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SUBSTRUCTURING PRECONDITIONERS FOR SADDLE-POINT PROBLEMS ARISING FROM MAXWELL'S EQUATIONS IN THREE DIMENSIONS

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Abstract. This paper is concerned with the saddle-point problems arising from edge element discretizations of Maxwell's equations in a general three dimensional nonconvex polyhedral domain. A new augmented technique is first introduced to transform the problems into equivalent augmented saddle-point systems so that they can be solved by some existing preconditioned iterative methods. Then some substructuring preconditioners are proposed, with very simple coarse solvers, for the augmented saddle-point systems. With the preconditioners, the condition numbers of the preconditioned systems are nearly optimal; namely, they grow only as the logarithm of the ratio between the subdomain diameter and the finite element mesh size.

1. Introduction

In the numerical simulation of electromagnetic models, one needs to repeatedly solve the following system [8], [9], [11], [14], [25], [29], [30]:

$$\begin{cases} \mathbf{curl}(\alpha\,\mathbf{curl}\,\mathbf{u}) + \gamma_0\beta\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}(\beta\mathbf{u}) = g \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

with the following boundary condition:

$$\mathbf{u}\times\mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

Here Ω is an open, simply connected and Lipschiz domain in $\mathbf{R}^3$, and $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$. The source functions $\mathbf{f} \in L^2(\Omega)^3$ and $g \in L^2(\Omega)$ satisfy the compatibility condition $\gamma_0\, g = \nabla\cdot\mathbf{f}$. The coefficients $\alpha(x)$ and $\beta(x)$ are two positive bounded functions in Ω. In applications, we have $\alpha(x)/\beta(x) = c(x)$ with $c(x)$ being the velocity of light. The constant γ_0 is nonnegative, i.e., $\gamma_0 \geq 0$, and it is allowed to be identically zero. It is this extreme case that causes the most troublesome technical difficulty to be dealt with in the paper.

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For the numerical solution of the system (1.1)-(1.2), the edge finite element methods have been widely used in recent years; see, for example, [8], [9], [21]. It is important to note that the algebraic systems arising from the discretization by the edge element methods are very different from the ones arising from the discretization by the standard nodal finite element methods. Thus the construction of the nonoverlapping domain decomposition preconditioners for the nodal element systems, which has been well developed for the second order elliptic problems in the past two decades (see the survey article [32]), does not work for the edge element discretization of the equations (1.1)-(1.2) in general, especially in three dimensions. Recently, there has been a rapidly growing interest in domain decomposition methods for solving Maxwell's equations. A substructuring domain decomposition method was discussed in [30] for Maxwell's equations in two dimensions, and an overlapping Schwarz method was studied in [13] and [29] for Maxwell's equations in three dimensions. Also, a nonoverlapping domain decomposition method with two subdomains was proposed in [2] for Maxwell's equations in three dimensions. To our knowledge, there exists no work in the literature, which studies nonoverlapping domain decomposition methods for Maxwell's equations in three dimensions for the case with general multiple subdomains. This paper intends to make an initial effort in this direction, and certainly there are still many problems which remain open. As we shall see, for the three-dimensional case with multiple nonoverlapping subdomains, not only the construction of the coarse subspace but also the estimates of the condition numbers of the preconditioned systems are much more difficult and technical than in the two-dimensional case or the three dimensional case with overlapping subdomains.

We will propose an efficient substructuring preconditioner for the saddle-point system arising from the edge element discretizations of the problem (1.1)-(1.2). The most difficult technical issue here lies in the following observation: in the saddle-point system, the block stiffness matrix corresponding to the operator $\mathbf{curl}(\alpha\mathbf{curl}\,\cdot)$ for the prime variable $\mathbf{u}$ is singular when $\gamma_0 = 0$ in (1.1); in fact, it is positive semi-definite. How to construct an efficient preconditioner for such saddle-point systems is still an open problem. To overcome this difficulty, we shall first transform the saddle-point system into another equivalent saddle-point problem whose block stiffness matrix corresponding to the prime variable $\mathbf{u}$ is positive definite. The corresponding Schur complement matrix of the saddle-point system can be well preconditioned by some substructuring preconditioners. It will be shown that the resulting preconditioned system has a nearly optimal condition number; namely, it grows only as the logarithm of the ratio between the subdomain diameter and the finite element mesh size.

The outline of the paper is as follows. In Section 2, we describe the edge element discretization of the system (1.1)-(1.2) and introduce some basic formulae and definitions. The construction of nonoverlapping domain decomposition preconditioners and the main results of the paper are discussed in Section 3. Section 4 presents a series of auxiliary lemmata, which will be used to deal with the technical difficulties in the estimates of the condition numbers in Section 5.

2. Edge element discretization and domain decomposition

This section is devoted to the introduction of the edge element discretization of the system (1.1)-(1.2) and the nonoverlapping domain decomposition.

2.1. **Edge element discretization.** The primary goal of this paper is to study the edge element discretization of the equations (1.1)-(1.2) and then to solve the resulting discrete system by a preconditioned iterative Uzawa method with a nonoverlapping domain decomposition preconditioner. First, we shall state the weak formulation of the equations. For this, we need the Sobolev space $H(\mathbf{curl};\Omega)$, a space with all square integrable functions whose curl's are also square integrable in Ω. To cope with the boundary condition (1.2), we introduce the following subspaces of $H(\mathbf{curl};\Omega)$:

$$H_0(\mathbf{curl};\Omega) = \left\{ \mathbf{v} \in H(\mathbf{curl};\Omega);\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \right\}.$$

Now, by introducing a Lagrange multiplier p and integration by parts we derive the following variational saddle-point problem associated with the system (1.1)-(1.2):

Find $(\mathbf{u}, p) \in H_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$ such that

$$\begin{cases} (\alpha \mathbf{curl}\ \mathbf{u}, \mathbf{curl}\ \mathbf{v}) + \gamma_0(\beta\mathbf{u}, \mathbf{v}) + (\nabla p, \beta\mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in H_0(\mathbf{curl};\Omega), \\ (\beta\mathbf{u}, \nabla q) = (g, q), \qquad \forall q \in H_0^1(\Omega). \end{cases} \tag{2.1}$$

Here and in what follows, $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$ or $L^2(\Omega)^3$.

Next, we introduce the domain decomposition and the triangulation of the domain Ω, and then we discuss the edge element discretization of the saddle-point problem (2.1).

Domain decomposition. We first decompose Ω into N nonoverlapping tetrahedral subdomains $\{\Omega_i\}_i^N$, with each Ω_i of size d (see [4] and [32]). The faces and vertices of the subdomains will be denoted by F and V, respectively. The common face of the subdomains Ω_i and Ω_j is denoted by Γ_{ij}. Also, $\Gamma = \bigcup \Gamma_{ij}$, and $\Gamma_i = \Gamma \cap \partial\Omega_i$. Γ will be called *the interface*. For definiteness, a unique unit normal direction $\mathbf{n}$ is assigned on each face F of Γ, and this normal vector is used whenever a unit normal direction is involved on any face in the subsequent analysis.

Finite element triangulation. We further divide each Ω_i into smaller tetrahedral elements of size h so that elements from two neighboring subdomains have an intersection which is either empty or a single nodal point or an edge or a face on the interface Γ. Let $\mathcal{T}_h$ be the resulting triangulation of the domain Ω, which we assume is quasi-uniform. By $\mathcal{E}_h$ and $\mathcal{N}_h$ we denote the set of edges of $\mathcal{T}_h$ and the set of nodes in $\mathcal{T}_h$, respectively. Then the Nédélec edge element space, of the lowest order, is a subspace of piecewise linear polynomials defined on $\mathcal{T}_h$ (cf. [12] and [22]):

$$V_h(\Omega) = \left\{ \mathbf{v} \in H_0(\mathbf{curl};\Omega);\ \mathbf{v}\,|_K \in R(K),\ \forall K \in \mathcal{T}_h \right\},$$

where $R(K)$ is a subset of all linear polynomials on the element K of the form:

$$R(K) = \left\{ \mathbf{a} + \mathbf{b} \times \mathbf{x};\ \mathbf{a}, \mathbf{b} \in \mathbf{R}^3,\ \mathbf{x} \in K \right\}.$$

It is well known that for any $\mathbf{v} \in V_h(\Omega)$, its tangential components are continuous on all edges of each element in the triangulation $\mathcal{T}_h$. Moreover, each edge element function $\mathbf{v}$ in $V_h(\Omega)$ is uniquely determined by its moments on each edge e of $\mathcal{T}_h$:

$$\left\{ \lambda_e(\mathbf{v}) = \int_e \mathbf{v} \cdot \mathbf{t}_e ds;\ e \in \mathcal{E}_h \right\}$$

where $\mathbf{t}_e$ denotes the unit vector on the edge e.

Let $\{L_e;\ e \in \mathcal{E}_h\}$ be the edge element basis functions of $V_h(\Omega)$ satisfying

$$\lambda_{e'}(L_e) = \begin{cases} 1, & \text{if } e' = e\,, \\ 0, & \text{if } e' \neq e. \end{cases}$$

Then each function $\mathbf{v}$ in $V_h(\Omega)$ can be expressed as

$$\mathbf{v}(\mathbf{x}) = \sum_{e \in \mathcal{E}_h} \lambda_e(\mathbf{v}) L_e(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

By $Z_h(\Omega)$ we denote the continuous piecewise linear finite element subspace of $H_0^1(\Omega)$ associated with the triangulation $\mathcal{T}_h$. Then the saddle-point system (2.1) may be approximated by the finite element problem: Find $(\mathbf{u}_h, p_h) \in V_h(\Omega) \times Z_h(\Omega)$ such that

(2.2)
$$\begin{cases} (\alpha \mathbf{curl}\ \mathbf{u}_h, \mathbf{curl}\ \mathbf{v}_h) + \gamma_0(\beta \mathbf{u}_h, \mathbf{v}_h) + (\nabla p_h, \beta \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h(\Omega), \\ (\beta \mathbf{u}_h, \nabla q_h) = (g, q_h), \qquad \forall q_h \in Z_h(\Omega). \end{cases}$$

2.2. **Some edge element spaces and discrete operators.** Before formulating the domain decomposition preconditioner to be used for solving the system (2.2), we introduce some useful notations and discrete operators.

Let G be either the entire interface Γ or the local interface Γ_i or a face F of Γ_i. We shall frequently use the restrictions of the tangential components of the functions in $V_h(\Omega)$ on G:

$$V_h(G) = \left\{ \boldsymbol{\psi} \in L^2(G)^3;\ \boldsymbol{\psi} = \mathbf{v} \times \mathbf{n} \text{ on } G \text{ for some } \mathbf{v} \in V_h(\Omega) \right\}.$$

The restrictions of $V_h(\Omega)$ on each subdomain Ω_i is denoted by $V_h(\Omega_i)$. The following local subspaces of $V_h(\Omega_i)$ and $V_h(\mathrm{F})$ will be important to our analysis:

$$\begin{aligned} V_h^0(\Omega_i) &= \left\{ \mathbf{v} \in V_h(\Omega_i);\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_i \right\}, \\ V_h^0(\mathrm{F}) &= \left\{ \Phi = \mathbf{v} \times \mathbf{n} \in V_h(\mathrm{F});\ \lambda_e(\mathbf{v}) = 0,\ \forall\, e \subset \partial\mathrm{F} \cap \mathcal{E}_h \right\}. \end{aligned}$$

The natural restriction operator from $V_h(\Gamma)$ onto $V_h(G)$ and the natural zero extension operator from $V_h(G)$ into $L^2(\Gamma)^3$ will be denoted as $\mathbf{I}_G$ and $\mathbf{I}_G^t$, respectively. For a face F, it is easy to see that $\mathbf{I}_{\mathrm{F}}^t \mathbf{v} \in V_h(\Gamma)$ if and only if $\mathbf{v} \in V_h^0(\mathrm{F})$, and $\mathbf{I}_G$ and $\mathbf{I}_G^t$ satisfy

$$\langle \mathbf{I}_G \Psi, \Phi \rangle_G = \langle \Psi, \mathbf{I}_G^t \Phi \rangle \quad \forall\, \Psi \in V_h(\Gamma),\ \Phi \in V_h(G).$$

Here and hereafter, $\langle \cdot, \cdot \rangle_G$ stands for the L^2-inner product in $L^2(G)$ or $L^2(G)^3$, and we will drop the subscript G when $G = \Gamma$. For simplicity, we shall often write $\mathbf{I}_{ij}^t = \mathbf{I}_{\Gamma_{ij}}^t$.

For any face F of Ω_i, we use F_b to denote the union of all $\mathcal{T}_h$-induced (closed) triangles on F, which have either one single vertex or one edge lying on $\partial\mathrm{F}$, and F_∂ to denote the open set $\mathrm{F} \backslash \mathrm{F}_b$; see Figure 1. For any subdomain Ω_i, define

$$\Delta_i = \bigcup_{\mathrm{F} \subset \Gamma_i} \mathrm{F}_b, \quad i = 1, \cdots, N.$$

From now on, the notation e, with $e \subset G \subset \Gamma_i$, always means that e is an edge of $\mathcal{T}_h$ and lies on G.

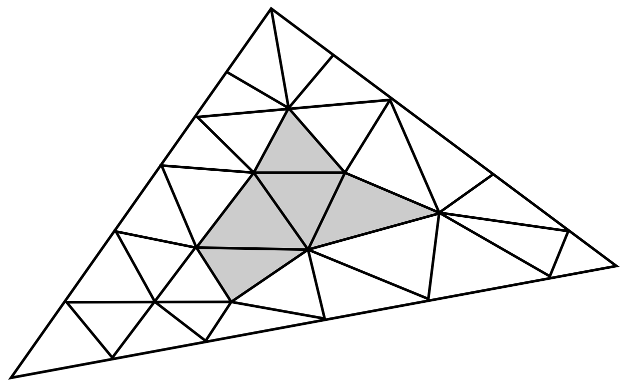

FIGURE 1. Face F, its boundary subset F_b (union of white triangles) and its interior subset F_∂ (union of shaded triangles)

By definition, for any $\Phi \in V_h(\Gamma_i)$, there exists a $\mathbf{v} \in V_h(\Omega_i)$ such that $\Phi = \mathbf{v} \times \mathbf{n}$ on Γ_i. Thus we can write

$$\Phi(\mathbf{x}) = \sum_{e \subset \Gamma_i} \lambda_e(\mathbf{v})(L_e \times \mathbf{n})(\mathbf{x}), \quad \mathbf{x} \in \Gamma_i. \tag{2.3}$$

Furthermore, for an open face $\mathrm{F} \subset \Gamma_i$, we define an operator $\mathbf{I}^0_{\mathrm{F}_\partial} : V_h(\Gamma_i) \to \mathbf{I}^t_{\mathrm{F}} V^0_h(\mathrm{F})$ by

$$(\mathbf{I}^0_{\mathrm{F}_\partial}\Phi)(\mathbf{x}) = \sum_{e \subset \mathrm{F}_\partial} \lambda_e(\mathbf{v})(L_e \times \mathbf{n})(\mathbf{x}), \quad \mathbf{x} \in \Gamma_i, \tag{2.4}$$

and an operator $\mathbf{I}^0_{\mathrm{F}_b}$ by

$$(\mathbf{I}^0_{\mathrm{F}_b}\Phi)(\mathbf{x}) = \sum_{e \subset \mathrm{F}_b} \lambda_e(\mathbf{v})\, \mathbf{I}^t_{\mathrm{F}}(L_e \times \mathbf{n})(\mathbf{x}), \quad \mathbf{x} \in \Gamma_i.$$

Similarly, we define for any $\mathbf{x} \in \Gamma_i$,

$$\mathbf{I}^0_{\mathrm{F}}\Phi(\mathbf{x}) = \sum_{e \subset \mathrm{F}} \lambda_e(\mathbf{v})\mathbf{I}^t_{\mathrm{F}}(L_e \times \mathbf{n})(\mathbf{x}); \quad \mathbf{I}^0_{\Delta_i}\Phi(\mathbf{x}) = \sum_{e \subset \Delta_i} \lambda_e(\mathbf{v})(L_e \times \mathbf{n})(\mathbf{x})\,.$$

Though our main focus in this paper is on the edge element spaces, we shall also make use of some nodal element spaces in the subsequent analyses. As defined earlier, $Z_h(\Omega)$ is the continuous piecewise linear finite element space of $H^1_0(\Omega)$ associated with $\mathcal{T}_h$. The restrictions of $Z_h(\Omega)$ on Γ and Γ_i, in each subdomain Ω_i and on each face F will be denoted by $Z_h(\Gamma)$, $Z_h(\Gamma_i)$, $Z_h(\Omega_i)$ and $Z_h(\mathrm{F})$, respectively. The operator $\mathrm{I}^t_{\mathrm{F}} : Z_h(\mathrm{F}) \to L^2(\Gamma)$ denotes the natural zero extension from F onto Γ.

For a subset G of Γ_i, we define a "local" subspace

$$Z^0_h(G) = \{v \in Z_h(\Gamma_i);\ v = 0 \text{ at all nodes on } \Gamma_i \backslash G\}.$$

For any open face $\mathrm{F} \subset \Gamma_i$, the operators $\mathrm{I}^0_{\mathrm{F}} : Z_h(\Gamma_i) \to Z^0_h(\mathrm{F})$ and $\mathrm{I}^0_{\partial\mathrm{F}} : Z_h(\Gamma_i) \to Z^0_h(\partial\mathrm{F})$ denote the following restriction operators:

$$\mathrm{I}^0_G w_h(x_i) = \begin{cases} w_h(x_i), & x_i \in G \cap \mathcal{N}_h \\ 0, & x_i \in (\Gamma_i \setminus G) \cap \mathcal{N}_h \end{cases} \qquad \text{for } G = \mathrm{F} \text{ or } \partial\mathrm{F} \text{ and any } \ w_h \in Z_h(\Gamma_i).$$

We end this section with the introduction of two frequently used extension operators. With each subdomain Ω_i, we define the local operator $A_i : V_h(\Omega_i) \to V_h(\Omega_i)$ by

$$(A_i\mathbf{u}, \mathbf{v}) = (\alpha\mathbf{curl}\ \mathbf{u}, \mathbf{curl}\ \mathbf{v})_{\Omega_i} + (\alpha\mathbf{u}, \mathbf{v})_{\Omega_i}, \quad \forall\mathbf{u},\ \mathbf{v} \in V_h(\Omega_i)\ (i = 1, 2, \cdots, N).$$

The first is the discrete A_i-extension operator $\mathbf{R}_h^i : V_h(\Gamma_i) \to V_h(\Omega_i)$ defined as follows: For any $\Phi \in V_h(\Gamma_i)$, $\mathbf{R}_h^i\Phi$ satisfies $\mathbf{R}_h^i\Phi \times \mathbf{n} = \Phi$ on Γ_i and solves

$$(A_i\mathbf{R}_h^i\Phi, \mathbf{v}_h) = 0, \quad \forall\mathbf{v}_h \in V_h^0(\Omega_i).$$

The second is the discrete harmonic extension operator $R_h^i : Z_h(\Gamma_i) \to Z_h(\Omega_i)$ defined as follows: For any $w_h \in Z_h(\Gamma_i)$, $R_h^i w_h \in Z_h(\Omega_i)$ satisfies $R_h^i w = w_h$ on Γ_i and solves

$$(\nabla R_h^i w_h, \nabla v_h) = 0, \quad \forall\, v_h \in Z_h(\Omega_i) \cap H_0^1(\Omega_i)\,.$$

3. Nonoverlapping domain decomposition methods

This section addresses how to solve the saddle-point problem (2.2) effectively. For convenience, we introduce two operators $\bar{A} : V_h(\Omega) \to V_h(\Omega)$ and $B : Z_h(\Omega) \to V_h(\Omega)$ by

$$\begin{aligned}(\bar{A}\mathbf{u}_h, \mathbf{v}_h) &= (\alpha\mathbf{curl}\ \mathbf{u}_h, \mathbf{curl}\ \mathbf{v}_h), \quad \forall\,\mathbf{u}_h, \mathbf{v}_h \in V_h(\Omega)\,,\\ (Bp_h, \mathbf{v}_h) &= (\nabla p_h, \beta\mathbf{v}_h), \quad \forall\, p_h \in Z_h(\Omega), \mathbf{v}_h \in V_h(\Omega).\end{aligned}$$

The dual operator $B^t : V_h(\Omega) \to Z_h(\Omega)$ of B can be defined by

$$(B^t\mathbf{u}_h, q_h) = (\beta\mathbf{u}_h, \nabla q_h), \quad \forall q_h \in Z_h(\Omega).$$

Let $\bar{\mathbf{f}}_h \in V_h(\Omega)$ and $g_h \in Z_h(\Omega)$ be the L^2-projections of $\mathbf{f}$ and g. Then, the system (2.2) can be written as

$$\begin{cases}(\bar{A} + \gamma_0\,\beta\, I)\mathbf{u}_h + Bp_h = \bar{\mathbf{f}}_h,\\ \quad B^t\mathbf{u}_h = g_h.\end{cases} \tag{3.1}$$

In recent years, there has been increasing interest in solving saddle-point problems like (3.1) by iterative methods; see, for example, [5], [6], [17], and [24]. But the most existing methods require the stiffness matrix corresponding to the primal variable $\mathbf{u}_h$ above to be nonsingular, so they cannot be applied to solve the saddle-point system (3.1) with $\gamma_0 = 0$, as the operator $\bar{A}$ is singular in the space $V_h(\Omega)$. To overcome this difficulty, we shall introduce another saddle-point system which has the same solution as problem (3.1) when $\gamma_0 = 0$, but which can be solved by existing preconditioned iterative methods.

3.1. **Augmented saddle-point system and Uzawa iterative methods.** Let $\hat{C} : Z_h(\Omega) \to Z_h(\Omega)$ be symmetric and positive definite and chosen as a preconditioner for the discrete Laplace operator on $Z_h(\Omega)$. Define

$$A = \begin{cases}\bar{A} + \gamma_0\,\beta\, I & \text{if } \gamma_0 \neq 0,\\ \bar{A} + r_0 B\hat{C}^{-1}B^t & \text{if } \gamma_0 = 0\,,\end{cases} \qquad \mathbf{f}_h = \begin{cases}\bar{\mathbf{f}}_h & \text{if } \gamma_0 \neq 0,\\ \bar{\mathbf{f}}_h + r_0 B\hat{C}^{-1}g_h & \text{if } \gamma_0 = 0\,,\end{cases}$$

where r_0 is some positive constant. One of the possible choices for r_0 is the average value of $c(x) = \alpha(x)/\beta(x)$.

Clearly, the system (3.1) has the same solution as the augmented saddle-point problem:

$$\begin{cases} A\mathbf{u}_h + Bp_h = \mathbf{f}_h, \\ B^t\mathbf{u}_h = g_h. \end{cases} \tag{3.2}$$

Let $\hat{A}$ be a preconditioner for A. Since the operator A is symmetric and positive definite, the system (3.2) can be solved by many existing iterative methods. Below is a recently developed Uzawa-type algorithm with variable relaxation parameters (see [17] and [18]):

Step 1. Choose a parameter ω_i and compute

$$\mathbf{u}_h^{i+1} = \mathbf{u}_h^i + \omega_i \hat{A}^{-1}[f_h - (A\mathbf{u}_h^i + Bp_h^i)].$$

Step 2. Choose a parameter τ_i and compute

$$p_h^{i+1} = p_h^i + \tau_i \hat{C}^{-1}(B^t\mathbf{u}_h^{i+1} - g_h).$$

Remark 3.1. Some choices of the parameters ω_i and τ_i are given in [17] and [18] to ensure the convergence of the algorithm. Note the fact that

$$f_h - A\mathbf{u}_h^i = \bar{f}_h - \bar{A}\mathbf{u}_h^i - r_0 B[\hat{C}^{-1}(B^t\mathbf{u}_h^i - g_h)]$$

in the case of $\gamma_0 = 0$, so the value $\hat{C}^{-1}(B^t\mathbf{u}_h^i - g_h)$ computed in Step 2 of the i-th iteration can be used in Step 1 of the $(i+1)$-th iteration. That is, the newly added term $r_0 B\hat{C}^{-1}B^t$ in the augmented saddle-point system (3.2) does not create any extra cost in the above Uzawa algorithm as the action of $\hat{C}^{-1}$ is needed only once at each iteration.

The convergence rate of the above Uzawa algorithm is completely determined by the condition numbers $\kappa(\hat{A}^{-1}A)$ and $\kappa(\hat{C}^{-1}B^t\hat{A}^{-1}B)$; see [17] and [18] for the detailed analyses. In the following we will construct an efficient preconditioner $\hat{A}$ which makes these two condition numbers to be nearly optimal.

3.2. **Construction of the preconditioner $\hat{A}$.** In the sequel, we shall frequently use the notation $\lesssim$ and $\eqsim$. For any two nonnegative quantities x and y, $x \lesssim y$ means that $x \le Cy$ for some constant C independent of mesh size h, subdomain size d and the related parameters; $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

The proofs of all results in this section will be given in Section 5.

Let $\tilde{A} : V_h(\Omega) \to V_h(\Omega)$ be an operator defined by

$$(\tilde{A}\mathbf{u}, \mathbf{v}) = (\alpha \mathbf{curl}\ \mathbf{u}, \mathbf{curl}\ \mathbf{v}) + (\alpha \mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V_h(\Omega)\,.$$

Theorem 3.1. *Let $G(\cdot) \ge 1$ be some given function, and let the operator $\hat{C}$ satisfy*

$$(\beta\nabla\phi, \nabla\phi) \lesssim (\hat{C}\phi, \phi) \lesssim G(d/h)(\beta\nabla\phi, \nabla\phi), \quad \forall \phi \in Z_h(\Omega)\,. \tag{3.3}$$

Then we have

$$\frac{1}{G(d/h)}(\tilde{A}\mathbf{v}_h, \mathbf{v}_h) \lesssim (A\mathbf{v}_h, \mathbf{v}_h) \lesssim (\tilde{A}\mathbf{v}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h(\Omega)\,.$$

With this theorem, it suffices to construct a preconditioner for $\tilde{A}$ instead of A.

We first define two subspaces of $V_h(\Omega)$:

$$V^p(\Omega) = \Big\{\mathbf{v} \in V_h(\Omega);\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\Big\} = \prod_{k=1}^{N} V_h^0(\Omega_k),$$

$$V^H(\Omega) = \Big\{\mathbf{v} \in V_h(\Omega);\ \mathbf{v} \text{ is the discrete } A_i\text{-extension of } \mathbf{v}|_{\partial\Omega_i} \text{ in each } \Omega_i\Big\}.$$

Obviously, $V_h(\Omega)$ has the orthogonal decomposition with respect to the inner product $(\tilde{A}\cdot, \cdot)$:

$$V_h(\Omega) = V^p(\Omega) \oplus V^H(\Omega). \tag{3.4}$$

Furthermore, we define two subspaces of $V^H(\Omega)$:

$$V^{ij}(\Omega) = \Big\{\mathbf{v} \in V^H(\Omega);\ \mathrm{supp}(\mathbf{v}) \subset \Omega_{ij} = \Omega_i \cup \Omega_j \cup \Gamma_{ij}\Big\},$$

$$V^0(\Omega) = \Big\{\mathbf{v} \in V^H(\Omega);\ \lambda_e(\mathbf{v}) = 0 \text{ for each } e \in \mathrm{F}_\partial \text{ with } \mathrm{F} \subset \Gamma\Big\}.$$

The subspace $V^0(\Omega)$ is called the *coarse* subspace. The introduction of such a coarse subspace is based on the following consideration: for any $\mathbf{v}_h \in V_h(\Omega)$, its tangential components are continuous on all *cross-edges*, namely, the edges which are shared by more than two fine elements (in the two-dimensional case, the tangential components have no definitions at the *cross-points*), but the moments on the *cross-edges* are not sufficient to determine the values of the tangential trace $\mathbf{v}_h \times \mathbf{n}$ on these edges.

It is easy to see that the space $V_h(\Omega)$ has the (nondirect sum) decomposition

$$V_h(\Omega) = V^p(\Omega) \oplus (V^0(\Omega) + \sum_{\Gamma_{ij}} V^{ij}(\Omega)). \tag{3.5}$$

Next, we define the corresponding solvers on the subspaces $V^p(\Omega)$, $V^0(\Omega)$ and $V^{ij}(\Omega)$.

Let $\hat{A}_p : V^p(\Omega) \to V^p(\Omega)$ and $\hat{A}_{ij} : V^{ij}(\Omega) \to V^{ij}(\Omega)$ be symmetric and positive definite operators such that

$$(\hat{A}_p\mathbf{v}, \mathbf{v}) \overset{=}{\sim} \sum_{k=1}^{N} (A_k\mathbf{v}_k, \mathbf{v}_k)_{\Omega_k}, \quad \forall \mathbf{v} \in V^p(\Omega),$$

where $\mathbf{v}_k = \mathbf{v}|_{\Omega_k}$ for $k = 1, 2, \cdots, N$ and

$$(\hat{A}_{ij}\mathbf{v}, \mathbf{v}) \overset{=}{\sim} (A_i\mathbf{v}_i, \mathbf{v}_i)_{\Omega_i} + (A_j\mathbf{v}_j, \mathbf{v}_j)_{\Omega_j}, \quad \forall \mathbf{v} \in V^{ij}(\Omega).$$

The global coarse solvers should be solvable in an efficient way on $V^0(\Omega)$, and their constructions are much more tricky and technical than the local solvers. To do so, we introduce the so-called tangential divergence $\mathrm{div}_\tau\, \Phi$ of any $\Phi \in V_h(\Gamma_i)$, as done in [1] and [2].

For ease of notation, we assume that $\alpha(x) = \alpha_i$ for $x \in \Omega_i$, with α_i being constants.

Then we define the coarse solver $\hat{A}_0 : V^0(\Omega) \to V^0(\Omega)$ as follows:

$$(\hat{A}_0\mathbf{v}, \mathbf{w}) = h[1 + \log(d/h)] \times \sum_{i=1}^{N} \alpha_i \Big\{\langle \mathrm{div}_\tau(\mathbf{v} \times \mathbf{n})|_{\Gamma_i}, \mathrm{div}_\tau(\mathbf{w} \times \mathbf{n})|_{\Gamma_i}\rangle_{\Delta_i} + \langle \mathbf{v} \times \mathbf{n}, \mathbf{w} \times \mathbf{n}\rangle_{\Delta_i}\Big\}.$$

Let $Q_p : V^p(\Omega) \to V^p(\Omega)$, $Q_0 : V_h(\Omega) \to V^0(\Omega)$ and $Q_{ij} : V_h(\Omega) \to V^{ij}(\Omega)$ denote the L^2-projections. The preconditioner for $\hat{A}$ can be defined as follows:

$$\hat{A}^{-1} = \hat{A}_p^{-1} Q_p + \hat{A}_0^{-1} Q_0 + \sum_{\Gamma_{ij}} \hat{A}_{ij}^{-1} Q_{ij}. \tag{3.6}$$

For this preconditioner, we have

Theorem 3.2. *The condition number of the preconditioned system can be estimated by*

$$\operatorname{cond}(\hat{A}^{-1} A) \lesssim G(d/h)[1 + \log(d/h)]^2. \tag{3.7}$$

Remark 3.2. It is known that $\operatorname{div}_\tau(L_e \times \mathbf{n})|_{\Gamma_i}$ vanishes for any interior edge e of Ω_i. Moreover, we have $\operatorname{div}_\tau(L_e \times \mathbf{n})|_{\Gamma_i} = (\mathbf{curl}\ L_e) \cdot \mathbf{n}|_{\Gamma_i}$ for each $e \subset \Gamma_i$. Thus, the entries of the stiffness matrix of $\hat{A}_0$ are of the form:

$$h[1 + \log(d/h)] \sum_{i=1}^{N} \alpha_i \Big\{ \langle (\mathbf{curl}\ L_e) \cdot \mathbf{n}, (\mathbf{curl}\ L_{e'}) \cdot \mathbf{n} \rangle_{\Delta_i} + \langle L_e \times \mathbf{n}, L_{e'} \times \mathbf{n} \rangle_{\Delta_i} \Big\},$$
$$e,\ e' \in \bigcup_{\mathrm{F} \subset \Gamma} \mathrm{F}_b.$$

The coarse solver $\hat{A}_0$ involves computations only on Δ_i, a very small fraction of the interface Γ. Also, $\hat{A}_0$ is rather simple in comparison with coarse solvers in many existing substructuring preconditioners for standard elliptic problems, where some optimizations are involved [4], [32]. The preconditioner $\hat{A}$ can be implemented as in [4] and [32].

For the preconditioned Schur complement, we have

Theorem 3.3. *The condition number of the preconditioned Schur complement system can be estimated by*

$$\operatorname{cond}(\hat{C}^{-1} B^t \hat{A}^{-1} B) \lesssim G(d/h)[1 + \log(d/h)]^2. \tag{3.8}$$

Remark 3.3. When $\hat{C}$ is chosen as the usual multigrid preconditioner, we have $G(d/h) = 1$; when $\hat{C}$ is chosen as the substructuring preconditioner (see [4], [32]), we have $G(d/h) = [1 + \log(d/h)]^2$.

4. Some auxiliary lemmata

As we shall see, the proof of Theorem 3.2, namely, the estimate of the condition number for the preconditioned system, is very technical. This section presents some basic properties of Sobolev spaces and some auxiliary lemmata, which will be used to deal with the technical difficulties in the proof of Theorem 3.2.

4.1. **Helmholz decomposition and edge element interpolation.** Denote by $H(\mathbf{curl}; \Omega_i)$ the restriction of $H_0(\mathbf{curl}; \Omega)$ on the subdomain Ω_i. It is known that the spaces $H_0(\mathbf{curl}; \Omega)$ and $H(\mathbf{curl}; \Omega_i)$ can be decomposed into (see [12])

$$H_0(\mathbf{curl}; \Omega) = \nabla H_0^1(\Omega) \oplus H_0^{\perp}(\mathbf{curl}; \Omega)\,, \tag{4.1}$$

$$H(\mathbf{curl}; \Omega_i) = \nabla H^1(\Omega_i) \oplus H^{\perp}(\mathbf{curl}; \Omega_i), \tag{4.2}$$

where

$$H_0^{\perp}(\mathbf{curl}; \Omega) = \{\mathbf{v} \in H_0(\mathbf{curl}; \Omega) : \operatorname{div} \mathbf{v} = 0\},$$

$$H^{\perp}(\mathbf{curl};\Omega_i) \quad = \quad \{\mathbf{v} \in H(\mathbf{curl};\Omega_i) : \operatorname{div} \mathbf{v} = 0, \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_i\}.$$

It follows from (Theorem 4.3, [2]) that $H_0^{\perp}(\mathbf{curl};\Omega) \subset H^{\delta}(\Omega)^3$ for some $1/2 < \delta < 1$, and

$$\|\mathbf{v}\|_{\delta,\Omega} \lesssim \|\mathbf{curl}\ \mathbf{v}\|_{0,\Omega}, \quad \forall \mathbf{v} \in \ H_0^{\perp}(\mathbf{curl};\Omega). \tag{4.3}$$

Since Ω_i is a convex polyhedron, $H^{\perp}(\mathbf{curl};\Omega_i) \subset H^1(\Omega_i)^3$ (see [12]). Moreover, we have

$$d^{-2}\|\mathbf{v}\|_{0,\Omega_i} + |\mathbf{v}|_{1,\Omega_i} \lesssim \|\mathbf{curl}\ \mathbf{v}\|_{0,\Omega_i}, \quad \forall \mathbf{v} \in \ H^{\perp}(\mathbf{curl};\Omega_i). \tag{4.4}$$

Here and in what follows, for any given domain D and each integer $m \geq 0$ we use $H^m(D)$ to denote the standard Sobolev space of real functions with their weak derivatives of order up to m in the Lebesgue space $L^2(D)$, and we use $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ to denote its norm and semi-norm. For a fractional number s, the Sobolev space $H^s(\Omega)$ is defined by the standard interpolation theory.

For a $\mathbf{v}_h \in V_h(\Omega)$, using (4.1) and (4.3) we can decompose $\mathbf{v}_h$ as follows:

$$\mathbf{v}_h = \nabla p \oplus \mathbf{w} \quad \text{in } \Omega \tag{4.5}$$

where $\mathbf{w} \in H_0^{\perp}(\mathbf{curl};\Omega) \cap H^{\delta}(\Omega)^3$ for some $\delta > \frac{1}{2}$ and $p \in H_0^1(\Omega)$ solves

$$(\nabla p, \nabla q) = (\mathbf{v}_h, \nabla q), \quad \forall q \in H_0^1(\Omega)\,.$$

Similarly, for a $\mathbf{v}_h \in V_h(\Omega_i)$, it follows from (4.2) and (4.4) that

$$\mathbf{v}_h = \nabla p \oplus \mathbf{w} \quad \text{in } \Omega_i \tag{4.6}$$

where $\mathbf{w} \in H^{\perp}(\mathbf{curl};\Omega_i) \cap H^1(\Omega_i)^3$, and $p \in H^1(\Omega_i)$ solves

$$(\nabla p, \nabla q)_{\Omega_i} = (\mathbf{v}_h, \nabla q)_{\Omega_i}, \quad \forall q \in H^1(\Omega_i).$$

Next, we present some interpolation results related to the finite element space $V_h(\Omega)$. We know from [3] (Lemma 4.7) that for any $\mathbf{v} \in H^{\delta}(\Omega)$ with $\mathbf{curl}\,\mathbf{v} \in L^p(\Omega)^3$ ($\delta > \frac{1}{2}$ and $p > 2$) we can define its interpolant $\mathbf{r}_h\,\mathbf{v}$ in $V_h(\Omega)$ by the relation

$$\lambda_e(\mathbf{r}_h\mathbf{v}) = \int_e \mathbf{v}\cdot\mathbf{t}_e ds, \quad \ \forall\ e \in \mathcal{E}_h\,.$$

Lemma 4.1 can be found in [9]:

Lemma 4.1. *Assume that* $\mathbf{w}$ *and* $\boldsymbol{curl}\mathbf{w}$ *are both in* $H^{\delta}(\Omega_i)$ *for some* $\delta > \frac{1}{2}$. *Then,*

$$\|\mathbf{r}_h\mathbf{w} - \mathbf{w}\|_{0,\Omega_i} + \|\boldsymbol{curl}(\mathbf{r}_h\mathbf{w} - \mathbf{w})\|_{0,\Omega_i} \lesssim h^{\delta}(\|\mathbf{w}\|_{\delta,\Omega_i} + \|\boldsymbol{curl}\ \mathbf{w}\|_{\delta,\Omega_i}). \tag{4.7}$$

Lemma 4.2. *For any* $\mathbf{v}_h \in V_h(\Omega)$ *which satisfies*

$$(\beta\mathbf{v}_h, \nabla q_h) = 0, \quad \forall q_h \in Z_h(\Omega)\,, \tag{4.8}$$

we have

$$\|\beta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega} \lesssim \|\beta^{\frac{1}{2}}\boldsymbol{curl}\ \mathbf{v}_h\|_{0,\Omega}. \tag{4.9}$$

Proof. Let $\mathbf{w} \in H_0^{\perp}(\mathbf{curl};\Omega)$ be defined by (4.5). Then, we have (see [12])

$$\mathbf{v}_h = \mathbf{r}_h\nabla p + \mathbf{r}_h\mathbf{w} = \nabla p_h + \mathbf{r}_h\mathbf{w}, \tag{4.10}$$

with $p_h \in Z_h(\Omega)$. Since $(\beta\mathbf{v}_h, \nabla p_h) = 0$, we infer from (4.10) that

$$\|\beta^{\frac{1}{2}}\nabla p_h\|_{0,\Omega} \leq \|\beta^{\frac{1}{2}}\mathbf{r}_h\mathbf{w}\|_{0,\Omega}.$$

This, together with (4.10), leads to

$$\begin{aligned} \|\beta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega} &\leq \|\beta^{\frac{1}{2}}\nabla p_h\|_{0,\Omega} + \|\beta^{\frac{1}{2}}\mathbf{r}_h\mathbf{w}\|_{0,\Omega} \\ &\lesssim \|\beta^{\frac{1}{2}}\mathbf{r}_h\mathbf{w}\|_{0,\Omega} \leq \|\beta^{\frac{1}{2}}\mathbf{w}\|_{0,\Omega} + \|\beta^{\frac{1}{2}}(\mathbf{r}_h\mathbf{w}-\mathbf{w})\|_{0,\Omega}. \end{aligned} \tag{4.11}$$

It follows from (4.3) that

$$\|\beta^{\frac{1}{2}}\mathbf{w}\|_{0,\Omega} \lesssim \|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{w}\|_{0,\Omega} = \|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}. \tag{4.12}$$

On the other hand, following the proof of Lemma 3.2 in [9], we derive

$$\begin{aligned} \|\beta^{\frac{1}{2}}(\mathbf{r}_h\mathbf{w}-\mathbf{w})\|_{0,K} &\lesssim h^{\delta}(\|\beta^{\frac{1}{2}}\mathbf{w}\|_{H^{\delta}(K)} + \|\beta^{\frac{1}{2}}\mathbf{curl}\,\mathbf{w}\|_{H^{\delta}(K)}) \\ &\lesssim \|\beta^{\frac{1}{2}}\mathbf{curl}\mathbf{v}_h\|_{0,K}, \qquad \forall\, K\in\mathcal{T}_h, \end{aligned}$$

where we have also used (4.3) and the inverse inequality. This with (4.11) gives the desired result. □

Lemma 4.3. *For any $\mathbf{v}_h \in V_h(\Omega_i)$, let $\mathbf{w}$ be defined as in* (4.6)*. Then*

$$\|\mathbf{r}_h\mathbf{w}-\mathbf{w}\|_{0,\Omega_i} \lesssim h\|\boldsymbol{curl}\mathbf{v}_h\|_{0,\Omega_i}. \tag{4.13}$$

Proof. For any element $K\in\mathcal{T}_h$, let $\mathbf{x} = F_K\hat{\mathbf{x}} = B_K\hat{\mathbf{x}} + b_K$ be the affine mapping between K and the reference element $\hat{K}$. In $\hat{K}$, define

$$\hat{\mathbf{w}} = B_K^t\mathbf{w}\circ F_K, \quad \hat{\mathbf{v}}_h = B_K^t\mathbf{v}_h\circ F_K.$$

Let $\hat{\mathbf{r}}_h$ be the interpolant on the reference element $\hat{K}$. One can show that (cf. [9])

$$\|\hat{\mathbf{r}}_h\hat{\mathbf{w}}-\hat{\mathbf{w}}\|_{0,\hat{K}} \lesssim (\|\hat{\mathbf{w}}\|^2_{\delta,\hat{K}} + \|\widehat{\mathbf{curl}}\,\hat{\mathbf{w}}\|^2_{\delta,\hat{K}})^{\frac{1}{2}}. \tag{4.14}$$

Since $\widehat{\mathbf{curl}}\,\hat{\mathbf{w}} = \widehat{\mathbf{curl}}\,\hat{\mathbf{v}}_h$, we have

$$\|\hat{\mathbf{r}}_h\hat{\mathbf{w}}-\hat{\mathbf{w}}\|_{0,\hat{K}} \lesssim (\|\hat{\mathbf{w}}\|^2_{1,\hat{K}} + \|\widehat{\mathbf{curl}}\,\hat{\mathbf{v}}_h\|^2_{0,\hat{K}})^{\frac{1}{2}}. \tag{4.15}$$

As the interpolation operator $\hat{\mathbf{r}}_h$ preserves constants, it follows by (4.15) that

$$\|\hat{\mathbf{r}}_h\hat{\mathbf{w}}-\hat{\mathbf{w}}\|_{0,\hat{K}} \lesssim (|\hat{\mathbf{w}}|^2_{1,\hat{K}} + \|\widehat{\mathbf{curl}}\,\hat{\mathbf{v}}_h\|^2_{0,\hat{K}})^{\frac{1}{2}}.$$

Now, by the standard scaling technique we obtain

$$\begin{aligned} \|\mathbf{r}_h\mathbf{w}-\mathbf{w}\|^2_{0,K} &\lesssim h\|\hat{\mathbf{r}}_h\hat{\mathbf{w}}-\hat{\mathbf{w}}\|^2_{0,\hat{K}} \\ &\lesssim h(|\hat{\mathbf{w}}|^2_{1,\hat{K}} + \|\widehat{\mathbf{curl}}\,\hat{\mathbf{v}}_h\|^2_{0,\hat{K}}) \\ &\lesssim h^2(|\mathbf{w}|^2_{1,K} + \|\mathbf{curl}\,\mathbf{v}_h\|^2_{0,K}), \end{aligned}$$

which, together with (4.4), immediately gives the desired result. □

Lemma 4.4. *For any $\mathbf{v}_h\in V_h(\Omega_i)$, let $\mathbf{w}$ be defined by* (4.6) *and let $p_h\in Z_h(\Omega_i)$ be defined by*

$$\mathbf{v}_h = \mathbf{r}_h(\nabla p\oplus\mathbf{w}) = \nabla p_h + \mathbf{r}_h\mathbf{w} \quad \text{in } \Omega_i. \tag{4.16}$$

Then we have

$$|p_h|_{1,\Omega_i} \lesssim \|\mathbf{v}_h\|_{0,\Omega_i} + \max\{h, d^2\}\,\|\boldsymbol{curl}\ \mathbf{v}_h\|_{0,\Omega_i}. \tag{4.17}$$

Proof. We know from (4.15) that for any $K \in \mathcal{T}_h$,

$$\|\hat{\mathbf{r}}_h \hat{\mathbf{w}}\|^2_{0,\hat{K}} \lesssim \|\hat{\mathbf{w}}\|^2_{1,\hat{K}} + \|\widehat{\mathbf{curl}}\, \hat{\mathbf{v}}_h\|^2_{0,\hat{K}}.$$

Therefore, we have

$$\|\mathbf{r}_h \mathbf{w}\|^2_{0,K} \lesssim h\, \|\hat{\mathbf{r}}_h \hat{\mathbf{w}}\|^2_{0,\hat{K}} \lesssim h^2\, \|\mathbf{curl}\ \mathbf{v}_h\|^2_{0,K} + h^2\, |\mathbf{w}|^2_{1,K} + \|\mathbf{w}\|^2_{0,K}\,,$$

which with (4.4) leads to

$$\begin{aligned} \|\mathbf{r}_h \mathbf{w}\|^2_{0,\Omega_i} &\lesssim h^2\, \|\mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega_i} + h^2\, |\mathbf{w}|^2_{1,\Omega_i} + \|\mathbf{w}\|^2_{0,\Omega_i} \\ &\lesssim \max\{h^2, d^4\}\, \|\mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega_i}. \end{aligned}$$

The desired estimate now follows from this and the triangle inequality

$$\|\nabla p_h\|_{0,\Omega_i} \le \|\mathbf{v}_h\|_{0,\Omega_i} + \|\mathbf{r}_h \mathbf{w}\|_{0,\Omega_i}\,. \qquad \square$$

4.2. **Some scaled norms and their estimates.** A large part of the condition number estimates in Section 5 will be carried out on the subdomains, for which we need some scaled norms. For the space $H^1(\Omega_i)^3$, we define a scaled norm by

$$\|\mathbf{v}\|_{1,\Omega_i} = \left\{ |\mathbf{v}|^2_{1,\Omega_i} + d^{-2}\|\mathbf{v}\|^2_{0,\Omega_i} \right\}^{\frac{1}{2}}, \quad \forall\, \mathbf{v} \in H^1(\Omega_i)^3,$$

while for the space $H(\mathbf{curl};\Omega_i)$, we define its scaled norm by

$$\|\mathbf{u}\|_{\mathbf{curl};\Omega_i} = \left\{ \|\mathbf{curl}\ \mathbf{u}\|^2_{0,\Omega_i} + d^{-2}\|\mathbf{u}\|^2_{0,\Omega_i} \right\}^{\frac{1}{2}}$$

and for each $\lambda \in H^{-\frac{1}{2}}(\Gamma_i)$, we define

$$\|\lambda\|_{-\frac{1}{2},\Gamma_i} = \sup_{v \in H^{\frac{1}{2}}(\Gamma_i)} \frac{\langle \lambda, v\rangle_{\Gamma_i}}{\|v\|_{\frac{1}{2},\Gamma_i}}$$

where

$$\|v\|_{\frac{1}{2},\Gamma_i} = (|v|^2_{\frac{1}{2},\Gamma_i} + d^{-1}\|v\|^2_{0,\Gamma_i})^{\frac{1}{2}}.$$

The same notation will be used for the norms in the space $H^{-\frac{1}{2}}(\Gamma_i)^3$.

For any $\Phi \in V_h(\Gamma_i)$, recall that $\mathrm{div}_\tau\, \Phi$ is the tangential divergence of Φ. We know that $\mathrm{div}_\tau\, \Phi \in H^{-\frac{1}{2}}(\Gamma_i)$ (cf. [1], [2]). This leads to the following important norm:

$$\|\Phi\|_{\mathcal{X}_{\Gamma_i}} = d^{-1}\|\Phi\|_{-\frac{1}{2},\Gamma_i} + \|\,\mathrm{div}_\tau\, \Phi\|_{-\frac{1}{2},\Gamma_i}.$$

The following two results about the norm $\|\cdot\|_{\mathcal{X}_{\Gamma_i}}$ can be found in [1], [2] (using the standard dilation from the reference domain):

Lemma 4.5. *Let $\mathbf{u} \in V_h(\Omega_i)$, which satisfies $\mathbf{u} \times \mathbf{n} = \Phi$ on Γ_i. Then*

$$\|\Phi\|_{\mathcal{X}_{\Gamma_i}} \lesssim \|\mathbf{u}\|_{\boldsymbol{curl};\Omega_i}. \tag{4.18}$$

Lemma 4.6. *The discrete A_i-extension $\mathbf{R}^i_h\Phi \in V_h(\Omega_i)$ satisfies*

$$\|\mathbf{R}^i_h\Phi\|_{0,\Omega_i} + \|\boldsymbol{curl}(\mathbf{R}^i_h\Phi)\|_{0,\Omega_i} \lesssim \|\Phi\|_{\mathcal{X}_{\Gamma_i}}. \tag{4.19}$$

The following results can be found in [10], while the factors d^α are derived by the standard scaling argument:

Lemma 4.7. *For any* $\mathbf{v} \in H(\boldsymbol{curl};\Omega_i) \cap H(\mathrm{div};\Omega_i)$*, if* $\mathbf{v}\times\mathbf{n}$ *or* $\mathbf{v}\cdot\mathbf{n}$ *is in* $L^2(\Gamma_i)$*, then*

$$|\mathbf{v}|_{\frac{1}{2},\Omega_i} \lesssim d^{-\frac{1}{2}}\|\mathbf{v}\|_{0,\Omega_i} + d^{\frac{1}{2}}\|\operatorname{div}\ \mathbf{v}\|_{0,\Omega_i} + d^{\frac{1}{2}}\|\boldsymbol{curl}\,\mathbf{v}\|_{0,\Omega_i} + \|\mathbf{v}\cdot\mathbf{n}\|_{0,\Gamma_i}\,, \tag{4.20}$$

$$\|\mathbf{v}\cdot\mathbf{n}\|_{0,\Gamma_i} \lesssim d^{-\frac{1}{2}}\|\mathbf{v}\|_{0,\Omega_i} + d^{\frac{1}{2}}\|\operatorname{div}\ \mathbf{v}\|_{0,\Omega_i} + d^{\frac{1}{2}}\|\boldsymbol{curl}\,\mathbf{v}\|_{0,\Omega_i} + \|\mathbf{v}\times\mathbf{n}\|_{0,\Gamma_i}. \tag{4.21}$$

For any $\Phi \in V_h(\Gamma_i)$, let $\omega(\Phi) \in H^1(\Omega_i)$ be the weak solution of the problem

$$\begin{cases} \triangle\omega(\Phi) = 0 & \text{in } \Omega_i, \\ \frac{\partial\omega(\Phi)}{\partial\mathbf{n}} = -\operatorname{div}_\tau \Phi & \text{on } \Gamma_i, \\ \int_{\Omega_i}\omega(\Phi)dx = 0. \end{cases} \tag{4.22}$$

Consider the equations

$$\begin{cases} \mathbf{curl}\ \mathbf{curl}\ \mathbf{q}(\Phi) - \nabla\operatorname{div}\ \mathbf{q}(\Phi) = \nabla\omega(\Phi) & \text{in } \Omega_i, \\ \mathbf{n}\times\mathbf{curl}\ \mathbf{q}(\Phi) = \Phi & \text{on } \Gamma_i, \\ \mathbf{q}(\Phi)\cdot\mathbf{n} = 0 & \text{on } \Gamma_i. \end{cases} \tag{4.23}$$

Lemma 4.8. *Let* $\mathbf{q}(\Phi) \in H(\boldsymbol{curl};\Omega_i)$ *be the solution of the equation* (4.23)*, and let* $\tilde{\mathbf{w}}(\Phi) = \boldsymbol{curl}\,\mathbf{q}(\Phi)$*. Then we have* (1) $\mathbf{n}\times\tilde{\mathbf{w}}(\Phi) = \Phi$ *on* Γ_i*;* (2) $\tilde{\mathbf{w}}(\Phi), \boldsymbol{curl}\,\tilde{\mathbf{w}} \in H^{\frac{1}{2}+\delta}(\Omega_i)$ *for some* $\delta \in [0,\frac{1}{2})$*;* (3) $\boldsymbol{curl}\,\tilde{\mathbf{w}}(\Phi)\cdot\mathbf{n} = -\operatorname{div}_\tau\Phi$ *on* Γ_i*;* (4) $\boldsymbol{curl}\ \boldsymbol{curl}\,\tilde{\mathbf{w}}(\Phi) = 0$*;* (5) $\operatorname{div}\tilde{\mathbf{w}}(\Phi) = 0$*; and*

$$\|\tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i} \lesssim \|\Phi\|_{-\frac{1}{2},\Gamma_i} + d\|\operatorname{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i}\,, \tag{4.24}$$

$$\|\boldsymbol{curl}\,\tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i} \lesssim \|\operatorname{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i}. \tag{4.25}$$

Proof. The inferences (1) and (5) are obvious. It is shown in [1] that $\operatorname{div}\mathbf{q}(\Phi) = 0$ and

$$\|\nabla\omega(\Phi)\|_{0,\Omega_i} \lesssim \|\operatorname{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i},$$

and so by (4.22) and (4.23) we obtain (3), (4) and (4.25). Since $\Phi|_{\Gamma_i} \in H^\delta(\Gamma_i)^3$ and $\operatorname{div}_\tau(\Phi|_{\Gamma_i}) \in H^\delta(\Gamma_i)$ for any $\delta \in [0,\frac{1}{2})$, the inference (2) follows by (1), (3) and Theorem 4.4 in [2]. Finally for (4.24), using (4.23) and Green's formulas, we have (note that $\operatorname{div}\mathbf{q}(\Phi) = 0$)

$$\|\mathbf{curl}\ \mathbf{q}(\Phi)\|^2_{0,\Omega_i} = (\nabla\omega(\Phi), \mathbf{q}(\Phi))_{\Omega_i} - \langle\Phi, \mathbf{q}(\Phi)\rangle_{\Gamma_i}.$$

Now (4.24) follows from the Cauchy-Schwarz inequality and (4.4). □

Lemma 4.9 plays a key role in the construction of our coarse solver.

Lemma 4.9. *For any* $\Phi \in V_h(\Gamma_i)$ *satisfying*

$$\|\Phi\|_{-\frac{1}{2},\Gamma_i} + \|\operatorname{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i} \lesssim h^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}(\|\Phi\|_{0,\Gamma_i} + \|\operatorname{div}_\tau\Phi\|_{0,\Gamma_i})\,, \tag{4.26}$$

we have

$$\|\mathbf{R}^i_h\Phi\|_{0,\Omega_i} + \|\boldsymbol{curl}\,\mathbf{R}^i_h\Phi\|_{0,\Omega_i} \lesssim h^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}(\|\Phi\|_{0,\Gamma_i} + \|\operatorname{div}_\tau\Phi\|_{0,\Gamma_i}). \tag{4.27}$$

Proof. Let $\tilde{\mathbf{w}}(\Phi) \in H^\delta(\Omega_i)$, also with $\mathbf{curl}\,\tilde{\mathbf{w}} \in H^\delta(\Omega_i)$ for some $\delta > \frac{1}{2}$, be defined as in Lemma 4.8. It follows from Lemma 4.8(1) that $\mathbf{r}_h\tilde{\mathbf{w}}(\Phi)\times\mathbf{n} = \Phi$ on Γ_i. By the minimum energy property of the discrete A_i-extension, we have

$$\|\mathbf{R}^i_h\Phi\|_{0,\Omega_i} + \|\mathbf{curl}(\mathbf{R}^i_h\Phi)\|_{0,\Omega_i} \le \|\mathbf{r}_h\tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i} + \|\mathbf{curl}(\mathbf{r}_h\tilde{\mathbf{w}}(\Phi))\|_{0,\Omega_i}.$$

It suffices to prove
(4.28)
$$\|\mathbf{r}_h\tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}(\mathbf{r}_h\tilde{\mathbf{w}}(\Phi))\|_{0,\Omega_i}\lesssim h^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}(\|\Phi\|_{0,\Gamma_i}+\|\operatorname{div}_\tau\Phi\|_{0,\Gamma_i}).$$

By (4.2), $\tilde{\mathbf{w}}(\Phi)$ can be decomposed as follows:
$$\tilde{\mathbf{w}}(\Phi)=\nabla p(\Phi)\oplus\mathbf{w}(\Phi),\quad \mathbf{w}(\Phi)\in H^{\perp}(\mathbf{curl};\Omega_i) \tag{4.29}$$
with $p(\Phi)$ satisfying the equations (cf. Lemma 4.8(5))
$$\begin{cases}\triangle p(\Phi)=\operatorname{div}\ \tilde{\mathbf{w}}(\Phi)=0 & \text{in } \Omega_i,\\ \frac{\partial p(\Phi)}{\partial\mathbf{n}}=\tilde{\mathbf{w}}(\Phi)\cdot\mathbf{n} & \text{on } \Gamma_i,\\ \int_{\Omega_i}p(\Phi)dx=0.\end{cases}$$
We have by Lemma 4.1 that
$$\|\mathbf{r}_h\mathbf{w}(\Phi)-\mathbf{w}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}(\mathbf{r}_h\mathbf{w}(\Phi)-\mathbf{w}(\Phi))\|_{0,\Omega_i}\lesssim h^{\frac{1}{2}}\|\mathbf{curl}\ \mathbf{w}(\Phi)\|_{\frac{1}{2},\Omega_i}.$$
This implies
$$\begin{aligned}&\|\mathbf{r}_h\mathbf{w}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}(\mathbf{r}_h\mathbf{w}(\Phi))\|_{0,\Omega_i}\\ \lesssim\ &\|\mathbf{w}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}\ \mathbf{w}(\Phi)\|_{0,\Omega_i}+h^{\frac{1}{2}}\|\mathbf{curl}\ \mathbf{w}(\Phi)\|_{\frac{1}{2},\Omega_i}.\end{aligned} \tag{4.30}$$
Using (4.29), we see that
$$\|\mathbf{w}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}\ \mathbf{w}(\Phi)\|_{0,\Omega_i}\leq\|\tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}\ \tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i},$$
which, together with (4.25) and (4.24), yields
$$\|\mathbf{w}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}\ \mathbf{w}(\Phi)\|_{0,\Omega_i}\lesssim\|\Phi\|_{-\frac{1}{2},\Gamma_i}+\|\operatorname{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i}. \tag{4.31}$$
By (4.20), (4.29) and Lemma 4.8[(4),(3)], we derive
$$\begin{aligned}\|\mathbf{curl}\ \mathbf{w}(\Phi)\|_{\frac{1}{2},\Omega_i} &= \|\mathbf{curl}\ \tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i}+|\mathbf{curl}\ \tilde{\mathbf{w}}(\Phi)|_{\frac{1}{2},\Omega_i}\\ &\lesssim d^{-\frac{1}{2}}\|\mathbf{curl}\ \tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}\ \tilde{\mathbf{w}}(\Phi)\cdot\mathbf{n}\|_{0,\Gamma_i}\\ &\lesssim d^{-\frac{1}{2}}\|\operatorname{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i}+\|\operatorname{div}_\tau\Phi\|_{0,\Gamma_i}.\end{aligned}$$
Substituting this inequality and (4.31) into (4.30), gives
$$\|\mathbf{r}_h\mathbf{w}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}(\mathbf{r}_h\mathbf{w}(\Phi))\|_{0,\Omega_i}\lesssim\|\Phi\|_{-\frac{1}{2},\Gamma_i}+\|\operatorname{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i}+h^{\frac{1}{2}}\|\operatorname{div}_\tau\Phi\|_{0,\Gamma_i}.$$
This, together with (4.26), leads to
(4.32)
$$\|\mathbf{r}_h\mathbf{w}(\Phi)\|_{0,\Omega_i}+\|\mathbf{curl}(\mathbf{r}_h\mathbf{w}(\Phi))\|_{0,\Omega_i}\lesssim h^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}(\|\Phi\|_{0,\Gamma_i}+\|\operatorname{div}_\tau\Phi\|_{0,\Gamma_i}).$$

On the other hand, it follows by Lemma 4.8(2) that $(\tilde{\mathbf{w}}(\Phi)\cdot\mathbf{n})|_{\Gamma_i}\in H^{\delta}(\Gamma_i)$. Thus, from the definition of $p(\Phi)$ we infer that $p(\Phi)\in H^{\frac{3}{2}+\delta}(\Omega_i)$ and $\nabla p(\Phi)\in H^{\frac{1}{2}+\delta}(\mathbf{curl};\Omega_i)$. Let $\pi_h: C(\Omega_i)\to Z_h(\Omega_i)$ be the nodal interpolation operator associated with $\mathcal{T}_h$. Then, $\mathbf{r}_h(\nabla p(\Phi))=\nabla(\pi_h p(\Phi))$, and so
$$\begin{aligned}&\|\mathbf{r}_h(\nabla p(\Phi))-\nabla p(\Phi)\|_{0,\Omega_i}\lesssim h^{\frac{1}{2}}\|p(\Phi)\|_{\frac{3}{2},\Omega_i}\\ \lesssim\ &h^{\frac{1}{2}}(\|\triangle p(\Phi)\|_{0,\Omega_i}+\|\frac{\partial p(\Phi)}{\partial\mathbf{n}}\|_{0,\Gamma_i})=h^{\frac{1}{2}}\|\tilde{\mathbf{w}}(\Phi)\cdot\mathbf{n}\|_{0,\Gamma_i}.\end{aligned}$$
Hence,
$$\|\mathbf{r}_h(\nabla p(\Phi))\|_{0,\Omega_i}\lesssim\|\nabla p(\Phi)\|_{0,\Omega_i}+h^{\frac{1}{2}}\|\tilde{\mathbf{w}}(\Phi)\cdot\mathbf{n}\|_{0,\Gamma_i}. \tag{4.33}$$

By (4.29) and (4.24), we have

$$\|\nabla p(\Phi)\|_{0,\Omega_i} \leq \|\tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i} \lesssim \|\Phi\|_{-\frac{1}{2},\Gamma_i} + \|\operatorname{div}_\tau \Phi\|_{-\frac{1}{2},\Gamma_i}. \tag{4.34}$$

Then it follows by (4.21), Lemma 4.8[(5),(1)] and (4.24)-(4.25) that

$$\begin{aligned} \|\tilde{\mathbf{w}}(\Phi)\cdot\mathbf{n}\|_{0,\Gamma_i} &\lesssim d^{-\frac{1}{2}}\|\tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i} + d^{\frac{1}{2}}\|\mathbf{curl}\ \tilde{\mathbf{w}}(\Phi)\|_{0,\Omega_i} + \|\tilde{\mathbf{w}}(\Phi)\times\mathbf{n}\|_{0,\Gamma_i} \\ &\lesssim d^{-\frac{1}{2}}(\|\Phi\|_{-\frac{1}{2},\Gamma_i} + \|\operatorname{div}_\tau \Phi\|_{-\frac{1}{2},\Gamma_i}) + \|\Phi\|_{0,\Gamma_i}. \end{aligned}$$

Plugging this and (4.34) into (4.33) and using (4.26) yield

$$\|\mathbf{r}_h(\nabla p(\Phi))\|_{0,\Omega_i} \lesssim h^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}(\|\Phi\|_{0,\Gamma_i} + \|\operatorname{div}_\tau \Phi\|_{0,\Gamma_i}).$$

This, together with (4.29), (4.32) and the fact that $\mathbf{curl}(\mathbf{r}_h\nabla p(\Phi)) = 0$, gives (4.28). □

4.3. **Some estimates with the norms $\|\cdot\|_{1/2,\Gamma_i}$, $\|\cdot\|_{-1/2,\Gamma_i}$ and $\|\cdot\|_{*,\mathrm{F}_b}$.** This section summarizes the results which will be used in the condition number estimates in Section 5. Detailed proofs are omitted here but can be found in [19] and [20].

For any subdomain Ω_i, by $\mathcal{W}_i$ we denote the set of the edges of Ω_i, which also belong to at least two other local interfaces Γ_j, $j \neq i$. For any given subset G of Γ_i and a function $\varphi \in L^2(G)$, we use $\gamma_G(\varphi)$ to denote the average value of φ on G. For any $\varphi \in Z_h(\Gamma_i)$, we define $\pi_0^i\varphi \in Z_h(\Gamma_i)$ as follows:

$$\pi_0^i\varphi(\mathbf{x}) = \begin{cases} \varphi(\mathbf{x}), & \text{for } \mathbf{x} \in \mathcal{W}_i \cap \mathcal{N}_h, \\ \gamma_{\mathrm{F}}(\varphi), & \text{for } \mathbf{x} \in \mathrm{F} \cap \mathcal{N}_h \ (\mathrm{F} \subset \Gamma_i). \end{cases} \tag{4.35}$$

It is easy to see that $\mathrm{I}_{\mathrm{F}}^t(\varphi - \pi_0^i\varphi) = \mathrm{I}_{\mathrm{F}}^0(\varphi - \gamma_{\mathrm{F}}(\varphi))$ for any $\mathrm{F} \subset \Gamma_i$, and we have

Lemma 4.10. *For any $\varphi \in Z_h(\Gamma_i)$ and any $\mathrm{F} \subset \Gamma_i$, we have (cf.* [4] [32] [19]*)*

$$\|\varphi\|_{0,\partial\mathrm{F}} \lesssim [1+\log(d/h)]^{\frac{1}{2}}\|\varphi\|_{\frac{1}{2},\Gamma_i}, \tag{4.36}$$

$$\|\mathrm{I}_{\mathrm{F}}^0\varphi\|_{\frac{1}{2},\Gamma_i} \lesssim [1+\log(d/h)]\|\varphi\|_{\frac{1}{2},\Gamma_i}, \tag{4.37}$$

$$|\mathrm{I}_{\partial\mathrm{F}}^0\varphi|_{\frac{1}{2},\mathrm{F}} \lesssim [1+\log(d/h)]^{\frac{1}{2}}\|\varphi\|_{\frac{1}{2},\Gamma_i}, \tag{4.38}$$

$$|\mathrm{I}_{\mathrm{F}}^t(\varphi - \pi_0^i\varphi)|_{\frac{1}{2},\Gamma_i} \lesssim [1+\log(d/h)]|\varphi|_{\frac{1}{2},\Gamma_i}. \tag{4.39}$$

For any face F of Γ_i, we introduce a quantity (not a norm) on F_b as follows:

$$\|\Phi\|_{*,\mathrm{F}_b} = \Big\{\sum_{e\in\mathrm{F}_b}\|\mathbf{v}\|_{0,e}^2\Big\}^{\frac{1}{2}}, \quad \forall\, \Phi = (\mathbf{v}\times\mathbf{n})|_{\Gamma_i} \in V_h(\Gamma_i).$$

Lemma 4.11. *Let F be a given face of Ω_i. Then for any $\mathbf{v}_h \in Z_h(\Omega_i)^3$ we have*

$$\|\mathbf{v}_h\times\mathbf{n}\|_{*,\mathrm{F}_b} \lesssim [1+\log(d/h)]^{\frac{1}{2}}\|\mathbf{v}_h\|_{\frac{1}{2},\Gamma_i}, \tag{4.40}$$

while for any $\mathbf{v}_h \in V_h(\Omega_i)$ with $\mathbf{w}$ defined as in (4.6) *we have*

$$\|(\mathbf{r}_h\mathbf{w})\times\mathbf{n}\|_{*,\mathrm{F}_b} \lesssim [1+\log(d/h)]^{\frac{1}{2}}\|\boldsymbol{curl}\,\mathbf{v}_h\|_{0,\Omega_i}, \tag{4.41}$$

$$d^{-2}\|\mathbf{r}_h\mathbf{w}\|_{0,\Omega_i}^2 \lesssim [1+\log(d/h)]\|\boldsymbol{curl}\,\mathbf{v}_h\|_{0,\Omega_i}^2. \tag{4.42}$$

Lemma 4.12. *For any $\Phi \in V_h(\Gamma_i)$ and any face* F *of Γ_i, we have (cf. [2] [19])*

$$\|\Phi\|_{0,\Gamma_i} \lesssim h^{-\frac{1}{2}}\|\Phi\|_{-\frac{1}{2},\Gamma_i}, \quad \|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|_{0,\mathrm{F}} \lesssim h^{\frac{1}{2}}\|\Phi\|_{*,\mathrm{F}_b}, \tag{4.43}$$

$$\|\mathbf{I}^0_{\mathrm{F}_\partial}\Phi\|_{-\frac{1}{2},\Gamma_i} \lesssim [1+\log(d/h)]\|\Phi\|_{-\frac{1}{2},\Gamma_i} + h^{\frac{1}{2}}\|\Phi\|_{*,\mathrm{F}_b}, \tag{4.44}$$

$$\|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|_{-\frac{1}{2},\mathrm{F}} \lesssim h^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}\|\mathbf{I}^0_{\mathrm{F}_b}\Phi\|_{0,\mathrm{F}}. \tag{4.45}$$

Lemma 4.13. *Let $\varphi \in L^2(\Gamma_i)$ be piecewise constant with respect to the $\mathcal{T}_h$-induced triangulation $\mathcal{T}_{h,i}$ on Γ_i. Then we have $\|\varphi\|_{0,\Gamma_i} \lesssim h^{-\frac{1}{2}}\|\varphi\|_{-\frac{1}{2},\Gamma_i}$.*

Lemma 4.14. *For any $\Phi = \mathbf{v}\times\mathbf{n} \in V_h(\Gamma_i)$, we have*

$$\|\operatorname{div}_\tau(\mathbf{I}^0_{\mathrm{F}}\Phi)\|_{-\frac{1}{2},\Gamma_i} \lesssim [1+\log(d/h)]\|\operatorname{div}_\tau\Phi\|_{-\frac{1}{2},\Gamma_i} + [1+\log(d/h)]^{\frac{1}{2}}\|\Phi\|_{*,\mathrm{F}_b}, \tag{4.46}$$

while for any $\mathbf{v}_0 \in V^0(\Omega)$, let $\Phi^i_0 = (\mathbf{v}_0\times\mathbf{n})|_{\Gamma_i} \in V_h(\Gamma_i)$. Then we have

$$\|\Phi^i_0\|_{-\frac{1}{2},\Gamma_i} + \|\operatorname{div}_\tau\Phi^i_0\|_{-\frac{1}{2},\Gamma_i} \lesssim h^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}(\|\Phi^i_0\|_{0,\Gamma_i} + \|\operatorname{div}_\tau\Phi^i_0\|_{0,\Gamma_i}). \tag{4.47}$$

The estimate in the following lemma indicates that the norm $\|\mathbf{I}^0_{\mathrm{F}}\Phi\|_{\mathcal{X}_{\Gamma_i}}$ cannot be bounded by $\|\Phi\|_{\mathcal{X}_{\Gamma_i}}$ only (compare with the estimate (4.37)).

Lemma 4.15. *Let $\mathbf{w}$ and $\mathbf{v}_h$ be the same as specified in Lemma 4.3, and let $\Phi = \mathbf{r}_h\mathbf{w}\times\mathbf{n}$ on Γ_i. Then, for any face $\mathrm{F}\subset\Gamma_i$ we have*

$$\|\mathbf{I}^0_{\mathrm{F}_\partial}\Phi\|_{\mathcal{X}_{\Gamma_i}} \lesssim [1+\log(d/h)](\|\Phi\|_{\mathcal{X}_{\Gamma_i}} + \|\boldsymbol{curl}\,\mathbf{v}_h\|_{0,\Omega_i}). \tag{4.48}$$

We end this section with some relation between the edge element space $V_h(\Omega_i)$ and the nodal element space $Z_h(\Omega_i)$:

Lemma 4.16. *Let t be an edge of Ω_i. For any $q \in Z_h(\Omega_i)$, if it vanishes on t, then its gradient $\nabla q \in V_h(\Omega_i)$ and $\lambda_e(\nabla q) = 0$ for any fine edge $e \subset \mathcal{E}_h \cap t$.*

5. Proofs of the main results

5.1. **Proof of Theorem 3.1.** For any $\mathbf{v}_h \in V_h(\Omega)$, we first decompose it as follows:

$$\mathbf{v}_h = \nabla q_h \oplus \mathbf{w}_h, \tag{5.1}$$

where $q_h \in Z_h(\Omega)$ solves

$$(\beta\nabla q_h, \nabla\psi_h) = (\beta\mathbf{v}_h, \nabla\psi_h), \quad \forall\psi_h \in Z_h(\Omega)$$

and $\mathbf{w}_h$ is orthogonal to ∇q_h in the scalar product $(\beta\cdot,\cdot)$. By Cauchy-Schwarz inequality,

$$\|\beta^{\frac{1}{2}}\nabla q_h\|_{0,\Omega} \le \|\beta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}. \tag{5.2}$$

Moreover, we apply Lemma 4.2 for $\mathbf{w}_h$ to obtain

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega} \lesssim \|\beta^{\frac{1}{2}}\mathbf{curl}\,\mathbf{w}_h\|_{0,\Omega} = \|\beta^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}. \tag{5.3}$$

Let $J : Z_h(\Omega) \to Z_h(\Omega)$ be the operator defined by

$$(J\phi_h, \psi_h) = (\beta\nabla\phi_h, \nabla\psi_h), \quad \forall\phi,\ \psi \in Z_h(\Omega). \tag{5.4}$$

Then, by the definitions of q_h, B^t and J, we have

$$\begin{aligned}(\beta\nabla q_h, \nabla q_h) &= (\beta\mathbf{v}_h, \nabla q_h) = (B^t\mathbf{v}_h, q_h)\\ &= (Jq_h, J^{-1}B^t\mathbf{v}_h) = (\beta\nabla q_h, \nabla(J^{-1}B^t\mathbf{v}_h))\\ &= (\beta\mathbf{v}_h, \nabla(J^{-1}B^t\mathbf{v}_h)) = (B^t\mathbf{v}_h, J^{-1}B^t\mathbf{v}_h).\end{aligned}$$

Therefore, we derive

$$(BJ^{-1}B^t\mathbf{v}_h, \mathbf{v}_h) = (\beta\nabla q_h, \nabla q_h). \tag{5.5}$$

This, together with (5.2), leads to

$$(BJ^{-1}B^t\mathbf{v}_h, \mathbf{v}_h) \le (\beta\mathbf{v}_h, \mathbf{v}_h).$$

Now it follows from (3.3) that

$$\begin{aligned}(A\mathbf{v}_h, \mathbf{v}_h) &\lesssim (\alpha\mathbf{curl}\ \mathbf{v}_h, \mathbf{curl}\ \mathbf{v}_h) + r_0(BJ^{-1}B^t\mathbf{v}_h, \mathbf{v}_h)\\ &\le (\alpha\mathbf{curl}\,\mathbf{v}_h, \mathbf{curl}\,\mathbf{v}_h) + r_0(\beta\mathbf{v}_h, \mathbf{v}_h)\\ &\lesssim (\tilde{A}\mathbf{v}_h, \mathbf{v}_h).\end{aligned} \tag{5.6}$$

On the other hand, using (5.5), (5.3) and (3.3) yields

$$\begin{aligned}(\alpha\mathbf{v}_h, \mathbf{v}_h) &= r_0(\beta\mathbf{v}_h, \mathbf{v}_h) = r_0[(\beta\nabla q_h, \nabla q_h) + (\beta\mathbf{w}_h, \mathbf{w}_h)]\\ &\lesssim r_0[(BJ^{-1}B^t\mathbf{v}_h, \mathbf{v}_h) + (\beta\mathbf{curl}\ \mathbf{v}_h, \mathbf{curl}\ \mathbf{v}_h)]\\ &\lesssim \max\{1, G(d/h)\}(A\mathbf{v}_h, \mathbf{v}_h) \lesssim G(d/h)(A\mathbf{v}_h, \mathbf{v}_h),\end{aligned}$$

and so

$$(\tilde{A}\mathbf{v}_h, \mathbf{v}_h) \lesssim G(d/h)(A\mathbf{v}_h, \mathbf{v}_h).$$

This, together with (5.6), gives the desired result. □

5.2. **Proof of Theorem 3.2.** This subsection is devoted to the estimate of the condition number of the preconditioned system $\hat{A}^{-1}A$; see (3.7). The following lemma reduces this task to the estimates of two positive constants C_1 and C_2. This framework can be regarded as a variant of the additive Schwarz theory associated with the space decomposition (3.5) (refer to [27] and [31]), and the proof is standard (cf. [15] and [28]). Here, we have used the orthogonality between $V^p(\Omega)$ and $V^H(\Omega) = V^0(\Omega) + \sum\limits_{\Gamma_{ij}} V^{ij}(\Omega)$.

Lemma 5.1. *Assume that the following two conditions hold:*

(i) *For any $\mathbf{v} \in V^H(\Omega)$ there is a decomposition*

$$\mathbf{v} = \mathbf{v}_0 + \sum_{\Gamma_{ij}} \mathbf{v}_{ij}$$

with $\mathbf{v}_0 \in V^0(\Omega)$ and $\mathbf{v}_{ij} \in V^{ij}(\Omega)$, such that

$$(\hat{A}_0\mathbf{v}_0, \mathbf{v}_0) + \sum_{\Gamma_{ij}}(\hat{A}_{ij}\mathbf{v}_{ij}, \mathbf{v}_{ij}) \le C_1(\tilde{A}\mathbf{v}, \mathbf{v}). \tag{5.7}$$

(ii) *For any $\mathbf{w}_0 \in V^0(\Omega)$ and $\mathbf{w}_{ij} \in V^{ij}(\Omega)$, we have*

$$(\tilde{A}(\mathbf{w}_0 + \sum_{\Gamma_{ij}}\mathbf{w}_{ij}),\ \mathbf{w}_0 + \sum_{\Gamma_{ij}}\mathbf{w}_{ij}) \le C_2\Big\{(\hat{A}_0\mathbf{w}_0, \mathbf{w}_0) + \sum_{\Gamma_{ij}}(\hat{A}_{ij}\mathbf{w}_{ij}, \mathbf{w}_{ij})\Big\}. \tag{5.8}$$

Then we have the following estimate

$$\mathrm{cond}(\hat{A}^{-1}\tilde{A}) \le C_1C_2.$$

By this lemma and Theorem 3.1, we obtain

$$\mathrm{cond}(\hat{A}^{-1}A) \le C_1C_2G(d/h).$$

For the proof of Theorem 3.2, it suffices to estimate the constants C_1 and C_2 in Lemma 5.1.

To do so, we first introduce some notation. For any $\mathbf{v}_h \in V^H(\Omega)$, let $\mathbf{v}_h^i = \mathbf{v}_h|_{\Omega_i}$. From the discussions in Subsection 4.1, there exist $p^i \in H^1(\Omega_i)$, $p_h^i \in Z_h(\Omega_i)$ and $\mathbf{w}^i \in H^1(\Omega_i)^3$ such that

$$\mathbf{v}_h^i = \nabla p^i \oplus \mathbf{w}^i\,, \tag{5.9}$$

$$\mathbf{v}_h^i = \nabla p_h^i + \mathbf{r}_h \mathbf{w}^i = \nabla p_h^i + \mathbf{w}_h^i \tag{5.10}$$

with $\mathbf{w}_h^i = \mathbf{r}_h \mathbf{w}^i \in V_h(\Omega_i)$. By (5.9) and (5.10), we know

$$\mathbf{curl}\ \mathbf{w}_h^i = \mathbf{curl}\ \mathbf{w}^i = \mathbf{curl}\ \mathbf{v}_h^i. \tag{5.11}$$

We are now ready to show Theorem 3.2 using Lemma 5.1 and to divide the proof into four steps.

Step 1: Establish a suitable decomposition for $\mathbf{v}_h \in V^H(\Omega)$.

We introduce $p_{h0}^i \in Z_h(\Omega_i)$ and $\mathbf{w}_{h0}^i \in V_h(\Omega_i)$ by

$$p_{h0}^i = R_h^i \pi_0^i(p_h^i|_{\Gamma_i}), \quad \mathbf{w}_{h0}^i = \mathbf{R}_h^i \mathbf{I}_{\Delta_i}^0(\mathbf{w}_h^i \times \mathbf{n})|_{\Gamma_i}, \quad i = 1, \cdots, N.$$

Define

$$\Phi_0^i = (\nabla p_{h0}^i + \mathbf{w}_{h0}^i) \times \mathbf{n}|_{\Gamma_i}$$

and

$$\Phi_0 = \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i} + \sqrt{\alpha_j}} \Phi_0^i + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i} + \sqrt{\alpha_j}} \Phi_0^j \quad \text{on } \Gamma_{ij} \subset \Gamma.$$

Set $\Phi = (\mathbf{v}_h \times \mathbf{n})|_\Gamma$. It follows from Lemma 4.16 that $(\Phi_0 - \Phi)|_{\mathcal{W}_i} = 0$ for every i. Thus, Φ_0 is defined uniquely on all the edges of the interface Γ. In particular, Φ_0 equals Φ on these edges. Define $\mathbf{v}_0 \in V^0(\Omega)$ such that $\mathbf{v}_0|_{\Omega_i}$ is the discrete A_i-extension of $\Phi_0|_{\Gamma_i}$. For each $\Gamma_{ij} \subset \Gamma$ define $\mathbf{v}_{ij} \in V_{ij}(\Omega)$, such that

$$(\mathbf{v}_{ij} \times \mathbf{n})|_{\Gamma_{ij}} = (\mathbf{v}_h - \mathbf{v}_0) \times \mathbf{n}|_{\Gamma_{ij}}.$$

It is easy to see that

$$\mathbf{v}_h = \mathbf{v}_0 + \sum_{\Gamma_{ij}} \mathbf{v}_{ij} \quad \text{in } \Omega.$$

Step 2: Derive the estimate

$$(\hat{A}_0 \mathbf{v}_0, \mathbf{v}_0) \lesssim [1 + \log(d/h)]^2 (\tilde{A} \mathbf{v}_h, \mathbf{v}_h). \tag{5.12}$$

We first estimate the term $h[1+\log(d/h)] \sum_{i=1}^N \alpha_i \|\operatorname{div}_\tau(\mathbf{v}_0 \times \mathbf{n})|_{\Gamma_i}\|_{0,\Delta_i}^2$.

By Lemma 6.2 in the Appendix and by the triangle inequality, we obtain for $\mathrm{F} = \Gamma_i \cap \Gamma_j$ that

$$\begin{aligned} \|\operatorname{div}_\tau(\mathbf{v}_0 \times \mathbf{n})|_{\Gamma_i}\|_{0,\mathrm{F}_b} &\lesssim \|\operatorname{div}_\tau(\mathbf{v}_h^i \times \mathbf{n})|_{\Gamma_i}\|_{0,\mathrm{F}_b} \\ &+ \Big\|\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i} + \sqrt{\alpha_j}} \operatorname{div}_\tau \mathbf{I}_{\mathrm{F}\partial}^0(\mathbf{w}_h^i \times \mathbf{n})|_{\Gamma_i} + \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i} + \sqrt{\alpha_j}} \operatorname{div}_\tau \mathbf{I}_{\mathrm{F}\partial}^0(\mathbf{w}_h^j \times \mathbf{n})|_{\Gamma_j}\Big\|_{0,\mathrm{F}_b} \\ &\lesssim \|\operatorname{div}_\tau(\mathbf{v}_h^i \times \mathbf{n})|_{\Gamma_i}\|_{0,\Gamma_i} + \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i} + \sqrt{\alpha_j}} \|\operatorname{div}_\tau \mathbf{I}_{\mathrm{F}\partial}^0(\mathbf{w}_h^i \times \mathbf{n})|_{\Gamma_i}\|_{0,\mathrm{F}_b} \\ &+ \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i} + \sqrt{\alpha_j}} \|\operatorname{div}_\tau \mathbf{I}_{\mathrm{F}\partial}^0(\mathbf{w}_h^j \times \mathbf{n})|_{\Gamma_j}\|_{0,\mathrm{F}_b}. \end{aligned} \tag{5.13}$$

By Lemma 4.13 and the inequality (2.11) in [1], we have

$$\|\operatorname{div}_\tau(\mathbf{v}_h^i \times \mathbf{n})|_{\Gamma_i}\|_{0,\Gamma_i} \lesssim h^{-\frac{1}{2}} \|\operatorname{div}_\tau(\mathbf{v}_h^i \times \mathbf{n})|_{\Gamma_i}\|_{-\frac{1}{2},\Gamma_i} \lesssim h^{-\frac{1}{2}} \|\mathbf{curl}\ \mathbf{v}_h^i\|_{0,\Omega_i}. \tag{5.14}$$

Moreover, similarly to the proof of Lemma 4.14 (see [19]), one can show

$$\begin{aligned}\|\operatorname{div}_\tau \mathbf{I}^0_{\mathrm{F}_\partial}(\mathbf{w}^i_h\times\mathbf{n})|_{\Gamma_i}\|_{0,\mathrm{F}_b} &\lesssim h^{-\frac12}\|\mathbf{w}^i_h\times\mathbf{n}\|_{*,\mathrm{F}_b},\\ \|\operatorname{div}_\tau \mathbf{I}^0_{\mathrm{F}_\partial}(\mathbf{w}^j_h\times\mathbf{n})|_{\Gamma_i}\|_{0,\mathrm{F}_b} &\lesssim h^{-\frac12}\|\mathbf{w}^j_h\times\mathbf{n}\|_{*,\mathrm{F}_b}.\end{aligned}$$

Substituting these two estimates and (5.14) into (5.13) and using Lemma 4.11 lead to

$$\begin{aligned}&h\|\operatorname{div}_\tau(\mathbf{v}_0\times\mathbf{n})|_{\Gamma_i}\|^2_{0,\mathrm{F}_b}\lesssim\|\mathbf{curl}\ \mathbf{v}^i_h\|^2_{0,\Omega_i}\\ &\quad+[1+\log(d/h)]\Big\{\Big(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\Big)^2\|\mathbf{curl}\ \mathbf{v}^i_h\|^2_{0,\Omega_i}+\Big(\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\Big)^2\|\mathbf{curl}\ \mathbf{v}^j_h\|^2_{0,\Omega_j}\Big\}.\end{aligned}$$

A similar estimate holds on Γ_j as well. Then using $\alpha_i+\alpha_j<(\sqrt{\alpha_i}+\sqrt{\alpha_j})^2$, we derive

$$\begin{aligned}&h\Big\{\alpha_i\|\operatorname{div}_\tau(\mathbf{v}_0\times\mathbf{n})|_{\Gamma_i}\|^2_{0,\mathrm{F}_b}+\alpha_j\|\operatorname{div}_\tau(\mathbf{v}_0\times\mathbf{n})|_{\Gamma_j}\|^2_{0,\mathrm{F}_b}\Big\}\\ \lesssim\ &[1+\log(d/h)]\Big\{\alpha_i\|\mathbf{curl}\ \mathbf{v}^i_h\|^2_{0,\Omega_i}+\alpha_j\|\mathbf{curl}\ \mathbf{v}^j_h\|^2_{0,\Omega_j}\Big\}.\end{aligned}$$

This yields

(5.15)

$$\begin{aligned}&h[1+\log(d/h)]\sum_{i=1}^N\alpha_i\|\operatorname{div}_\tau(\mathbf{v}_0\times\mathbf{n})|_{\Gamma_i}\|^2_{0,\Delta_i}\\ &=[1+\log(d/h)]\sum_{\Gamma_{ij}\subset\Gamma}h\Big\{\alpha_i\|\operatorname{div}_\tau(\mathbf{v}_0\times\mathbf{n})|_{\Gamma_i}\|^2_{0,\mathrm{F}_b}+\alpha_j\|\operatorname{div}_\tau(\mathbf{v}_0\times\mathbf{n})|_{\Gamma_j}\|^2_{0,\mathrm{F}_b}\Big\}\\ &\lesssim[1+\log(d/h)]^2\sum_{i=1}^N\alpha_i\|\mathbf{curl}\ \mathbf{v}^i_h\|^2_{0,\Omega_i}.\end{aligned}$$

Next, we estimate the term $h[1+\log(d/h)]\sum\limits_{i=1}^N\alpha_i\|\mathbf{v}_0\times\mathbf{n}\|^2_{0,\Delta_i}$.

From the definition of $\mathbf{v}_0$ and the triangle inequality, we have ($\mathrm{F}=\Gamma_{ij}$)

$$\begin{aligned}\|\mathbf{v}_0\times\mathbf{n}\|_{0,\mathrm{F}_b} &\lesssim \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\|\nabla p^i_{h0}\times\mathbf{n}\|_{0,\mathrm{F}_b}+\|\mathbf{w}^i_{h0}\times\mathbf{n}\|_{0,\mathrm{F}_b})\\ &+\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\|\nabla p^j_{h0}\times\mathbf{n}\|_{0,\mathrm{F}_b}+\|\mathbf{w}^j_{h0}\times\mathbf{n}\|_{0,\mathrm{F}_b}).\end{aligned}\tag{5.16}$$

By Lemma 6.2 (see the Appendix) and (4.36), we can deduce

$$\begin{aligned}\|\nabla p^i_{h0}\times\mathbf{n}\|_{0,\mathrm{F}_b} &\lesssim h^{-\frac12}\|p^i_h-\gamma_{\mathrm{F}}(p^i_h)\|_{0,\partial\mathrm{F}}\\ &\lesssim h^{-\frac12}[1+\log(d/h)]^{\frac12}\|p^i_h-\gamma_{\mathrm{F}}(p^i_h)\|_{\frac12,\Gamma_i}\\ &\lesssim h^{-\frac12}[1+\log(d/h)]^{\frac12}|p^i_h|_{\frac12,\Gamma_i}\\ &\lesssim h^{-\frac12}[1+\log(d/h)]^{\frac12}|p^i_h|_{1,\Omega_i}.\end{aligned}\tag{5.17}$$

Similarly, we have

$$\|\nabla p^j_{h0}\times\mathbf{n}\|_{0,\mathrm{F}_b}\lesssim h^{-\frac12}[1+\log(d/h)]^{\frac12}|p^j_h|_{1,\Omega_j}.\tag{5.18}$$

On the other hand, it follows from (4.43) and Lemma 4.11 that

$$\|\mathbf{w}^i_{h0}\times\mathbf{n}\|_{0,\mathrm{F}_b}\lesssim h^{\frac12}[1+\log(d/h)]^{\frac12}\|\mathbf{curl}\ \mathbf{v}^i_h\|_{0,\Omega_i}$$

and

$$\|\mathbf{w}_{h0}^j \times \mathbf{n}\|_{0,\mathrm{F}_b} \lesssim h^{\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}\|\mathbf{curl}\ \mathbf{v}_h^j\|_{0,\Omega_j}.$$

Plugging these and (5.17) and (5.18) into (5.16) yields

$$\begin{aligned}\|\mathbf{v}_0 \times \mathbf{n}\|_{0,\mathrm{F}_b} &\lesssim h^{-\frac{1}{2}}[1+\log(d/h)]^{\frac{1}{2}}[\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(|p_h^i|_{1,\Omega_i} + \|\mathbf{curl}\ \mathbf{v}_h^i\|_{0,\Omega_i}) \\ &+ \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(|p_h^j|_{1,\Omega_j} + \|\mathbf{curl}\ \mathbf{v}_h^j\|_{0,\Omega_j})].\end{aligned}$$

Thus ($\mathrm{F} = \Gamma_{ij}$),

$$\begin{aligned}&h[1+\log(d/h)]\sum_{i=1}^N \alpha_i\|\mathbf{v}_0\times\mathbf{n}\|_{0,\Delta_i}^2 = h[1+\log(d/h)]\sum_{\mathrm{F}\subset\Gamma}(\alpha_i+\alpha_j)\|\mathbf{v}_0\times\mathbf{n}\|_{0,\mathrm{F}_b}^2 \\ &\quad\lesssim [1+\log(d/h)]^2 \\ &\qquad\times\sum_{\Gamma_{ij}\subset\Gamma}[\alpha_i(|p_h^i|_{1,\Omega_i} + \|\mathbf{curl}\ \mathbf{v}_h^i\|_{0,\Omega_i}) + \alpha_j(|p_h^j|_{1,\Omega_j} + \|\mathbf{curl}\ \mathbf{v}_h^j\|_{0,\Omega_j})] \\ &\quad= [1+\log(d/h)]^2\sum_{i=1}^N \alpha_i(|p_h^i|_{1,\Omega_i} + \|\mathbf{curl}\ \mathbf{v}_h^i\|_{0,\Omega_i}).\end{aligned}$$

This, together with (5.15) and (4.17), leads to (5.12).

Step 3: Prove the estimate

$$\sum_{\Gamma_{ij}}(\hat{A}_{ij}\mathbf{v}_{ij}, \mathbf{v}_{ij}) \lesssim [1+\log(d/h)]^2(\tilde{A}\mathbf{v}_h, \mathbf{v}_h). \tag{5.19}$$

Since $\mathbf{v}_h^i = \mathbf{v}_h^j$ on Γ_{ij}, we have

$$\mathbf{v}_h = \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{v}_h^i + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{v}_h^j \quad \text{on } \Gamma_{ij}.$$

Hence, $\mathbf{v}_{ij}$ on Γ_{ij} can be written as

$$\begin{aligned}\mathbf{v}_{ij} &= \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\nabla(p_h^i - p_{h0}^i)\times\mathbf{n} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\nabla(p_h^j - p_{h0}^j)\times\mathbf{n} \\ &\quad+\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{w}_h^i - \mathbf{w}_{h0}^i)\times\mathbf{n} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{w}_h^j - \mathbf{w}_{h0}^j)\times\mathbf{n}.\end{aligned}$$

Define

$$\begin{aligned}&p_{ij}^i = R_h^i \mathrm{I}_{ij}^t\Big\{\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(p_h^i - p_{h0}^i)|_{\Gamma_{ij}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(p_h^j - p_{h0}^j)|_{\Gamma_{ij}}\Big\} \in Z_h(\Omega_i), \\ &\mathbf{w}_{ij}^i = \mathbf{R}_h^i \mathbf{I}_{ij}^t\Big\{\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{w}_h^i - \mathbf{w}_{h0}^i)\times\mathbf{n}|_{\Gamma_{ij}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{w}_h^j - \mathbf{w}_{h0}^j)\times\mathbf{n}|_{\Gamma_{ij}}\Big\} \\ &\qquad\qquad\in V_h(\Omega_i), \\ &\mathbf{v}_{ij}^i = \nabla p_{ij}^i + \mathbf{w}_{ij}^i \in V_h(\Omega_i).\end{aligned}$$

One can verify by Lemma 4.16 that

$$\mathbf{v}_{ij}^i \times \mathbf{n} = \mathbf{v}_{ij}\times\mathbf{n} \quad \text{on } \Gamma_i.$$

Thus, we obtain by the minimum **curl**-energy property of the discrete A_i-extension that

$$
\begin{aligned}
(A_i(\mathbf{v}_{ij}|_{\Omega_i}), \mathbf{v}_{ij}|_{\Omega_i})_{\Omega_i} &\le (A_i\mathbf{v}_{ij}^i, \mathbf{v}_{ij}^i) = \|\alpha^{\frac{1}{2}}\,\mathbf{curl}\,\mathbf{w}_{ij}^i\|_{0,\Omega_i}^2 + \|\alpha^{\frac{1}{2}}\mathbf{v}_{ij}^i\|_{0,\Omega_i}^2 \\
&\lesssim \alpha_i(\|\nabla p_{ij}^i\|_{0,\Omega_i}^2 + \|\mathbf{curl}\ \mathbf{w}_{ij}^i\|_{0,\Omega_i}^2 + \|\mathbf{w}_{ij}^i\|_{0,\Omega_i}^2).
\end{aligned}
\tag{5.20}
$$

As $p_{h0}^i = \pi_0^i(p_h^i|_{\Gamma_i})$ on Γ_i, we have

$$
\mathrm{I}_{ij}^t[(p_h^i - p_{h0}^i)|_{\Gamma_{ij}}] = \mathrm{I}_{ij}^0(p_h^i|_{\Gamma_i} - \pi_0^i(p_h^i|_{\Gamma_i}))\,.
$$

Then using (4.39) and the trace theorem, we obtain

$$
\begin{aligned}
&\|\nabla p_{ij}^i\|_{0,\Omega_i}^2 = |p_{ij}^i|_{1,\Omega_i}^2 \lesssim |(p_{ij}^i|_{\Gamma_{ij}})|_{\frac{1}{2},\Gamma_i}^2 \\
&\lesssim |\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathrm{I}_{ij}^0(p_h^i|_{\Gamma_i} - \pi_0^i(p_h^i|_{\Gamma_i}))|_{\frac{1}{2},\Gamma_i}^2 + |\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathrm{I}_{ij}^0(p_h^j|_{\Gamma_j} - \pi_0^j(p_h^j|_{\Gamma_j}))|_{\frac{1}{2},\Gamma_j}^2 \\
&\lesssim [1+\log(d/h)]^2\Big\{\Big(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\Big)^2|p_h^i|_{\frac{1}{2},\Gamma_i}^2 + \Big(\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\Big)^2|p_h^j|_{\frac{1}{2},\Gamma_j}^2\Big\} \\
&\lesssim [1+\log(d/h)]^2\Big\{\Big(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\Big)^2|p_h^i|_{1,\Omega_i}^2 + \Big(\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\Big)^2|p_h^j|_{1,\Omega_j}^2\Big\}.
\end{aligned}
\tag{5.21}
$$

We next estimate $\mathbf{w}_{ij}^i$. It follows by Lemma 4.6 that

$$
\begin{aligned}
\|\mathbf{curl}\ \mathbf{w}_{ij}^i\|_{0,\Omega_i}^2 + \|\mathbf{w}_{ij}^i\|_{0,\Omega_i}^2 &\lesssim \|(\mathbf{w}_{ij}^i \times \mathbf{n})|_{\Gamma_i}\|_{\mathcal{X}_{\Gamma_i}}^2 \\
&\lesssim \|\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{I}_{ij}^t[(\mathbf{w}_h^i - \mathbf{w}_{h0}^i)\times\mathbf{n}|_{\Gamma_{ij}}]\|_{\mathcal{X}_{\Gamma_i}}^2 \\
&\quad + \|\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{I}_{ij}^t[(\mathbf{w}_h^j - \mathbf{w}_{h0}^j)\times\mathbf{n}|_{\Gamma_{ij}}]\|_{\mathcal{X}_{\Gamma_j}}^2.
\end{aligned}
\tag{5.22}
$$

For each (open) common face $\mathrm{F} = \Gamma_{ij}$ shared by Ω_i and Ω_j, it follows from the definitions of $\mathbf{w}_{h0}^i$ that

$$
\lambda_e(\mathbf{w}_h^i - \mathbf{w}_{h0}^i) = \begin{cases} 0, & \text{if } e \subset \mathrm{F}_b, \\ \lambda_e(\mathbf{w}_h^i), & \text{if } e \subset \mathrm{F}_\partial\,. \end{cases}
$$

Then we derive by using Lemma 4.15 that

$$
\begin{aligned}
&\|\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{I}_{ij}^t[(\mathbf{w}_h^i - \mathbf{w}_{h0}^i)\times\mathbf{n}|_{\Gamma_{ij}}]\|_{\mathcal{X}_{\Gamma_i}}^2 \\
&\quad = (\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}})^2\|\mathbf{I}_{\mathrm{F}_\partial}^0[(\mathbf{w}_h^i\times\mathbf{n})|_{\Gamma_{ij}}]\|_{\mathcal{X}_{\Gamma_i}}^2 \\
&\quad \lesssim [1+\log(d/h)]^2(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}})^2(\|\mathbf{w}_h^i\times\mathbf{n}\|_{\mathcal{X}_{\Gamma_i}}^2 + \|\mathbf{curl}\ \mathbf{v}_h^i\|_{0,\Omega_i}^2).
\end{aligned}
\tag{5.23}
$$

On the other hand, for the term $\mathbf{w}_h^i \times \mathbf{n}$ we have by Lemma 4.5 and (5.9) that

$$
\begin{aligned}
\|\mathbf{w}_h^i\times\mathbf{n}\|_{\mathcal{X}_{\Gamma_i}}^2 &\lesssim \|\mathbf{w}_h^i\|_{\mathbf{curl};\Omega_i}^2 \\
&= \|\mathbf{curl}\,\mathbf{w}_h^i\|_{0;\Omega_i}^2 + d^{-2}\|\mathbf{w}_h^i\|_{0;\Omega_i}^2 \\
&= \|\mathbf{curl}\,\mathbf{v}_h^i\|_{0;\Omega_i}^2 + d^{-2}\|\mathbf{w}_h^i\|_{0;\Omega_i}^2.
\end{aligned}
$$

Combining this with (5.23) and using Lemma 4.11 give

$$\begin{aligned}
&\|\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{I}^t_{ij}[(\mathbf{w}^i_h-\mathbf{w}^i_{h0})\times\mathbf{n}|_{\Gamma_{ij}}]\|^2_{\mathcal{X}_{\Gamma_i}}\\
&\qquad\lesssim[1+\log(d/h)]^2(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}})^2\|\mathbf{curl}\,\mathbf{v}^i_h\|^2_{0,\Omega_i}.
\end{aligned}$$

A similar estimate holds on Γ_j as well. Substituting these inequalities into (5.22) yields

$$\begin{aligned}
&\|\mathbf{curl}\ \mathbf{w}^i_{ij}\|^2_{0,\Omega_i}+\|\mathbf{w}^i_{ij}\|^2_{0,\Omega_i}\\
&\quad\lesssim[1+\log(d/h)]^2[(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}})^2\|\mathbf{curl}\,\mathbf{v}^i_h\|^2_{0,\Omega_i}+(\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}})^2\|\mathbf{curl}\,\mathbf{v}^j_h\|^2_{0,\Omega_j}].
\end{aligned}$$

With this estimate, (5.20) and (5.21), we come to

$$\begin{aligned}
&(A_i(\mathbf{v}_{ij}|_{\Omega_i}),\mathbf{v}_{ij}|_{\Omega_i})_{\Omega_i}\\
&\quad\lesssim[1+\log(d/h)]^2\alpha_i[(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}})^2(|p^i_h|^2_{1,\Omega_i}+\|\mathbf{curl}\,\mathbf{v}^i_h\|^2_{0,\Omega_i})\\
&\qquad+(\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}})^2(|p^j_h|^2_{1,\Omega_j}+\|\mathbf{curl}\,\mathbf{v}^j_h\|^2_{0,\Omega_j})].
\end{aligned}\tag{5.24}$$

Similarly, we have

$$\begin{aligned}
&(A_j(\mathbf{v}_{ij}|_{\Omega_j}),\mathbf{v}_{ij}|_{\Omega_j})_{\Omega_j}\\
&\quad\lesssim[1+\log(d/h)]^2\alpha_j[(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}})^2(|p^i_h|^2_{1,\Omega_i}+\|\mathbf{curl}\,\mathbf{v}^i_h\|^2_{0,\Omega_i})\\
&\qquad+(\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}})^2(|p^j_h|^2_{1,\Omega_j}+\|\mathbf{curl}\,\mathbf{v}^j_h\|^2_{0,\Omega_j})].
\end{aligned}$$

Summing this inequality with (5.24) and noting that

$$\alpha_i+\alpha_j<(\sqrt{\alpha_i}+\sqrt{\alpha_j})^2,$$

we obtain

$$\begin{aligned}
(\hat{A}_{ij}\mathbf{v}_{ij},\mathbf{v}_{ij})\quad\lesssim\quad&[1+\log(d/h)]^2[\alpha_i(|p^i_h|^2_{1,\Omega_i}+\|\mathbf{curl}\,\mathbf{v}^i_h\|^2_{0,\Omega_i})\\
&+\alpha_j(|p^j_h|^2_{1,\Omega_j}+\|\mathbf{curl}\,\mathbf{v}^j_h\|^2_{0,\Omega_j})]
\end{aligned}$$

or

$$\sum_{\Gamma_{ij}}(\hat{A}_{ij}\mathbf{v}_{ij},\mathbf{v}_{ij})\lesssim[1+\log(d/h)]^2\sum_{i=1}^N\alpha_i(|p^i_h|^2_{1,\Omega_i}+\|\mathbf{curl}\,\mathbf{v}^i_h\|^2_{0,\Omega_i}).\tag{5.25}$$

This, together with (4.17), implies

$$\begin{aligned}
\sum_{\Gamma_{ij}}(\hat{A}_{ij}\mathbf{v}_{ij},\mathbf{v}_{ij})\quad\lesssim\quad&[1+\log(d/h)]^2\sum_{i=1}^N\alpha_i(\|\mathbf{curl}\,\mathbf{v}^i_h\|^2_{0,\Omega_i}+\|\mathbf{v}^i_h\|^2_{0,\Omega_i})\\
=\quad&[1+\log(d/h)]^2\sum_{i=1}^N(A_i\mathbf{v}^i_h,\mathbf{v}^i_h)_{\Omega_i}\\
=\quad&[1+\log(d/h)]^2(\tilde{A}\mathbf{v}_h,\mathbf{v}_h).
\end{aligned}$$

The estimates (5.12) and (5.19) indicate that the constant C_1 in (5.7) can be bounded by $C[1+\log(d/h)]^2$.

Step 4: Estimate the constant C_2 in (5.8).

Using the triangle inequality, (4.47) and Lemma 4.9 yields

$$
\begin{aligned}
&(\tilde{A}(\mathbf{w}_0 + \sum_{\Gamma_{ij}} \mathbf{w}_{ij}),\ \mathbf{w}_0 + \sum_{\Gamma_{ij}} \mathbf{w}_{ij}) \\
&\lesssim (\tilde{A}\mathbf{w}_0, \mathbf{w}_0) + (\tilde{A} \sum_{\Gamma_{ij}} \mathbf{w}_{ij}, \sum_{\Gamma_{ij}} \mathbf{w}_{ij}) \\
&\lesssim (\hat{A}_0\mathbf{w}_0, \mathbf{w}_0) + \sum_{k=1}^{N} (A_k(\sum_{\Gamma_{ij}} \mathbf{w}_{ij})|_{\Omega_k}, (\sum_{\Gamma_{ij}} \mathbf{w}_{ij})|_{\Omega_k})_{\Omega_k}. \qquad (5.26)
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
&\sum_{k=1}^{N} (A_k(\sum_{\Gamma_{ij}} \mathbf{w}_{ij})|_{\Omega_k}, (\sum_{\Gamma_{ij}} \mathbf{w}_{ij})|_{\Omega_k})_{\Omega_k} \\
&= \sum_{k=1}^{N} \sum_{\Gamma_{ij}} (A_k(\mathbf{w}_{ij}|_{\Omega_k}), \sum_{\Gamma_{ij}} (\mathbf{w}_{ij}|_{\Omega_k}))_{\Omega_k} \\
&\lesssim \sum_{k=1}^{N} \sum_{\Gamma_{ij} \subset \Gamma_k} (A_k(\mathbf{w}_{ij}|_{\Omega_k}), \mathbf{w}_{ij}|_{\Omega_k})_{\Omega_k}.
\end{aligned}
$$

As each face Γ_{ij} is shared by only two subdomains Ω_i and Ω_j, we have

$$\sum_{k=1}^{N} (A_k(\sum_{\Gamma_{ij}} \mathbf{w}_{ij})|_{\Omega_k}, (\sum_{\Gamma_{ij}} \mathbf{w}_{ij})|_{\Omega_k})_{\Omega_k} \lesssim \sum_{\Gamma_{ij}} (\hat{A}_{ij}\mathbf{w}_{ij}, \mathbf{w}_{ij}).$$

This, together with (5.26), indicates that the constant C_2 in (5.8) is bounded by a constant independent of h and d. □

5.3. **Proof of Theorem 3.3.** One can verify that (cf. [5])

$$(B^t \tilde{A}^{-1} Bq, q) = \sup_{\mathbf{v} \in V_h(\Omega)} \frac{(\mathbf{v}, Bq)^2}{(\tilde{A}\mathbf{v}, \mathbf{v})}, \quad q \in Z_h(\Omega).$$

By the definitions of B, we have

$$(\mathbf{v}, Bq) = r_0(\beta\mathbf{v}, \nabla q) \eqsim (\alpha\mathbf{v}, \nabla q),$$

so we have

$$(B^t \tilde{A}^{-1} Bq, q) \eqsim \sup_{\mathbf{v} \in V_h(\Omega)} \frac{(\alpha\mathbf{v}, \nabla q)^2}{(\tilde{A}\mathbf{v}, \mathbf{v})}, \quad q \in Z_h(\Omega). \qquad (5.27)$$

If we choose $\mathbf{v} = \nabla q$, then $(\tilde{A}\mathbf{v}, \mathbf{v}) = (\alpha\nabla q, \nabla q)$, and

$$(B^t \tilde{A}^{-1} Bq, q) \geq \frac{(\alpha\nabla q, \nabla q)^2}{(\alpha\nabla q, \nabla q)} = \|\alpha^{\frac{1}{2}} \nabla q\|_{0,\Omega}. \qquad (5.28)$$

On the other hand, by the Cauchy-Schwarz inequality and the definition of $\tilde{A}$ we obtain

$$\frac{(\alpha\mathbf{v}, \nabla q)^2}{(\tilde{A}\mathbf{v}, \mathbf{v})} \leq \frac{\|\alpha^{\frac{1}{2}}\mathbf{v}\|_{0,\Omega}^2 \cdot \|\alpha^{\frac{1}{2}}\nabla q\|_{0,\Omega}^2}{\|\alpha^{\frac{1}{2}}\mathbf{v}\|_{0,\Omega}^2} = \|\alpha^{\frac{1}{2}}\nabla q\|_{0,\Omega}^2, \ \forall \mathbf{v} \in V_h(\Omega).$$

Thus by (5.27) we have

$$(B^t\tilde{A}^{-1}Bq,q)\lesssim\|\alpha^{\frac{1}{2}}\nabla q\|^2_{0,\Omega}.$$

This, together with (5.28), leads to

$$(B^t\tilde{A}^{-1}Bq,q)\eqsim\|\alpha^{\frac{1}{2}}\nabla q\|^2_{0,\Omega},\quad\forall q\in Z_h(\Omega).$$

Therefore, we infer from (3.3) that

$$\mathrm{cond}(\hat{C}^{-1}B^t\tilde{A}^{-1}B)\lesssim G(d/h),$$

which, together with Theorem 3.2, yields (3.8). □

6. Appendix

Lemma 6.1. *For any $p^i_h\in Z_h(\Omega_i)$, let $p^i_{h0}=R^i_h\pi^i_0(p^i_h|_{\Gamma_i})\in Z_h(\Omega_i)$. We have*

$$\|\nabla p^i_{h0}\times\mathbf{n}\|_{0,\mathrm{F}_b}\lesssim h^{-\frac{1}{2}}\|p^i_h-\gamma_{\mathrm{F}}(p^i_h)\|_{0,\partial\mathrm{F}},\quad\mathrm{F}\subset\Gamma_i. \tag{6.1}$$

Proof. Let G be a fine triangle in F_b, and let A_i $(i=1,2,3)$ be the vertices of G. Without loss of generality, we assume that $A_1,A_2\in\partial\mathrm{F}$. Let K be the element of Ω_i which has a face G, and let $A_4\in\Omega_i$ be another vertex of K. Let $\lambda_i(x,y,z)$ denote the nodal basis function at the node A_i. Then

$$p^i_h(x,y,z)=\sum_{k=1}^{4}p^i_h(A_k)\lambda_k(x,y,x),\quad\forall(x,y,z)\in K.$$

By the definition of p^i_{h0}, we have

$$p^i_{h0}-\gamma_{\mathrm{F}}(p^i_h)=\sum_{k=1}^{2}(p^i_h(A_k)-\gamma_{\mathrm{F}}(p^i_h))\lambda_k+\hat{a}_4\lambda_4\quad\text{on }K.$$

It follows from Lemma 4.16 that $\nabla\lambda_4\times\mathbf{n}=0$ on G. Thus,

$$\nabla p^i_{h0}\times\mathbf{n}|_G=\nabla(p^i_{h0}-\gamma_{\mathrm{F}}(p^i_h))\times\mathbf{n}|_G=\sum_{k=1}^{2}(p^i_h(A_k)-\gamma_{\mathrm{F}}(p^i_h))\nabla\lambda_k\times\mathbf{n}|_G. \tag{6.2}$$

It can be verified that

$$\|\nabla\lambda_k\times\mathbf{n}\|_{0,G}\lesssim 1,\quad k=1,\ 2,$$

which, together with (6.2), yields

$$\|\nabla p^i_{h0}\times\mathbf{n}\|_{0,G}\lesssim\sum_{k=1}^{2}|p^i_h(A_k)-\gamma_{\mathrm{F}}(p^i_h)|.$$

Summing this over all $G\subset\mathrm{F}_b$ leads to

$$\|\nabla p^i_{h0}\times\mathbf{n}\|^2_{0,\mathrm{F}_b}\lesssim\sum_{A\in\partial\mathrm{F}}|p^i_h(A)-\gamma_{\mathrm{F}}(p^i_h)|^2\,,$$

where A denotes a node of the fine mesh on $\partial\mathrm{F}$. Now, (6.1) follows from the equivalence between the L^2-norm and the discrete L^2-norm on $\partial\mathrm{F}$. □

Lemma 6.2. *Let $\mathbf{v}_0 \in V^0(\Omega)$ be defined as in subsection 5.2. Then we have the identity*

$$\begin{aligned}&\mathrm{div}_\tau(\mathbf{v}_0\times\mathbf{n})|_{\Gamma_i} = \mathrm{div}_\tau(\mathbf{v}_h^i\times\mathbf{n})|_{\Gamma_i}\\ &-\sum_{\mathrm{F}\subset\Gamma_i}[\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\,\mathrm{div}_\tau\,\mathbf{I}^0_{\mathrm{F}\partial}(\mathbf{w}_h^i\times\mathbf{n})|_{\Gamma_i} + \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\,\mathrm{div}_\tau\,\mathbf{I}^0_{\mathrm{F}\partial}(\mathbf{w}_h^j\times\mathbf{n})|_{\Gamma_j}].\end{aligned}\tag{6.3}$$

Proof. It is easy to see that

$$\begin{aligned}\nabla p_{h0}^i + \mathbf{w}_{h0}^i &= \mathbf{v}_h^i + \nabla(p_{h0}^i - p_h^i) + \mathbf{w}_{h0}^i - \mathbf{w}_h^i,\\ \nabla p_{h0}^j + \mathbf{w}_{h0}^j &= \mathbf{v}_h^j + \nabla(p_{h0}^j - p_h^j) + \mathbf{w}_{h0}^j - \mathbf{w}_h^j.\end{aligned}$$

Let $\mathrm{F}=\Gamma_{ij}$. Then we have directly by the definition of $\mathbf{v}_0$ that

$$\begin{aligned}(\mathbf{v}_0\times\mathbf{n})|_{\mathrm{F}} &= \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{v}_h^i\times\mathbf{n})|_{\mathrm{F}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{v}_h^j\times\mathbf{n})|_{\mathrm{F}}\\ &+ \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\nabla(p_{h0}^i - p_h^i)\times\mathbf{n}|_{\mathrm{F}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\nabla(p_{h0}^j - p_h^j)\times\mathbf{n}|_{\mathrm{F}}\\ &+ \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{w}_{h0}^i - \mathbf{w}_h^i)\times\mathbf{n}|_{\mathrm{F}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{w}_{h0}^j - \mathbf{w}_h^j)\times\mathbf{n}|_{\mathrm{F}}.\end{aligned}\tag{6.4}$$

But note that $\mathbf{v}_h^i\times\mathbf{n} = \mathbf{v}_h^j\times\mathbf{n}$ on F. Then

$$\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{v}_h^i\times\mathbf{n}) + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{v}_h^j\times\mathbf{n}) = \mathbf{v}_h^i\times\mathbf{n}\quad\text{on F}.$$

This implies

$$\sum_{\mathrm{F}\subset\Gamma_i}\mathbf{I}^t_{\mathrm{F}}[\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{v}_h^i\times\mathbf{n})|_{\mathrm{F}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{v}_h^j\times\mathbf{n})|_{\mathrm{F}}] = (\mathbf{v}_h^i\times\mathbf{n})|_{\Gamma_i}.\tag{6.5}$$

Again noting the fact that

$$\begin{aligned}(p_{h0}^i - p_h^i) - (R_h^i\mathrm{I}^t_{\mathrm{F}}(p_{h0}^i - p_h^i)|_{\mathrm{F}}) = 0\quad\text{on F},\\ (p_{h0}^j - p_h^j) - (R_h^i\mathrm{I}^t_{\mathrm{F}}(p_{h0}^j - p_h^j)|_{\mathrm{F}}) = 0\quad\text{on F},\end{aligned}$$

we derive by Lemma 6.1 that

$$\begin{aligned}&\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\nabla(p_{h0}^i - p_h^i)\times\mathbf{n}|_{\mathrm{F}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\nabla(p_{h0}^j - p_h^j)\times\mathbf{n}|_{\mathrm{F}}\\ &\quad= \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\nabla R_h^i\mathrm{I}^t_{\mathrm{F}}(p_{h0}^i - p_h^i)|_{\mathrm{F}})\\ &\quad\quad\times\mathbf{n}|_{\mathrm{F}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\nabla R_h^i\mathrm{I}^t_{\mathrm{F}}(p_{h0}^j - p_h^j)|_{\mathrm{F}})\times\mathbf{n}|_{\mathrm{F}}\\ &\quad= \Big[\nabla R_h^i\mathrm{I}^t_{\mathrm{F}}\Big(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(p_{h0}^i - p_h^i)|_{\mathrm{F}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(p_{h0}^j - p_h^j)|_{\mathrm{F}}\Big)\Big]\times\mathbf{n}|_{\mathrm{F}}.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{\mathrm{F}\subset\Gamma_i}\mathbf{I}^t_{\mathrm{F}}[\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\nabla(p_{h0}^i - p_h^i)\times\mathbf{n}|_{\mathrm{F}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\nabla(p_{h0}^j - p_h^j)\times\mathbf{n}|_{\mathrm{F}}]\\ &\quad= [\nabla R_h^i\sum_{\mathrm{F}\subset\Gamma_i}\mathrm{I}^t_{\mathrm{F}}(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(p_{h0}^i - p_h^i)|_{\mathrm{F}} + \frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(p_{h0}^j - p_h^j)|_{\mathrm{F}})]\times\mathbf{n}|_{\Gamma_i}.\end{aligned}\tag{6.6}$$

In addition, it is easy to see that

$$\begin{aligned}(6.7)\qquad &\sum_{F\subset\Gamma_i}\mathbf{I}^t_{F}[\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{w}^i_{h0}-\mathbf{w}^i_h)\times\mathbf{n}|_F+\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(\mathbf{w}^j_{h0}-\mathbf{w}^j_h)\times\mathbf{n}|_F]\\ &=-\sum_{F\subset\Gamma_i}[\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{I}^0_{F\partial}(\mathbf{w}^i_h\times\mathbf{n})|_{\Gamma_i}+\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{I}^0_{F\partial}(\mathbf{w}^j_h\times\mathbf{n})|_{\Gamma_j}].\end{aligned}$$

Summing (6.4) over $F\subset\Gamma_i$ and using (6.5)-(6.7) yield

$$\begin{aligned}&(\mathbf{v}_0\times\mathbf{n})|_{\Gamma_i}=\sum_{F\subset\Gamma_i}\mathbf{I}^t_F(\mathbf{v}_0\times\mathbf{n})|_F=(\mathbf{v}^i_h\times\mathbf{n})|_{\Gamma_i}\\ &+\Big[\nabla R^i_h\sum_{F\subset\Gamma_i}\mathrm{I}^t_F\Big(\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(p^i_{h0}-p^i_h)|_F+\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}(p^j_{h0}-p^j_h)|_F\Big)\Big]\times\mathbf{n}|_{\Gamma_i}\\ &-\sum_{F\subset\Gamma_i}[\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{I}^0_{F\partial}(\mathbf{w}^i_h\times\mathbf{n})|_{\Gamma_i}+\frac{\sqrt{\alpha_j}}{\sqrt{\alpha_i}+\sqrt{\alpha_j}}\mathbf{I}^0_{F\partial}(\mathbf{w}^j_h\times\mathbf{n})|_{\Gamma_j}].\end{aligned}$$

Now (6.3) follows from the fact that $\mathrm{div}_\tau(\nabla q_h\times\mathbf{n})|_{\Gamma_i}=0$ for any $q_h\in Z_h(\Omega_i)$. □

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