_{K \in \mathcal{T}_h} | \cdot |^2_{1,K}}, \quad || \cdot ||_{1,h} = \sqrt{|| \cdot ||^2 + | \cdot |^2_{1,h}}. \tag{3.4}$$

Hypothesis (H1). The equality

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{q}_h \cdot \boldsymbol{n}_K \, v_h = 0 \quad \forall (v_h, \boldsymbol{q}_h) \in U_h \times X_h \tag{3.5}$$

holds, where $\boldsymbol{n}_K$ is the unit outward normal vector to K with boundary ∂K.

Hypothesis (H2). For $u \in U \cap H^2(\Omega)$ and $\boldsymbol{p} \in X \cap H^1(\mathrm{div};\Omega)$, there exist two interpolants $I_h \, u \in U_h$ and $\Pi_h \boldsymbol{p} \in X_h$ such that

$$||u - I_h \, u|| + h|u - I_h \, u|_{1,h} \le C \, h^2 \, ||u||_2, \tag{3.6}$$

$$||\boldsymbol{p} - \Pi_h \boldsymbol{p}|| \le C \, h \, |\boldsymbol{p}|_1, \quad ||\operatorname{div}(\boldsymbol{p} - \Pi_h \boldsymbol{p})|| \le C \, h \, |\operatorname{div} \boldsymbol{p}|_1. \tag{3.7}$$

Hypothesis (H3). There exists a constant $C_1 > 0$, independent of h, such that

$$||v_h|| \le C_1 \, |v_h|_{1,h}, \quad \forall v_h \in U_h. \tag{3.8}$$

Theorem 3.1. *Under Hypotheses (H1) and (H3), there exists a constant $C > 0$, independent of h, such that*

$$L_h(v_h, \boldsymbol{q}_h; v_h, \boldsymbol{q}_h) \geq C\,\{||v_h||_{1,h}^2 + \|\boldsymbol{q}_h\|_{H(\mathrm{div};\Omega)}^2\} \quad \forall (v_h, \boldsymbol{q}_h) \in U_h \times X_h. \tag{3.9}$$

Proof. Given a constant $\alpha > 0$, we have

$$||\mathrm{div}\,\boldsymbol{q}_h||^2 = ||\mathrm{div}\,\boldsymbol{q}_h - \alpha\, v_h||^2 + 2\,\alpha\,(\mathrm{div}\,\boldsymbol{q}_h, v_h) - \alpha^2\,||v_h||^2, \tag{3.10}$$

where, in the light of Hypothesis (H1), we have

$$\begin{aligned}(\mathrm{div}\,\boldsymbol{q}_h, v_h) &= -\sum_{K\in\mathcal{T}_h} (\boldsymbol{q}, \bigtriangledown v_h)_{0,K} + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{q}_h \cdot \boldsymbol{n}_K\, v_h \\ &= -\sum_{K\in\mathcal{T}_h} (\boldsymbol{q}, \bigtriangledown v_h)_{0,K}.\end{aligned} \tag{3.11}$$

Since

$$\begin{aligned}&-2\,\alpha \sum_{K\in\mathcal{T}_h} (\boldsymbol{q}, \bigtriangledown v_h)_{0,K} + ||\boldsymbol{q}_h + A\bigtriangledown_h v_h||^2 \\ &\quad = ||\boldsymbol{q}_h + (A - \alpha\,E)\bigtriangledown_h v_h||^2 + 2\,\alpha\,(A\bigtriangledown_h v_h, \bigtriangledown_h v_h) - \alpha^2\,|v_h|_{1,h}^2,\end{aligned} \tag{3.12}$$

where E is the identity matrix, by (2.2) we have

$$(A\bigtriangledown_h v_h, \bigtriangledown_h v_h) \geq C_2\,|v_h|_{1,h}^2. \tag{3.13}$$

Therefore, from (3.10)–(3.13) and Hypothesis (H3) we get

$$\begin{aligned}L_h(v_h, \boldsymbol{q}_h; v_h, \boldsymbol{q}_h) &= ||\boldsymbol{q}_h + A\bigtriangledown_h v_h||^2 + ||\mathrm{div}\,\boldsymbol{q}_h||^2 \\ &= ||\mathrm{div}\,\boldsymbol{q}_h - \alpha\,v_h||^2 + ||\boldsymbol{q}_h + (A - \alpha\,E)\bigtriangledown_h v_h||^2 \\ &\qquad - \alpha^2\,||v_h||^2 + 2\,\alpha\,(A\bigtriangledown_h v_h, \bigtriangledown_h v_h) - \alpha^2\,|v_h|_{1,h}^2 \\ &\geq \alpha\,(2\,C_2 - \alpha\,(1 + C_1^2))\,|v_h|_{1,h}^2.\end{aligned} \tag{3.14}$$

Putting $\alpha = \dfrac{C_2}{1 + C_1^2}$, we get

$$\alpha\,(2\,C_2 - \alpha\,(1 + C_1^2)) = \frac{C_2^2}{1 + C_1^2}, \tag{3.15}$$

$$L_h(v_h, \boldsymbol{q}_h; v_h, \boldsymbol{q}_h) \geq C\,|v_h|_{1,h}^2, \tag{3.16}$$

from which we can obtain (3.9), using (3.8) and the triangle inequality. □

Theorem 3.2. *Let $(u, \boldsymbol{p} = -A\bigtriangledown u) \in (U \cap H^2(\Omega)) \times (X \cap H^1(\mathrm{div};\Omega))$ be the exact solutions, and let $(u_h, \boldsymbol{p}_h) \in U_h \times X_h$ be the finite element solution of* (3.2)*. Under Hypotheses (H1)–(H3), we have*

$$||u - u_h||_{1,h} + ||\boldsymbol{p} - \boldsymbol{p}_h||_{H(\mathrm{div};\Omega)} \leq C\,h\,\{||u||_2 + |\mathrm{div}\boldsymbol{p}|_1\}. \tag{3.17}$$

Proof. For any $v_h \in U_h, \boldsymbol{q}_h \in X_h$, we have

$$
\begin{aligned}
&||u-u_h||_{1,h} + ||\boldsymbol{p}-\boldsymbol{p}_h||_{H(\mathrm{div};\Omega)} \\
&\quad \le ||u-v_h||_{1,h} + ||\boldsymbol{p}-\boldsymbol{q}_h||_{H(\mathrm{div};\Omega)} \\
&\qquad + ||v_h-u_h||_{1,h} + ||\boldsymbol{q}_h-\boldsymbol{p}_h||_{H(\mathrm{div};\Omega)},
\end{aligned} \tag{3.18}
$$

where

$$
\begin{aligned}
&||v_h-u_h||^2_{1,h} + ||\boldsymbol{q}_h-\boldsymbol{p}_h||^2_{H(\mathrm{div};\Omega)} \\
&\quad \le L_h(v_h-u_h, \boldsymbol{q}_h-\boldsymbol{p}_h; v_h-u_h, \boldsymbol{q}_h-\boldsymbol{p}_h) \\
&\quad = L_h(v_h-u, \boldsymbol{q}_h-\boldsymbol{p}; v_h-u_h, \boldsymbol{q}_h-\boldsymbol{p}_h) \\
&\qquad + L_h(u-u_h, \boldsymbol{p}-\boldsymbol{p}_h; v_h-u_h, \boldsymbol{q}_h-\boldsymbol{p}_h)
\end{aligned} \tag{3.19}
$$

and

$$
\begin{aligned}
&L_h(u, \boldsymbol{p}; v_h-u_h, \boldsymbol{q}_h-\boldsymbol{p}_h) \\
&\quad = (\boldsymbol{p} + A \bigtriangledown u, \boldsymbol{q}_h-\boldsymbol{p}_h + A \bigtriangledown_h (v_h-u_h)) \\
&\qquad + (\mathrm{div}\, \boldsymbol{p}, \mathrm{div}\, (\boldsymbol{q}_h-\boldsymbol{p}_h)) \\
&\quad = (f, \mathrm{div}\, (\boldsymbol{q}_h-\boldsymbol{p}_h)) \\
&\quad = L_h(u_h, \boldsymbol{p}_h; v_h-u_h, \boldsymbol{q}_h-\boldsymbol{p}_h).
\end{aligned} \tag{3.20}
$$

Hence, from (3.19) and (3.20) we get

$$
||v_h-u_h||_{1,h} + ||\boldsymbol{q}_h-\boldsymbol{p}_h||_{H(\mathrm{div};\Omega)} \le C\{||v_h-u||_{1,h} + ||\boldsymbol{q}_h-\boldsymbol{p}||_{H(\mathrm{div};\Omega)}\}. \tag{3.21}
$$

Combining (3.18) and (3.21), from (3.6) and (3.7) we have

$$
\begin{aligned}
&||u-u_h||_{1,h} + ||\boldsymbol{p}-\boldsymbol{p}_h||_{H(\mathrm{div};\Omega)} \\
&\quad \le C \inf_{(v_h,\boldsymbol{q}_h)\in U_h\times X_h} \{||u-v_h||_{1,h} + ||\boldsymbol{p}-\boldsymbol{q}_h||_{H(\mathrm{div};\Omega)}\} \\
&\quad \le C\, h\, \{||u||_2 + |\mathrm{div}\boldsymbol{p}|_1\}.
\end{aligned} \tag{3.22}
$$

□

Remark 3.1. Clearly, the standard argument of analyzing nonconforming error is lost in proving Theorem 3.2, since the term of inconsistent error does not exist. Nevertheless, the *patch test* (i.e., Hypothesis (H1)) is still indispensable to obtain the coerciveness.

Remark 3.2. Let Ω be a polygonal (or polyhedral) bounded domain, with $\Omega = \bigcup_{K\in\mathcal{T}_h} K$. Let S be the set of all element-edges (or faces) in the triangulation $\mathcal{T}_h$ and let S^0 be the set of all internal element-edges (or faces). Let $S^\partial = S - S^0$ and S^∂_D be the set of element-edges (or faces) on Γ_D, and let $S^\partial_N = S^\partial - S^\partial_D$. Denote by $[v] = v_{|_{K_1}} - v_{|_{K_2}}$ the jump of v across the interelement boundaries.

Moreover, let $U(K)$ be the nonconforming element of the CR linear element or the rotated $\mathcal{Q}_1$ element, and let $X(K)$ be the normal continuous element of the $RTN_0(K)$ triangular element or $RTN_{[0]}(K)$ rectangular element of lowest order.

Define

$$U_h = \{v \in L^2(\Omega);\ v_{|_K} \in U(K),\ \forall K \in \mathcal{T}_h,\quad \int_e [v] = 0, \forall e \in S^0, \int_e v = 0, \forall e \in S_D^{\partial}\}, \tag{3.23}$$

$$X_h = \{\boldsymbol{q} \in X; \boldsymbol{q}_{|_K} \in X(K), \forall K \in \mathcal{T}_h\}. \tag{3.24}$$

Then Hypotheses (H1) and (H2) hold, while (H3) can be derived from (H1) (see [27]).

3.2. **Method (II).** If X_h is replaced by

$$X_h^c \subset X \cap (H^1(\Omega))^d, \tag{3.25}$$

which is a continuous subspace in the usual sense [20], [22], then it is not obvious whether Theorem 3.1 holds or not, since Hypothesis (H1) is not necessarily valid.

In order to show that Theorem 3.1 is still true with X_h and U_h given by (3.25) and (3.1), respectively, we replace Hypothesis (H1) by Hypothesis (H1′) and give an additional Hypothesis (H4) as follows:
Hypothesis (H1′). There exists a constant $C > 0$, independent of h, such that

$$|\sum_{K\in\mathcal{T}_h} \int_{\partial K} v_h\, \boldsymbol{\chi} \cdot \boldsymbol{n}_K| \le C\, h\, ||\boldsymbol{\chi}||_1\, |v_h|_{1,h} \quad \forall (v_h, \boldsymbol{\chi}) \in U_h \times (X \cap (H^1(\Omega))^d). \tag{3.26}$$

Hypothesis (H4). There exists an interpolation operator $I_h^c : (H^1(\Omega))^d \to X_h^c$ such that

$$||I_h^c\, \boldsymbol{\chi} - \boldsymbol{\chi}|| \le C\, h\, |\boldsymbol{\chi}|_1, \quad |I_h^c\, \boldsymbol{\chi}|_1 \le C\, |\boldsymbol{\chi}|_1. \tag{3.27}$$

If additionally $\boldsymbol{\chi} \in (H^2(\Omega))^d$, then

$$||I_h^c \boldsymbol{\chi} - \boldsymbol{\chi}|| + h\, |I_h^c\, \boldsymbol{\chi} - \boldsymbol{\chi}|_1 \le C\, h^2\, ||\boldsymbol{\chi}||_2. \tag{3.28}$$

Theorem 3.3. *Under Hypotheses (H1′) and (H4), we have*

$$||\boldsymbol{q}_h + A \bigtriangledown_h v_h||^2 \ge C\, \{||\boldsymbol{q}_h||^2 + ||v_h||_{1,h}^2\} \quad \forall (\boldsymbol{q}_h, v_h) \in X_h^c \times U_h, \tag{3.29}$$

so long as h is sufficiently small.

Proof. For any $\boldsymbol{s}_h \in X_h^c$, we have

$$\begin{aligned} ||\boldsymbol{s}_h + A \bigtriangledown_h v_h||^2 = {} & ||\boldsymbol{s}_h + (A - \alpha\, E) \bigtriangledown_h v_h||^2 + 2\, \alpha\, (\boldsymbol{s}_h, \bigtriangledown_h v_h) \\ & + 2\, \alpha\, (A \bigtriangledown_h v_h, \bigtriangledown_h v_h) - \alpha^2\, |v_h|_{1,h}^2, \end{aligned} \tag{3.30}$$

where $\alpha > 0$ is a constant to be specified. Choosing $\boldsymbol{q}^* \in (X \cap (H^1(\Omega))^d)$ such that

$$\operatorname{div} \boldsymbol{q}^* = -v_h, \quad ||\boldsymbol{q}^*||_1 \le C\, ||v_h||, \tag{3.31}$$

we have

$$2\, \alpha\, (\boldsymbol{s}_h, \bigtriangledown_h v_h) = 2\, \alpha\, (\boldsymbol{s}_h - \boldsymbol{q}^*, \bigtriangledown_h v_h) + 2\, \alpha\, (\boldsymbol{q}^*, \bigtriangledown_h v_h), \tag{3.32}$$

where, from (3.31) and Hypothesis (H1′)

(3.33)
$$
\begin{aligned}
2\,\alpha\,(\boldsymbol{q}^*, \bigtriangledown_h v_h) &= -2\alpha\,(\operatorname{div}\boldsymbol{q}^*, v_h) + 2\alpha \sum_{K\in\mathcal{C}_h} \int_{\partial K} \boldsymbol{q}^*\cdot\boldsymbol{n}_K\, v_h \\
&\geq 2\alpha\,||v_h||^2 - 2\alpha\, C\, h\, ||\boldsymbol{q}^*||_1\, |v_h|_{1,h} \\
&\geq 2\alpha\,\{C_3\,||v_h||^2 - C_4\,h^2\,|v_h|_{1,h}^2\}.
\end{aligned}
$$

Since

$$
2\,\alpha\,(A\bigtriangledown_h v_h, \bigtriangledown_h v_h) - \alpha^2\,|v_h|_{1,h}^2 \geq \alpha\,(2\,C_2 - \alpha)\,|v_h|_{1,h}^2, \tag{3.34}
$$

then, if we put $0 < \alpha < 2\,C_2$, we have

(3.35)
$$
\begin{aligned}
||\boldsymbol{s}_h + A\bigtriangledown_h v_h||^2 &\geq ||\boldsymbol{s}_h + (A - \alpha\, E)\bigtriangledown_h v_h||^2 + \alpha(2C_2 - \alpha)\,|v_h|_{1,h}^2 \\
&\quad + 2\alpha\, C_3\,||v_h||^2 - 2\alpha\, C_4\, h^2\,|v_h|_{1,h}^2 - 2\alpha\,||\boldsymbol{s}_h - \boldsymbol{q}^*||\,|v_h|_{1,h}.
\end{aligned}
$$

Then, taking the infimum in (3.35) with respect to $\boldsymbol{s}_h$, we have

$$
\begin{aligned}
&\inf_{\boldsymbol{s}_h\in X_h^c} ||\boldsymbol{s}_h + A\bigtriangledown_h v_h||^2 + 2\alpha\,|v_h|_{1,h} \inf_{\boldsymbol{s}_h\in X_h^c} ||\boldsymbol{s}_h - \boldsymbol{q}^*|| \\
&\qquad \geq 2\alpha\, C_3\,||v_h||^2 + \alpha\,(2C_2 - \alpha - 2C_4\,h^2)\,|v_h|_{1,h}^2.
\end{aligned} \tag{3.36}
$$

It follows that

$$
\begin{aligned}
&\inf_{\boldsymbol{s}_h\in X_h^c} ||\boldsymbol{s}_h + A\bigtriangledown_h v_h||^2 \\
&\quad \geq -2\alpha\,|v_h|_{1,h} \inf_{\boldsymbol{s}_h\in X_h^c} ||\boldsymbol{s}_h - \boldsymbol{q}^*|| \\
&\qquad + 2\alpha\, C_3\,||v_h||^2 + \alpha\,(2C_2 - \alpha - 2C_4\,h^2)\,|v_h|_{1,h}^2 \\
&\quad \geq -2\alpha\,|v_h|_{1,h}\,||I_h^c\,\boldsymbol{q}^* - \boldsymbol{q}^*|| \\
&\qquad + 2\alpha\, C_3\,||v_h||^2 + \alpha\,(2C_2 - \alpha - 2C_4\,h^2)\,|v_h|_{1,h}^2 \\
&\quad \geq 2\alpha\, C_3\,||v_h||^2 + \alpha\,(2C_2 - \alpha - 2C_4\,h^2 - 2\,C\,h)\,|v_h|_{1,h}^2,
\end{aligned} \tag{3.37}
$$

where we have used (3.27) in Hypothesis (H4) and the second inequality in (3.31). Choosing h such that

$$
2C_2 - \alpha > 2C_4\,h^2 + 2\,C\,h, \tag{3.38}
$$

we have

$$
||\boldsymbol{q}_h + A\bigtriangledown_h v_h||^2 \geq \inf_{\boldsymbol{s}_h\in X_h^c} ||\boldsymbol{s}_h + A\bigtriangledown_h v_h||^2 \geq C\,||v_h||_{1,h}^2. \tag{3.39}
$$

Using the triangle inequality, we get

(3.40)
$$
C\,||\boldsymbol{q}_h||^2 \leq ||\boldsymbol{q}_h + A\bigtriangledown_h v_h||^2 + |v_h|_{1,h}^2 \leq C\,||\boldsymbol{q}_h + A\bigtriangledown_h v_h||^2,
$$

which completes the proof. □

Corollary 3.1. *Under Hypotheses (H1′), (H4) and* (3.6)*, if h is sufficiently small, then*

(3.41)
$$
L_h(v_h, \boldsymbol{q}_h; v_h, \boldsymbol{q}_h) \geq C\,\{||\boldsymbol{q}_h||_{H(\operatorname{div};\Omega)}^2 + ||v_h||_{1,h}^2\}, \quad \forall (v_h, \boldsymbol{q}_h) \in U_h \times X_h^c,
$$

$$
||u - u_h||_{1,h} + ||\boldsymbol{p} - \boldsymbol{p}_h||_{H(\operatorname{div};\Omega)} \leq C\,h\,\{||u||_2 + ||\boldsymbol{p}||_2\}, \tag{3.42}
$$

where $(u, \boldsymbol{p} = -A\nabla u) \in (U \cap H^2(\Omega)) \times (X \cap (H^2(\Omega))^d)$ *and* $(u_h, \boldsymbol{p}_h) \in U_h \times X_h^c$ *are the exact and the finite element solutions, respectively.* $\square$

Remark 3.3. Define

$$X_h^c = \{\boldsymbol{q} \in X \cap (H^1(\Omega))^d; \boldsymbol{q}_{|_K} \in (\mathcal{R}_1(K))^d, \forall K \in \mathcal{T}_h\}, \tag{3.43}$$

where $\mathcal{R}_1(K)$ denotes $\mathcal{P}_1(K)$ (the space of linear polynomials) or $\mathcal{Q}_1(K)$ (the space of bilinear polynomials), while U_h is still defined by (3.23). Then (H1$'$) (cf. [11]), (H4) and (3.6) hold, where I_h^c can be taken as the well-known Clément interpolation operator [21], [22].

Remark 3.4. Our method can be applied for other choices of X_h and U_h. For example, on triangles, we define (cf. [11], [36], [19])

$$\begin{aligned} U_h = \{v \in L^2(\Omega); v_{|_K} \in \mathcal{P}_3(K), K \in \mathcal{T}_h \\ \int_e [v]\, w = 0, w \in \mathcal{P}_2(e), e \in S^0, \int_e v\, w = 0, w \in \mathcal{P}_2(e), e \in S_D^\partial\}, \end{aligned} \tag{3.44}$$

$$X_h = \{\boldsymbol{q} \in X; \boldsymbol{q}_{|_K} \in BDFM_3(K), RTN_2(K), K \in \mathcal{T}_h\}, \tag{3.45}$$

$$X_h^c = \{\boldsymbol{q} \in (H^1(\Omega))^2 \cap X; \boldsymbol{q}_{|_K} \in (\mathcal{P}_2(K))^2, K \in \mathcal{T}_h\}, \tag{3.46}$$

it can be easily verified that Hypotheses (H1)–(H3) (or (H1$'$), (H4)) hold. Therefore, using similar arguments, we can obtain the coerciveness and the error bound $O(h^3)$ (or $O(h^2)$), with (3.44) and (3.45) (or (3.44) and (3.46)).

Remark 3.5. We can further consider the *Robin*-boundary value problem:

$$\begin{aligned} -\mathrm{div}\,(A \nabla u) + \kappa\, u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_D, \\ \boldsymbol{n} \cdot A \nabla u + \rho\, u = 0 \quad \text{on } \Gamma_N, \end{aligned} \tag{3.47}$$

where if $\Gamma_D = \emptyset$, we require that either $\kappa(x)$ or $\rho(x)$ is bounded below away from zero; if $\Gamma_D \neq \emptyset$, we require that both $\kappa(x)$ and $\rho(x)$ are nonnegative functions. A is a sufficiently smooth, symmetric matrix of coefficients which satisfies (2.2).

We introduce

$$W_{0,N}(\Omega) = \{(\boldsymbol{p}, u) \in H(\mathrm{div};\Omega) \times U; -\boldsymbol{p} \cdot \boldsymbol{n} + \rho\, u = 0 \text{ on } \Gamma_N\}, \tag{3.48}$$

where $W_{0,N}(\Omega)$ is a Hilbert space with respect to the norm $||\boldsymbol{p}||_{H(\mathrm{div};\Omega)} + ||u||_1$ (cf. [23]).

We consider the following finite element method: Find $(u_h, \boldsymbol{p}_h) \in W_h \subset W_{0,N}(\Omega)$ such that

$$(\boldsymbol{p}_h + A\nabla_h u_h, \boldsymbol{q}_h + A\nabla_h v_h) + (\mathrm{div}\, \boldsymbol{p}_h + \kappa\, u_h, \mathrm{div}\, \boldsymbol{q}_h + \kappa\, v_h) = (f, \mathrm{div}\, \boldsymbol{q}_h + \kappa\, v_h) \tag{3.49}$$

for all $(v_h, \boldsymbol{q}_h) \in W_h$, where

(3.50)
$$\begin{aligned} W_h = \{ & (\boldsymbol{q}, v) \in H(\mathrm{div};\Omega) \times L^2(\Omega); \\ & \boldsymbol{q}_{|_K} \in X(K) \text{ (or } \boldsymbol{q} \in (H^1(\Omega))^d \text{ and } (\mathcal{R}_1(K))^d), \forall K \in \mathcal{T}_h, \\ & v_{|_K} \in U(K), \forall K \in \mathcal{T}_h, \int_e [v] = 0, e \in S^0, \int_e v = 0, e \in S_D^\partial, \\ & \boldsymbol{q}\cdot\boldsymbol{n} = \rho\, v \text{ on } S_N^\partial \}. \end{aligned}$$

Similarly, we can obtain the coerciveness and the optimal error bound $O(h)$.

4. Quadrilateral elements

Clearly, under Hypotheses (H1)–(H3) (or (H1′), (H4) and (3.6)), we have established both coerciveness and error bound for the first-order system least-squares nonconforming mixed finite element problem (3.2). However, in the previous section we only dealt with triangular (or rectangular) elements in $\Re^2$, or tetrahedral (or rectangular solid) elements in $\Re^3$. In this section we will construct quadrilateral elements satisfying (H1)-(H3) (or (H1′), (H4) and (3.6)). Due to the Clément interpolation, (H4) can still be easily verified.

We consider the quadrilateral triangulation $\mathcal{T}_h = \{K\}$ in the xy-plane, with K a quadrilateral whose diameter is h_K and whose four vertices are $(x_i, y_i), 1 \le i \le 4$, and $h = \sup_{K\in\mathcal{T}_h} h_K$ is the mesh size. $|e|$ denotes the length of any edge $e \in \partial K$, and $\boldsymbol{n}_e$ denotes the unit outward normal vector to e. In addition, in this section the curl operator $\mathrm{curl}\, v = (-\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x})^t$ is used. $|K|$ denotes the measurement of K.

Let $\hat{K}$ be the reference square on the $\xi\eta$-plane, and let $F_K : \hat{K} \to K$ be the invertible mapping, with inverse mapping $F_K^{-1} : K \to \hat{K}$. From the assigned function $\hat{v} : \hat{K} \to \Re$ we can get a corresponding function $v_K : K \to \Re$ by $v_K(x, y) := \hat{v}(F_K^{-1}(x, y))$.

Lemma 4.1. *For any given function $v \in H^1(K)$, if it has zero mean value over any given side $e \in \partial K$ ($\int_e v = 0$), or over K ($\int_K v = 0$), then, under the shape regular condition of $\mathcal{T}_h$,*

$$||v||_{0,K} \le C\, h_K\, |v|_{1,K}. \tag{4.1}$$

Proof. Since the restriction of F_K to any given side $e \in \partial K$ is an affine mapping, from $\int_e v = 0$ we know that $\int_{\hat{e}} \hat{v} = 0$. Then, applying the Poincaré inequality [14] to the reference element $\hat{K}$, we have $||\hat{v}||_{0,\hat{K}} \le C\, |\hat{v}|_{1,\hat{K}}$. As a result, $||v||_{0,K} \le C\, h_K\, ||\hat{v}||_{0,\hat{K}} \le C\, h_K\, |\hat{v}|_{1,\hat{K}} \le C\, h_K\, |v|_{1,K}$. If $\int_K v = 0$, it follows from the Poincaré inequality again that $||v||_{0,K} \le C\, h_K\, |v|_{1,K}$. □

4.1. **A quadrilateral nonconforming element.** Define

$$Y(K) = \mathrm{span}\{1, x, y, B_K(x, y)\}, \quad B_K(x, y) = (\xi^2 - \eta^2) \circ F_K^{-1}, \tag{4.2}$$

$$\begin{aligned} U_h = \{ & v \in L^2(\Omega); v_{|_K} \in Y(K), K \in \mathcal{T}_h, \\ & \int_e [v] = 0, e \in S^0, \int_e v = 0, e \in S_D^\partial \}. \end{aligned} \tag{4.3}$$

Remark 4.1. Unlike [18], here $Y(K)$ is not a parametric or nonparametric space, but a *part-parametric* one, and this allows for not only arbitrary shape-regular quadrilateral meshes, but also efficient calculation on the reference element $\hat{K}$.

Theorem 4.1. *If $w \in H^2(K)$, we can define a unique $I_K\, w \in Y(K)$ such that*

$$\int_e (I_K\, w - w) = 0 \quad \forall e \in \partial K, \tag{4.4}$$

where $I_K\, w$ satisfies

$$||w - I_K\, w||_{0,K} + h_K\, |w - I_K\, w|_{1,K} \leq C\, h_K^2\, ||w||_{2,K}, \tag{4.5}$$

$$h_K^{2m-2}\, ||w - I_K\, w||_{m,K} + |I_K\, w|_{1,K} \leq C\, |w|_{1,K} \quad (m = 0, 1). \tag{4.6}$$

Proof. To determine $I_K w \in Y(K)$, we only need to show that the following coefficient matrix is nonsingular:

$$\begin{pmatrix} |e_1|^{-1} \int_{e_1} & |e_1|^{-1} \int_{e_1} x & |e_1|^{-1} \int_{e_1} y & |e_1|^{-1} \int_{e_1} B_K(x,y) \\ |e_2|^{-1} \int_{e_2} & |e_2|^{-1} \int_{e_2} x & |e_2|^{-1} \int_{e_2} y & |e_2|^{-1} \int_{e_2} B_K(x,y) \\ |e_3|^{-1} \int_{e_3} & |e_3|^{-1} \int_{e_3} x & |e_3|^{-1} \int_{e_3} y & |e_3|^{-1} \int_{e_3} B_K(x,y) \\ |e_4|^{-1} \int_{e_4} & |e_4|^{-1} \int_{e_4} x & |e_4|^{-1} \int_{e_4} y & |e_4|^{-1} \int_{e_4} B_K(x,y) \end{pmatrix}, \tag{4.7}$$

where $1 \leq i \pmod 4 \leq 4$

$$|e_i|^{-1} \int_{e_i} = 1, \quad |e_i|^{-1} \int_{e_i} B_K = (-1)^i\, \frac{2}{3},$$

$$|e_i|^{-1} \int_{e_i} x = \frac{x_i + x_{i+1}}{2}, \quad |e_i|^{-1} \int_{e_i} y = \frac{y_i + y_{i+1}}{2}.$$

It can be seen that the determinant of (4.7) is $\dfrac{4\,|K|}{3} \neq 0$, which leads to the conclusion.

To show both (4.5) and (4.6), we let $I_K\, w(x,y) = a_1 + a_2 x + a_3 y + a_4 B_K(x,y)$ on each K, where the $a_i, 1 \leq i \leq 4$, are coefficients to be determined, and we let $b_i = \int_{e_i} w/|e_i|, e_i \in \partial\, K, 1 \leq i \leq 4$. By solving a standard algebraic linear system, we get

$$\begin{aligned} a_4 &= \frac{3}{8}\,(b_4 - b_3 + b_2 - b_1), \\ \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} &= \frac{1}{4\,|K|} \begin{pmatrix} y_4 - y_2 & y_1 - y_3 \\ x_2 - x_4 & x_3 - x_1 \end{pmatrix} \begin{pmatrix} b_2 - b_1 + b_3 - b_4 \\ b_3 - b_2 + b_4 - b_1 \end{pmatrix}, \end{aligned} \tag{4.8}$$

where $a_1 = b_1 - \dfrac{x_1 + x_2}{2} a_2 - \dfrac{y_1 + y_2}{2} a_3 + \dfrac{2}{3} a_4$.

Note that

$$\int_K \bigtriangledown (w - I_K\, w) = \int_{\partial\, K} (w - I_K\, w) = 0. \tag{4.9}$$

From Lemma 4.1 we have

$$\begin{aligned} &|w - I_K\, w|_{1,K} \leq C\, h_K\, |w - I_K\, w|_{2,K} \\ &\qquad \text{(with } \bigtriangledown (w - I_K\, w) \text{ vanishing average on } K), \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\|w - I_K\, w\|_{0,K} \leq C\, h_K\, |w - I_K\, w|_{1,K} \\ &\qquad \text{(with } w - I_K\, w \text{ vanishing average on } \partial\, K). \end{aligned} \tag{4.11}$$

In what follows, we shall show that

$$|I_K\, w|_{r,K} \le C\, |w|_{r,K} \quad (r=1,2). \tag{4.12}$$

By an easy but tedious calculation, from the standard trace theorem [20], [22] we have

$$|I_K\, w|_{1,K} \le C \left\{ \sum_{i=1}^{4} \left(|e_i|^{-1} \int_{e_i} w \right)^2 \right\}^{1/2} \le C\, \|\hat{w}\|_{1,\hat{K}}. \tag{4.13}$$

Due to the fact that for any constant polynomial $p_0 \in \mathcal{P}_0(K)$ the function $\tilde{v} \in Y(K)$ defined by $w + p_0$ is $I_K\, w + p_0$, we have

$$|I_K\, w|_{1,K} = |\tilde{v}|_{1,K} \le C \inf_{p_0 \in \mathcal{P}_0(K)} \{\|\hat{w} + p_0\|_{0,\hat{K}} + |\hat{w}|_{1,\hat{K}}\} \le C\, |\hat{w}|_{1,\hat{K}} \le C\, |w|_{1,K}. \tag{4.14}$$

Next, similarly to (4.13), from (4.8) and using the trace theorem again, we get

$$|I_K\, w|_{2,K} \le C\, h_K^{-1} \left\{ \sum_{i=1}^{4} \left(|e_i|^{-1} \int_{e_i} w \right)^2 \right\}^{1/2} \le C\, h_K^{-2} \{\|w\|_{0,K} + h_K\, |w|_{1,K}\}. \tag{4.15}$$

Also, since for any linear polynomial $p_1 \in \mathcal{P}_1(K)$ the function $\tilde{v} \in Y(K)$ defined by $w + p_1$ is $I_K\, w + p_1$, we have

$$|I_K\, w|_{2,K} = |\tilde{v}|_{2,K} \le C\, h_K^{-2} \inf_{p_1 \in \mathcal{P}_1(K)} \{\|w + p_1\|_{0,K} + h_K\, |w + p_1|_{1,K}\}. \tag{4.16}$$

Furthermore, it can be seen that

$$\begin{aligned} \inf_{p_1 \in \mathcal{P}_1(K)} \|w + p_1\|_{0,K} &\le \inf_{p_1 \in \mathcal{P}_1(K)} \inf_{p_0 \in \mathcal{P}_0(K)} \|w + p_1 + p_0\|_{0,K} \\ &\le C\, h_K \inf_{p_1 \in \mathcal{P}_1(K)} |w + p_1|_{1,K}, \end{aligned}$$

$$\begin{aligned} \inf_{p_1 \in \mathcal{P}_1(K)} |w + p_1|_{1,K} &= \inf_{p_1 \in \mathcal{P}_1(K)} \|\bigtriangledown w + \bigtriangledown p_1\|_{0,K} \\ &\le \|\bigtriangledown w + \boldsymbol{p}_0\|_{0,K} \le C\, h_K\, |\bigtriangledown w|_{1,K}, \end{aligned}$$

where we have chosen a special linear polynomial $p_1 = d_0 + d_1\, x + d_2\, y \in \mathcal{P}_1(K)$ with $\boldsymbol{p}_0 = (d_1, d_2)^t = -\int_K \bigtriangledown w/|K|$; that is to say, $\bigtriangledown w + \boldsymbol{p_0}$ has a vanishing average on K, and the above estimation is derived from Lemma 4.1.

Therefore, we have

$$\inf_{p_1 \in \mathcal{P}_1(K)} \{\|w + p_1\|_{0,K} + h_K\, |w + p_1|_{1,K}\} \le C\, h_K^2\, |w|_{2,K}. \tag{4.17}$$

It follows that $|I_K\, w|_{2,K} \le C\, |w|_{2,K}$. So (4.10)–(4.12) yield both (4.5) and (4.6). □

Remark 4.2. It is interesting to note that both (4.5) and (4.6) are derived from the standard Poincaré inequality, not from the Bramble-Hilbert lemma (cf. [20], [22]). As a matter of fact, since $\mathcal{P}_1(K) \circ F_K$ is only a proper subspace of $\mathcal{Q}_1(\hat{K})$ and $Y(K)$ is not an isoparametric space, the classical estimation on quadrilaterals is not available.

Corollary 4.1. *Define* $I_h : U \cap H^2(\Omega) \to U_h$ *as follows:*

$$I_h\, w(x,y) = I_K\, w(x,y) \quad \forall (x,y) \in K, \quad \forall K \in \mathcal{T}_h. \tag{4.18}$$

Then

$$||I_h\, w - w|| + h\, |w - I_h\, w|_{1,h} \le C\, h^2\, ||w||_2, \tag{4.19}$$

$$\left(\sum_{K \in \mathcal{T}_h} h_K^{2m-2}\, ||w - I_h\, w||_{m,K}^2 \right)^{1/2} + |I_h\, w|_{1,h} \le C\, |w|_1 \quad (m = 0, 1). \quad \square \tag{4.20}$$

4.2. **A quadrilateral normal continuous element.** Define

$$D(K) = \mathrm{span}\{(1,0)^t, (0,1)^t, (x,y)^t, \mathrm{curl}(N_1 \circ F_K^{-1})\}, \quad N_1(\xi,\eta) = (1+\xi)(1+\eta)/4, \tag{4.21}$$

$$X_h = \{\boldsymbol{q} \in X; \boldsymbol{q}_{|_K} \in D(K), K \in \mathcal{T}_h\}. \tag{4.22}$$

Remark 4.3. $D(K)$ is a new element for quadrilaterals, but is the same as the $\mathrm{RT}_{[0]}$ rectangular element [5], [19] when $\mathcal{T}_h$ is composed of rectangles.

Theorem 4.2. *Given a function* $\boldsymbol{\chi} \in H^1(\mathrm{div}; K)$*, we can find a unique interpolant* $\Pi_K\, \boldsymbol{\chi} \in D(K)$ *such that*

$$\int_e (\Pi_K\, \boldsymbol{\chi} - \boldsymbol{\chi}) \cdot \boldsymbol{n}_e = 0 \quad \forall e \in \partial K, \tag{4.23}$$

$$||\Pi_K\, \boldsymbol{\chi} - \boldsymbol{\chi}||_{0,K} \le C\, h_K\, |\boldsymbol{\chi}|_{1,K}, \tag{4.24}$$

$$||\mathrm{div}(\boldsymbol{\chi} - \Pi_K \boldsymbol{\chi})||_{0,K} \le C\, h_K\, |\mathrm{div}\boldsymbol{\chi}|_{1,K}, \tag{4.25}$$

$$\mathrm{div}\Pi_K \boldsymbol{\chi} = P_K \mathrm{div}\boldsymbol{\chi}, \quad ||\mathrm{div}\Pi_K \boldsymbol{\chi}||_{0,K} \le ||\mathrm{div}\boldsymbol{\chi}||_{0,K}, \tag{4.26}$$

where P_K *is the* L^2*-projection operator onto* $\mathcal{P}_0(K)$*, the space of constant polynomials.*

Proof. To show the existence and uniqueness of $\Pi_K\, \boldsymbol{\chi}$, we only show that the matrix generated from (4.23) is nonsingular. Let the outward unit normal vector to e_i be denoted by $\boldsymbol{n}_{e_i} = (y_i - y_{i+1}, x_{i+1} - x_i)^t / |e_i|$; then from (4.23) we get the following coefficient matrix:

$$\begin{pmatrix} y_1 - y_2 & x_2 - x_1 & y_1 x_2 - y_2 x_1 & 1 \\ y_2 - y_3 & x_3 - x_2 & y_2 x_3 - y_3 x_2 & 0 \\ y_3 - y_4 & x_4 - x_3 & y_3 x_4 - y_4 x_3 & 0 \\ y_4 - y_1 & x_1 - x_4 & y_4 x_1 - y_1 x_4 & -1 \end{pmatrix}. \tag{4.27}$$

By a direct calculation, we know that the determinant of (4.27) is

$$\begin{vmatrix} y_1 - y_3 & x_3 - x_1 \\ y_2 - y_4 & x_4 - x_2 \end{vmatrix} \begin{vmatrix} y_2 - y_3 & x_3 - x_2 \\ y_2 - y_4 & x_4 - x_2 \end{vmatrix} \neq 0,$$

since under the shape regular condition we have

$$\begin{vmatrix} x_1 & y_1 & 1 & 1 \\ x_2 & y_2 & 1 & 0 \\ x_3 & y_3 & 1 & 0 \\ x_4 & y_4 & 1 & 0 \end{vmatrix} \neq 0.$$

Next, we consider (4.24).

Consider two adjacent sides of K, say e_1, e_2, with outward unit vectors $\boldsymbol{n}_{e_1}, \boldsymbol{n}_{e_2}$. Under the shape regular condition, $\boldsymbol{n}_{e_1}, \boldsymbol{n}_{e_2}$ are linearly independent, and they can form a base in the plane. Then, any given vector-valued function $\boldsymbol{\chi}$ can be uniquely written as $\boldsymbol{\chi} = (\boldsymbol{\chi}\cdot\boldsymbol{n}_{e_1})\,\boldsymbol{n}_{e_1} + (\boldsymbol{\chi}\cdot\boldsymbol{n}_{e_2})\,\boldsymbol{n}_{e_2}$, and it can be easily seen that

$$||\boldsymbol{\chi}||_E \le |\boldsymbol{\chi}\cdot\boldsymbol{n}_{e_1}| + |\boldsymbol{\chi}\cdot\boldsymbol{n}_{e_2}| \le 2\,||\boldsymbol{\chi}||_E,$$

where $||\cdot||_E$ is the norm in 2-dimensional Euclidean space. Since $\Pi_K\,\boldsymbol{\chi}$ is defined by (4.23), i.e.,, $(\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi})\cdot\boldsymbol{n}_{e_i}$ has a vanishing average on $e\in\partial K$, by Lemma 4.1 we have

(4.28)
$$||(\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi})\cdot\boldsymbol{n}_{e_i}||_{0,K} \le C\,h_K\,|(\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi})\cdot\boldsymbol{n}_{e_i}|_{1,K} \le C\,h_K\,|\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi}|_{1,K}.$$

Then

(4.29)
$$\begin{aligned}||\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi}||_{0,K} &\le C\,\{||(\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi})\cdot\boldsymbol{n}_{e_1}||_{0,K} + ||(\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi})\cdot\boldsymbol{n}_{e_2}||_{0,K}\}\\ &\le C\,h_K\,\{|(\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi})\cdot\boldsymbol{n}_{e_1}|_{1,K} + |(\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi})\cdot\boldsymbol{n}_{e_2}|_{1,K}\}\\ &\le C\,h_K\,|\Pi_K\,\boldsymbol{\chi}-\boldsymbol{\chi}|_{1,K}.\end{aligned}$$

Now we will show that

$$|\Pi_K\,\boldsymbol{\chi}|_{1,K} \le C\,|\boldsymbol{\chi}|_{1,K}. \tag{4.30}$$

To that goal, define $b_i = \int_{e_i}\boldsymbol{\chi}\cdot\boldsymbol{n}_{e_i}$, $1\le i\le 4$, and $\Pi_K\,\boldsymbol{\chi} = (a,b)^t + c\,\boldsymbol{\varphi}_K + d\,\boldsymbol{\omega}_K$, with $\boldsymbol{\varphi}_K = (x,y)^t$ and $\boldsymbol{\omega}_K = \mathrm{curl}\,(N_1\circ F_K^{-1})$. From (4.23) we know that for any constant vector $\boldsymbol{p}_0\in(\mathcal{P}_0(K))^2$ we have $\Pi_K\,\boldsymbol{p}_0 \equiv \boldsymbol{p}_0$, and so

$$|c| \le C\,h_K^{-1}\,|\sum_{i=1}^4 b_i| = C\,h_K^{-1}\,|\int_K \mathrm{div}\,\boldsymbol{\chi}| \le C\,||\mathrm{div}\,\boldsymbol{\chi}||_{0,K}, \tag{4.31}$$

(4.32)
$$\begin{aligned}|d| \le C\sum_{i=1}^4 |b_i| &\le C\,h_K\,||\hat{\boldsymbol{\chi}}||_{1,\hat K} \le C\,h_K\,\{||\hat{\boldsymbol{\chi}}+\boldsymbol{p}_0||_{0,\hat K} + |\hat{\boldsymbol{\chi}}|_{1,\hat K}\}\\ &\le C\,h_K\,|\hat{\boldsymbol{\chi}}|_{1,\hat K} \le C\,h_K\,|\boldsymbol{\chi}|_{1,K}.\end{aligned}$$

Noting that

$$|\boldsymbol{\varphi}_K|_{1,K} \le C\,h_K, \quad |\boldsymbol{\omega}_K|_{1,K} \le C\,|N_1\circ F_K^{-1}|_{2,K} \le C\,h_K^{-1}, \tag{4.33}$$

we have

$$|\Pi_K\,\boldsymbol{\chi}|_{1,K} \le |c|\,|\boldsymbol{\varphi}_K|_{1,K} + |d|\,|\boldsymbol{\omega}_K|_{1,K} \le C\,|\boldsymbol{\chi}|_{1,K}. \tag{4.34}$$

Obviously, (4.24) follows from (4.29) and (4.30).

Moreover, from (4.23) and $\mathrm{div}\Pi_K\boldsymbol{\chi}\in\mathcal{P}_0(K)$, we have

$$\int_K (\mathrm{div}\,\boldsymbol{\chi} - P_K\mathrm{div}\,\boldsymbol{\chi}) = \int_K \mathrm{div}\,(\boldsymbol{\chi}-\Pi_K\,\boldsymbol{\chi}) = 0, \tag{4.35}$$

where P_K is the standard orthogonal L^2-projection operator onto $\mathcal{P}_0(K)$. Thus both (4.25) and (4.26) hold. The proof is finished. □

Corollary 4.2. *Let* $\Pi_h : H^1(\mathrm{div};\Omega)\to X_h$ *be defined by*

$$\Pi_h\,\boldsymbol{\chi}(x,y) = \Pi_K\boldsymbol{\chi}(x,y) \quad \forall (x,y)\in K, \quad \forall K\in\mathcal{T}_h. \tag{4.36}$$

Then

$$||\Pi_h \boldsymbol{\chi} - \boldsymbol{\chi}|| \leq C\, h\, |\boldsymbol{\chi}|_1, \quad ||\mathrm{div}\,(\Pi_h \boldsymbol{\chi} - \boldsymbol{\chi})|| \leq C\, h\, |\mathrm{div}\, \boldsymbol{\chi}|_1, \tag{4.37}$$

$$\mathrm{div}\Pi_h \boldsymbol{\chi} = P_h \mathrm{div}\boldsymbol{\chi}, \quad ||\mathrm{div}\Pi_h \boldsymbol{\chi}|| \leq ||\mathrm{div}\boldsymbol{\chi}||, \tag{4.38}$$

where P_h is the standard othogonal L^2-pojection operator onto M_h defined by

$$M_h = \{v \in L^2(\Omega); v_{|_K} \in \mathcal{P}_0(K), \forall K \in \mathcal{T}_h\}. \tag{4.39}$$

Lemma 4.2. *For any $\boldsymbol{s} \in D(K)$,*

$$\mathrm{div}\, \boldsymbol{s}_{|_K} = \text{constant}, \quad \boldsymbol{s} \cdot \boldsymbol{n}_e = \text{constant}, \quad \forall e \in \partial K. \tag{4.40}$$

Proof. Since the restrictions of $N_1 \circ F_K^{-1}$ to e_2 and e_3 are zero, it follows that

$$\mathrm{curl}\,(N_1 \circ F_K^{-1}) \cdot \boldsymbol{n}_{e_2} = \mathrm{curl}\,(N_1 \circ F_K^{-1}) \cdot \boldsymbol{n}_{e_3} = 0 \quad \text{on } e_2 \text{ and } e_3.$$

Next, let us consider, say, e_4, with the unit normal vector $\boldsymbol{n}_{e_4}$ and the unit tangent vector $\boldsymbol{\tau}_{e_4}$. Since

$$\mathrm{curl}\,(N_1 \circ F_K^{-1}) \cdot \boldsymbol{n}_{e_4} = \frac{\partial(N_1 \circ F_K^{-1})}{\partial \boldsymbol{\tau}_{e_4}}$$

and the restriction to e_4 of $N_1 \circ F_K^{-1}$ is a linear polynomial, we immediately know that $\mathrm{curl}\,(N_1 \circ F_K^{-1}) \cdot \boldsymbol{n}_{e_4} = \text{constant on } e_4$. □

Remark 4.4. Lemma 4.2 indicates that the interpolation $\Pi_K \boldsymbol{\chi}$ can be also determined by the mid-point values $\boldsymbol{\chi} \cdot \boldsymbol{n}$ on the four sides of ∂K.

Corollary 4.3. *There exists a constant $C > 0$, independent of h, such that*

$$\sup_{\boldsymbol{\chi} \in X_h} \frac{(v, \mathrm{div}\boldsymbol{\chi})}{||\boldsymbol{\chi}||_{H(\mathrm{div};\Omega)}} \geq C\, ||v||, \quad \forall v \in M_h. \tag{4.41}$$

Proof. For any given $v \in M_h$, let $\boldsymbol{s} \in (H^1(\Omega))^2$ be such that

$$\mathrm{div}\, \boldsymbol{s} = v, \quad ||\boldsymbol{s}||_1 + ||\boldsymbol{s}||_{H(\mathrm{div};\Omega)} \leq C\, ||v||. \tag{4.42}$$

We can define $\Pi_h\, \boldsymbol{s} \in X_h$ by

$$\int_e (\boldsymbol{s} - \Pi_h\, \boldsymbol{s}) \cdot \boldsymbol{n}_e = 0, \quad \forall e \in \partial K, \quad \forall K \in \mathcal{T}_h, \tag{4.43}$$

with

$$||\Pi_h\, \boldsymbol{s}|| \leq C\, |\boldsymbol{s}|_1, \quad ||\mathrm{div}\, \Pi_h\, \boldsymbol{s}|| \leq ||\mathrm{div}\, \boldsymbol{s}||. \tag{4.44}$$

Therefore, we get

$$\begin{aligned} \sup_{\boldsymbol{\chi} \in X_h} \frac{(v, \mathrm{div}\boldsymbol{\chi})}{||\boldsymbol{\chi}||_{H(\mathrm{div};\Omega)}} &\geq \frac{(v, \mathrm{div}\, \Pi_h\, \boldsymbol{s})}{||\Pi_h\, \boldsymbol{s}||_{H(\mathrm{div};\Omega)}} \\ &= \frac{(v, \mathrm{div}\, \boldsymbol{s})}{||\Pi_h\, \boldsymbol{s}||_{H(\mathrm{div};\Omega)}} + \frac{(v, \mathrm{div}\, (\Pi_h\, \boldsymbol{s} - \boldsymbol{s}))}{||\Pi_h\, \boldsymbol{s}||_{H(\mathrm{div};\Omega)}} \\ &= \frac{(v, \mathrm{div}\, \boldsymbol{s})}{||\Pi_h\, \boldsymbol{s}||_{H(\mathrm{div};\Omega)}} \geq C\, ||v||. \quad \square \end{aligned} \tag{4.45}$$

Remark 4.5. Corollaries 4.2 and 4.3 indicate that (X_h, M_h) can be used in the classical mixed finite element approximation for (2.5) of second-order elliptic problems, when using arbitrary quadrilateral meshes.

Remark 4.6. It is obvious that Hypotheses (H1)–(H3) hold with U_h and X_h defined by (4.3) and (4.22), respectively. Therefore, on the arbitrary quadrilateral triangulation, we can easily establish the coerciveness and the error bound $O(h)$ with both (4.3) and (4.22) used for solving (3.2).

Remark 4.7. For other quadrilateral versions of the $\mathrm{RTN}_{[0]}$ rectangular mixed element, (4.38) and the second inequality in (4.37) do not hold, since the *Piola*-transformation is involved (cf. [19], [25]). In the literature, (4.38) is referred to as the property of commuting diagram. This property is very important (cf. [19], [35], [37], [38]).

In what follows, we give other properties of X_h on quadrilateral meshes. To do so, we first define

$$V_h = \{v \in H^1(\Omega); v \circ F_K \in \mathcal{Q}_1(\hat{K}), K \in \mathcal{T}_h, v_{|\Gamma_N} = 0\}, \tag{4.46}$$

where $\mathcal{Q}_1(\hat{K}) = \mathrm{span}\{N_i(\xi, \eta), 1 \le i \le 4\}$ is the space of bilinear polynomials on $\hat{K}$, with $N_i(\xi, \eta) = (1 + \xi_i \xi)(1 + \eta_i \eta)/4, (\xi_i, \eta_i) \in \{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$.

Theorem 4.3. *We have*

$$\mathrm{curl}\, V_h \subset X_h. \tag{4.47}$$

Proof. For any given $v \in V_h$ we have $v_{|K} = \sum_{i=1}^4 v_i\, N_i \circ F_K^{-1}$, with the $v_i, 1 \le i \le 4$, being nodal values of v. To show (4.47), we only need to show that

$$\mathrm{curl}\,(N_i \circ F_K^{-1}) \in \mathrm{span}\{(1,0)^t, (0,1)^t, \mathrm{curl}\,(N_1 \circ F_K^{-1})\}.$$

Noting that

$$1 = \sum_{i=1}^4 N_i \circ F_K^{-1}, \quad x = \sum_{i=1}^4 x_i\, N_i \circ F_K^{-1}, \quad y = \sum_{i=1}^4 y_i\, N_i \circ F_K^{-1}, \tag{4.48}$$

we have

$$\left\{ \begin{array}{l} \sum_{i=1}^4 \mathrm{curl}\,(N_i \circ F_K^{-1}) = (0,0)^t, \\ \sum_{i=1}^4 x_i\, \mathrm{curl}\,(N_i \circ F_K^{-1}) = (0,1)^t, \\ \sum_{i=1}^4 y_i\, \mathrm{curl}\,(N_i \circ F_K^{-1}) = (-1,0)^t, \end{array} \right. \tag{4.49}$$

which can alternatively be written as

$$\left\{ \begin{array}{rcl} \sum_{i=2}^4 \mathrm{curl}\,(N_i \circ F_K^{-1}) & = & -\mathrm{curl}\,(N_1 \circ F_K^{-1}), \\ \sum_{i=2}^4 x_i\, \mathrm{curl}\,(N_i \circ F_K^{-1}) & = & (0,1)^t - x_1\, \mathrm{curl}\,(N_1 \circ F_K^{-1}), \\ \sum_{i=2}^4 y_i\, \mathrm{curl}\,(N_i \circ F_K^{-1}) & = & (-1,0)^t - y_1\, \mathrm{curl}\,(N_1 \circ F_K^{-1}). \end{array} \right. \tag{4.50}$$

By virtue of the shape-regularity of the partition, we know that

$$\begin{pmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{pmatrix} \tag{4.51}$$

is a nonsingular matrix, and by solving (4.50) we get

(4.52)
$$\operatorname{curl}(N_i \circ F_K^{-1}) = (a_i, d_i)^t + g_i \operatorname{curl}(N_1 \circ F_K^{-1}), \quad 2 \le i \le 4,$$

with (a_i, d_i, g_i) being constants. It immediately follows that (4.47) is true. □

Corollary 4.4. *For any $\boldsymbol{s} \in X_h$ $\operatorname{div} \boldsymbol{s} = 0$, there is an $v \in V_h$ such that*

(4.53)
$$\boldsymbol{s} = \operatorname{curl} v. \qquad \square$$

5. Conclusions

In this paper, in fact, we have shown that both coerciveness and optimal error bounds in energy norms still hold for the first-order system least-squares mixed method for second-order elliptic problems subject to various homogeneous boundary conditions, even if a nonconforming finite element is used for the displacement and a conforming element for the stress, since, in most cases, the nonconforming finite element space U_h satisfies Hypothesis (H1), while Hypotheses (H3) and (H1′) can be derived from (H1) (cf. [11], [27]), and Hypotheses (H2) and (H4) are trivial in the standard interpolation theory.

On arbitrary regular quadrilaterals [20], we have constructed two new elements. One is a new variant of the rotated $\mathcal{Q}_1$ nonconforming rectangular element, and the other is a new variant of the lowest-order $\mathrm{RTN}_{[0]}$ rectangular element. These two elements can also be used for other problems, such as the Stokes problem and the Reissner-Mindlin plate problem (cf. [38]).

This new nonconforming element can be easily generalized to 3-D. The analogy is

$$Y(K) = \operatorname{span}\{1, x, y, z, B_{1K}, B_{2K}\},$$

with $B_{1K}(x, y, z) = (\xi^2 - \eta^2) \circ F_K^{-1}, B_{2K}(x, y, z) = (\xi^2 - \zeta^2) \circ F_K^{-1}$. However, it seems difficult to construct an analogous normal continuous element in 3-D.

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