MATHEMATICS OF COMPUTATION
Volume 73, Number 245, Pages 425–449
S 0025-5718(03)01536-9
Article electronically published on May 21, 2003

CHEBYSHEV'S BIAS FOR COMPOSITE NUMBERS WITH RESTRICTED PRIME DIVISORS

PIETER MOREE

Abstract. Let $\pi(x; d, a)$ denote the number of primes $p \leq x$ with $p \equiv a(\mathrm{mod}\, d)$. Chebyshev's bias is the phenomenon for which "more often" $\pi(x; d, n) > \pi(x; d, r)$, than the other way around, where n is a quadratic non-residue mod d and r is a quadratic residue mod d. If $\pi(x; d, n) \geq \pi(x; d, r)$ for every x up to some large number, then one expects that $N(x; d, n) \geq N(x; d, r)$ for every x. Here $N(x; d, a)$ denotes the number of integers $n \leq x$ such that every prime divisor p of n satisfies $p \equiv a(\mathrm{mod}\, d)$. In this paper we develop some tools to deal with this type of problem and apply them to show that, for example, $N(x; 4, 3) \geq N(x; 4, 1)$ for every x.

In the process we express the so-called second order Landau-Ramanujan constant as an infinite series and show that the same type of formula holds for a much larger class of constants.

1. Introduction

Especially for small moduli d primes seem to have a preference for nonquadratic residue classes mod d over quadratic residue classes mod d. This phenomenon is called Chebysev's bias [4] (the term used in the older literature in this connection is the Shanks-Rényi primes race problem). For example, $\pi(x; 3, 1)$ does not exceed $\pi(x; 3, 2)$ for the first time until $x = 608981813029$, as was shown by Bays and Hudson [1]. On the other hand Littlewood [21] has shown that the function $\pi(x; 3, 2) - \pi(x; 3, 1)$ has infinitely many sign changes. The comparison of the behaviour of primes lying in various arithmetic progressions is the subject of comparative prime number theory, which was systematically developed in a series of papers by Knapowski and Turán; cf. [17, 40, 41]. Knapowski and Turán quantified Chebyshev's vague formulation by conjecturing that if $N(x)$ denotes the number of integers $m \leq x$ such that $\pi(m; 4, 1) \geq \pi(m; 4, 3)$, then $N(x) = o(x)$. After Knapowski and Turán's work a period of relative silence followed, until Kaczorowski revived this topic in a series of papers (for a recent survey see [16]). In [15] he showed that we cannot have $N(x) = o(x)$ under the Generalized Riemann Hypothesis (GRH).

Rubinstein and Sarnak [34] were the first to quantify some biases under GRH and the assumption that the nonnegative imaginary parts of the nontrivial zeros of all Dirichlet L-functions are linearly independent over the rationals. Define δ_{q,a_1,a_2}

Received by the editor November 7, 2001 and, in revised form, May 2, 2002.
2000 *Mathematics Subject Classification.* Primary 11N37, 11Y60; Secondary 11N13.
Key words and phrases. Comparative number theory, constants, primes in progression, multiplicative functions.

to be the logarithmic density of the set of real numbers x such that the inequality $\pi(x;q,a_1) > \pi(x;q,a_2)$ holds, where the logarithmic density of a set S is

$$\lim_{x\to\infty} \frac{1}{\log x} \int_{[2,x]\cap S} \frac{dt}{t},$$

assuming the limit exists. Under the aforementioned assumptions and assuming that $(\mathbb{Z}/q\mathbb{Z})^*$ is cyclic, Rubinstein and Sarnak showed that δ_{q,a_1,a_2} always exists and is strictly positive and, moreover, that $\delta_{q,n,r} > 0.5$ if and only if n is a nonsquare mod q and r is a square mod q. They calculated, amongst others, that $\delta_{4;3,1} = 0.9959\cdots$ and $\delta_{3;2,1} = 0.9990\cdots$. Thus Chebyshev's bias is not only an initial interval phenomenon. An important ingredient in their approach is a formula for the Fourier transform of a distribution function, which turns out to be an infinite product involving Bessel functions. This formula, however, was first derived by Wintner [43] (a simpler proof was given later by Bochner and Jessen [3]).

Let $g_{d,a}(n) = 0$ if n has a prime divisor p satisfying $p \not\equiv a (\operatorname{mod} d)$ and $g_{d,a}(n) = 1$ otherwise (note that $g_{d,a}(1) = 1$). We let $N(x;d,a) = \sum_{n\le x} g_{d,a}(n)$. The contribution of the small primes to the growth of $N(x;d,a)$ is much bigger than to $\pi(x;d,a)$ and hence we might expect that if $\pi(x;d,a) \ge \pi(x;d,b)$ up to some reasonable x, then actually $N(x;d,a) \ge N(x;d,b)$ for every x. In general, given two nonnegative multiplicative functions f and g such that f and g are equal in the primes on average to a positive constant τ and such that there is a bias towards f in the sense that $\sum_{p\le x} f(p) \ge \sum_{p\le x} g(p)$ for all x up to some rather large number, is it true that $\sum_{n\le x} f(n) \ge \sum_{n\le x} g(n)$ for *every* x ? The asymptotic behaviour of the latter type of sums is well understood and so proving that these types of results are true *asymptotically* is usually not difficult. We can, for example, invoke the following classical result due to Wirsing [44].

Theorem 1 (Wirsing [44]). *Let f be a multiplicative function satisfying $0 \le f(p^r) \le c_1 c_2^r$, $c_1 \ge 1$, $1 \le c_2 < 2$, and $\sum_{p\le x} f(p) = (\tau + o(1))x/\log x$, where τ, c_1 and c_2 are constants. Then, as $x \to \infty$,*

$$\sum_{n\le x} f(n) \sim \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p\le x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \frac{f(p^3)}{p^3} + \cdots\right),$$

where γ is Euler's constant and $\Gamma(\tau)$ denotes the gamma-function.

(Here and in the sequel the letter p is used to indicate primes.) We thus see that, for $i = 1$ and $i = 2$,

$$N(x;3,i) \sim \frac{e^{-\gamma/2}}{\sqrt{\pi}} \frac{x}{\log x} \prod_{\substack{p\le x \\ p\equiv i (\operatorname{mod} 3)}} \left(1 - \frac{1}{p}\right)^{-1}, \tag{1}$$

showing clearly the strong influence of the smaller primes. By [42, Theorem 2] we deduce from the latter formula that $N(x;3,i) \sim C_{3,i} x/\sqrt{\log x}$, with

$$C_{3,1} = \frac{3^{1/4}}{\pi\sqrt{2}} \prod_{p\equiv 1 (\operatorname{mod} 3)} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{3^{\frac{5}{4}}} \prod_{p\equiv 2 (\operatorname{mod} 3)} \left(1 - \frac{1}{p^2}\right)^{\frac{1}{2}}, \tag{2}$$

where in the derivation of (2) we used Euler's identity $\pi^2/6 = \prod_p (1 - p^{-2})^{-1}$ (another, self-contained, derivation of (2) is given in Section 6). Using Mertens' theorem or [42, Theorem 2] again, we easily infer that $C_{3,2} = 2/(3\pi C_{3,1})$. Restricting to the primes $p \leq 29$, we compute that $C_{3,1} < 0.302$ and $C_{3,2} > 0.703$ (for more precise numerical evaluations see Section 6). We thus infer that $N(x;3,2) \geq N(x;3,1)$ for every sufficiently large x. If we want to make this effective, the extensive literature (cf. [30]) on multiplicative functions satisfying conditions as in Wirsing's theorem appears to offer no help as nobody seems to have been concerned with proving effective results in this area, which is precisely what the Chebyshev bias problem for composites challenges us to do. In this paper we develop some tools for this and apply them to prove:

Theorem 2. *The inequalities $N(x;3,2) \geq N(x;3,1)$, $N(x;4,3) \geq N(x;3,1)$, $N(x;3,2) \geq N(x;4,1)$ and $N(x;4,3) \geq N(x;4,1)$ hold for every x.*

Not surprisingly the Chebyshev bias problem for composites is rather computational in nature and this appears to prohibit one from proving more general results.

The counting functions appearing in Theorem 2 can be shown to satisfy more precise asymptotic estimates than (1). Theorem 3 together with the prime number theorem for arithmetic progressions, show there exist constants $C_{d,a}, C_{d,a}(1), C_{d,a}(2), \ldots$ such that for each integer $m \geq 0$ we have

$$N(x;d,a) = \frac{C_{d,a}x}{\log^{1-1/\varphi(d)} x}\left(1 + \sum_{j=1}^{m} \frac{C_{d,a}(j)}{\log^j x} + O\left(\frac{1}{\log^{m+1} x}\right)\right),$$

where the implied constant may depend on m, a, d and $C_{d,a} > 0$. Thus $N(x;d,a)$ satisfies an asymptotic expansion in the sense of Poincaré in terms of $\log x$. The most famous example of such an expression states that for $B(x)$, the counting function of the integers that can be represented as a sum of two integer squares, we have

$$B(x) = \frac{Kx}{\sqrt{\log x}}\left(1 + \sum_{j=1}^{m} \frac{K_{j+1}}{\log^j x} + O\left(\frac{1}{\log^{m+1} x}\right)\right), \tag{3}$$

where K is the Landau-Ramanujan constant and K_2 the second order Landau-Ramanujan constant. The Landau-Ramanujan constant is named after Landau [19], who proved in 1908, using contour integration, that $B(x) \sim Kx/\sqrt{\log x}$ and Ramanujan, who in his first letter to Hardy claimed he could prove that

$$B(x) = K \int_2^x \frac{dt}{\sqrt{\log t}} + O(x^{1/2+\epsilon}); \tag{4}$$

cf. [25]. Ramanujan's claim implies $K_2 = 1/2$ by partial integration, which was shown to be false by Shanks [38]. Indeed, we have

$$K = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{-1/2} = 0.764223653589220662990698731250\cdots$$

and

$$K_2 = \frac{1}{2} - \frac{\gamma}{4} - \frac{1}{4}\frac{L'}{L}(1,\chi_{-4}) + \frac{\log 2}{4} + \frac{1}{2}\sum_{p \equiv 3 \pmod 4} \frac{\log p}{p^2 - 1} = 0.58194865931729\cdots,$$

where for any fundamental discriminant D, χ_D denotes Kronecker's extension (D/n) of the Legendre symbol [10, Chapter 5]. In his "unpublished" manuscript on the partition and tau-functions Ramanujan [2] made claims similar to (4), for various primes p, regarding the functions counting the integers $1 \le n \le x$ with $p \nmid \tau(n)$, where τ denotes Ramanujan's tau-function. Here again the asymptotics turn out to be correct, but the second order constants are not conforming to reality [24].

It was a folklore result that $B(x)$ should satisfy (3), which was written down by Serre [37], who gave some nice applications to Fourier coefficients of modular forms as well.

Let f be a nonnegative multiplicative function. Suppose there exists a positive constant τ such that

$$\sum_{n\le x} f(n) = \lambda_1(f) x \log^{\tau-1} x \left(1 + (1+o(1))\frac{\lambda_2(f)}{\log x}\right), \qquad x\to\infty.$$

We then define $\lambda_2(f)$ to be the *generalized second order Landau-Ramanujan constant*. In Theorem 4 we will identify a subclass of multiplicative functions for which this constant exists and express it as an infinite series.

The second order generalized Landau-Ramanujan constant $\lambda_2(f)$ is closely related to the constant B_f appearing in the proof of Lemma 1 (the key lemma in the proof of Theorem 2). (I suggest reading the next section first before going further.) Lemma 1 yields an effective estimate for $m_f(x)$, provided we can find constants τ, C_- and C_+ satisfying (8). For the functions f associated to the quantities in Theorem 2 we find admissible values of these constants in Section 8, which requires effective estimates for counting functions of squarefree numbers of a certain type (Section 7). (At the end of Section 8 we show that under GRH finding C_- and C_+ is much easier.) In Section 4 we show how to obtain effective estimates for $M_f(x)$ from effective estimates for $m_f(x)$. In Section 9 we show how to prove certain subcases of Theorem 2 for every x up to some large x_0 using existing numerical work on the associated Chebyshev prime biases. All these ingredients then come together in Section 10, where a proof of Theorem 2 is given.

In Section 5 we find an infinite series expansion for the constant B_f appearing in Lemma 2 (we have $C_- \le B_f \le C_+$) and relate it to the generalized second order Landau-Ramanujan constant. Section 6 contains a numerical study of some of the constants appearing in this paper.

In [26] the methods developed in this paper are somewhat refined and then are used to prove Schmutz Schaller's conjecture (see [35, p. 201] or the introduction of [8]) that the hexagonal lattice is "better" than the square lattice. More precisely, let $0 < h_1 < h_2 < \cdots$ be the positive integers, listed in ascending order, which can be written as $h_i = x^2 + 3y^2$ for integers x and y. Let $0 < q_1 < q_2 < \cdots$ be the positive integers, listed in ascending order, which can be written as $q_i = x^2 + y^2$ for integers x and y. Then Schmutz Schaller's conjecture is that $q_i \le h_i$ for $i = 1, 2, 3, \ldots$.

2. Notation

Let f be a nonnegative real-valued multiplicative function. We define $M_f(x) = \sum_{n\le x} f(n)$, $m_f(x) = \sum_{n\le x} f(n)/n$ and $\lambda_f(x) = \sum_{n\le x} f(n)\log n$. We denote the formal Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ associated to f by $L_f(s)$. If $f(p)$ equals $\tau > 0$ on average at primes p, it can be shown that $\lim_{s\to 1+0}(s-1)^{\tau} L_f(s)$ exists, under some mild additional conditions on f. In that case we let

$$C_f = \frac{1}{\Gamma(\tau)} \lim_{s\to 1+0} (s-1)^{\tau} L_f(s).$$

We have $C_f > 0$. We define $\Lambda_f(n)$ by

$$-\frac{L_f'(s)}{L_f(s)} = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s}.$$

Notice that

$$f(n)\log n = \sum_{d|n} f(d)\Lambda_f(\frac{n}{d}). \tag{5}$$

The notation suggests that $\Lambda_f(n)$ is an analogue of the von Mangoldt function. Indeed, if $f = \mathbf{1}$, then $L_f(s) = \zeta(s)$ and $\Lambda_f(n) = \Lambda(n)$. From (5) we infer by Möbius inversion the well-known formula

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d}. \tag{6}$$

In general, on writing $L_f(s)$ as an Euler product, one easily sees that $\Lambda_f(n)$ is zero if n is not a prime power. If f is the characteristic function of a subsemigroup of the natural integers with $(1 <)q_1 < q_2 < \cdots$ as generators, then it can be shown that $\Lambda_f(n) = \log q_i$ if n equals a positive power of a generator q_i and $\Lambda_f(n) = 0$ otherwise. Thus, for example, if $f = g_{d,a}$, then $\Lambda_{g_{d,a}}(n) = \log p$ if $n = p^r$, $r \ge 1$ and $p \equiv a (\bmod d)$, and $\Lambda_{g_{d,a}}(n) = 0$ otherwise.

From property (5) of $\Lambda_f(n)$, we easily infer that

$$\lambda_f(x) = \sum_{n \le x} f(n)\psi_f(\frac{x}{n}), \tag{7}$$

where $\psi_f(x) = \sum_{n\le x} \Lambda_f(n)$. For some further properties of $\Lambda_f(n)$ the reader is referred to [25, §2.2]. Our usage of $\Lambda_f(n)$ is inspired by the work of Levin and Fainleib; cf. [20].

The notation x_0, α and β is used to indicate inessential local constants; their values might be different in different contexts.

3. Effective estimates for $m_f(x)$

The following result will play a crucial rôle. It uses some ideas from the proof of Theorem A in [39].

Lemma 1. *Let f be a nonnegative multiplicative arithmetic function. Suppose that there exist constants $\tau (> 0), C_-$ and C_+ such that*

$$C_- \le \sum_{n\le x} \frac{\Lambda_f(n)}{n} - \tau \log x \le C_+ \quad \text{for every } x \ge 1. \tag{8}$$

Then, for $x > \exp(C_+)$, we have

$$\frac{C_f}{\tau} \log^{\tau} x \frac{\left(1 - \frac{C_+}{\log x}\right)^{\tau+1}}{1 - \frac{C_-}{\log x}} \le m_f(x) \le \frac{C_f}{\tau} \log^{\tau} x \frac{\left(1 - \frac{C_-}{\log x}\right)^{\tau+1}}{1 - \frac{C_+}{\log x}}, \tag{9}$$

where

$$C_f = \frac{1}{\Gamma(\tau)} \lim_{s \to 1+0} (s-1)^{\tau} L_f(s). \tag{10}$$

Remark. An alternative expression for C_f is given by

$$C_f = \frac{1}{\Gamma(\tau)} \lim_{s\to 1+0} \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}}\right)\left(1 - \frac{1}{p^s}\right)^{\tau}.$$

Proof of Lemma 1. Let B_f be an arbitrary constant and write

$$\sum_{n\le x} \frac{\Lambda_f(n)}{n} = \tau \log x + B_f + E_f(x). \tag{11}$$

(This is unnecessary for this proof, but it is needed in the proof of Lemma 3, so we do this now to save some space later.) We have

$$\begin{aligned}\sum_{n\le x} \frac{f(n)\log n}{n} &= \sum_{n\le x}\sum_{d|n} \frac{f(d)}{d}\frac{\Lambda_f(n/d)}{\frac{n}{d}} = \sum_{d\le x}\frac{f(d)}{d}\sum_{k\le \frac{x}{d}}\frac{\Lambda_f(k)}{k}\\ &= \tau \sum_{n\le x}\frac{f(n)}{n}\log(\frac{x}{n}) + B_f m_f(x) + \sum_{n\le x}\frac{f(n)}{n}E_f(\frac{x}{n}).\end{aligned}$$

We write this equality in the form

$$-\sum_{n\le x}\frac{f(n)}{n}\log(\frac{x}{n}) + m_f(x)\log x = \tau\sum_{n\le x}\frac{f(n)}{n}\log(\frac{x}{n}) + B_f m_f(x) + \sum_{n\le x}\frac{f(n)}{n}E_f(\frac{x}{n}).$$

This inequality in turn can be written, using that

$$\sum_{n\le x}\frac{f(n)}{n}\log\frac{x}{n} = \sum_{n\le x}\frac{f(n)}{n}\int_n^x \frac{dt}{t} = \int_1^x \frac{m_f(t)}{t}dt,$$

as

$$m_f(x)\log x - (\tau+1)\int_1^x \frac{m_f(v)}{v}dv = B_f m_f(x) + \sum_{n\le x}\frac{f(n)}{n}E_f(\frac{x}{n}). \tag{12}$$

Let $\sigma_f(x) = \int_1^x \frac{m_f(v)}{v}dv$. By assumption $C_- \le B_f + E_f(x) \le C_+$ for $x \ge 1$. Using (12) we then deduce that $m_f(x) = (\tau+1)\sigma_f(x)/\log x + m_f(x)\epsilon_f(x)$, where $C_- \le \epsilon_f(x)\log x \le C_+$. Solving this for $m_f(x)$, we find that

$$m_f(x) = \frac{1}{1-\epsilon_f(x)}\frac{\tau+1}{\log x}\sigma_f(x), \qquad x \ge x_0, \tag{13}$$

where $x_0 = \exp((1+\delta)C_+)$, and $\delta > 0$ is arbitrary and fixed. In the rest of the proof we assume that $x \ge x_0$. Let

$$R_f(t) = \log\left(\frac{\tau+1}{\log^{\tau+1} t}\sigma_f(t)\right).$$

Note that, for $t \ge x_0$,

$$R_f'(t) = \frac{\tau+1}{t\log t}\frac{\epsilon_f(t)}{[1-\epsilon_f(t)]} \tag{14}$$

and hence $R_f'(t) = O(t^{-1}\log^{-2} t)$. Thus $\int_x^{\infty} R_f'(t)dt$ converges absolutely, and therefore $\int_x^{\infty} R_f'(t)dt = A_f - R_f(x)$, for some constant A_f not depending on x. On writing $D_f = \exp(A_f)$, we obtain

$$\frac{\tau+1}{\log^{\tau+1} x}\sigma_f(x) = \exp(R_f(x)) = D_f \exp\left(-\int_x^{\infty} R_f'(t)dt\right). \tag{15}$$

Using (14) and $C_- \le \epsilon_f(x)\log x \le C_+$, we see that

$$\int_x^\infty \frac{C_-(\tau+1)}{t\log t[\log t - C_-]}dt \le \int_x^\infty R_f'(t)dt \le \int_x^\infty \frac{C_+(\tau+1)}{t\log t[\log t - C_+]}dt.$$

Thus

$$-(\tau+1)\log\left(1-\frac{C_-}{\log x}\right) \le \int_x^\infty R_f'(t)dt \le -(\tau+1)\log\left(1-\frac{C_+}{\log x}\right).$$

On combining (15) with (14), we deduce that

$$D_f\left(1-\frac{C_+}{\log x}\right)^{\tau+1} \le \frac{\tau+1}{\log^{1+\tau} x}\sigma_f(x) \le D_f\left(1-\frac{C_-}{\log x}\right)^{\tau+1}. \tag{16}$$

We will now show that $D_f = C_f/\tau$. The inequalities (16) in combination with (13) imply in particular that

$$m_f(x) = D_f \log^\tau x + O(\log^{\tau-1} x). \tag{17}$$

By partial integration and using the well-known integral expression for the gamma function, we find that

$$\begin{aligned} L_f(s) &= (s-1)\int_1^\infty \frac{m_f(t)}{t^s}dt = (s-1)\int_1^\infty \frac{D_f\log^\tau t + O(\log^{\tau-1} t)}{t^s}dt \\ &= D_f\frac{\Gamma(\tau+1)}{(s-1)^\tau} + O\left(\frac{s-1}{(s-1)^\tau}\right) \end{aligned}$$

and thus $D_f = C_f/\tau$. The inequalities (16) together with (13) yield (9), on using that $D_f = C_f/\tau$ and $C_- \le \epsilon_f(x)\log x \le C_+$. □

The convolutional nature of $\sum_{n\le x} E(x/n)f(n)/n$ forces us to require that $x \ge 1$ in (8) (whereas we would like to replace it with $x \ge x_0$). Nevertheless we can invoke the following easy lemma to improve (9).

Lemma 2. *Suppose that there exist constants D_- and D_+ such that for every $x \ge x_0$,*

$$D_- m_f(x) \le B_f m_f(x) + \sum_{n\le x}\frac{f(n)}{n}E_f(\frac{x}{n}) \le D_+ m_f(x). \tag{18}$$

Then we have, for $x > \max\{x_0, \exp(D_+)\}$,

$$\frac{C_f}{\tau}\log^\tau x\frac{\left(1-\frac{D_+}{\log x}\right)^{\tau+1}}{1-\frac{D_-}{\log x}} \le m_f(x) \le \frac{C_f}{\tau}\log^\tau x\frac{\left(1-\frac{D_-}{\log x}\right)^{\tau+1}}{1-\frac{D_+}{\log x}}. \tag{19}$$

Proof. Follows easily on closer scrutiny of the previous proof. □

We now give an example of how Lemma 2 can be used. By assumption we have $E_f(x) \le C_+ - B_f$ for every $x \ge 1$. Suppose that $E_f(x) \le C_+' - B_f$ for $x \ge n_0$, where $C_+' < C_+$. An upper bound for the inner term in (18) is then given by

$$C_+ m_f(x) - (C_+ - C_+')m_f(\frac{x}{n_0}).$$

Using the explicit bounds in (9), we can then find an x_0 and $D_+ < C_+$ such that the conditions of Lemma 2 are satisfied (note that $D_+ > C_+'$). By applying (19) instead of (9), a better value for D_+ can then be obtained. Then iterate.

By making an assumption on $E_f(x)$, we will, not surprisingly, be able to do better than both Lemma 1 and Lemma 2.

Lemma 3. *Let f be a nonnegative multiplicative arithmetic function and suppose that* (11) *holds with*

$$|E_f(x)| \le \frac{c_0}{\max\{1, \log x\}} \tag{20}$$

for every $x \ge 1$, where c_0 is some explicit constant. Then there exist effectively computable constants α, β and x_0 such that

$$m_f(x) = \frac{C_f}{\tau} \log^\tau x - C_f B_f \log^{\tau-1} x + \mathcal{E}_f(x),$$

where $\alpha \log^{\tau-1/2} x \le \mathcal{E}_f(x) \le \beta \log^{\tau-1/2} x$ for every $x \ge x_0$.

Proof. We denote the right-hand side of (20) by $h(x)$ and let $s(x) = x/e^{\sqrt{\log x}}$. Let $x_0 \ge e$. Using Lemma 1, it is not difficult to see that

$$\beta_f(x_0) = \sup_{x \ge x_0} \sqrt{\log x} \left\{ 1 - \frac{m_f(s(x))}{m_f(x)} \left(1 - \frac{1}{\sqrt{\log x}} \right) \right\}$$

and that it is finite and can be effectively computed (note that $\beta_f(x_0) \ge \tau + 1$).

Clearly

$$\Big| \sum_{n \le x} \frac{f(n)}{n} E_f(\frac{x}{n}) \Big| \le \sum_{n \le s(x)} \frac{f(n)}{n} h(\frac{x}{n}) + \sum_{s(x) < n \le x} \frac{f(n)}{n} h(\frac{x}{n}).$$

Denote the latter two sums by I_1 and I_2. We have $I_1 \le c_0 m_f(s(x))/\sqrt{\log x}$ and $I_2 \le c_0(m_f(x) - m_f(s(x)))$. We thus find that (13) holds with

$$|\epsilon_f(x) - \frac{B_f}{\log x}| \le \frac{c_0 \beta_f(x_0)}{\log^{3/2} x}, \qquad x \ge x_0.$$

Proceeding as in the proof of Lemma 1, but with this improved error estimate, the result then easily follows. □

4. Relating $m_f(x)$ to $M_f(x)$

Given an effective estimate for $m_f(t)$, we can derive an effective estimate for $M_f(t)$ on using that

$$M_f(x) - M_f(x_0) = \int_{x_0}^{x} t \, dm_f(t). \tag{21}$$

Suppose that

$$\frac{C_f}{\tau} \log^\tau x \left(1 + \frac{\alpha}{\log x} \right) \le m_f(x) \le \frac{C_f}{\tau} \log^\tau x \left(1 + \frac{\alpha + \beta}{\log x} \right),$$

for some constants α and β and every $x \ge x_0$ (if the conditions of Lemma 1 are met, such α, β and x_0 can certainly be determined). This leads to an upper bound for $M_f(x)$ that is asymptotically equal to $C_f(1+\beta/\tau) x \log^{\tau-1} x$ and a lower bound that is asymptotically equal to $\max\{0, C_f(1-\beta/\tau) x \log^{\tau-1} x\}$. These estimates are too weak for our purposes.

Write $m_f(x) = C_f \log^\tau x/\tau - C_f B_f \log^{\tau-1} x + \mathcal{E}_f(x)$ (cf. Lemma 3) and suppose that $\mathcal{E}_f^-(x) \le \mathcal{E}_f(x) \le \mathcal{E}_f^+(x)$ for every $x \ge x_0$, where $\mathcal{E}_f^+(x)$ and $\mathcal{E}_f^-(x)$ are effectively computable. (This supposition is certainly true if the conditions of Lemma

3 are satisfied.) Let $C_f(x_0) = M_f(x_0) - x_0\mathcal{E}_f(x_0) - C_f x_0 \log^\tau x_0$. Then an easy computation shows that for every $x \geq x_0$,

$$M_f(x) \leq C_f x \log^{\tau-1} x + (1-\tau)C_f(1+B_f)\int_{x_0}^x \log^{\tau-2} t\ dt + C_f(x_0) + R_f(x),$$

where

$$x\mathcal{E}_f^-(x) - \int_{x_0}^x \mathcal{E}_f^+(t)\ dt \leq R_f(t) \leq x\mathcal{E}_f^+(x) - \int_{x_0}^x \mathcal{E}_f^-(t)\ dt.$$

There are various problems with this approach, one of the major ones being getting a good estimate for c_0 in Lemma 3.

An alternative approach starts with the observation that, for $x \geq 2$,

$$M_f(x) = \int_{2-}^x \frac{d\lambda_f(t)}{\log t} = \frac{\lambda_f(x)}{\log x} + \int_2^x \frac{\lambda_f(t)}{t\log^2 t}\ dt \tag{22}$$

and that if we have explicit bounds of the type $\alpha x < \psi_f(x) < \beta x$, then $\lambda_f(x)$ can be related to $xm_f(x)$ by (7). Note in particular that if $\lambda_f(x) \geq \lambda_g(x)$ for every $x \geq 2$, then $M_f(x) \geq M_g(x)$ for every x (the reverse implication is not always true in general). The disadvantage of proving something stronger is hopefully compensated for by the fact that $\lambda_f(x)$ can be easily related to $m_f(x)$.

5. The generalized second order Ramanujan-Landau constant

In Theorem 4 we will identify a subclass of multiplicative functions for which the generalized Landau-Ramanujan constant (defined in Section 1) exists and relate it to an infinite series involving $\Lambda_f(n)$. The following result will play an essential rôle in this.

Theorem 3. [25, Theorem 6] *Let f be a multiplicative function satisfying*

$$0 \leq f(p^r) \leq c_1 c_2^r, \qquad c_1 \geq 1,\ 1 \leq c_2 < 2, \tag{23}$$

and $\sum_{p\leq x} f(p) = \tau\,\mathrm{Li}(x) + O\left(x\log^{-2-\rho} x\right)$, where τ and ρ are positive real fixed numbers and $\mathrm{Li}(x)$ denotes the logarithmic integral. Then there exists a constant B_f such that (11) holds with $E_f(x) = O(\log^{-\rho} x)$. Moreover, for every $\epsilon > 0$,

$$\sum_{n\leq x} \frac{f(n)}{n} = \sum_{0\leq\nu<\rho+1} a_\nu \log^{\tau-\nu} x + O(\log^{\tau-1-\rho+\epsilon} x), \tag{24}$$

where the implied constant depends at most on f and ϵ. In case f is completely multiplicative, condition (23) can be weakened to $\sum_{p,r\geq 2,\ p^r>x} (f(p)/p)^r \log p = O(\log^{-\rho} x)$.

Proof. This result is just Theorem 6 of [25], except for the claim regarding $E_f(x)$, the truth of which is, however, established in the course of the proof of Theorem 6 of [25]. $\square$

The next result shows that the second order Landau-Ramanujan constant is closely related to the constant B_f appearing in (11).

Theorem 4. *Let f be a multiplicative function satisfying the conditions of Theorem (3) with $\rho > 1$. Then $\lambda_2(f)$, the generalized second order Landau-Ramanujan constant, equals*

$$\lambda_2(f) = (1-\tau)\left(1 + \tau\gamma + \sum_{n=1}^\infty \frac{\Lambda_f(n)-\tau}{n}\right),$$

or alternatively $\lambda_2(f) = (1-\tau)(1+B_f)$, *where*

$$B_f = \lim_{x\to\infty}\left(\sum_{n\le x}\frac{\Lambda_f(n)}{n} - \tau\log x\right).$$

Proof. Since by assumption $\rho > 1$, we have by (24)

$$m_f(x) = a_0\log^{\tau} x + a_1\log^{\tau-1} x + a_2\log^{\tau-2} x + O(\log^{\tau-2-\delta} x),$$

for some $\delta > 0$. Theorem 3 implies that B_f exists. Using that $\log x = \sum_{n\le x} 1/n - \gamma + o(1)$, we see that it suffices to prove that $\lambda_2(f) = (1-\tau)(1+B_f)$. Theorem 3 yields that $E_f(x) = O(\log^{-1} x)$, hence the conditions of Lemma 3 are satisfied and it follows that $a_0 = \tau C_f$ and $a_1 = -C_f B_f$. On using that $M_f(x) = xm_f(x) - \int_1^x m_f(t)dt$, it follows by partial integration that $\lambda_1(f) = C_f$ and $\lambda_2(f) = (1-\tau)(1+B_f)$, as required. $\square$

Example. Let b_1 be the characteristic function of the set of natural numbers that can be written as a sum of two integer squares. This is a subsemigroup of the natural numbers that is generated by the primes p with $p \equiv 1(\mathrm{mod}\, 4)$, $p = 2$ and the squares of the remaining prime numbers (this result goes back to Fermat). From what has been said in Section 2, it then follows that

$$\Lambda_{b_1}(n) = \begin{cases} 2\log p & \text{if } n = p^{2r},\ r\ge 1 \text{ and } p\equiv 3(\mathrm{mod}\,4);\\ \log p & \text{if } n = p^{r},\ r\ge 1 \text{ and } p\equiv 1(\mathrm{mod}\,4) \text{ or } p=2;\\ 0 & \textit{otherwise}.\end{cases}$$

Application of Theorem 4 yields the following two formulae for the second order Landau-Ramanujan constant K_2 (cf. Section 1):

$$K_2 = \frac{1}{2}\left(1 + \frac{\gamma}{2} + \sum_{n=1}^{\infty}\frac{\Lambda_{b_1}(n) - \frac{1}{2}}{n}\right) = \frac{1}{2}\lim_{x\to\infty}\left(1 + \sum_{n\le x}\frac{\Lambda_{b_1}(n)}{n} - \frac{1}{2}\log x\right).$$

6. Numerical evaluation of certain constants

In order to prove Theorem 2 we need to evaluate certain constants with enough precision. For some of them this has been done before; cf. [13].

We first consider the evaluation of $C_{3,1}$ and $C_{3,2}$. We have, for $\Re(s) > 1$, $L_{g_{3,1}}(s) = \prod_{p\equiv 1(\mathrm{mod}\,3)}(1-p^{-s})^{-1}$. Note that

$$L_{g_{3,1}}(s)^2 = \zeta(s)L(s,\chi_{-3})(1-3^{-s})\prod_{p\equiv 2(\mathrm{mod}\,3)}(1-p^{-2s}). \tag{25}$$

From this, (10), $\lim_{s\to 1+0}(s-1)\zeta(s) = 1$ and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we obtain

$$C_{3,1}^2 = \frac{2L(1,\chi_{-3})}{3\pi}\prod_{p\equiv 2(\mathrm{mod}\,3)}\left(1-\frac{1}{p^2}\right).$$

If χ is a real primitive character modulo k and $\chi(-1) = -1$, then

$$L(1,\chi) = -\frac{\pi}{k^{3/2}}\sum_{n=1}^{k} n\chi(n),$$

by Dirichlet's celebrated class number formula (cf. equation (17) of Chapter 6 of [10]). We infer that $L(1,\chi_{-3}) = \pi/\sqrt{27}$. Using that $C_{g_{3,1}} > 0$ and $\zeta(2) = \pi^2/6$, we then deduce (2). Using that $L_{g_{3,2}}(s)L_{g_{3,1}}(s)(1-3^{-s})^{-1}=\zeta(s)$, we infer that $C_{3,2}=2/(3\pi C_{3,1})$.

In order to compute $C_{3,2}$ and $C_{3,1}$ with many decimal places of accuracy, we proceed as in Shanks [38, p. 78]. We note that, for $\Re(s)>1/2$,

$$\prod_{p\equiv 2(\mathrm{mod}\,3)}(1-p^{-2s})^2=\frac{L(2s,\chi_{-3})}{\zeta(2s)(1-3^{-2s})}\prod_{p\equiv 2(\mathrm{mod}\,3)}(1-p^{-4s}),\tag{26}$$

from which we infer by recursion that

$$C_{3,1}=\frac{\sqrt{2}}{3^{\frac{5}{4}}}\prod_{n=1}^{\infty}\left(\frac{L(2^n,\chi_{-3})}{(1-3^{-2^n})\zeta(2^n)}\right)^{\frac{1}{2^{n+1}}}.$$

Because of the lacunary character of this expression, it can be calculated quickly up to high precision, which yields $C_{3,1}=0.3012165544749342124\cdots$ and $C_{3,2}=0.7044984335\cdots$. Similarly one can show that $C_{4,3}=1/(2\pi C_{4,1})$ and

$$C_{4,1}=\frac{1}{2\sqrt{2}}\prod_{p\equiv 3(\mathrm{mod}\,4)}\left(1-\frac{1}{p^2}\right)^{1/2}=\frac{1}{\pi}\prod_{p\equiv 1(\mathrm{mod}\,4)}\left(1-\frac{1}{p^2}\right)^{-1/2}.$$

Using Shanks' trick, we then infer that $C_{4,1}=0.32712936694102638240023 28\cdots$ and $C_{4,3}=0.4865198883\cdots$.

On noting that, for $\Re(s)\geq 1$,

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)-1}{n^s}=-\frac{\zeta'(s)}{\zeta(s)}-\zeta(s)$$

and using that $\zeta(s)=1/(s-1)+\gamma+O(s-1)$ is the Taylor series for $\zeta(s)$ around $s=1$ (see, e.g., [28, pp. 162-164]), one infers that

$$\sum_{n\leq x}\frac{\Lambda(n)}{n}=\sum_{n\leq x}\frac{1}{n}-2\gamma+o(1)=\log x-\gamma+o(1).\tag{27}$$

Taking the logarithmic derivative of (25), one obtains that

$$-2\frac{L'_{g_{3,1}}}{L_{g_{3,1}}}(s)=-\frac{\zeta'}{\zeta}(s)-\frac{L'}{L}(s,\chi_{-3})-\frac{\log 3}{3^s-1}-2\sum_{p\equiv 2(\mathrm{mod}\,3)}\frac{\log p}{p^{2s}-1},$$

from which one easily infers that

$$2\sum_{n\leq x}\frac{\Lambda_{g_{3,1}}(n)}{n}=\sum_{n\leq x}\frac{\Lambda(n)}{n}-\frac{L'}{L}(1,\chi_{-3})-\frac{\log 3}{2}-2\sum_{p\equiv 2(\mathrm{mod}\,3)}\frac{\log p}{p^2-1}+o(1),$$

which yields, on invoking (27),

$$2B_{g_{3,1}}=-\gamma-\frac{L'}{L}(1,\chi_{-3})-\frac{\log 3}{2}-2\sum_{p\equiv 2(\mathrm{mod}\,3)}\frac{\log p}{p^2-1}.$$

Similarly we deduce that

$$2B_{g_{4,1}}=-\gamma-\frac{L'}{L}(1,\chi_{-4})-\log 2-2\sum_{p\equiv 3(\mathrm{mod}\,4)}\frac{\log p}{p^2-1}.$$

As to the numerical evaluation of $B_{g_{3,1}}$ and $B_{g_{4,1}}$, we note that

$$2\sum_{p\equiv 2(\bmod 3)}\frac{\log p}{p^2-1}=-\frac{d}{ds}\log\prod_{p\equiv 2(\bmod 3)}\left(\frac{1}{1-p^{-2s}}\right)\Big|_{s=1}.$$

Then, applying (26) m times, we obtain

$$\sum_{p\equiv 2(\bmod 3)}\frac{\log p}{p^2-1} = \sum_{p\equiv 2(\bmod 3)}\frac{\log p}{p^{2^{m+1}}-1}+\frac{1}{2}\sum_{n=1}^{m}\left\{\frac{L'}{L}(2^m,\chi_{-3})-\frac{\zeta'}{\zeta}(2^m)-\frac{\log 3}{3^{2^m}-1}\right\}.$$

Now L-functions and their derivatives can be computed with high accuracy using, for example, PARI (cf. [5, Section 10.3]). On doing so, we find that the prime sum in the left-hand side of the latter formula equals $0.35164781326380875601577 90\cdots$. Similarly we have

$$\sum_{p\equiv 3(\bmod 4)}\frac{\log p}{p^2-1} = \sum_{p\equiv 3(\bmod 4)}\frac{\log p}{p^{2^{m+1}}-1}+\frac{1}{2}\sum_{n=1}^{m}\left\{\frac{L'}{L}(2^m,\chi_{-4})-\frac{\zeta'}{\zeta}(2^m)-\frac{\log 2}{2^{2^m}-1}\right\}.$$

We thus find that the sum on the left-hand side equals $0.2287363531940324576\cdots$. For more on evaluating infinite sums or products involving primes, we refer to [6] and [23].

For the logarithmic derivative $(L'/L)(1,\chi_D)$ we find, for $D=-3$ and $D=-4$,

$$\frac{L'}{L}(1,\chi_{-3})=0.36828161597014784263323790407578664254876430999\cdots,$$
$$\frac{L'}{L}(1,\chi_{-4})=0.24560958477731417238881662617906251843353378295 49\cdots.$$

An alternative way of evaluating the latter two logarithmic derivatives is by relating them to the gamma function or the arithmetic-geometric-mean (AGM). We have (Berger (1883), Lerch (1897), de Séguier (1899) and Landau [18]),

$$\frac{L'}{L}(1,\chi_{-4})=\log\left(M(1,\sqrt{2})^2\frac{e^\gamma}{2}\right),$$

where $M(1,\sqrt{2})$ denotes the limiting value of Lagrange's AGM algorithm $a_{n+1}=(a_n+b_n)/2$, $b_{n+1}=\sqrt{a_nb_n}$ with starting values $a_0=1$ and $b_0=\sqrt{2}$. It can be shown that $M(1,\sqrt{2})=\sqrt{\frac{2}{\pi}}\Gamma(\frac{3}{4})^2$. Gauss showed (in his diary) (cf. [9]) that

$$\frac{1}{M(1,\sqrt{2})}=\frac{2}{\pi}\int_0^1\frac{dx}{\sqrt{1-x^4}}.$$

The total arclength of the lemniscate $r^2=\cos(2\theta)$ is given by $2L$, where $L=\pi/M(1,\sqrt{2})$ is the so-called *lemniscate constant*. If $\chi=\chi_{-3}$, we have similarly, with $z=\sin(\frac{\pi}{12})=\frac{(\sqrt{3}-1)}{\sqrt{8}}$,

$$\frac{L'}{L}(1,\chi_{-3})=\log\left(\frac{2^{\frac{4}{3}}M(1+z,1-z)^2e^\gamma}{3}\right),\qquad M(1+z,1-z)=\frac{2^{\frac{4}{3}}\pi^2}{3^{\frac{1}{4}}\Gamma(\frac{1}{3})^3},$$

and

$$\frac{1}{M(1+z,1-z)}=\frac{3^{\frac{1}{4}}}{\pi}\int_0^1\frac{dx}{\sqrt{x(1-x^3)}}.$$

The values of $L'(1,\chi_{-4})$ and $L'(1,\chi_{-3})$ can also be determined using generalized Euler constants for arithmetical progressions; see Examples 1 and 2 of [11]. For general nontrivial real χ the quotients $(L'/L)(1,\chi)$ "feel" the zeroes of $L(s,\chi)$ close to 1 (see [10, pp. 80-83] for a quantitative version) and a study of their average behaviour might throw some light on the (non)-existence of the Landau-Siegel zeros; cf. [22].

Putting our subcomputations together, we find that

$$B_{g_{3,1}}=-1.0990495258666765304844653683056 1\cdots.$$

On noting that for $3\nmid n$ we have $\Lambda(n)=\Lambda_{g_{3,1}}(n)+\Lambda_{g_{3,2}}(n)$, it is easily deduced that $B_{g_{3,1}}=-\gamma-\frac{\log 3}{2}-B_{g_{3,2}}$, and using this we compute

$$B_{g_{3,2}}=-.0274722833689111758196693402380 5\cdots.$$

Similarly, we find $B_{g_{4,1}}=-0.9867225683134286288516284\cdots$ and

$$B_{g_{4,3}}=-\log 2-\gamma-B_{g_{4,1}}=-0.2836402771480495411721157\cdots.$$

The constant $B_{g_{d,a}}$ can be alternatively calculated by invoking the formula

$$\varphi(d)\sum_{\substack{n\le x\\ n\equiv a(\operatorname{mod} d)}}\frac{\Lambda(n)}{n}=\log x-\gamma-\sum_{p|d}\frac{\log p}{p-1}-\sum_{\chi\ne\chi_0}\chi(\bar a)\frac{L'}{L}(1,\chi)+o(1),$$

where a and d are coprime integers, the last sum is over the characters mod d different from the principal character and $\bar a$ is any integer such that $a\bar a\equiv 1(\operatorname{mod} d)$. The latter formula is derived by elementary means in [29].

Using Theorem 4, we are now in the position to compute some second order Landau-Ramanujan constants. They are simply given by $\lambda_2(f)=(1+B_f)/2$ for $f\in\{g_{3,1},g_{3,2},g_{4,1},g_{4,3}\}$.

7. Effective estimates for squarefree integers

In the sequel we will establish some effective estimates for certain number theoretic functions of a real variable. The general procedure is to establish the estimates for every $x\ge x_0$ for some x_0. The following lemma can then often be used to show that there exists a number $x_1<x_0$ such that the estimates in fact hold for every $x\ge x_1$. It reduces a seemingly continuous problem to a discrete one.

Lemma 4. *Let $y_1>y_0$ be arbitrary real numbers. Let F and r be nondecreasing real-valued functions such that, moreover, F changes its value only at integers. Let $x_1,x_2,\ldots,x_n$ be the integers in (y_0,y_1) where F changes its value. Let $x_0=y_0$ and $x_{n+1}=y_1$. Then*

$$\sup_{y_0\le x\le y_1}\{F(x)-r(x)\}=\max_{0\le i\le n}\{F(x_i)-r(x_i)\}$$

and

$$\inf_{y_0\le x\le y_1}\{F(x)-r(x)\}=\min_{0\le i\le n}\{F(x_i)-r(x_{i+1})\}.$$

In our proof of Theorem 2 we need effective estimates for $Q_{\chi_{-3}}(x)$ and $Q_{\chi_{-4}}(x)$, where $Q_\chi(x)$ denotes the number of integers $n \le x$ such that $\mu(n)\chi(n) \neq 0$. Note that $Q_{\chi_{-4}}$ merely counts the odd squarefree numbers and hence we will use the more suggestive notation Q_{odd} for it. There are two obvious approaches in estimating these functions: relating them to $Q(x)$, where $Q(x)$ denotes the number of squarefree integers not exceeding x, and an ab initio approach. We demonstrate both.

Let $R(x) = Q(x) - 6x/\pi^2$. It was shown by Moser and MacLeod [27] that $|R(x)| < \sqrt{x}$ for all x and that $|R(x)| < \sqrt{x}/2$ for $x \ge 8$. Cohen and Dress [7] showed that $|R(x)| < 0.1333\sqrt{x}$ for $x \ge 1664$.

Lemma 5. *For $x \ge 0$ we have*

$$\left|Q_{\chi_{-3}}(x) - \frac{9}{2\pi^2}x\right| \le 0.3154\sqrt{x} + 17.2$$

and

$$\left|Q_{\chi_{-3}}(x) - \frac{9}{2\pi^2}x\right| \le \frac{1}{2}\sqrt{x} + 1. \tag{28}$$

Proof. We clearly have $Q(x) = Q_{\chi_{-3}}(x) + Q_{\chi_{-3}}(x/3)$, from which we infer that

$$Q_{\chi_{-3}}(x) = \sum_{i=0}^{\infty}(-1)^i Q(\frac{x}{3^i}). \tag{29}$$

Let $x_0 = 1664$. On applying Lemma 4 with $y_0 = x_0/3$ and $y_1 = x_0$, we find that $|R(x)| \le 0.15\sqrt{x}$ in the interval $(x_0/3, x_0]$. Similarly we compute that $|R(x)| \le 0.29\sqrt{x}$ in the interval $(x_0/27, x_0/3]$. These estimates yield when combined with identity (29) and the quoted bounds for $|R(x)|$:

$$\left|Q_{\chi_{-3}}(x) - \frac{9}{2\pi^2}x\right| \le \alpha\frac{\sqrt{3x}}{\sqrt{3}-1} + (0.15-\alpha)\sqrt{x_0} + (0.29-\alpha)\left(\sqrt{\frac{x_0}{3}} + \sqrt{\frac{x_0}{9}}\right)$$
$$+(0.5-\alpha)\left(\sqrt{\frac{x_0}{27}} + \sqrt{\frac{x_0}{81}}\right) + (1-\alpha)(\sqrt{9} + \sqrt{3} + 1 + \frac{1}{\sqrt{3}} + \frac{1}{3} + \cdots),$$

where $\alpha = 0.1333$. The latter bound does not exceed $0.3154\sqrt{x} + 17.2$. From this bound we then infer that (28) holds for every $x \ge 10000$. We now apply Lemma 4 with $y_0 = 0$ and $y_1 = 10000$ to establish the validity of (28) in the remaining range. □

Using that $|R(x)| \le 0.15\sqrt{x}$ in the interval $(x_0/2, x_0]$ and $|R(x)| \le 0.29\sqrt{x}$ in the interval $(x_0/32, x_0/2]$, we deduce, proceeding as in the proof of Lemma 5, that $|Q_{odd}(x) - 4x/\pi^2| \le 0.4552\sqrt{x} + 26.5$. Although the latter bound is sharp enough for our purposes, we present a self-contained proof of a slightly sharper bound (which uses ideas from [27]).

Lemma 6. *For $x \ge 0$ we have*

$$\left|Q_{odd}(x) - \frac{4}{\pi^2}x\right| \le \frac{1}{2}\sqrt{x} + 1 \quad \text{and} \quad \left|Q_{odd}(x) - \frac{4}{\pi^2}x\right| \le (\frac{2}{\pi^2} + \frac{1}{4})\sqrt{x} + \frac{1}{4}x^{\frac{1}{4}} + 2.$$

Proof. We have

$$Q_{odd}(x) = \sum_{\substack{n \le x \\ n \text{ odd}}} |\mu(n)| = \sum_{\substack{n \le x \\ n \text{ odd}}} \sum_{d^2|n} \mu(d) = \sum_{\substack{d \le x \\ d \text{ odd}}} \mu(d)\left[\frac{x}{2d^2} + \frac{1}{2}\right].$$

Let $R_{odd}(x) = Q_{odd}(x) - 4x/\pi^2$. On noting that $\sum_{d \text{ odd}} \mu(d)/d^2 = 8/\pi^2$, we find that

$$|R_{odd}(x)| \le \left| \sum_{\substack{d^2 \le x \\ d \text{ odd}}} \mu(d) \left(\frac{x}{2d^2} - \left[\frac{x}{2d^2} + \frac{1}{2} \right] \right) \right| + x \left| \sum_{\substack{d^2 > x \\ d \text{ odd}}} \frac{\mu(d)}{2d^2} \right|.$$

Since $|x - [x + 1/2]| \le 1/2$ for every x, we deduce that

$$|R_{odd}(x)| \le \frac{Q_{odd}(\sqrt{x})}{2} + x \left| \sum_{\substack{d^2 > x \\ d \text{ odd}}} \frac{\mu(d)}{2d^2} \right|. \tag{30}$$

Suppose that $x > 4$, then

$$\sum_{\substack{d^2 > x \\ d \text{ odd}}} \frac{1}{d^2} \le \sum_{m > \frac{\sqrt{x}-1}{2}} \frac{1}{(2m+1)(2m-1)} = \sum_{m > \frac{\sqrt{x}-1}{2}} \left[\frac{1}{4m-2} - \frac{1}{4m+2} \right] \le \frac{1}{2\sqrt{x} - 4}.$$

On using this and the trivial estimate $Q_{odd}(x) \le (x+1)/2$, we deduce that $|R_{odd}(x)| \le \frac{1}{2}\sqrt{x} + 1$ on applying Lemma 4 with $y_0 = 0$ and $y_1 = 36$. Using the latter bound for $Q_{odd}(x)$ in (30), one then easily obtains the second stated bound in the formulation of the lemma on applying Lemma 4 with $y_0 = 0$ and $y_1 = 9$. □

8. On the difference $\sum_{n \le x} \frac{\Lambda_f(n)}{n} - \tau \log x$

In order to use Lemma 1, we need to find finite constants C_+ and C_- such that

$$C_- \le \sum_{n \le x} \frac{\Lambda_f(n)}{n} - \tau \log x \le C_+$$

for every $x \ge 1$. Recall that $\psi_f(x) = \sum_{n \le x} \Lambda_f(n)$. Suppose that $\psi_f(x) = \tau x + \mathcal{E}_f(x)$, where $|\mathcal{E}_f(x)| \le c_\epsilon \log^{-1-\epsilon} x$ for $x \ge x_0$. Then

$$\sum_{n \le x} \frac{\Lambda_f(n)}{n} = \tau \log x + B_f + \frac{\mathcal{E}_f(x)}{x} - \int_x^\infty \frac{\mathcal{E}_f(t)}{t^2} dt,$$

and thus, for $x \ge x_0$,

$$\left| \sum_{n \le x} \frac{\Lambda_f(n)}{n} - \tau \log x - B_f \right| \le \frac{c_\epsilon}{\log^\epsilon x} \left(\frac{1}{\epsilon} + \frac{1}{\log x} \right). \tag{31}$$

Let $f = \mathbf{1}$, $\psi(x) = \psi_{\mathbf{1}}(x)$ and $\theta(x) = \sum_{p \le x} \log p$. It is known that $|\theta(x) - x| \le 3.965x/\log^2 x$ for $x > 1$ [12, p.14]. Using this with the bound $\psi(x) - \theta(x) < 1.43\sqrt{x}$ [33, Theorem 13], we can compute C_+ and C_- in case $f = \mathbf{1}$. Instead of carrying this out along these lines, we proceed slightly differently as this will result in a sharper bound for the difference in (31).

Lemma 7. *For $x \ge 97$ we have*

$$-\frac{1}{2\log x} + \frac{1}{2\sqrt{x}} \le \sum_{n \le x} \frac{\Lambda(n)}{n} - \log x + \gamma \le \frac{2}{\sqrt{x}} + \frac{1}{2\log x}.$$

The upper bound holds even for every $x > 1$.

Proof. By [33, Theorem 6] we have, for $x \geq 319$,

$$\left| \sum_{p \leq x} \frac{\log p}{p} - \log x - E \right| < \frac{1}{2 \log x},$$

where $E = -\gamma - \sum_p \log p \sum_{k \geq 2} p^{-k}$. Notice that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \sum_{p \leq x} \frac{\log p}{p} + \sum_{k \geq 2} \sum_{p} \frac{\log p}{p^k} - \sum_{\substack{p^k > x \\ k \geq 2}} \frac{\log p}{p^k}.$$

By partial integration we find that

$$\sum_{\substack{p^k > x \\ k \geq 2}} \frac{\log p}{p^k} = \frac{\theta(x) - \psi(x)}{x} + \int_x^\infty \frac{\psi(t) - \theta(t)}{t^2} dt. \tag{32}$$

Suppose that $\alpha\sqrt{t} \leq \psi(t) - \theta(t) \leq \beta\sqrt{t}$ for $t \geq x_0$. Then, for $x \geq x_0$ the sum in (32) is in the interval $(\frac{2\alpha - \beta}{\sqrt{x}}, \frac{2\beta - \alpha}{\sqrt{x}})$. By Theorems 13 and 14 of [33] we can take $\alpha = 0.98$ and $\beta = 1.4262$ when $x_0 = 319$. On combining the various estimates, the result follows after some numerical analysis in the interval $(1, 319)$. □

From Lemma 7 and Lemma 4 with $y_0 = 1$ and $y_1 = 215$ it is easily deduced that

$$\sup_{x \geq 1} \left\{ \sum_{n \leq x} \frac{\Lambda(n)}{n} - \log x \right\} = -\frac{\log 2}{2} = -0.34657359 \cdots$$

and

$$\inf_{x \geq 1} \left\{ \sum_{n \leq x} \frac{\Lambda(n)}{n} - \log x \right\} = \frac{\log 2}{2} - \log 3 = -0.75203869 \cdots .$$

Other than for $f = \mathbf{1}$, the author is unaware of cases where an unconditional effective upper bound for $\mathcal{E}_f(x)$ of order $\log^{-1-\epsilon} x$ is known. Thus in order to obtain admissible values for C_+ and C_- in the case $f \in \{g_{3,1}, g_{3,2}, g_{4,1}, g_{4,3}\}$, we have to follow another approach. To this end notice that

$$2 \sum_{n \leq x} \frac{\Lambda_{g_{3,1}}(n)}{n} = \sum_{n \leq x} \frac{(1 + \chi_{-3}(n))\Lambda(n)}{n} - 2 \sum_{\substack{p^r \leq \sqrt{x} \\ p \equiv 2 (\bmod 3)}} \frac{\log p}{p^{2r}} - \sum_{1 < 3^r \leq x} \frac{\log 3}{3^r}. \tag{33}$$

The latter two sums are easily explicitly estimated and we already explicitly estimated $\sum_{n \leq x} \Lambda(n)/n$. If we can explicitly estimate $\sum_{n \leq x} \chi_{-3}(n)\Lambda(n)/n$, we are done. In order to do so, we need a few lemmas.

Lemma 8. *Let h be a completely multiplicative function with $h(1) = 1$. Then if $g(x) = \sum_{n \leq x} h(n) f(\frac{x}{n})$ for every x, it follows that $f(x) = \sum_{n \leq x} h(n)\mu(n) g(\frac{x}{n})$.*

Proof. Substitute the expression $\sum_{mn \leq x} h(m) f(x/mn)$ for $g(x/n)$ in the sum $\sum_{n \leq x} h(n)\mu(n)g(x/n)$. The resulting expression simplifies to $f(x)$. □

Lemma 9. *Let χ be a nonprincipal character and $m_0 > 1$ be the smallest integer > 1 such that $\chi(m_0) \neq 0$. Then*

$$\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n}+\frac{L'}{L}(1,\chi)=O\left(\frac{L'}{L}(1,\chi)\frac{m_0}{x}Q_\chi(\frac{x}{m_0})\right)+O\left(\frac{1}{x}\sum_{\substack{d\le x/m_0\\ \mu(d)\chi(d)\neq 0}}\log\frac{x}{d}\right).$$

Proof. On using (6) and writing $n = dd_1$, we obtain, for an arbitrary character χ,

$$\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n}=\sum_{d\le x/m_0}\frac{\chi(d)\mu(d)}{d}\sum_{d_1\le x/d}\frac{\chi(d_1)\log d_1}{d_1}. \tag{34}$$

On inserting

$$\sum_{d_1\le x/d}\frac{\chi(d_1)\log d_1}{d_1}=-L'(1,\chi)+O\left(\frac{\log(x/d)}{x/d}\right)$$

in this, we obtain

$$\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n}=-L'(1,\chi)\sum_{d\le x/m_0}\frac{\chi(d)\mu(d)}{d}+O\left(\frac{1}{x}\sum_{\substack{d\le x/m_0\\ \mu(d)\chi(d)\neq 0}}\log\frac{x}{d}\right). \tag{35}$$

We apply Lemma 8 with $h(n)=\frac{\chi(n)}{n}$ and $f(n) = 1$ together with $\sum_{n\le x}\chi(n)/n = L(1,\chi)+O(1/x)$ and obtain

$$\begin{aligned}1 &= \sum_{n\le x/m_0}\frac{\chi(n)\mu(n)}{n}\left(L(1,\chi)+O(\frac{nm_0}{x})\right)\\ &= L(1,\chi)\sum_{n\le x/m_0}\frac{\chi(n)\mu(n)}{n}+O\left(\frac{m_0}{x}Q_\chi(\frac{x}{m_0})\right).\end{aligned}$$

Combining the latter equation with (35) and using that $L(1,\chi)\neq 0$ (a well-known fact), the result then follows. □

Remark. By using more refined elementary methods [29] one can show that actually, as x tends to infinity,

$$\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n}+\frac{L'}{L}(1,\chi)=o(1).$$

Let us consider the case where $\chi=\chi_{-3}$ or $\chi=\chi_{-4}$. Then, for $x>0$,

$$\left|\sum_{n\le x}\frac{\chi(n)}{n}-L(1,\chi)\right|\le\frac{1}{x}, \tag{36}$$

where we use the fact that the nonzero terms in the sum are alternating in sign and monotonically decreasing. The function $\log x/x$ is only decreasing for $x>e$ and a similar argument then shows that, for $x>e$,

$$\left|\sum_{n\le x}\frac{\chi(n)\log n}{n}+L'(1,\chi)\right|\le\frac{\log x}{x}. \tag{37}$$

A numerical analysis shows, however, that (37) is still valid for every $x \geq 2$. The implication of these estimates is that for these characters and $x \geq 1$ all the implied constants in the latter lemma and its proof are ≤ 1. Note that for $x \geq 1$

$$\sum_{\substack{d \leq x \\ \chi(d)\mu(d)\neq 0}} \log \frac{x}{d} = \sum_{\substack{d \leq x \\ \chi(d)\mu(d)\neq 0}} \int_d^x \frac{dt}{t} = \int_1^x \frac{Q_\chi(t)}{t} dt$$

and thus, for $x \geq m_0$,

$$\sum_{\substack{d \leq x/m_0 \\ \chi(d)\mu(d)\neq 0}} \log \frac{x}{d} = \int_1^{\frac{x}{m_0}} \frac{Q_\chi(t)}{t} dt + Q_\chi(\frac{x}{m_0}) \log m_0.$$

We thus find that, for $x \geq m_0$,

$$x \left| \sum_{n \leq x} \frac{\chi(n)\Lambda(n)}{n} + \frac{L'}{L}(1,\chi) \right| \leq \left(\frac{L'}{L}(1,\chi) m_0 + \log m_0 \right) Q_\chi(\frac{x}{m_0}) + \int_1^{\frac{x}{m_0}} \frac{Q_\chi(t)}{t} dt.$$

For $\chi = \chi_{-3}$ we see, using (28), that the right-hand side is bounded by

$$\left(\frac{L'}{L}(1,\chi_{-3}) + \frac{\log 2}{2} + \frac{1}{2} \right) \frac{9}{2\pi^2} + \frac{1}{\sqrt{2x}} \left(\frac{L'}{L}(1,\chi_{-3}) + \frac{\log 2}{2} + 1 \right)$$
$$+ \frac{\log(x/2)}{x} + \frac{1}{x} \left(2\frac{L'}{L}(1,\chi_{-3}) + \log 2 \right).$$

For $\chi = \chi_{-4}$ we see, using that $Q_{\chi_{-4}}(t) \leq 4t/\pi^2 + \sqrt{t}/2 + 1$ (Lemma 6), that the right-hand side is bounded by

$$\left(\frac{L'}{L}(1,\chi_{-4}) + \frac{\log 3}{3} + \frac{1}{3} \right) \frac{4}{\pi^2} + \frac{1}{\sqrt{3x}} \left(\frac{3}{2}\frac{L'}{L}(1,\chi_{-4}) + \frac{\log 3}{2} + 1 \right)$$
$$+ \frac{\log(x/3)}{x} + \frac{1}{x} \left(3\frac{L'}{L}(1,\chi_{-4}) + \log 3 \right).$$

In the case where $\chi = \chi_{-3}$, it remains to explicitly estimate the latter two sums in (33). We have

$$\sum_{\substack{p^r > \sqrt{x} \\ p \equiv a (\mathrm{mod}\, d)}} \frac{\log p}{p^{2r}} \leq \sum_{p^r > \sqrt{x}} \frac{\log p}{p^{2r}} = -\frac{\psi(\sqrt{x})}{x} + 2\int_{\sqrt{x}}^\infty \frac{\psi(t)}{t^3} dt.$$

Using that $0.8t \leq \psi(t) \leq 1.04t$ for $t \geq 17$ (this easily follows from Theorem 10 and Theorem 12 from [33]), we find that

$$\sum_{\substack{p^r > \sqrt{x} \\ p \equiv a (\mathrm{mod}\, d)}} \frac{\log p}{p^{2r}} \leq \frac{1.3}{\sqrt{x}} \quad \text{for } x \geq 289. \tag{38}$$

Furthermore, for every fixed $v > 1$ and every $x > 0$,

$$\frac{\log v}{v-1}(1 - \frac{v}{x}) \leq \sum_{1 < v^r \leq x} \frac{\log v}{v^r} \leq \frac{\log v}{v-1},$$

where the sum is over the integral powers of v not exceeding x. (These two estimates can also be used in the case where $\chi = \chi_{-4}$.)

Let us define

$$C_+(f) = \sup_{x\geq 1}\left(\sum_{n\leq x}\frac{\Lambda_f(n)}{n} - \tau\log x\right) = B_f + \sup_{x\geq 1} E_f(x),$$

and let $C_-(f)$ be similarly defined, with sup replaced by inf. Let $\epsilon > 0$ be fixed. Note that the sharpest result the method we followed here allows us to prove, with enough numerical computation, is

$$\lim_{x\to\infty}|E_f(x)| \leq \left(\frac{L'}{L}(1,\chi_{-3}) + \frac{\log 2}{2} + \frac{1}{2}\right)\frac{9}{4\pi^2} + \epsilon = 0.2769537767\cdots + \epsilon$$

and

$$\lim_{x\to\infty}|E_g(x)| \leq \left(\frac{L'}{L}(1,\chi_{-4}) + \frac{\log 3}{3} + \frac{1}{3}\right)\frac{2}{\pi^2} + \epsilon = 0.1915268284\cdots + \epsilon, \tag{39}$$

where $f \in \{g_{3,1}, g_{3,2}\}$ and $g \in \{g_{4,1}, g_{4,3}\}$.

On putting the various effective bounds together, we arrive at the following result, after numerical calculations not going beyond the interval $[1, 10^5]$.

Theorem 5. *We have*

(a) $C_-(g_{4,1}) > -1.202$ *and* $C_+(g_{4,1}) = 0$.
(b) $C_-(g_{4,3}) = \frac{\log 3}{3} - \frac{\log 7}{2} = -0.606750\cdots$ *and* $C_+(g_{4,3}) = 0$.
(c) $C_-(g_{3,1}) > -1.4$ *and* $C_+(g_{3,1}) = 0$.
(d) $C_-(g_{3,2}) = -\frac{\log 2}{2} = -0.34657\cdots$ *and* $C_+(g_{3,2}) < 0.2764$.

On GRH it is much easier to find the C_+ and C_- satisfying (8), which is what will be demonstrated now. By RH(d) we indicate the hypothesis that for every character χ mod d every nontrivial zero of $L(s,\chi)$ is on the critical line. Let

$$H(x;d,a) = \sum_{\substack{1<p^r\leq x\\ p\equiv a (\mathrm{mod}\, d)}}\frac{\log p}{p^r} \text{ and } \psi(x;d,a) = \sum_{\substack{n\leq x\\ n\equiv a(\mathrm{mod}\, d)}}\Lambda(n).$$

Lemma 10. *For $d \leq 432$ and $(a,d) = 1$, there exists a constant $c_{d,a}$ such that for $x \geq 224$ we have, on RH(d), that*

$$\left|\sum_{\substack{n\leq x\\ n\equiv a(\mathrm{mod}\, d)}}\frac{\Lambda(n)}{n} - \frac{\log x}{\varphi(d)} - c_{d,a}\right| \leq \frac{11}{32\pi\sqrt{x}}\{3\log^2 x + 8\log x + 16\}. \tag{40}$$

Proof. In [12] it is proved that for $d \leq 432$ and $x \geq 224$ we have, on RH(d), that

$$|\psi(x;d,a) - \frac{x}{\varphi(d)}| \leq \frac{11}{32\pi}\sqrt{x}\log^2 x. \tag{41}$$

Using the latter estimate and partial integration, the lemma follows. □

Using the latter lemma, the exact values of $C_-(g_{4,1}), C_+(g_{3,1})$ and $C_+(g_{3,2})$ can be computed under GRH.

Theorem 6. *We have*

(a) $C_-(g_{4,1}) = H(197;4,1) - \frac{\log(229)}{2} = -0.99076124051235\cdots$, *on RH*(4).
(b) $C_-(g_{3,1}) = H(3121;3,1) - \frac{\log(3163)}{2} = -1.100304022673\cdots$, *on RH*(3).
(c) $C_+(g_{3,2}) = H(5;3,2) - \frac{\log 5}{2} = \frac{3}{4}\log 2 - \frac{3}{10}\log 5 = 0.03702\cdots$, *on RH*(3).

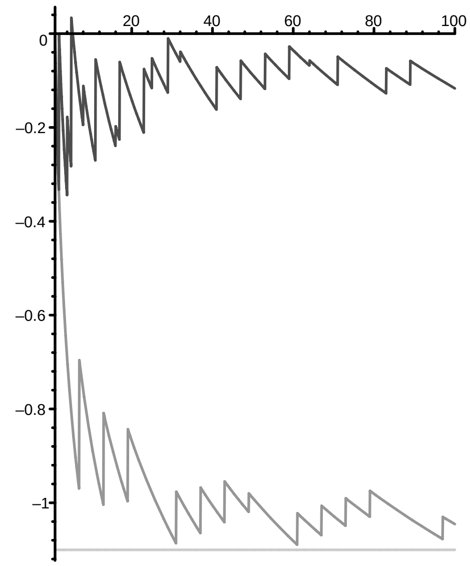

FIGURE 1. $H(x;3,2) - \frac{\log x}{2}$ (topline) versus $H(x;3,1) - \frac{\log x}{2}$. The line is at -1.10031; the x-range is $[1,100]$.

Proof. (a) Note that $c_{4,1} = B_{g_{4,1}}$. On applying Lemma 10 with $d = 4$ and $a = 1$, Lemma 4, (38) and using the numerical value for $B_{g_{4,1}}$ given in Section 6, we deduce that $C_-(g_{4,1}) = \min_{q_i \le 1.79 \times 10^9}(H(v_i;4,1) - \log(v_{i+1})/2)$, where $5 = v_1 < v_2 < \cdots$ are the consecutive prime powers p^r with $p \equiv 1(\bmod 4)$.

(b) In this case we have $C_-(g_{3,1}) = \min_{q_i \le 2.935 \times 10^{10}}(H(q_i;3,1) - \log(q_{i+1})/2)$, where $7 = q_1 < q_2 < \cdots$ are the consecutive prime powers p^r with $p \equiv 1(\bmod 3)$.

(c) Now $C_+(g_{3,2}) = \max_{w_i \le 1582079}(H(w_i;3,2) - \log(w_i)/2)$, where $2 = w_1 < w_2 < \cdots$ are the consecutive prime powers p^r with $p \equiv 2(\bmod 3)$. □

9. Connections with Chebyshev's bias for primes

In this section we make some observations that allow us to prove, for example, that $N(x;3,2) \ge N(x;3,1)$ for every $x \le x_0$ for some large x_0, using known numerical observations regarding $\pi(x;3,2)$ and $\pi(x;3,1)$.

Let $Q_1 = \{q_1, q_2, q_3, \dots\}$ and $Q_2 = \{v_1, v_2, v_3, \dots\}$ be sets of pairwise coprime prime powers that satisfy $q_1 < q_2 < q_3 < \cdots$ and $v_1 < v_2 < v_3 < \cdots$. Let S_1 denote the set of integers of the form $q_1^{e_1} \cdots q_s^{e_s}$ with $q_i \in Q_1$ and $e_i \in \mathbb{Z}_{\ge 0}$ for $1 \le i \le s$. Let S_2 be similarly defined, but with Q_1 replaced by Q_2. Let $\pi_1(x)$, $\pi_2(x)$ count the number of elements in Q_1, respectively Q_2, up to x. If $n = q_1^{e_1} \cdots q_s^{e_s} \in S_1$, then $m = v_1^{e_1} \cdots v_s^{e_s}$ is said to be its *associate* in S_1. Let $h : \mathbb{N} \to \mathbb{R}_{\ge 0}$ be a nonincreasing function. Let $V_1(x) = \sum_{n \in S_1} h(n)$ and $V_2(x) = \sum_{n \in S_2} h(n)$. In the rest of this section x_0 denotes some arbitrary number.

Lemma 11. *We have*

(a) *If $\pi_1(x) \ge \pi_2(x)$ for $x \ge 0$, then $V_1(x) \ge V_2(x)$ for $x \ge 0$.*
(b) *If $\pi_1(x) \ge \pi_2(x)$ for $x \le x_0$, then $V_1(x) \ge V_2(x)$ for $x \le x_0$.*

Proof. (a) The assumption implies that if $m \in S_2$, then its associate $n \in S_1$ satisfies $n \leq m$ and $h(n) \geq h(m)$. Thus clearly $V_1(x) \geq V_2(x)$. The proof of part (b) will be obvious to the reader now. □

Corollary 1. *If $\pi(x;d,a) \geq \pi(x;d,b)$ for $x \leq x_0$, then for $x \leq x_0$ we have both $N(x;d,a) \geq N(x;d,b)$ and $m_{g_{d,a}}(x) \geq m_{g_{d,b}}(x)$.*

The hypothesis in the corollary is in general not strong enough to infer that $\lambda_{g_{d,a}}(x) \geq \lambda_{g_{d,b}}(x)$ if $x \leq x_0$. However, we have the following easy result.

Lemma 12. *If $M_f(x) \geq M_g(x)$ and $\psi_f(x) \geq \psi_g(x)$ for every $x \leq x_0$, then $\lambda_f(x) \geq \lambda_g(x)$ for $x \leq x_0$.*

Proof. Use (7). □

Corollary 2. *If*

$$\pi(x;d,a) \geq \pi(x;d,b) \quad \text{and} \quad \sum_{\substack{1<p^r\leq x\\ p\equiv a(\mathrm{mod}\, d)}} \log p \geq \sum_{\substack{1<p^r\leq x\\ p\equiv b(\mathrm{mod}\, d)}} \log p$$

for every $x \leq x_0$, then $\lambda_{g_{d,a}}(x) \geq \lambda_{g_{d,b}}(x)$ for $x \leq x_0$.

In the proof of Theorem 7 we will use Corollary 2 a few times.

10. The proof of Theorem 2

The proof of Theorem 2 will easily follow from the following theorem.

Theorem 7. *For every x we have $\lambda_{g_{3,2}}(x) \geq \lambda_{g_{3,1}}(x)$, $\lambda_{g_{3,2}}(x) \geq \lambda_{g_{4,1}}(x)$ and $\lambda_{g_{4,3}}(x) \geq \lambda_{g_{3,1}}(x)$. For $x \geq 7$ we have $\lambda_{g_{4,3}}(x) \geq \lambda_{g_{4,1}}(x)$.*

Note that

$$e^{\lambda_{g_{d,a}}(x)} = \prod_{\substack{n\leq x\\ p|n\Rightarrow p\equiv a(\mathrm{mod}\, d)}} n.$$

In the proof of Theorem 7 we will make use of the following lemma.

Lemma 13. *We have $\psi_{g_{3,1}}(x) \leq 0.50456x$ for $x \geq 0$, $\psi_{g_{3,2}}(x) \geq 0.335x$ for $x \geq 5$, $\psi_{g_{4,1}}(x) \leq 0.50456x$ for $x \geq 0$ and $\psi_{g_{4,3}}(x) \geq 0.48508x$ for $x \geq 127$.*

Proof. Let $d \leq 13$ and $(a,d) = 1$. Then $|\psi(x;d,a) - x/\varphi(d)| \leq \sqrt{x}$ for $224 \leq x \leq 10^{10}$ by [32, Theorem 1] and $|\psi(x;d,a) - \frac{x}{\varphi(d)}| < 0.004560\frac{x}{\varphi(d)}$ for $x \geq 10^{10}$ by [32, Theorem 5.2.1]. From these inequalities the lemma follows after some computation. □

In our proof we consider inequalities of the form

$$\log^\tau(\frac{x}{r})\frac{\left(1-\frac{C_+}{\log(x/r)}\right)^{\tau+1}}{\left(1-\frac{C_-}{\log(x/r)}\right)} \geq c_1 \log^\tau(\frac{x}{s})\frac{\left(1-\frac{C'_-}{\log(x/s)}\right)^{\tau+1}}{\left(1-\frac{C'_+}{\log(x/s)}\right)}, \tag{42}$$

where all variables and constants are real numbers with τ, r, s and c_1 positive, $C_- \leq C_+$, $C'_- \leq C'_+$ and $x \geq x_0 = \max\{\exp(C'_+)s, \exp(C_+)r\}$. This inequality can be rewritten as

$$(43)\qquad 1+\frac{C'_{-}-C_{+}+\log(s/r)}{\log(x/s)-C'_{-}}\geq\left[c_1\frac{\left(1+\frac{C'_{+}-C'_{-}}{\log(x/s)-C'_{+}}\right)}{1+C_{-}-C_{+}\log(x/r)-C_{-}}\right]^{\frac{1}{\tau}}.$$

Note that for $x>x_0$ the right-hand side is a nonincreasing function of x. If $C'_{-}+\log s\leq C_{+}+\log r$, the left-hand side is nondecreasing, whereas if the latter inequality is not satisfied, the left-hand side asymptotically decreases to 1. We thus arrive at the following conclusion.

Lemma 14. *If $\log s+C'_{-}\leq C_{+}+\log r$ and* (42) *is satisfied for some $x_1>x_0$, then* (42) *is satisfied for every $x\geq x_1$. If $\log s+C'_{-}>C_{+}+\log r$, and the right-hand side of* (43) *does not exceed* 1 *for some $x_1>x_0$, then* (42) *is satisfied for every $x\geq x_1$.*

Proof of Theorem 7. $\lambda_{g_{3,2}}(x)$ versus $\lambda_{g_{3,1}}(x)$. Using Lemma 13, we infer that

$$\lambda_{g_{3,2}}(x)\geq\sum_{n\leq\frac{x}{5}}g_{3,2}(n)\psi_{g_{3,2}}\left(\frac{x}{n}\right)\geq 0.335m_{g_{3,2}}\left(\frac{x}{5}\right),$$

and that

$$\lambda_{g_{3,1}}(x)=\sum_{n\leq x}g_{3,1}(n)\psi_{g_{3,1}}\left(\frac{x}{n}\right)=\sum_{n\leq\frac{x}{7}}g_{3,1}(n)\psi_{g_{3,1}}\left(\frac{x}{n}\right)\leq 0.50456m_{g_{3,1}}\left(\frac{x}{7}\right).$$

With $d=3$, $a=2$ and $b=1$ the conditions of Corollary 2 are satisfied for every $x<196699$ (but not for $x=196699$ as $\psi_{g_{3,1}}(196699)>\psi_{g_{3,2}}(196699)$). Thus we certainly may assume that $x>1900$. Using the estimates $C_{3,1}<0.302$ and $C_{3,2}>0.703$, we then deduce from Lemma 1, Theorem 5 and Lemma 14 that $0.335m_{g_{3,2}}(x/7)>0.50456m_{g_{3,1}}(x/7)$.

$\lambda_{g_{3,2}}(x)$ versus $\lambda_{g_{4,1}}(x)$. The conditions of Corollary 2 are now satisfied for every $x\leq 10^7$ (the smallest x for which the conditions are not satisfied is not known, but must be less than 10^{82} by [14]). Thus we certainly may assume that $x>4600$. Then reasoning as before, we infer that $\lambda_{g_{3,2}}(x)\geq 0.335m_{g_{3,2}}(x/5)\geq 0.50456m_{g_{4,1}}(x/5)\geq\lambda_{g_{4,1}}(x)$.

$\lambda_{g_{3,2}}(x)$ versus $\lambda_{g_{4,1}}(x)$. The conditions of Corollary 2 are now satisfied for every $x\leq 10^7$ (the smallest x for which the conditions are not satisfied is not known, but must be less than 10^{82} by [14]). Thus we may assume that $x>199000$. Then it is seen that $\lambda_{g_{4,3}}(x)\geq 0.4594m_{g_{4,3}}(x/59)\geq 0.50456m_{g_{3,1}}(x/5)\geq\lambda_{g_{3,1}}(x)$.

$\lambda_{g_{4,3}}(x)$ versus $\lambda_{g_{4,1}}(x)$. For $7\leq x\leq 1.1\times 10^6$ one directly verifies the inequality (note, however, that Corollary 2 cannot be used this time). For $x>1.1\times 10^6$ one deduces, proceeding as before, that $\lambda_{g_{4,3}}(x)\geq 0.48508m_{g_{4,3}}(x/127)\geq 0.50456m_{g_{4,1}}(x/5)\geq\lambda_{g_{4,1}}(x)$. □

It remains to establish Theorem 2.

Proof of Theorem 2. We only deal with $N(x;4,3)$ versus $N(x;4,1)$, the other cases following at once from Theorem 7 and (22). Let $\delta(x)=\lambda_{g_{4,3}}(x)-\lambda_{g_{4,1}}(x)$. By

Theorem 7 we have $\delta(x) \geq 0$ for $x \geq 7$. Using this and (22) we infer that

$$\begin{aligned} N(x;4,3) - N(x;4,1) &= \frac{\delta(x)}{\log x} + \int_2^7 \frac{\delta(t)dt}{t\log^2 t} + \int_7^x \frac{\delta(t)dt}{t\log^2 t} \\ &\geq \int_2^7 \frac{\delta(t)dt}{t\log^2 t} = \frac{\log 5 - \log 3}{\log 7} > 0, \end{aligned}$$

for $x \geq 7$. For $x < 7$ the result is clearly true. □

Acknowledgments

T. Dokshitzer was so kind as to redo some of the computations (mainly carried out in Maple) in PARI. K. Ford, R. Hudson and M. Rubinstein helpfully provided me with some numerical data regarding Chebyshev's bias (and with a preprint of [14]). G. Martin pointed out some errors in an earlier version. I'd like to thank O. Ramaré for pointing out reference [12]. He also informed me of his paper [31] in which he roughly halves the values given in (39) (which is not enough to give an unconditional proof of any of the claims in Theorem 6). H. te Riele kindly did the computations required to validate parts (a) and (b) of Theorem 6, taking, respectively, 20 and 5 minutes of CPU time on a 300MHz SGI processor. Finally the author thanks the referee for helpful comments regarding the Chebyshev bias literature, in particular for pointing out the relevance of [3].

References

[1] C. Bays and R.H. Hudson, Details of the first region of integers x with $\pi_{3,2}(x) < \pi_{3,1}(x)$, *Math. Comp.* **32** (1978), 571-576. MR **57:**16175

[2] B.C. Berndt and K. Ono, Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary, *The Andrews Festschrift* (Maratea, 1998), (Eds.) D. Foata and G.N. Han, 2001, 39–110. MR **2000i:**01027

[3] S. Bochner and B. Jessen, Distribution functions and positive definite functions, *Ann. Math.* **35** (1934), 252-257.

[4] P.L. Chebyshev, Lettre de M. le Professeur Tchébychev à M. Fuss sur un nouveaux théorème relatif aux nombres premiers contenus dans les formes $4n+1$ ets $4n+3$, *Bull. Classe Phys. Acad. Imp. Sci. St. Petersburg* **11** (1853), 208.

[5] H. Cohen, *Advanced topics in computational number theory*, GTM **193**, Springer-Verlag, New York, 2000. MR **2000k:**11144

[6] H. Cohen, High precision computation of Hardy-Littlewood constants, draft of a preprint, available at http://www.math.u-bordeaux.fr/~cohen/.

[7] H. Cohen and F. Dress, Estimations numériques du reste de la fonction sommatoire relative aux entiers sans facteur carré, *Publ. Math. Orsay* 88/02 (1988), 73-76. MR **89i:**11103

[8] J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, Third edition, Springer-Verlag, New York, 1999. MR **2000b:**11077

[9] D.A. Cox, The arithmetic-geometric mean of Gauss, *Enseign. Math.* **30** (1984), 275-330. MR **86a:**01027

[10] H. Davenport, *Multiplicative number theory*, Third revised edition, Springer-Verlag, New York, 2000. MR **2001f:**11001

[11] K. Dilcher, Generalized Euler constants for arithmetical progressions, *Math. Comp.* **59** (1992), 259-282. MR **92k:**11145

[12] P. Dusart, Autour de la fonction qui compte le nombre de nombres premiers, PhD thesis, Université de Limoges, 1998.

[13] S.R. Finch, Mathematical Constants, Cambridge University Press, (2003); website http://pauillac.inria.fr/algo/bsolve/constant/constant.html

[14] K. Ford and R.H. Hudson, Sign changes in $\pi_{q,a}(x) - \pi_{q,b}(x)$, *Acta Arith.* **100** (2001), 297-314.

[15] J. Kaczorowski, The boundary values of generalized Dirichlet series and a problem of Chebyshev, *Astérisque* **209** (1992), 227–235. MR **94a:**11144

[16] J. Kaczorowski, Boundary values of Dirichlet series and the distribution of primes, European Congress of Mathematics, Vol. I (Budapest, 1996), 237-254, Progr. Math., 168, Birkhäuser, Basel, 1998. MR **99f:**11119

[17] S. Knapowski and P. Turán, Further developments in the comparative prime-number theory. I, *Acta Arith.* **9** (1964), 23-40. MR **29:**75

[18] E. Landau, Über die zu einem algebraischen Zahlkörper gehörige Zetafunktion und die Ausdehnung der Tschebyschefschen Primzahlentheorie aus das Problem der Verteilung der Primideale, *J. Reine Angew. Math.* **125** (1903), 64-188.

[19] E. Landau, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der mindest Anzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, *Arch. der Math. und Phys.* **13** (1908), 305-312.

[20] B.V. Levin and A.S. Fainleib, Application of some integral equations to problems of number theory, *Russian Math. Surveys* **22** (1967), 119-204. MR **37:**5174

[21] J.E. Littlewood, Distribution des nombres premiers, *C.R. Acad. Sci. Paris* **158** (1914), 1869-1872.

[22] S.D. Miller and G. Moore, Landau-Siegel zeroes and black hole entropy, *Asian J. Math.* **4** (2000), 183-211. MR **2002i:**11063

[23] P. Moree, Approximation of singular series and automata, *Manuscripta Math.* **101** (2000), 385-399. MR **2001f:**11204

[24] P. Moree, On some claims in Ramanujan's 'unpublished' manuscript on the partition and tau functions, arXiv:math.NT/0201265, to appear in *The Ramanujan Journal*.

[25] P. Moree and J. Cazaran, On a claim of Ramanujan in his first letter to Hardy, *Exposition. Math.* **17** (1999), 289-311. MR **2001c:**11103

[26] P. Moree and H.J.J. te Riele, The hexagonal versus the square lattice, arXiv:math.NT/0204332, to appear in *Math. Comp.*

[27] L. Moser and R.A. MacLeod, The error term for the squarefree integers, *Canad. Math. Bull.* **9** (1966), 303-306. MR **34:**150

[28] W. Narkiewicz, *The development of prime number theory. From Euclid to Hardy and Littlewood*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000. MR **2001c:**11098

[29] V. Nevanlinna, On constants connected with the prime number theorem for arithmetic progressions, *Ann. Acad. Sci. Fenn. Ser. A I* **539** (1973), 11 pp. MR **49:**7219

[30] A.G. Postnikov, *Introduction to analytic number theory*, AMS translations of mathematical monographs 68, AMS, Providence, Rhode Island, 1988. MR **89a:**11001

[31] O. Ramaré, Sur un théorème de Mertens, *Manuscripta Math.* **108** (2002), 495–513.

[32] O. Ramaré and R. Rumely, Primes in arithmetic progressions, *Math. Comp.* **65** (1996), 397-425. MR **97a:**11144

[33] J.B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois Journal Math.* **6** (1962), 64-94. MR **25:**1139

[34] M. Rubinstein and P. Sarnak, Chebyshev's bias, *Experiment. Math.* **3** (1994), 173-197. MR **96d:**11099

[35] P. Schmutz Schaller, Geometry of Riemann surfaces based on closed geodesics, *Bull. Amer. Math. Soc. (N.S.)* **35** (1998), 193-214. MR **99b:**11098

[36] L. Schoenfeld, Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II, *Math. Comp.* **30** (1976), 337-360. MR **56:**15881b

[37] J.-P. Serre, Divisibilité de certaines fonctions arithmétiques, *Enseign. Math.* **22** (1976), 227-260. MR **55:**7958

[38] D. Shanks, The second order term in the asymptotic expansion of $B(x)$, *Math. Comp.* **18** (1964), 75-86. MR **28:**2391

[39] J.M. Song, Sums over nonnegative multiplicative functions over integers without large prime factors. I, *Acta Arith.* **97** (2001), 329-351. MR **2002f:**11130

[40] P. Turán, *On a new method of analysis and its applications*, John Wiley and Sons, Inc., New York, 1984. MR **86b:**11059

[41] P. Turán, *Collected papers of Paul Turán*, Vol. 1-3, Ed. P. Erdös. Akadémiai Kiadó, Budapest, 1990. MR **91i:**01145

[42] K.S. Williams, Mertens' theorem for arithmetic progressions, *J. Number Theory* **6** (1974), 353-359. MR **51:**392

[43] A. Wintner, Upon a statistical method in the theory of diophantine approximations, *Amer. J. Math.* **55** (1933), 309-331.

[44] E. Wirsing, Das asymptotische Verhalten von Summen über multiplikative Funktionen, *Math. Ann.* **143** (1961), 75-102. MR **24:**A1241

KdV Institute, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands
E-mail address: `moree@science.uva.nl`