

CLASS NUMBERS OF SOME ABELIAN EXTENSIONS OF RATIONAL FUNCTION FIELDS

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ABSTRACT. Let P be a monic irreducible polynomial. In this paper we generalize the determinant formula for $h(K_{P^n}^+)$ of Bae and Kang and the formula for $h^-(K_{P^n})$ of Jung and Ahn to any subfields K of the cyclotomic function field K_{P^n} . By using these formulas, we calculate the class numbers $h^-(K), h(K^+)$ of all subfields K of K_P when q and $\deg(P)$ are small.

1. INTRODUCTION

Let $\mathbb{A} = \mathbb{F}_q[T]$ be the ring of polynomials over the finite field \mathbb{F}_q with q elements, and $k = \mathbb{F}_q(T)$. For each polynomial $N \in \mathbb{A}$, one uses the Carlitz module ρ to construct a field extension K_N , called the N -th cyclotomic function field, and its maximal real subfield K_N^+ . Let K be any subfield of some K_N , and $K^+ = K \cap K_N^+$. It is well known that the divisor class number $h(K)$ of K is divisible by the divisor class number $h(K^+)$ of K^+ . The relative divisor class number of K , denoted by $h^-(K)$, is given by $h(K)/h(K^+)$. In a recent paper Bae and Kang [BK] have obtained determinant formulas for $h^-(K_{P^n})$ and $h(K_{P^n}^+)$, where P is a monic irreducible polynomial in \mathbb{A} . In [JA1], Jung and Ahn have obtained a determinant formula for $h^-(K_{P^n})$. In this paper we generalize the Bae-Kang determinant formula for $h(K_{P^n}^+)$ and the Jung-Ahn formula for $h^-(K_{P^n})$ to any subfield K of K_{P^n} . Our proof is much simpler than the proofs of [BK] and [JA1].

In the classical case Girstmair [G] generalized the Maillet determinant and expressed the relative class number $h^-(K)$ of an imaginary abelian number field K in terms of the generalized Maillet determinant. He also computed the relative class number $h^-(K)$ when K has a prime conductor and its degree over \mathbb{Q} is small.

The layout of this paper is as follows. In section 2, we give some notation for cyclotomic extensions and their subfields of rational function fields, and we also state some earlier results needed in this paper. In section 3, we extend Kučera's lemma [Ku, Lemma 2] to the function field case. By using this lemma, we obtain determinant class number formulas for $h^-(K)$ and $h(K^+)$ (Theorem 3.2). In section 4, we consider the case of a prime conductor P . Since K_P/k is a cyclic extension, all subfields of K_P can be characterized by their degrees over k . We obtain reduced determinant class number formulas (Propositions 4.1 and 4.3) by using a primitive root modulo P . In the final section, these reduced formulas are used to calculate

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the class numbers $h^-(K)$ and $h(K^+)$ of all subfields K of K_P when q and $\deg(P)$ are small.

Throughout the paper, K is assumed to be contained in some cyclotomic function field.

2. BASIC FACTS AND NOTATION

For any nonzero polynomial $A \in \mathbb{A}$, let $\text{sgn}(A)$ be the leading coefficient of A . A nonzero polynomial $A \in \mathbb{A}$ is called *monic* if $\text{sgn}(A) = 1$. We fix a monic polynomial N . For any polynomial $A \in \mathbb{A}$, we denote by \bar{A} the unique polynomial such that $\bar{A} \equiv A \pmod{N}$ and $\deg(\bar{A}) < \deg(N)$. We also define $\text{sgn}_N(A) = \text{sgn}(\bar{A})$ and $\deg_N(A) = \deg(\bar{A})$.

It is well known that the Galois group of K_N over k is canonically isomorphic to $(\mathbb{A}/N)^*$. Explicitly, there is an isomorphism given as follows:

$$\Psi : (\mathbb{A}/N)^* \rightarrow G_N, \quad A \pmod{N} \mapsto \sigma_A,$$

where $\sigma_A(\lambda_N) = \rho_A(\lambda_N)$ for any N -torsion element λ_N of the Carlitz module. Moreover, $\Psi(\mathbb{F}_q^*) = J = \text{Gal}(K_N/K_N^+)$. Let us denote by \mathbb{M}_N the set of polynomials in \mathbb{A} with degree less than $\deg(N)$ and prime to N , and by \mathbb{M}_N^+ the subset of \mathbb{M}_N consisting of monic polynomials.

For any $A \in \mathbb{M}_N$, $Z_N(s, A)$ denotes the partial zeta function associated to the class of A in $Cl(\mathbb{A})$. For $A \in \mathbb{A}$, we let $\langle A \rangle_N = 1$ if $\text{sgn}_N(A) = 1$ and $\langle A \rangle_N = 0$ otherwise. Then from [JA2, Lemma 3.1], we have

$$(2.1) \quad Z_N(0, A) = \langle A \rangle_N - \frac{1}{q-1}.$$

Let K be a finite abelian extension of k which is contained in some cyclotomic function field. By the conductor of K , we mean the monic polynomial $N \in \mathbb{A}$ such that K_N is the smallest cyclotomic function field which contains K . Let $K^+ = K \cap K_N^+$ be the maximal real subfield of K . Let G_K be the Galois group of K/k and $J_K = \text{Gal}(K/K^+)$. Let \widehat{G}_K be the set of characters of G_K with values in \mathbb{C} . We denote by χ_0 the trivial character of G_K . A character $\chi \in \widehat{G}_K$ is called *real* if $\chi(J_K) = 1$ and *nonreal* otherwise. Denote by \widehat{G}_K^+ the set of all real characters of G_K , and put $\widehat{G}_K^- = \widehat{G}_K \setminus \widehat{G}_K^+$. Let $\mathbb{Q}[G_K]$ be the group ring with coefficients in \mathbb{Q} . For any subset H of G_K , let $s(H) = \sum_{\sigma \in H} \sigma \in \mathbb{Q}[G_K]$. Let $e^+ = s(J_K)/|J_K|$ and $e^- = 1 - e^+$. Let $\mathbb{Q}[G_K]^\pm = \{x \in \mathbb{Q}[G_K] : e^\mp x = 0\}$. There are well known class number formulas [Y, BK]

$$(2.2) \quad h^-(K) = \prod_{\chi \in \widehat{G}_K^-} \sum_{A \in \mathbb{M}_N} Z_N(0, A) \bar{\chi}(A)$$

and

$$(2.3) \quad h(K^+) = \prod_{\substack{\chi \in \widehat{G}_K^+ \\ \chi \neq \chi_0}} (- \sum_{A \in \mathbb{M}_N^+} \deg(A) \chi(A)).$$

3. DETERMINANT CLASS NUMBER FORMULAS

Let K be a finite abelian extension of k with conductor $N = P^n$, a power of a monic irreducible polynomial P . Fix a system C of representatives of G_K/J_K . For any $\tau \in G_K$, we denote by $\tilde{\tau}$ the unique element of C such that $\tau \tilde{\tau}^{-1} \in J_K$. Let

$\chi \in \widehat{G}_K$. Then χ induces a homomorphism $\mathbb{C}[G_K] \rightarrow \mathbb{C}$ in an obvious way. We denote it also by χ .

Lemma 3.1. *Let C be any system of representatives of G_K/J_K .*

(i) *For any $\theta = \sum_{\sigma \in G_K} t_\sigma \sigma^{-1} \in \mathbb{Q}[G_K]^-$, we have*

$$(3.1) \quad \det(t_{\sigma\tau} - t_{\sigma\bar{\tau}} : \sigma, \tau \notin C) = \pm \prod_{\chi \in \widehat{G}_K^-} \chi(\theta).$$

(ii) *For any $\eta = \sum_{\sigma \in G_K} s_\sigma \sigma^{-1} \in \mathbb{Q}[G_K]^+$, we have*

$$(3.2) \quad \det(s_{\sigma\tau} : \sigma, \tau \in C) = \pm |J_K|^{-[K^+:k]} \prod_{\chi \in \widehat{G}^+} \chi(\eta).$$

Proof. For each $\chi \in \widehat{G}$, let $e_\chi = \frac{1}{|G_K|} \sum_{\sigma \in G_K} \chi(\sigma) \sigma^{-1}$ be the idempotent element associated to χ . For (i) we consider the linear transformation T of $\mathbb{C}[G_K]^-$ into $\mathbb{C}[G_K]^-$ defined as multiplication by θ . Clearly the matrix of T with respect to the basis $\{e_\chi : \chi \in \widehat{G}_K^-\}$ is given as $\text{diag}(\chi(\theta) : \chi \in \widehat{G}_K^-)$. One can easily show that $X = \{(1-j)\sigma^{-1} : 1 \neq j \in J_K, \sigma \in C\}$ forms a basis of $\mathbb{C}[G_K]^-$. Since $C' = \{\sigma^{-1} : \sigma \in C\}$ is also a system of representatives of G_K/J_K , it follows that $X' = \{(1-j)\sigma : 1 \neq j \in J_K, \sigma \in C\}$ is also a basis of $\mathbb{C}[G_K]^-$. Then the matrix of T from X' to X is given by $(t_{\sigma\tau} - t_{\sigma\bar{\tau}} : \sigma, \tau \notin C)$. Since X and X' are bases of the free abelian group $\mathbb{Z}[G_K]^-$, the transition matrix from X to X' has determinant ± 1 , so we get (i).

For (ii) we consider the linear transformation S of $\mathbb{C}[G_K]^+$ into $\mathbb{C}[G_K]^+$ defined by multiplication by η . Clearly S has the matrix $\text{diag}(\chi(\theta) : \chi \in \widehat{G}_K^+)$ with respect to the basis $\{e_\chi : \chi \in \widehat{G}_K^+\}$. It is easy to show that $Y = \{s(J_K)\sigma^{-1} : \sigma \in C\}$ and $Y' = \{s(J_K)\sigma : \sigma \in C\}$ constitute bases of $\mathbb{Q}[G_K]^+$, and the matrix of S from Y' to Y is given by $(|J_K|s_{\sigma\tau} : \sigma, \tau \in C)$. Since Y and Y' are also bases of the free abelian group $\mathbb{Z}[G_K]^+$, the transition matrix from Y to Y' has determinant ± 1 , so we get (ii). \square

For $A \in \mathbb{M}_N$, let σ'_A be the restriction of σ_A to K . By using Lemma 3.1 and the class number formulas (2.2) and (2.3), we obtain the following determinant class number formulas.

Theorem 3.2. *Let K/k be any finite abelian extension with conductor $N = P^n$. Let C be any system of representatives of G_K/J_K .*

(i) *For $\sigma \in G_K$, let $t_\sigma = \sum_{A \in \mathbb{M}_N, \sigma'_A = \sigma} Z_N(0, A)$. Then*

$$(3.3) \quad h^-(K) = |\det(t_{\sigma\tau} - t_{\sigma\bar{\tau}} : \sigma, \tau \notin C)|.$$

(ii) *For $\sigma \in G_K$, let $s_\sigma = -\frac{|J_K|}{q-1} \sum_{A \in \mathbb{M}_N, \sigma'_A = \sigma} \text{deg}(A)$. Then*

$$(3.4) \quad h(K^+) = \frac{1}{d(N)} |\det(s_{\sigma\tau} : \sigma, \tau \in C)|,$$

where $d(N) = \sum_{A \in \mathbb{M}_N^+} \text{deg}(A)$.

Proof. (i) Take

$$\theta = \sum_{\sigma \in G_K} \left(\sum_{A \in \mathbb{M}_N, \sigma'_A = \sigma} Z_N(0, A) \right) \sigma^{-1}.$$

We note that $\theta \in \mathbb{Q}[G_K]^-$ and $\chi(\sigma) = \chi(A)$ for any $A \in \mathbb{M}_N$ with $\sigma'_A = \sigma$. Thus for any $\chi \in \widehat{G}_K^-$, we have

$$\chi(\theta) = \sum_{\sigma \in G_K} \bar{\chi}(\sigma) \sum_{\substack{A \in \mathbb{M}_N \\ \sigma'_A = \sigma}} Z_N(0, A) = \sum_{A \in \mathbb{M}_N} Z_N(0, A) \bar{\chi}(A).$$

By Lemma 3.1 (i) and the class number formula (2.2), we get (3.3).

(ii) Take $\eta = \sum_{\sigma \in G_K} s_\sigma \sigma^{-1}$ with $s_\sigma = -\frac{|J_K|}{q-1} \sum_{A \in \mathbb{M}_N, \sigma'_A = \sigma} \deg(A)$. For any $\chi \in \widehat{G}_K^+$, we have

$$\begin{aligned} \chi(\theta) &= -\frac{|J_K|}{q-1} \sum_{\sigma \in G_K} \bar{\chi}(\sigma) \sum_{\substack{A \in \mathbb{M}_N \\ \sigma'_A = \sigma}} \deg(A) \\ &= -\frac{|J_K|}{q-1} \sum_{A \in \mathbb{M}_N} \deg(A) \bar{\chi}(A) = -|J_K| \sum_{A \in \mathbb{M}_N^+} \deg(A) \bar{\chi}(A). \end{aligned}$$

In particular, if $\chi = \chi_0$, then $\chi(\theta) = -|J_K|d(N)$. Thus by Lemma 3.1 (ii) and the class number formula (2.3), we get (3.4). \square

Remark 3.3. (i) One can easily see that Proposition 3.2 (i) and (ii) are generalizations of [JA1, Proposition 2.5] and [BK, Theorem 4], respectively.

(ii) Let $\bar{t}_\sigma = \sum_{A \in \mathbb{M}_N, \sigma'_A = \sigma} \langle A \rangle_N$. Then by (2.1), Theorem 3.2 can be read as

$$h^-(K) = |\det(\bar{t}_{\sigma\tau} - \bar{t}_{\sigma\bar{\tau}} : \sigma, \tau \notin C)|.$$

4. THE CASE OF PRIME CONDUCTOR

In this section we assume that K has the conductor P , where P is a monic irreducible polynomial of degree d in \mathbb{A} . Let $n = [K : k], n_0 = [K^+ : k]$. Choose a primitive root Q modulo P . Clearly we may assume that $\deg(Q) < \deg(P)$. For each $i \in \mathbb{Z}$, let $Q_i \in \mathbb{M}_P$ be the unique element such that $Q_i \equiv Q^i \pmod{P}$. Thus $Q_i = Q_{i'}$ if $i \equiv i' \pmod{q^d - 1}$. Let $\sigma \in G_K$ be the restriction of $\sigma_Q \in G_P$ to K . Then G_K is a cyclic group of order n generated by σ . Since $\text{Gal}(K_P/K) = \{\sigma_{Q_{n_j}} : 0 \leq j \leq \frac{q^d - 1 - n}{n}\}$ and $\text{Gal}(K_P/K^+) = \{\sigma_{Q_{n_0j}} : 0 \leq j \leq \frac{q^d - 1 - n_0}{n_0}\}$, the sign group J_K is generated by σ^{n_0} . Thus $C = \{\sigma^i : 0 \leq i \leq n_0 - 1\}$ is a system of representatives of G_K/J_K . For each $i \in \mathbb{Z}$, we define i^* to be the unique integer such that $i \equiv i^* \pmod{n_0}$ and $0 \leq i^* < n_0$. Then $\tilde{\sigma}^i = \sigma^{i^*}$. For each $i \in \mathbb{Z}$, we let $t_i = t_{\sigma^i}$ and $s_i = s_{\sigma^i}$. Clearly $t_i = t_{i'}, s_i = s_{i'}$ if $i \equiv i' \pmod{n}$. Then we have

$$t_i = \sum_j Z_P(0, Q_{i+jn}), \quad s_i = -\frac{|J_K|}{q-1} \sum_j \deg(Q_{i+jn}),$$

where $0 \leq j \leq \frac{q^d - 1 - n}{n}$. We note that by (2.1) we have

$$t_i = \sum_j \langle Q_{i+jn} \rangle_P - (q^d - 1)/n(q - 1).$$

In this case Theorem 3.2 can be written as follows.

Proposition 4.1. *Let K/k be a finite abelian extension with prime conductor P . With the same notation as above, we have the following class numbers formulas:*

- (i) $h^-(K) = |\det(t_{i+j} - t_{i+j^*} : n_0 \leq i, j \leq n-1)|$,
- (ii) $h(K^+) = \frac{1}{d(P)} |\det(s_{i+j} : 0 \leq i, j \leq n_0-1)|$.

Since $\text{Gal}(K_P/K_P^+) = \langle \sigma_Q^{(q^d-1)/(q-1)} \rangle$ and $\text{Gal}(K_P/K) = \langle \sigma_Q^n \rangle$, we have

$$\text{Gal}(K_P/K^+) = \langle \sigma_Q^{q^d-1/q-1}, \sigma_Q^n \rangle.$$

Thus $n_0 = \gcd(n, \frac{q^d-1}{q-1})$. Since K_P/k is a cyclic extension, we can completely determine all subfields, their sign groups and their class numbers by Proposition 4.1.

Lemma 4.2. *For each $i, j \in \{0, \dots, n-1\}$, we have*

- (i) $\sum_{0 \leq e \leq (n/n_0)-1} t_{i+en_0} = 0$,
- (ii) $s_i = s_{i'}$ if $i \equiv i' \pmod{n_0}$,
- (iii) $s_0 + \dots + s_{n_0-1} = -d(P)$.

Proof. Let $S_i = \{Q_{i+jn} : j = 0, \dots, \frac{q^d-1-n}{n}\}$ for $i = 0, \dots, n-1$. Then $\{S_i : i = 0, \dots, n-1\}$ forms a partition of \mathbb{M}_P . It is easy to see that if $\sigma'_A = \sigma^e$ for some $A \in \mathbb{M}_P$, then multiplication by A followed by reduction modulo P defines a one-to-one map from S_i to S_{i+e} for each i . Suppose that $Z_P(0, A)$ appears in the summation of (i) with nonzero coefficient. Since J_K is generated by σ^{n_0} and $\text{res}_{K_P/K}(\text{Gal}(K_P/K_P^+)) = J_K$, for any $\alpha \in \mathbb{F}_q^*$ we have $\sigma'_\alpha = \sigma^{en_0}$, and so $Z_P(0, \alpha A)$ also appears in the summation of (i) exactly once. Since $\sum_{\alpha \in \mathbb{F}_q^*} Z_P(0, \alpha A) = 0$ by (2.1), we get (i). Suppose that (ii) is true. Then we have

$$\begin{aligned} s_0 + \dots + s_{n_0-1} &= -\frac{|J_K|}{q-1} \sum_{0 \leq i \leq n_0-1} \sum_{0 \leq j \leq \frac{q^d-1-n}{n}} \deg(Q_{i+jn}) \\ &= -\frac{1}{q-1} \sum_{0 \leq i \leq n-1} \sum_{0 \leq j \leq \frac{q^d-1-n}{n}} \deg(Q_{i+jn}) = -d(P), \end{aligned}$$

so we get (iii). It remains to prove (ii). Suppose that $i' = i + en_0$ for some e . Then there exists $\alpha \in \mathbb{F}_q^*$ with $\sigma'_\alpha = \sigma^{en_0}$, and so we have

$$s_{i'} = -\frac{|J_K|}{q-1} \sum_j \deg(\alpha Q_{i+jn}) = s_i. \quad \square$$

Now we give reduced matrices of the matrices given in Proposition 4.1. Define

$$\mathbb{S}_K = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ s_0 & s_1 & s_2 & \dots & s_{n_0-2} & s_{n_0-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ s_{n_0-2} & s_{n_0-1} & s_0 & \dots & s_{n_0-4} & s_{n_0-3} \end{pmatrix}.$$

Proposition 4.3. (i) $h^-(K) = |J_K|^{[K^+:k]} |\det(t_{i+j} : n_0 \leq i, j \leq n-1)|$.
 (ii) $h(K^+) = |\det(\mathbb{S}_K)|$.

Proof. Consider the matrix $(t_{i+j} - t_{i+j^*} : n_0 \leq i, j \leq n - 1)$ in Proposition 4.1 (i). For each $0 \leq j^* \leq n_0 - 1$, add all $(j^* + un_0)$ -columns with $u = 2, \dots, (n/n_0) - 1$ to the $(j^* + n_0)$ -column. Then the $(i, j^* + n_0)$ -entry becomes

$$\begin{aligned} & \sum_{1 \leq u \leq (n/n_0) - 1} t_{i+j^*+un_0} - ((n/n_0) - 1)t_{i+j^*} \\ &= \sum_{0 \leq u \leq (n/n_0) - 1} t_{i+j^*+un_0} - (n/n_0)t_{i+j^*} \\ &= -(n/n_0)t_{i+j^*} \end{aligned}$$

by Lemma 4.2 (i). By using Lemma 4.2 (i) again, we get (i). From Lemma 4.2 (ii), (iii) and elementary row operations, we can change the matrix in Proposition 4.1 (ii) to

$$\begin{pmatrix} -d(P) & -d(P) & -d(P) & \dots & -d(P) & -d(P) \\ s_0 & s_1 & s_2 & \dots & s_{n_0-2} & s_{n_0-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ s_{n_0-2} & s_{n_0-1} & s_0 & \dots & s_{n_0-4} & s_{n_0-3} \end{pmatrix}.$$

Thus we get (ii). □

Remark 4.4. Let \mathcal{O}_K be the integral closure of \mathbb{A} in K and \mathcal{O}_K^* the unit group of \mathcal{O}_K . Let $h(\mathcal{O}_K)$ and $h(\mathcal{O}_{K^+})$ be the ideal class numbers of \mathcal{O}_K and \mathcal{O}_{K^+} , and let $h^-(\mathcal{O}_K) = h(\mathcal{O}_K)/h(\mathcal{O}_{K^+})$ be the relative ideal class number of \mathcal{O}_K . Then from [BJA, (2.1)], we have $h^-(K) = |J_K|^{[K^+:k]-1} h^-(\mathcal{O}_K)$. Thus by Proposition 4.3,

$$(4.1) \quad h^-(\mathcal{O}_K) = |J_K| \cdot |\det(t_{i+j} : n_0 \leq i, j \leq n - 1)|.$$

Let ℓ be a prime divisor of $q - 1$. We consider the subfields K of K_P of degree ℓ over k , which are characterized as follows [An, Lemma 3.2].

Lemma 4.5. *Let K/k be a finite abelian extension with prime conductor P , whose degree over k is ℓ .*

- (i) *If $d \equiv 0 \pmod{\ell}$. Then $K = k(\sqrt[\ell]{P})$ and it is contained in K_P^+ .*
- (ii) *If $d \not\equiv 0 \pmod{\ell}$, let $e, 1 \leq e \leq \ell - 1$, be such that $ed \equiv 1 \pmod{\ell}$. Then $K = k(\sqrt[\ell]{-P^e})$ and it is not contained in K_P^+ .*

We calculate the class number $h(K)$ by using Proposition 4.3. We note that if $d \not\equiv 0 \pmod{\ell}$, then $K^+ = k, G_K = J_K, h(K) = h^-(K)$, and if $d \equiv 0 \pmod{\ell}$, then $K^+ = K, J_K = \{1\}, h(K) = h(K^+)$. Define

$$\mathbb{D}_K = \begin{pmatrix} t_0 & t_1 & t_2 & \dots & t_{\ell-3} & t_{\ell-2} \\ t_1 & t_2 & t_3 & \dots & t_{\ell-2} & t_{\ell-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_{\ell-2} & t_{\ell-1} & t_0 & \dots & t_{\ell-5} & t_{\ell-4} \end{pmatrix}, \quad \text{if } d \not\equiv 0 \pmod{\ell},$$

and $\mathbb{D}_K = \mathbb{S}_K$ if $d \equiv 0 \pmod{\ell}$.

Corollary 4.6. *We let $\varepsilon = 1$ if $d \not\equiv 0 \pmod{\ell}$, and $\varepsilon = 0$ otherwise. Then $h(K) = \ell^\varepsilon |\det(\mathbb{D}_K)|$.*

Proof. We only prove the case that $d \not\equiv 0 \pmod{\ell}$. By using the fact that $\sum_{i=0}^{\ell-1} t_i = \sum_{A \in \mathbb{M}_N} Z_N(0, A) = 0$ and by elementary row operations, we can change the matrix $(t_{i+j} : 1 \leq i, j \leq \ell - 1)$ in Proposition 4.3 (i) to \mathbb{D}_K up to ± 1 . Thus we get the result. \square

Assume that q is odd. Then the class number of the quadratic subfield of K_P can be obtained in a simple form as follows.

Corollary 4.7. *Let q be odd. Let X_0 and X_1 be the subsets of \mathbb{M}_P consisting of quadratic residues modulo P and quadratic nonresidues modulo P , respectively.*

(i) *If $\deg(P)$ is odd, then the class number of $k(\sqrt{-P})$ is*

$$\left| 2S - \frac{q^d - 1}{q - 1} \right|,$$

where $S = |X_0 \cap \mathbb{M}_P^+|$.

(ii) *If $\deg(P)$ is even, then the class number of $k(\sqrt{P})$ is*

$$\frac{1}{q - 1} \left| (q - 1)d(P) - 2 \sum_{A \in X_0} \deg(A) \right|.$$

In particular, if $\deg(P) = 2$, then $h(k(\sqrt{P})) = 1$.

Proof. Let K be the unique quadratic subfield of K_P . Assume that $d \not\equiv 0 \pmod{2}$. Note that for any $A \in \mathbb{M}_P$, $\sigma'_A = 1$ if and only if A is a quadratic residue modulo P . Then by Corollary 4.6, we have $h(K) = 2|t_0|$, where $t_0 = \sum_{A \in \mathbb{M}_P \cap X_0} Z_P(0, A)$. By the definition of S , we have $t_0 = S - \frac{q^d - 1}{2(q - 1)}$, which implies (i).

Now assume that $d \equiv 0 \pmod{2}$. Then by Lemma 4.2 (iii) and Corollary 4.6, we have $h(K) = |s_0 - s_1| = |2s_0 + d(P)|$. Since

$$s_0 = -\frac{1}{q - 1} \sum_{1 \leq j \leq \frac{q^d - 3}{2}} \deg(Q_{2j}) = -\frac{1}{q - 1} \sum_{A \in X_0} \deg(A),$$

we get (ii). \square

5. SOME NUMERICAL EXAMPLES

Let K/k be a finite abelian extension with conductor P . In this section, we give some numerical examples for the class numbers $h^-(K)$ and $h(K^+)$ of K using Proposition 4.3. Here we used MAPLE for the computation. As the k -isomorphisms $T \mapsto T + \alpha$ with $\alpha \in \mathbb{F}_q^*$ send a monic irreducible polynomial to another monic irreducible polynomial, it suffices to consider only the monic irreducible polynomials up to these k -isomorphisms. In [Ar], Artin calculated the class number of quadratic function fields. One can see that our numerical data coincide with Artin's [Ar, p. 80-81] in the real quadratic function field case.

Example 5.1. Assume $q = 3$. There is only one monic irreducible polynomial T of degree 1 up to the above k -isomorphisms. It is already known that $h^-(K_T) = h(K_T^+) = 1$.

TABLE 1. $q = 3$, $\deg P = 2$

P	$[K : k]$	$[K^+ : k]$	$h^-(K)$	$h(K^+)$
$T^2 + 1$	2	2	1	1
	4	4	1	1
	8	4	2^3	1

TABLE 2. $q = 3$, $\deg P = 3$

P	$[K : k]$	$[K^+ : k]$	$h^-(K)$	$h(K^+)$
$T^3 + 2T + 1$	2	1	1	1
	13	13	1	3^9
	26	13	$2^{12} \cdot 3^6$	3^9
$T^3 + 2T + 2$	2	1	7	1
	13	13	1	3^9
	26	13	$2^{12} \cdot 3^3 \cdot 3^7$	3^9
$T^3 + T^2 + 2$	2	1	5	1
	13	13	1	$53 \cdot 313$
	26	13	$2^{12} \cdot 5 \cdot 79$	$53 \cdot 313$
$T^3 + 2T^2 + 1$	2	1	3	1
	13	13	1	$53 \cdot 313$
	26	13	$2^{12} \cdot 3 \cdot 131$	$53 \cdot 313$

TABLE 3. $q = 3$, $P = T^4 + T + 2$

P	$[K : k]$	$[K^+ : k]$	$h^-(K)$	$h(K^+)$
$T^4 + T + 2$	2	2	1	2^2
	4	4	1	$2^3 \cdot 17$
	5	5	1	71
	8	8	1	$2^4 \cdot 17 \cdot 97$
	10	10	1	$2^2 \cdot 71 \cdot 491$
	16	8	$2^7 \cdot 881$	$2^4 \cdot 17 \cdot 97$
	20	20	1	$2^3 \cdot 11^2 \cdot 17 \cdot 71 \cdot 491 \cdot 541$
	40	40	1	$2^4 \cdot 11^2 \cdot 17 \cdot 41 \cdot 71 \cdot 97 \cdot 491 \cdot 541$ $\cdot 881 \cdot 1564361$
	80	40	$2^{39} \cdot 241 \cdot 641 \cdot 881$ $\cdot 532611841$	$2^4 \cdot 11^2 \cdot 17 \cdot 41 \cdot 71 \cdot 97 \cdot 491 \cdot 541$ $\cdot 881 \cdot 1564361$

There is also one monic irreducible polynomial $T^2 + 1$ of degree 2 up to the above k -isomorphisms. Table 1 gives the degrees $[K : k]$, $[K^+ : k]$ and class numbers $h^-(K)$, $h(K^+)$ for all subfields K of K_{T^2+1} .

There are four monic irreducible polynomials $T^3 + 2T + 1$, $T^3 + 2T + 2$, $T^3 + T^2 + 2$ and $T^3 + 2T^2 + 1$ of degree 3 up to the above k -isomorphisms. The table for these polynomials are given in Table 2.

There are six monic irreducible polynomials of degree 4 up to the above k -isomorphisms. We only give the table for $T^4 + T + 2$ (Table 3).

TABLE 4. $q = 5, P = T^2 + 2$

P	$[K : k]$	$[K^+ : k]$	$h^-(K)$	$h(K^+)$
$T^2 + 2$	2	2	1	1
	3	3	1	1
	4	2	$2 \cdot 5$	1
	6	6	1	1
	8	2	$2^2 \cdot 5 \cdot 17$	1
	12	6	$2^5 \cdot 5$	1
	24	6	$2^{10} \cdot 5 \cdot 7^2 \cdot 17$	1

Example 5.2. Assume $q = 5$. For $P = T^2 + 2$, it is easy to check that $Q = T + 1$ is a primitive root modulo $T^2 + 2$. Note that $[K_P : k] = 24$. We consider the unique subfield K of K_P with degree $[K : k] = 12$. Then $[K^+ : k] = 6$ and $|J_K| = 2$. The matrix $(t_{i+j} : n_0 \leq i, j \leq n - 1)$ in Proposition 4.3 becomes

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{pmatrix}.$$

Then we have $h^-(K) = 2^5 \cdot 5$ and $h^-(\mathcal{O}_K) = 5$. The matrix S_K in Proposition 4.3 becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Thus we have $h(K^+) = 1$. Moreover, the table for $T^2 + 2$ is Table 4.

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