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A NOTE ON A PAPER BY G. MASTROIANNI AND G. MONEGATO

G. CRISCUOLO

Abstract. Recently, Mastroianni and Monegato derived error estimates for a numerical approach to evaluate the integral

$$\int_a^b \int_{-1}^1 \frac{f(x,y)}{x-y} dxdy,$$

where $(a,b) \equiv (-1,1)$ or $(a,b) \equiv (a,-1)$ or $(a,b) \equiv (1,b)$ and $f(x,y)$ is a smooth function (see G. Mastroianni and G. Monegato, *Error estimates in the numerical evaluation of some BEM singular integrals*, Math. Comp. **70** 2001, 251–267). The error bounds for the quadrature rule approximating the inner integral given in Theorems 3, 4 of that paper are not correct according to the proof. However, following a different approach, we are able to improve the pointwise error estimates given in that paper.

1. Introduction

Following a recent numerical approach, Mastroianni and Monegato have suggested approximating the integral

$$H(f;y) := \int_{-1}^1 \frac{f(x,y)}{x-y} dx, \tag{1.1}$$

whenever $y \in (-1,1)$ or $y \notin (-1,1)$ by a quadratue rule of interpolatory type based on the zeros of suitable orthogonal polynomials (see [6]). When $y \in (-1,1)$, the integral $H(f;y)$ is defined in the Cauchy principal value sense. An accurate calculation of (1.1) may be useful for many applications, for instance, to approximate the two-dimensional integrals of type

$$\int_a^b \int_{-1}^1 \frac{f(x,y)}{x-y} dxdy, \tag{1.2}$$

where $(a,b) \equiv (-1,1)$ or $(a,b) \equiv (a,-1)$ or $(a,b) \equiv (1,b)$. Such integrals arise in some applications of Galerkin boundary element methods (see also [6] and the references given therein). Furthermore, the estimate of the error in the numerical approximation of (1.1) can be used in the numerical solution of singular integral equations by a collocation method. Assuming that a symmetric Jacobi weight

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function $w^\alpha(x) := (1-x^2)^\alpha, \alpha > -1$, is also present, the authors of [6] consider the quadrature formula

$$H^\alpha(f;y) := \int_{-1}^{1} \frac{f(x,y)}{x-y} w^\alpha(x)dx = \sum_{i=1}^{n} w_{n,i}(y) f(x_{n,i}, y) + R_n^\alpha(f;y), \tag{1.3}$$

whenever $y \in (-1,1)$ or $y \notin (-1,1)$. It is of interpolatory type and it is obtained by replacing, for any given y, $f(x,y)$ by its Lagrange interpolation (with respect to x) polynomial $\mathcal{L}_n^\alpha(f,y;x)$ of degree $n-1$, based on the zeros $x_{n,n} < \cdots < x_{n,1}$ of the nth-degree Jacobi polynomial $P_n(w^\alpha)$ corresponding to the weight w^α.

Defining the modulus of smoothness $\omega^r_{I,\varphi}(f;.)$ as in [4, (12.1.2)], where $r \geq 1$ is an integer, $I = [-1,1]^2$ and $\varphi(t) = \sqrt{1-t^2}$, and setting

$$E_k(f)_\infty := \inf_{p_k \in \Pi_k} \|f - p_k\|_\infty = \inf_{p_k \in \Pi_k} \sup_{[-1,1]^2} |f(x,y) - p_k(x,y)|,$$

where Π_k denotes the set of all polynomials of degree k in each variable, Mastroianni and Monegato prove their main result on the quadrature rule (1.3) in the following

Theorem 1.1 ([6], Theorem 3)**.** *Let* $|\alpha| \leq \frac{1}{2}$ *and* $y \in (-1,1)$. *Given any* $f \in \mathrm{C}([-1,1]^2)$, *for the remainder term in* (1.3) *we have*

$$|R_n^\alpha(f;y)| \leq c\ \overline{h}_\alpha(y) \left[E_{n-1}(f)_\infty \log n + \int_0^{\frac{1}{n-1}} \frac{\omega^r_{I,\varphi}(f;u)}{u} du \right], \quad n > 1, \tag{1.4}$$

where

$$\overline{h}_\alpha(y) = \begin{cases} w^{\frac{\alpha}{2}-\frac{1}{4}}(y), & \alpha < 0, \\ w^{-\frac{1}{4}}(y) \log \frac{1}{1-y^2}, & \alpha = 0, \\ w^{-\frac{1}{4}}(y), & \alpha > 0, \end{cases}$$

and $c = c(\alpha)$ *depends only upon* α.

Unfortunately, the error bound of the Theorem 1.1 is not correct according to the proof given in [6]. For instance, to bound $R_n^\alpha(f;y)$ the authors of [6] have to bound

$$A_n(w^\alpha;y) = A_n^*(w^\alpha;y)\mathcal{L}_n^\alpha(e_m,y;y) \tag{1.5}$$

with

$$A_n^*(w^\alpha;y) = \int_{-1}^{1} \frac{w^\alpha(x)}{x-y} dx - \sum_{i=1, i \neq i_c}^{n} \frac{\lambda_{n,i}}{x_{n,i} - y}, \tag{1.6}$$

where $\lambda_{n,i}, i = 1,\ldots,n$, are the Christoffel constants corresponding to the weight w^α, x_{n,i_c} is the closest node to y, and $e_m = f - p_m$ with p_m the best uniform approximation polynomial of degree $m-1$ with respect to each variable. By using a known result on $A_n^*(w^\alpha;y)$ and a weighted bound for the Lagrange operator (see [2, 5], respectively), Mastroianni and Monegato deduce an estimate for $A_n(w^\alpha;y)$ from which (1.4) does not follow for $\alpha > 0$. Following the proof in [6], in order to deduce (1.4), it is necessary to define $\overline{h}_\alpha(y) = w^{-\frac{\alpha}{2}-\frac{1}{4}}(y)$, for $\alpha > 0$.

The boundedness of $R_n^I(f;y)$ when $y \notin (-1,1)$ is also studied in [6]. We remark that the related result given in Theorem 4 of [6] trivially fails because in the proof the authors make use of the bound $|f(x,y)| \leq \|f\|_\infty = \sup_{[-1,1]^2} |f(x,y)|$ while $y \notin (-1,1)$ as in the assumptions of that theorem.

We shall improve and generalize the above results.

2. Main result

In order to bound $R_n^\alpha(f;y)$, the authors of [6] examine the boundedness of the operators H^α and $\mathcal{L}_n^\alpha$ in unrelated ways. For instance, an estimate for $A_n(w^\alpha;y)$ defined in (1.5) is required to bound $R_n^\alpha(f;y)$. As we shall see, it is possible to improve and generalize the results of [6] following the standard technique to deal with interpolation rules; particularly, we shall make use of good bounds on the functions of the second kind.

Defining the amplification coefficient

$$K_n^\alpha(y) := \sum_{i=1}^{n} |w_{n,i}(y)|\,,$$

whenever $y \in (-1,1)$ or $y \notin (-1,1)$, the following Theorems 2.1 and 2.3 give an accurate estimate of $K_n^\alpha(y)$.

Theorem 2.1. *When $\alpha > -1$ and $y \in (-1,1)$, we have*

$$K_n^\alpha(y) \le c\ \log n\ \overline{\overline{h}}_\alpha(y), \qquad n > 1, \tag{2.1}$$

where

$$\overline{\overline{h}}_\alpha(y) = \begin{cases} w^\alpha(y), & \alpha < -\frac{1}{2}, \\ w^{\frac{\alpha}{2}-\frac{1}{4}}(y), & 0 < |\alpha| \le \frac{1}{2}, \\ w^{-\frac{1}{4}}(y)\log\frac{1}{1-y^2}, & \alpha = 0, \\ 1, & \alpha > \frac{1}{2}, \end{cases} \tag{2.2}$$

and $c = c(\alpha)$ depends only upon α.

To derive this result, some preliminary lemmas are needed.

If we define the function

$$S_n^\rho(y) := \sum_{i=1, i\ne i_c}^{n} \frac{(1-x_{n,i}^2)^\rho}{n|y-x_{n,i}|}, \qquad y \in (-1,1), \tag{2.3}$$

where $x_{n,i}, i = 1,\dots,n$, are the Jacobi zeros corresponding to the weight $w^\alpha(x) = (1-x^2)^\alpha, \alpha > -1,\ \ i_c$ denotes the index corresponding to the closest node to $y \in (-1,1)$ and ρ is a real number, we have the following result.

Lemma 2.1. *For every $y \in (-1,1)$, we have*

$$S_n^\rho(y) \le c\ \log n \begin{cases} w^{\rho-\frac{1}{2}}(y), & |\rho| \le \frac{1}{2}, \\ 1, & \rho > \frac{1}{2}, \end{cases}$$

where $c = c(\rho)$ depends only upon ρ.

For the proof see Lemmas 3.1 and 3.3 in [1].

Next we define the functions of the second kind Q_n^α associated with the weight w^α by

$$Q_n^\alpha(y) := \int_{-1}^{1} \frac{P_n^\alpha(x)}{x-y} w^\alpha(x)dx, \qquad n = 0,1,\dots,$$

where

$$P_n^\alpha(x) = P_n(w^\alpha;x) = \gamma_n x^n + \text{lower degree terms},$$

is the nth–degree Jacobi orthonormal polynomial.

The following propositions are the key to proving the main result of this paper.

Lemma 2.2. *For every $y \in (-1,1)$, we have*

$$|Q_n^\alpha(y)| \le c \begin{cases} w^{\frac{\alpha}{2}-\frac{1}{4}}(y), & -1<\alpha\le\frac{1}{2},\ \alpha\ne 0, \\ w^{-\frac{1}{4}}(y)\log\frac{1}{1-y^2}, & \alpha=0, \\ 1, & \alpha>\frac{1}{2}, \end{cases}$$

where $c=c(\alpha)$ depends only upon α.

Lemma 2.3. *For every $y \in (-1,1)$, we have*

$$|A_n^*(w^\alpha;y)| \le c\ h_\alpha(y),$$

where

$$h_\alpha(y) = \begin{cases} w^\alpha(y), & \alpha<0, \\ \log\frac{1}{1-y^2}, & \alpha=0, \\ 1, & \alpha>0, \end{cases} \tag{2.4}$$

$A_n^*(w^\alpha;y)$ is the function defined in (1.6) *and $c=c(\alpha)$ depends only upon α.*

Lemmas 2.2 and 2.3 are particular cases of more general results (see [3], Theorem 2.1 and Lemma 3.2, respectively).

We remark that we have for the coefficients of (1.3)

$$w_{n,i}(y) = \frac{Q_n^\alpha(y)}{P_n'(w^\alpha;x_{n,i})(y-x_{n,i})} - \frac{\lambda_{n,i}}{y-x_{n,i}}, \qquad i=1,2,\dots,n. \tag{2.5}$$

We also derive

$$w_{n,i_c}(y) = A_n^*(w^\alpha;y)\ell_{n,i_c}(y) + \frac{\ell_{n,i_c}(y)}{y-x_{n,i_c}}\lambda_{n,i_c}, \qquad y\ne x_{n,i}, i=1,2,\dots,n, \tag{2.6}$$

$$w_{n,i_c}(x_{n,i_c}) = A_n^*(w^\alpha;x_{n,i_c}) + \ell_{n,i_c}'(x_{n,i_c})\lambda_{n,i_c}, \tag{2.7}$$

where x_{n,i_c} is the closest node to y, and

$$\ell_{n,i_c}(y) = \frac{P_n(w^\alpha;y)}{P_n'(w^\alpha;x_{n,i_c})(y-x_{n,i_c})},$$

(see [2], (2.9)–(2.11)).

Lemma 2.4. *For every $y \in (-1,1)$, we have*

$$|w_{n,i_c}(y)| \le c\ h_\alpha(y), \tag{2.8}$$

where h_α is the function defined in (2.4) *and i_c denotes the index corresponding to the closest node to y and $c=c(\alpha)$ depends only upon α.*

Proof. For simplicity we examine only (2.6), since the case (2.7) is very similar. We recall that

$$|\ell_{n,i_c}(y)| \sim 1, \tag{2.9}$$

where $A\sim B$ means that there exist two positive constants c_1, c_2 such that $|A^{-1}B|\le c_1$ and $|AB^{-1}|\le c_2$ (see [7], proof of Theorem 33, p. 171). Thus, by applying Lemma 2.3,

$$|A_n^*(w^\alpha;y)\ell_{n,i_c}(y)| \le c\ h_\alpha(y). \tag{2.10}$$

On the other hand, we have

$$\left|\frac{\ell_{n,i_c}(y)}{y-x_{n,i_c}}\right| = \left|\ell_{n,i_c}'(\xi_{i_c})\right| \le c\ n(1-\xi_{i_c})^{-\frac{1}{2}}, \qquad |y-\xi_{i_c}|<|x_{n,i_c}-y|,$$

having used the Bernstein inequality and $|\ell_{n,i_c}(\xi_{i_c})| \sim 1$, with x_{n,i_c} the node closest to ξ_{i_c}. Now, if $|y| \leq (1 + x_{n,1})/2$, then $1 \pm \xi_{i_c} \sim 1 \pm x_{n,i_c} \sim 1 \pm y$. Thus, recalling that

$$\lambda_{n,i} \sim \frac{\sqrt{1 - x_{n,i}^2}}{n} w^\alpha(x_{n,i}), \qquad i = 1, 2, \ldots, n, \tag{2.11}$$

(see [7], Theorem 6.3.28, p. 120), we have

$$\lambda_{n,i_c} \left| \frac{\ell_{n,i_c}(y)}{y - x_{n,i_c}} \right| \leq c\, w^\alpha(y), \qquad |y| \leq \frac{1 + x_{n,1}}{2}. \tag{2.12}$$

In the case $(1 + x_{n,1})/2 < |y| < 1$, recalling that $1 - x_{n,1} \sim n^{-2}$ (see [7], Theorem 9.22, p. 166) and the symmetry of the nodes $x_{n,i}$ with respect to 0, we have

$$\left| \frac{\ell_{n,i_c}(y)}{y - x_{n,i_c}} \right| \leq c\, n^2 \left| \ell_{n,i_c}(y) \right|.$$

So, by (2.9) and (2.11)

$$\lambda_{n,i_c} \left| \frac{\ell_{n,i_c}(y)}{y - x_{n,i_c}} \right| \leq c\, w^\alpha(x_{n,1}) \leq c \begin{cases} w^\alpha(y), & \alpha < 0, \\ 1, & \alpha \geq 0, \end{cases} \qquad \frac{1 + x_{n,1}}{2} < |y| < 1. \tag{2.13}$$

Combining (2.10), (2.12) and (2.13), we finally obtain (2.8). □

Proof of Theorem 2.1. To bound $K_n^\alpha(y)$, we remark that

$$K_n^\alpha(y) \leq c \left\{ |Q_n^\alpha(y)| \sum_{i=1, i \neq i_c}^{n} \frac{1}{|P_n'(w^\alpha; x_{n,i})|\, |y - x_{n,i}|} + \sum_{i=1, i \neq i_c}^{n} \frac{\lambda_{n,i}}{|y - x_{n,i}|} + h_\alpha(y) \right\},$$

having used (2.5)–(2.7) and Lemma 2.4. Then, taking into account that

$$\left[P_n'(w^\alpha; x_{n,i})\right]^{-1} = \frac{\gamma_{n-1}}{\gamma_n} \lambda_{n,i} P_{n-1}^\alpha(x_{n,i}) \sim \frac{1}{n}(1 - x_{n,i}^2)^{\frac{\alpha}{2} + \frac{3}{4}}, \qquad i = 1, 2, \ldots, n,$$

(see [7], Theorem 9.31, p. 170), by (2.11) we deduce

$$K_n^\alpha(y) \leq c \left\{ |Q_n^\alpha(y)| S_n^{\frac{\alpha}{2} + \frac{3}{4}}(y) + S_n^{\alpha + \frac{1}{2}}(y) + h_\alpha(y) \right\},$$

where $S_n^\rho(y), \rho \in \{\frac{\alpha}{2} + \frac{3}{4}, \alpha + \frac{1}{2}\}$ are the functions defined by (2.3). So, by applying Lemmas 2.1 and 2.2, we deduce (2.1). □

The following theorem generalizes and improves the corresponding result by Mastroianni and Monegato in [6] about the boundedness of the remainder term $R_n^\alpha(f; y)$ of (1.3).

Theorem 2.2. *Let $\alpha > -1$ and $y \in (-1, 1)$. Given any $f \in \mathrm{C}([-1, 1]^2)$, for the remainder term in* (1.3) *we have*

$$|R_n^\alpha(f; y)| \leq c\, \overline{\overline{h}}_\alpha(y) \left[E_{n-1}(f)_\infty \log n + \int_0^{\frac{1}{n-1}} \frac{\omega_{I,\varphi}^r(f; u)}{u} du \right], \quad n > 1, \tag{2.14}$$

where $\overline{\overline{h}}_\alpha(y)$ is defined as in (2.2) *and $c = c(\alpha)$ depends only upon α.*

Proof. Since $R_n^\alpha(f;y) = 0$ whenever f is a polynomial of degree $\le n-1$, we have

$$\begin{aligned} |R_n^\alpha(f;y)| &\le \left| \int_{-1}^{1} \frac{f(x,y) - p_{n-1}(x,y)}{x-y} w^\alpha(x)dx \right| \\ &\quad + \sum_{i=1}^{n} |w_{n,i}(y)| \; |f(x_{n,i},y) - p_{n-1}(x_{n,i},y)| \\ &=: I_1 + I_2, \end{aligned} \tag{2.15}$$

where p_{n-1} is the best uniform approximation polynomial of degree $n-1$. By Theorem 2 in [6] we immediately derive

$$I_1 \le c\; h_\alpha(y) \left[E_{n-1}(f)_\infty \log n + \int_0^{\frac{1}{n-1}} \frac{\omega_{I,\varphi}^r(f,u)}{u} du \right], \qquad y \in (-1,1), \tag{2.16}$$

where $h_\alpha(y)$ is the function defined in (2.4). To bound I_2, we remark that

$$I_2 \le c\; E_{n-1}(f)_\infty K_n^\alpha(y) \le c\; E_{n-1}(f)_\infty \overline{\overline{h}}_\alpha(y) \log n, \tag{2.17}$$

having used Theorem 2.1. Combining (2.16), (2.17) with (2.15), we deduce (2.14). $\square$

Next we examine the situation $|y| > 1$. The derivation of the corresponding results is fairly simple. As we have already remarked, the proof of the Theorem 4 in [6] is incorrect. In any case, following the technique used in [6], it is possible to derive

$$K_n^\alpha(y) \le c\; \log n \begin{cases} (y^2-1)^{\frac{\alpha}{2}-\frac{1}{4}}, & -\frac{1}{2} < \alpha < \frac{1}{2}, \\ \log \frac{1}{y^2-1}, & \alpha = \frac{1}{2}, \\ 1 & \alpha > \frac{1}{2}, \end{cases} \qquad |y| \to 1+0. \tag{2.18}$$

Estimate (2.18) can be obtained making use again of a weighted bound for the Lagrange operator as in [6]. The next theorem proves that (2.18) can be improved.

Theorem 2.3. *When $\alpha > -1$ and $|y| \to 1+0$, we have*

$$K_n^\alpha(y) \le c \begin{cases} (y^2-1)^{\frac{3}{2}\alpha+\frac{1}{4}} \log n, & \alpha < -\frac{1}{2}, \\ (y^2-1)^{\alpha} \log n, & -\frac{1}{2} < \alpha < 0, \\ \log \frac{1}{y^2-1} \log n, & \alpha = 0, \\ 1 & \alpha > 0, \end{cases} \tag{2.19}$$

and $c = c(\alpha)$ depends only upon α.

To derive this result, some other preliminary lemmas are needed.

If we define the function

$$\overline{S}_n^\rho(y) := \sum_{i=1}^{n} \frac{(1-x_{n,i}^2)^\rho}{n|y-x_{n,i}|}, \qquad y \notin (-1,1), \tag{2.20}$$

where $x_{n,i}, i = 1, \ldots, n$, are the Jacobi zeros corresponding to the weight $w^\alpha(x) = (1-x^2)^\alpha, \alpha > -1$, and ρ is a real number, we have the following result.

Lemma 2.5. *For every $y \notin (-1,1)$, we have*

$$\overline{S}_n^\rho(y) \le c \begin{cases} (y^2-1)^{\rho-\frac{1}{2}} \log n, & |\rho| \le \frac{1}{2}, \\ 1, & \rho > \frac{1}{2}, \end{cases} \tag{2.21}$$

where $c = c(\rho)$ depends only upon ρ.

Proof. Let $y > 1$, since the case $y < -1$ is very similar. First assume that $\rho > \frac{1}{2}$. Taking into account that $y - x_{n,i} > 1 - x_{n,i}$ and recalling that $1 - x_{n,i} \sim \frac{i^2}{n^2}$, $1 + x_{n,i} \sim \frac{(n-i+1)^2}{n^2}$, $i = 1, \ldots, n$, we deduce

$$\overline{S}_n^\rho(y) \leq c\, n^{1-4\rho} \sum_{i=1, i\neq n+1}^{n+2} i^{2\rho-2} |i - (n+1)|^{2\rho}.$$

Applying Lemma 9 in [7, p. 109], we obtain (2.21). Now, if $|\rho| \leq \frac{1}{2}$, we have

$$\begin{aligned}
\overline{S}_n^\rho(y) &= n^{-1}(y-1)^{\rho-\frac{1}{2}} \sum_{i=1}^{n} \left(\frac{y-1}{y-x_{n,i}}\right)^{\frac{1}{2}-\rho} \frac{(1-x_{n,i}^2)^\rho}{(y-x_{n,i})^{\rho+\frac{1}{2}}} \\
&\leq n^{-1}(y-1)^{\rho-\frac{1}{2}} \sum_{i=1}^{n} \frac{(1-x_{n,i}^2)^\rho}{(y-x_{n,i})^{\rho+\frac{1}{2}}} \\
&\leq n^{-1}(y-1)^{\rho-\frac{1}{2}} \sum_{i=1}^{n} (1-x_{n,i})^{-\frac{1}{2}} (1+x_{n,i})^\rho \\
&\leq c\, n^{-2\rho}(y-1)^{\rho-\frac{1}{2}} \sum_{i=1, i\neq n+1}^{n+2} i^{-1} |i-(n+1)|^{2\rho},
\end{aligned}$$

since $y - 1 < y - x_{n,i}$ and $\frac{1}{2} - \rho > 0$. Again, by the Lemma 9 of [7], we obtain (2.21) also in this case. □

Lemma 2.6. *For* $|y| \to 1 + 0$*, we have*

$$Q_n^\alpha(y) \leq c \begin{cases} (y^2-1)^\alpha, & \alpha < 0, \\ \log(y^2-1), & \alpha = 0, \\ 1, & \alpha > 0, \end{cases}$$

where $c = c(\alpha)$ *depends only upon* α*.*

The above lemma follows from a well-known result on the behaviour of the functions of the second kind Q_n^α (see [8], Theorem 4.62.1, p. 77).

Proof of Theorem 2.3. Proceeding as in the proof of the Theorem 2.1, by (2.5) we have

$$K_n^\alpha(y) \leq c\left\{ |Q_n^\alpha(y)| \overline{S}_n^{\frac{\alpha}{2}+\frac{3}{4}}(y) + \overline{S}_n^{\alpha+\frac{1}{2}}(y) \right\},$$

where $\overline{S}_n^\rho(y), \rho \in \{\frac{\alpha}{2}+\frac{3}{4}, \alpha+\frac{1}{2}\}$, are the functions defined by (2.20). So, by applying Lemmas 2.5 and 2.6, we deduce (2.19). □

Finally, we remark that proceeding as in the proof of the Theorem 2.2, since

$$\begin{aligned}
\int_{-1}^{1} \frac{w^\alpha(x)}{|x-y|} dx &= \left| \int_{-1}^{1} \frac{w^\alpha(x)}{x-y} dx \right| \\
&= |Q_0(y)| \leq c \begin{cases} (y^2-1)^\alpha, & \alpha < 0, \\ \log(y^2-1), & \alpha = 0, \\ 1, & \alpha > 0, \end{cases} \qquad |y| \to 1+0,
\end{aligned}$$

where $c = c(\alpha)$ depends only upon α (see [8], (4.62.2), p. 76), we can deduce the behaviour of $R_n^\alpha(f; y)$, $y \notin (-1, 1)$, depending on the best uniform approximation error defined on the domain of $f(x, y)$.

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Dipartimento di Matematica e Applicazioni, Universitá degli Studi di Napoli "Federico II", Complesso Monte Sant'Angelo, Edificio T, Via Cintia, 80126 Napoli, Italy

E-mail address: criscuo@matna2.dma.unina.it