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EXPLICIT DIFFUSIVE KINETIC SCHEMES FOR NONLINEAR DEGENERATE PARABOLIC SYSTEMS

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ABSTRACT. We design numerical schemes for nonlinear degenerate parabolic systems with possibly dominant convection. These schemes are based on discrete BGK models where both characteristic velocities and the source-term depend singularly on the relaxation parameter. General stability conditions are derived, and convergence is proved to the entropy solutions for scalar equations.

1. INTRODUCTION

In this paper we study discrete kinetic schemes for systems of conservation laws with possibly degenerate diffusion

(1.1)
$$\partial_t u_k + \sum_{d=1}^{D} \partial_{x_d}[A_{kd}(u)] = \Delta_x[B_k(u)], \qquad (x,t) \in \mathbb{R}^D \times]0,\infty[, \qquad 1 \le k \le K,$$

and with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^D. \tag{1.2}$$

We assume that the functions A and B are Lipschitz continuous and satisfy the following conditions, for all u lying in some fixed rectangle I:

- For all $\xi \in \mathbb{R}^D$, $\sum_{d=1}^{D} \xi_d A'_d(u)$ has real eigenvalues and is diagonalizable.
- The real parts of the eigenvalues of $B'(u)$ are nonnegative.

The theory of such systems is far from complete, except in the scalar case [29, 6, 25]; see also the overview in [10]. For the theory of general parabolic systems we refer to [9, 32], and we refer to [8] for some recent results on degenerate parabolic systems.

From the numerical viewpoint, there has been new interest in these problems for a few years. For classical and recent schemes in the nondegenerate case let us refer just to Morton's recent book [26] and references therein. A first study of convergence of monotone numerical schemes in the fully degenerate case can be found in [11]. Concerning systems, some results are presented in the paper of Karlsen *et al.* [19] where numerical schemes based on a splitting between the hyperbolic and the parabolic parts of the problem are under consideration. Quite

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effective central schemes have been recently proposed by Kurganov and Tadmor [21].

Our approach is inspired on one hand by relaxation schemes for hyperbolic conservation laws [18, 1] and on the other hand by kinetic approximations of hydrodynamic equations, in the hyperbolic setting; see for instance [30]. For other diffusive kinetic models and approximations, see [23, 17, 16]. Here, we design numerical approximations of (1.1) by considering diffusive BGK models of the following form:

$$(1.3)\quad \begin{cases} \partial_t f_l^\epsilon + \sum_{d=1}^D \lambda_{ld}\partial_{x_d} f_l^\epsilon = \dfrac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right), & 1 \le l \le N, \\ \partial_t f_{N+m}^\epsilon + \gamma^\epsilon \displaystyle\sum_{d=1}^D \sigma_{md}\partial_{x_d} f_{N+m}^\epsilon = \dfrac{1}{\epsilon}\left(\dfrac{B(u^\epsilon)}{N'\theta^2} - f_{N+m}^\epsilon\right), & 1 \le m \le N', \end{cases}$$

where $u^\epsilon(x,t) = \sum_{l=1}^{N+N'} f_l^\epsilon(x,t)$, each f_l^ϵ and M_l take values in $\mathbb{R}^K$, ϵ is a positive parameter, the λ_{ld} are some fixed real constants, $\gamma^\epsilon = \mu + \frac{\theta\sqrt{N'}}{\sqrt{\epsilon}}$, $\mu \ge 0$, $\theta > 0$, $N' \ge D+1$, and $\left\{\sigma^{(1)}, \dots, \sigma^{(D)}\right\}$ is an orthonormal family in $\left\{X \in \mathbb{R}^{N'}, \sum_{m=1}^{N'} X_m = 0\right\}$.

Systems (1.1) and (1.3) are linked by compatibility conditions which are investigated in Section 2. By analogy with the BGK framework, the function M is called a local Maxwellian function. In the scalar case, this model is a particular case of the ones proposed in [5].

An advantage of a model like (1.3) is that the nonlinearity inside the derivatives in (1.1) is replaced by a semilinearity: the differential part is linear and diagonal, and all the nonlinearity is concentrated in the source-term. Moreover, the ordinary differential system associated to the source-term owns an exact solution. These features allow us to design Riemann solver free schemes for (1.1) by discretizing (1.3) and taking ϵ small enough. Another nice property is that we naturally deal with the diffusion degeneracy, so that hyperbolic-parabolic systems are treated without additional complications. Due to the globality of the approach, it may happen that the hyperbolic numerical fluxes depend on the diffusion B. However, two difficulties appear: both the source-term and the characteristic velocities depend singularly on ϵ. The first difficulty already appears in the purely hyperbolic case and has been succesfully treated in [2]. In the more general context of conservation laws with relaxation this problem was considered in [15]. The second difficulty is specifically connected to the parabolic nature of (1.1). In somewhat different contexts, answers have been brought up; see for example [17] and references therein. Here we propose two different methods, based on two ways of splitting the system (1.1). When dealing with cartesian meshes, which is the only configuration we look for in the present article, both methods lead to similar schemes. However there exists a natural extension of our second method, without any dimensional splitting, to unstructured grids. This aspect will be treated in a future paper. Our schemes depend on three main factors: the choice of the kinetic model (1.3), the space discretization, and the time discretization. As far as we are concerned with the choice of the space discretization, we only have to construct schemes for linear hyperbolic systems, so that our method may be thought of as a way of transferring, *via* the nonlinear Maxwellian functions, the approximation of linear systems to the one of nonlinear convection-diffusion systems.

The plan of the paper is as follows. In Section 2, we give compatibility conditions between our BGK models and (1.1). A necessary stability condition is interpreted as a generalized subcharacteristic condition (see [24]). Related numerical schemes are constructed in Section 3. We begin by explicit first order in time discretizations and then reach different time approximations, such as higher order or implicit schemes; see subsection 3.3. Section 4 is devoted to the study of numerical stability. In the scalar case we have convergence to the entropy solution of the problem. Our result can actually be related only to the paper by S. Evje and K.H. Karlsen [11], where however an additional hypothesis on the regularity of the data is assumed to prove the convergence of some particular monotone schemes. Here, thanks to our more general framework, we do not need such an assumption. In this sense our paper, although mainly focused on the approximation of general nonlinear systems, presents even in the scalar case a first general result of convergence for numerical schemes for problems with degenerate diffusion and convection. Numerical experiments are presented in Section 5, and comparisons are made mainly with the central schemes proposed in [21].

2. Compatibility and stability

In the sequel we denote by I a domain in $\mathbb{R}^K$ such that $u(x,t) \in I$ for all (x,t). If (1.1) is scalar, we can usually take $I = [-\|u_0\|_\infty, \|u_0\|_\infty]$. Systems (1.1) and (1.3) are linked by the following compatibility conditions, for all $u \in I$:

$$\sum_{l=1}^{N} M_l(u) = u - \frac{B(u)}{\theta^2}\ , \quad \sum_{l=1}^{N} \lambda_{ld} M_l(u) = A_d(u),\ \ 1 \le d \le D. \tag{2.1}$$

It is easy to see that if the sequence u^ϵ converges to some limit function u in a suitable (strong) topology, then the limit function is a weak solution to equation (1.1). In the purely hyperbolic case, this framework reduces to the one developed in [28] and [2].

To insure stability, it is essential to deal with monotone Maxwellian functions.

Definition 2.1. A local Maxwellian function M is a monotone Maxwellian function (MMF) (respectively strictly monotone Maxwellian function (SMMF)) if for all $l \in \{1, \dots, N\}$, for all $\epsilon \in]0,1]$ and for all $u \in I$, the real parts of the eigenvalues of $M_l'(u)$ are nonnegative (respectively positive).

In [5] it is proved in the scalar case that convergence holds if M is a MMF. In [22], the convergence is also proved for a special class of one dimensional strongly parabolic systems, with SMMF. A necessary condition for M to be a MMF is that $B' \le \theta^2 I$. In fact the monotonicity condition can be read as a generalized subcharacteristic condition. The following result easily follows from conditions (2.1).

Proposition 2.1. *Assume that M is a MMF satisfying* (2.1) *on I. Take $\xi \in \mathbb{R}^D$, $|\xi| = 1$, and suppose that for all $u \in I$, there exists a basis of common eigenvectors for the matrices $\sum_{d=1}^D \xi_d A_d'(u)$, $M_l'(u)$ $(l = 1, \dots, N)$ and $B'(u)$. Then for all $u \in I$ and for all $k = 1, \dots, K$, it holds that*

$$\min_{1\le l\le N} \left(\sum_{d=1}^{D} \xi_d \lambda_{ld}\right)\left(1 - \frac{\theta_k^2(u)}{\theta^2}\right) \le a_k(u,\xi) \le \max_{1\le l\le N} \left(\sum_{d=1}^{D} \xi_d \lambda_{ld}\right)\left(1 - \frac{\theta_k^2(u)}{\theta^2}\right) \tag{2.2}$$

where $a_k(u,\xi)$ and $\theta_k^2(u)$ are respectively the eigenvalues of $\sum_{d=1}^D \xi_d A_d'(u)$ and the real parts of the eigenvalues of $B'(u)$.

All the models designed in [2] for purely hyperbolic problems can be extended to models of the form (1.3). Let us write an extension of Jin and Xin's relaxation system [18], which we call DRM2, since its hyperbolic version is different from the Diagonal Relaxation Model (DRM1) studied in [2]. We detail only the choices on the N first equations. The choices of the N' last ones are done at the numerical level in Section 3.

We take $N = 2D$ and we work with velocities parallel to the axes:

$$\lambda_{jd} = \delta_{jd}\lambda_{md}, \quad \lambda_{D+j,d} = \delta_{jd}\lambda_{pd}, \quad j = 1, \dots, D, \quad d = 1, \dots, D,$$

where δ_{jd} is the Kroenecker symbol and $\lambda_{md} < \lambda_{pd}$. The Maxwellian functions are

$$\left\{ \begin{array}{lcl} M_d(u) & = & \dfrac{1}{\lambda_{pd} - \lambda_{md}} \left(\dfrac{\lambda_{pd}}{D} \left(u - \dfrac{B(u)}{\theta^2} \right) - A_d(u) \right), \\ & & \\ M_{D+d}(u) & = & \dfrac{1}{\lambda_{pd} - \lambda_{md}} \left(-\dfrac{\lambda_{md}}{D} \left(u - \dfrac{B(u)}{\theta^2} \right) + A_d(u) \right), \end{array} \right. \qquad d = 1, \dots, D. \tag{2.3}$$

Let us examine the case $K = D = 1$ with $B(u) \in \mathbb{R}$. The function M is monotone if, for all $u \in I$,

$$\lambda_m\left(1 - \frac{B'(u)}{\theta^2}\right) \leq A'(u) \leq \lambda_p\left(1 - \frac{B'(u)}{\theta^2}\right), \tag{2.4}$$

which is equivalent to condition (2.2).

This relation shows that the characteristic velocities of the kinetic model approximate more sharply A' than the choice $\lambda_m = -\lambda_p$, letting us hope that the related numerical approximation is better. In practice, condition (2.4) implies that, for all $u \in I$,

$$1 - \frac{B'(u)}{\theta^2} > 0. \tag{2.5}$$

Thus, we fix $\alpha \in]0, 1[$ and we first take θ such that

$$0 < \alpha \leq 1 - \frac{\sup_{u \in I} B'(u)}{\theta^2} \leq 1. \tag{2.6}$$

This parameter being fixed, we take λ_m, λ_p such that relation (2.4) holds true. Following the same ideas, it is not a difficult task to design a diffusive extension of DRM1 or of the flux decomposition method (FDM) or of orthogonal velocity method (OVM), developed in [2].

3. Numerical schemes

In this section we use the diffusive kinetic models introduced in Section 2 to design numerical schemes for the convection-diffusion equation (1.1). The general idea is to construct a discretization of (1.3) in order to obtain, for ϵ sufficiently small, a scheme for (1.1). We present two different methods. First we use a kinetic splitting and we connect ϵ with the discretization parameter Δx so that this process provides consistent approximations of (1.1) (subsection 3.1). Another method, available for a continuous range $[0, +\infty[$ of values of ϵ, and inspired by the ideas of [17], is presented in the second subsection.

Let us first introduce some notation. In this paper we restrict ourselves to cartesian uniform grids:

$$\mathbb{R}^D = \bigcup_{\alpha \in \mathbb{Z}^D} \mathrm{I}_\alpha, \quad [0,T] = \bigcup_{0 \le n \le N-1} [t_n, t_{n+1}].$$

Set $\alpha = (\alpha_d)_{1 \le d \le D} \in \mathbb{Z}^D$ and let e_d be the canonical d^{th} vector in $\mathbb{R}^D$.

As usual we denote by x_α the center of I_α, by Δx_d the length of I_α in the direction d, $\Delta t = t_{n+1} - t_n$, $\Delta x = (\Delta x_d)_{1 \le d \le D}$. Finally we set

$$\begin{aligned} f^\epsilon_\Delta(x,t) &= \sum_{n=0}^{N-1} \sum_{\alpha \in \mathbb{Z}^D} f^{\epsilon,n}_\alpha \mathbf{1}_{\mathrm{I}_\alpha}(x) \mathbf{1}_{]t_n,t_{n+1}[}(t), \\ f^{\epsilon,n}_\Delta(x) &= \sum_{\alpha \in \mathbb{Z}^D} f^{\epsilon,n}_\alpha \mathbf{1}_{\mathrm{I}_\alpha}(x), \\ f^{\epsilon,n} &= (f^{\epsilon,n}_\alpha)_{\alpha \in \mathbb{Z}^D}. \end{aligned}$$

If

$$u^0_\alpha = (\mathrm{vol}(\mathrm{I}_\alpha))^{-1} \int_{\mathrm{I}_\alpha} u_0(x)dx, \tag{3.1}$$

then f^ϵ_0 is approximated by

$$f^{\epsilon,0}_\alpha = M(u^0_\alpha). \tag{3.2}$$

We use for u the same notation as for f above. Now, our goal is to obtain a stable and consistent approximation of (1.1) in the conservation form

$$\begin{aligned} u^{n+1}_\alpha = u^n_\alpha &- \sum_{d=1}^D \frac{\Delta t}{\Delta x_d} \left(\mathcal{A}^n_{\alpha+e_d/2} - \mathcal{A}^n_{\alpha-e_d/2} \right) \\ &+ \sum_{d=1}^D \frac{\Delta t}{\Delta x_d^2} \left(\mathcal{B}^n_{\alpha+e_d,d} - 2\mathcal{B}^n_{\alpha,d} + \mathcal{B}^n_{\alpha-e_d,d} \right), \end{aligned} \tag{3.3}$$

with

$$\mathcal{A}^n_{\alpha+e_d/2} = \mathcal{A}_d \left(u^n_{\alpha-a+e_d}, \dots, u^n_{\alpha+a} \right), \quad a \in (\mathbb{N}^*)^D, \tag{3.4}$$

where the function $\mathcal{A}_d$ of $2a_d \cdot \Pi_{j \ne d}(2a_j+1)$ variables is Lipschitz continuous and satisfies $\mathcal{A}_d(u, \dots, u) = A_d(u)$ for all u, and

$$\mathcal{B}^n_{\alpha,d} = \mathcal{B}_d \left(u^n_{\alpha-b}, \dots, u^n_{\alpha+b} \right), \quad b \in \mathbb{N}^D, \tag{3.5}$$

where the function $\mathcal{B}_d$ of $\Pi_d(2b_d+1)$ variables is Lipschitz continuous and satisfies $\mathcal{B}_d(u, \dots, u) = B(u)$.

In subsection 3.3 we shall discuss the way of constructing schemes, based on the same space discretization, which are high-order in time and/or implicit.

3.1. **A kinetic splitting method.**

We begin by recalling two well-known results which will be useful in the sequel.

Lemma 3.1. *Consider the scalar transport equation*

$$\partial_t f + \sum_{d=1}^D \gamma_d \partial_{x_d} f = 0 \tag{3.6}$$

and a related linear $D \times 5$ point discretization:

$$f_\alpha^{n+1} = f_\alpha^n - \sum_{d=1}^{D} \gamma_d \frac{\Delta t}{\Delta x_d} \left(F^n_{\alpha+e_d/2} - F^n_{\alpha-e_d/2} \right), \tag{3.7}$$

where $F^n_{\alpha+e_d/2} = \sum_{k=-1}^{2} a_{kd} f^n_{\alpha+ke_d}$ and $\sum_{k=-1}^{2} a_{kd} = 1$. The scheme (3.7) *is monotone and consistent with* (3.6) *if and only if for all d*

$$\gamma_d a_{0d} \geq \gamma_d a_{-1d} \geq 0 \geq \gamma_d a_{2d} \geq \gamma_d a_{1d} \tag{3.8}$$

and

$$\sum_{d=1}^{D} \gamma_d (a_{0d} - a_{1d}) \frac{\Delta t}{\Delta x_d} \leq 1. \tag{3.9}$$

In that case, for all convex functions S, and setting

$$S^n_{\alpha+e_d/2} = \sum_{k=-1}^{2} a_{kd} S(f^n_{\alpha+ke_d}),$$

it holds that

$$S(f_\alpha^{n+1}) \leq S(f_\alpha^n) - \sum_{d=1}^{D} \gamma_d \frac{\Delta t}{\Delta x_d} \left(S^n_{\alpha+e_d/2} - S^n_{\alpha-e_d/2} \right). \tag{3.10}$$

Lemma 3.2. *Consider the scalar nonlinear equation*

$$\partial_t u - \Delta B(u) = 0, \quad B'(u) \geq 0, \tag{3.11}$$

and a related linear consistent $D \times 5$ point discretization

$$u_\alpha^{n+1} = u_\alpha^n + \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d^2} \left(\mathcal{B}^n_{\alpha-e_d,d} - 2\mathcal{B}^n_{\alpha,d} + \mathcal{B}^n_{\alpha+e_d,d} \right), \tag{3.12}$$

where $\mathcal{B}^n_{\alpha,d} = \sum_{k=-1}^{1} c_{kd} B(u^n_{\alpha+ke_d})$ and $\sum_{k=-1}^{1} c_{kd} = 1$. The scheme (3.12) *is monotone and symmetric if and only if $\mathcal{B}^n_{\alpha,d}$ can be put in the form*

$$\mathcal{B}^n_{\alpha,d} = m_d B(u^n_{\alpha-e_d}) + (1-2m_d) B(u^n_\alpha) + m_d B(u^n_{\alpha+e_d}), \quad m_d \in [0; 1/4], \tag{3.13}$$

and Δt satisfies the time step restriction

$$\Delta t \sum_{d=1}^{D} \frac{2(1-3m_d) \max_{u \in I} B'(u)}{\Delta x_d^2} \leq 1. \tag{3.14}$$

In the following, we denote $\Delta_{m,\alpha} = \sum_{d=1}^{D} \dfrac{\Delta t}{\Delta x_d^2} \left(\mathcal{B}^n_{\alpha-e_d,d} - 2\mathcal{B}^n_{\alpha,d} + \mathcal{B}^n_{\alpha+e_d,d} \right)$, with $\mathcal{B}^n_{\alpha,d}$ given by (3.13).

3.1.1. *One space dimension.* We consider the one dimensional version of (1.3):

$$
\left\{
\begin{array}{ll}
\partial_t f_l^\epsilon + \lambda_l \partial_x f_l^\epsilon = \dfrac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right), & 1 \le l \le N, \\
\partial_t f_{N+1}^\epsilon - \left(\dfrac{\mu}{\sqrt{2}} + \dfrac{\theta}{\sqrt{\epsilon}}\right)\partial_x f_{N+1}^\epsilon = \dfrac{1}{\epsilon}\left(\dfrac{B(u^\epsilon)}{2\theta^2} - f_{N+1}^\epsilon\right), & \\
\partial_t f_{N+2}^\epsilon + \left(\dfrac{\mu}{\sqrt{2}} + \dfrac{\theta}{\sqrt{\epsilon}}\right)\partial_x f_{N+2}^\epsilon = \dfrac{1}{\epsilon}\left(\dfrac{B(u^\epsilon)}{2\theta^2} - f_{N+2}^\epsilon\right), & \\
u^\epsilon(x,t) = \displaystyle\sum_{l=1}^{N+2} f_l^\epsilon(x,t). &
\end{array}
\right.
\tag{3.15}
$$

We split (3.15) into its hyperbolic diagonal linear part and an ordinary differential system. For a given $f_\Delta^{\epsilon,n}$, the function $f_\Delta^{\epsilon,n+1/2}$ is an approximate solution at time t_{n+1} of the problem

$$
\partial_t f + \Gamma^\epsilon \partial_x f = 0, \tag{3.16}
$$

$$
f(t_n) = f_\Delta^{\epsilon,n}\ , \tag{3.17}
$$

where $\Gamma^\epsilon = \operatorname{diag}\left(\lambda_1, \ldots, \lambda_N, -\dfrac{\mu}{\sqrt{2}} - \dfrac{\theta}{\sqrt{\epsilon}}, \dfrac{\mu}{\sqrt{2}} + \dfrac{\theta}{\sqrt{\epsilon}}\right)$. Since the system is diagonal, we may consider each equation separately. We suppose that the scheme is stable and can be put in conservation form

$$
f_\alpha^{\epsilon,n+1/2} = f_\alpha^{\epsilon,n} - \Gamma^\epsilon \frac{\Delta t}{\Delta x}\left(f_{\alpha+\frac{1}{2}}^{\epsilon,n} - f_{\alpha-\frac{1}{2}}^{\epsilon,n}\right). \tag{3.18}
$$

The first N equations are approximated by any high-order (for example MUSCL type) scheme

$$
f_{\alpha+\frac{1}{2},l}^{\epsilon,n} = F_l\left(f_{\alpha-k_l+1,l}^{\epsilon,n}, \ldots, f_{\alpha+k_l,l}^{\epsilon,n}\right), \quad F_l(g,\ldots,g) = g, \quad 1 \le l \le N, \tag{3.19}
$$

with F_l Lipschitz continuous. The last two equations are discretized by a linear monotone scheme. We detail here five-point schemes and we suppose that the corresponding numerical flux functions F_l do not depend on ϵ and satisfy the following symmetry property, for all $g \in \mathbb{R}^4$:

$$
F_{N+1}\left(g_{-1}, \ldots, g_2\right) = F_{N+2}\left(g_2, \ldots, g_{-1}\right). \tag{3.20}
$$

Then, we write, setting $F_{N+2}\left(g_{-1}, \ldots, g_2\right) = \sum_{k=-1}^{2} a_k g_k$,

$$
f_{\alpha+\frac{1}{2},N+2}^{\epsilon,n} = F_{N+2}\left(f_{\alpha-1,N+2}^{\epsilon,n}, \ldots, f_{\alpha+2,N+2}^{\epsilon,n}\right), \tag{3.21}
$$

with $\sum_{k=-1}^{2} a_k = 1$ and the conditions (3.8). As a particular case we recover three-point monotone schemes for which $a_{-1} = a_2 = 0$, $a_0 \ge 0 \ge a_1$ and $a_0 + a_1 = 1$.

For these last two equations the CFL condition (3.9) reads

$$
\left(\frac{\mu}{\sqrt{2}} + \frac{\theta}{\sqrt{\epsilon}}\right)(a_0 - a_1)\frac{\Delta t}{\Delta x} \le 1. \tag{3.22}
$$

In the following, the scheme on the linear part will be referred to as (HS) (*homogeneous scheme*) and the associated evolution operator will be denoted by H_Δ^ϵ, i.e.,

$$
f_\Delta^{\epsilon,n+1/2} = H_\Delta^\epsilon(\Delta t) f_\Delta^{\epsilon,n}\ .
$$

To take into account the contribution of the singular perturbation term on the right-hand side, we solve on $[t_n, t_{n+1}]$ the ordinary differential system

$$F' = \frac{1}{\epsilon}\left(M(u) - F\right), \tag{3.23}$$

where, as usual, $u = \sum_{l=1}^{N+2} F_l$, and for the initial data

$$F(t_n) = f_\alpha^{\epsilon,n+1/2}\ ,$$

for all $\alpha \in \mathbb{Z}$. By (2.1), we obtain $u' = 0$, so that we have an explicit solution for (3.23), and taking in this solution the limit $\epsilon = 0$, we can set

$$f_\alpha^{\epsilon,n+1} = M(u_\alpha^{\epsilon,n+1/2}).$$

The macroscopic variable u is given by

$$u_\alpha^{\epsilon,n+1} = u_\alpha^{\epsilon,n+1/2} = u_\alpha^{\epsilon,n} - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{\alpha+1/2}^{\epsilon,n} - \mathcal{F}_{\alpha-1/2}^{\epsilon,n}\right), \tag{3.24}$$

where $\mathcal{F}_{\alpha+1/2}^{\epsilon,n} = \sum_{l=1}^{N+2} \gamma_l^\epsilon F_l\left(M_l\left(u_{\alpha-k_l+1}^{\epsilon,n}\right), \ldots, M_l\left(u_{\alpha+k_l}^{\epsilon,n}\right)\right)$. Clearly, ϵ cannot vanish here. We have to consider ϵ as a discretization parameter linked to Δx.

From now on, we shall use the notation $F_\alpha^{\epsilon,n} = F(u_\alpha^{\epsilon,n})$ for all the functions of u, like M and B. We are going to show that for a certain value $\epsilon = \epsilon(\Delta x)$, we can find functions $\mathcal{A}$ and $\mathcal{B}$ such that

$$\sum_{l=1}^{N+2} \gamma_l^\epsilon F_l\left(M_{\alpha-k_l+1,l}^{\epsilon,n}, \ldots, M_{\alpha+k_l,l}^{\epsilon,n}\right) = \mathcal{A}_{\alpha+1/2}^n - \frac{\mathcal{B}_{\alpha+1}^n - \mathcal{B}_\alpha^n}{\Delta x}. \tag{3.25}$$

Actually we have

(3.26)

$$\begin{aligned}\mathcal{F}_{\alpha+1/2}^{\epsilon,n} &= \sum_{l=1}^{N} \lambda_l F_l\left(M_{\alpha-k_l+1,l}^{\epsilon,n}, \ldots, M_{\alpha+k_l,l}^{\epsilon,n}\right)\\ &\quad + \frac{\gamma_{N+2}^\epsilon}{2\theta^2}\left[F_{N+2}\left(B_{\alpha-s+1}^{\epsilon,n}, \ldots, B_{\alpha+s}^{\epsilon,n}\right) - F_{N+2}\left(B_{\alpha+s}^{\epsilon,n}, \ldots, B_{\alpha-s+1}^{\epsilon,n}\right)\right].\end{aligned}$$

We can define the numerical flux function associated to A, by setting

$$\mathcal{A}_{\alpha+1/2}^n = \sum_{l=1}^{N} \lambda_l F_l\left(M_{\alpha-k_l+1,l}^{\epsilon,n}, \ldots, M_{\alpha+k_l,l}^{\epsilon,n}\right), \tag{3.27}$$

which is consistent, thanks to (2.1). It remains to find $\mathcal{B}_\alpha^n$ and ϵ such that

(3.28)

$$\begin{aligned}\mathcal{B}_{\alpha+1}^n - 2\quad \mathcal{B}_\alpha^n + \mathcal{B}_{\alpha-1}^n = \frac{\gamma_{N+2}^\epsilon \Delta x}{2\theta^2}&\left[B_{\alpha-2}^n(a_{-1} - a_2) + B_{\alpha-1}^n(a_2 - a_1 - a_{-1} + a_0)\right.\\ &\left. -2B_\alpha^n(a_0 - a_1) + B_{\alpha+1}^n(a_2 - a_1 - a_{-1} + a_0) + B_{\alpha+2}^n(a_{-1} - a_2)\right].\end{aligned}$$

The above equality implies that we are reaching a family of linear central five-point discretizations of $\partial_{xx} B$. It is easy to prove the following result.

Proposition 3.1. *Given $\{a_k,\ -1 \le k \le 2\}$ satisfying* (3.8)*, there exists $\mathcal{B}_\alpha^n$ and $\epsilon > 0$ such that* (3.28) *is satisfied. $\mathcal{B}_\alpha^n$ is given by*

$$\begin{cases} \mathcal{B}_\alpha^n = mB_{\alpha-1}^n + (1-2m)B_\alpha^n + mB_{\alpha+1}^n, \\ m = \dfrac{-a_2 + a_{-1}}{a_0 - a_1 - 3a_2 + 3a_{-1}}, \end{cases} \tag{3.29}$$

and ϵ is such that

$$\left(\frac{\mu}{\sqrt{2}}+\frac{\theta}{\sqrt{\epsilon}}\right)\frac{\Delta x}{2\theta^2}=\frac{1-3m}{a_0-a_1}\,. \tag{3.30}$$

Moreover, $m \in [0,1/4]$.

Of course μ must be small enough but this is always possible; we may take for example $\mu=0$. The obtained ϵ is positive and of order Δx^2. The CFL condition (3.22) may be expressed as

$$\Delta t \leq \frac{\Delta x^2}{2(1-3m)\theta^2}. \tag{3.31}$$

Finally, we have designed an approximation of (1.1) in the form (3.3). The hyperbolic flux $\mathcal{A}^n_{\alpha+1/2}$ depends only on the (possibly high-order) discretization of the N first equations of (3.15) and is given by (3.27). Note that even for $l \leq N$, M_l depends on B, so that also $\mathcal{A}^n_{\alpha+1/2}$ does. We obtain a central linear approximation of $\partial_{xx}B$. The time step restriction for this scheme is the one imposed by the CFL condition on H^ϵ_Δ:

$$\Delta t \leq \min\left\{\frac{\Delta x^2}{2\theta^2(1-3m)},\ \frac{\Delta x}{|\lambda_l|}\ ,\ 1\leq l\leq N\right\}. \tag{3.32}$$

The following proposition gives the reciprocal. By using our methods, we are able to reach all monotone linear central five-point discretizations of $\partial_{xx}B$. The proof follows easily and will be omitted.

Proposition 3.2. *We suppose that the flux function $\mathcal{A}$ is given by* (3.27)*. Consider a central five-point discretization of $\partial_{xx}B(u)$ in the form $\dfrac{\mathcal{B}_{\alpha+1}-2\mathcal{B}_\alpha+\mathcal{B}_{\alpha-1}}{\Delta x^2}$ with* (3.13)*. There exist $\epsilon>0$ and $\{a_k,\ -1\leq k\leq 2\}$ satisfying $\sum_{k=-1}^{2}a_k=1$ and* (3.8)*, such that the scheme takes the form* (3.3)*. Moreover the time step condition can be expressed independently of the a_k as* (3.32)*.*

3.1.2. *Higher space dimensions.* By using the ideas presented for the one space dimension, one may develop the higher dimensional case. We approximate the transport part by a high-order scheme on the first N equations and a linear scheme on the last N' ones (see Lemma 3.1). Then we solve exactly the ODEs associated to the RHS and put $\epsilon=0$ in this last step. This yields

$$u^{\epsilon,n+1}_\alpha=u^{\epsilon,n}_\alpha-\sum_{d=1}^{D}\frac{\Delta t}{\Delta x_d}\left(\mathcal{F}^{\epsilon,n}_{\alpha+e_d/2}-\mathcal{F}^{\epsilon,n}_{\alpha-e_d/2}\right). \tag{3.33}$$

We retrieve the numerical flux function associated to A with the first N equations

$$\mathcal{A}^n_{d,\alpha+e_d/2}=\sum_{l=1}^{N}\lambda_{ld}F_{ld}\left(M_l\left(u^{\epsilon,n}_{\alpha-k_l+e_d}\right),\dots,M_l\left(u^{\epsilon,n}_{\alpha+k_l}\right)\right),\quad d=1,\dots,D \tag{3.34}$$

with obvious notations. Concerning the diffusion, we write the generalization of (3.28), and for a three-point scheme we find that $\mathcal{B}^n_{\alpha,d}=B^n_\alpha$, if we take ϵ such that

$$\gamma^\epsilon:=\mu+\frac{\theta\sqrt{N'}}{\sqrt{\epsilon}}=\frac{N'\theta^2}{s_d\Delta x_d(a_{0d}-a_{1d})}, \tag{3.35}$$

where $s_d=\sum_{m,\sigma_{md}>0}\sigma_{md}$. This shows that there is an essential difference with the one dimensional case: an arbitrary choice of a_{jd} is not possible because γ^ϵ must not depend on d. Nevertheless, and this is most important, we are able to give a generalization of Proposition 3.2. The proof will be omitted.

Proposition 3.3. *Take*

$$N' = 2D \text{ and } \sigma_{md} = \begin{cases} -1/\sqrt{2} & \text{if } m = 2d-1, \\ 1/\sqrt{2} & \text{if } m = 2d, \\ 0 & \text{otherwise.} \end{cases} \tag{3.36}$$

Consider a monotone central five-point discretization of $\Delta B(u)$ in the form $\Delta_{m,\alpha} = \sum_{d=1}^{D} \frac{\mathcal{B}_{\alpha+e_d,d} - 2\mathcal{B}_\alpha + \mathcal{B}_{\alpha-e_d,d}}{\Delta x_d^2}$ with (3.13). *There exist $\epsilon > 0$ and $\{a_{kd},\ -1 \le k \le 2,\ 1 \le d \le D\}$ satisfying $\sum_{k=-1}^{2} a_{kd} = 1$ and* (3.8)*, such that the numerical scheme* (3.33) *takes the form* (3.3)*. Moreover the CFL condition can be expressed independently of the a_{kd}, by the inequality*

$$\Delta t \le \min \left\{ \left(\sum_{d=1}^{D} \frac{|\lambda_{ld}|}{\Delta x_d} \right)^{-1}, \left(\sum_{d=1}^{D} \frac{2D(1-3m_d)\theta^2}{\Delta x_d^2} \right)^{-1} \right\}. \tag{3.37}$$

For instance, if we apply this method to DRM2 with upwind scheme on each transport equation, we obtain a parabolic version of the HLL scheme, which for $\lambda_m \le 0 \le \lambda_p$ reads

(3.38)
$$\begin{aligned} u_\alpha^{n+1} &= u_\alpha^n - \frac{\Delta t}{\Delta x} \frac{-\lambda_m A_{\alpha+1}^n + (\lambda_m + \lambda_p) A_\alpha^n - \lambda_p A_{\alpha-1}^n}{\lambda_p - \lambda_m} + \Delta t\, \Delta_{m,\alpha} B^n \\ &\quad + \frac{\Delta t}{\Delta x} \left| \frac{\lambda_m \lambda_p}{\lambda_p - \lambda_m} \right| \left[u_{\alpha+1}^n - \frac{B_{\alpha+1}^n}{\theta^2} - 2\left(u_\alpha^n - \frac{B_\alpha^n}{\theta^2} \right) + u_{\alpha-1}^n - \frac{B_{\alpha-1}^n}{\theta^2} \right]. \end{aligned}$$

Remark 3.1. This kinetic splitting method can also be used for other models, where the Maxwellian functions depend on ϵ. For the three velocity one dimensional model presented in [5], we obtain, at the first order, the scheme

$$u_\alpha^{n+1} = u_\alpha^n - \Delta t \frac{A(u_{\alpha+1}^n) - A(u_{\alpha-1}^n)}{2\Delta x} + \Delta t\, \Delta_{m,\alpha} B^n.$$

Here, we can see clearly the coherence between the monotonicity condition, which imposes the nondegeneracy of the diffusion, and the related discretization which becomes unstable in this case.

3.2. **A semi-conservative splitting method.** Here, we adopt a different approach, which consists in constructing a scheme for the kinetic model (1.3) and then making ϵ tend to zero to obtain a so-called relaxed scheme for (1.1). As seen in the previous section, this is not possible if one keeps the kinetic variables. The following alternative approach is partially inspired by the spirit of [17]. Nevertheless, we point out that our schemes, due to a different and more rigorous analysis of the stability conditions, in general do not coincide with those proposed in that paper.

Let us consider a multidimensional model in the form (1.3) with $N' = D + 1$. We transform this system in a linear way, keeping the unknowns $f_1^\epsilon, \dots, f_N^\epsilon$ and replacing $f_{N+1}^\epsilon, \dots, f_{N+N'}^\epsilon$ by

$$u^\epsilon = \sum_{l=1}^{N+N'} f_l^\epsilon \quad \text{and} \quad v_j^\epsilon = \sum_{l=1}^{N} \lambda_{lj} f_l^\epsilon + \gamma^\epsilon \sum_{m=1}^{D+1} \sigma_{mj} f_{N+m}^\epsilon, \quad 1 \le j \le D, \tag{3.39}$$

where $\gamma^\epsilon = \mu + \frac{\theta\sqrt{D+1}}{\sqrt{\epsilon}}$. This transformation is regular. We obtain

$$
\left\{
\begin{array}{lcl}
\partial_t f_l^\epsilon & + & \displaystyle\sum_{d=1}^{D} \lambda_{ld}\partial_{x_d} f_l^\epsilon = \frac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right), \quad 1 \le l \le N, \\
\partial_t u^\epsilon & + & \displaystyle\sum_{d=1}^{D} \partial_{x_d} v_d^\epsilon = 0, \\
\partial_t v_j^\epsilon & + & \displaystyle\sum_{d=1}^{D}\left[\sum_{l=1}^{N} \lambda_{lj}\lambda_{ld}\partial_{x_d} f_l^\epsilon + \mu^2 \sum_{m=1}^{N'} \sigma_{mj}\sigma_{md}\partial_{x_d} f_{N+m}^\epsilon\right] = \frac{1}{\epsilon}\left(A_j - v_j\right) \\
 & - & \displaystyle\frac{1}{\epsilon}(\sqrt{\epsilon}2\mu\theta\sqrt{N'} + \theta^2 N')\sum_{d=1}^{D}\sum_{m=1}^{N'} \sigma_{mj}\sigma_{md}\partial_{x_d} f_{N+m}^\epsilon, \quad 1 \le j \le D.
\end{array}
\right.
\tag{3.40}
$$

The numerical approximation is constructed as follows. We fix the initial data $U^0 = (f_1^0, \dots, f_N^0, u^0, v^0)$ by (3.1), (3.2), and(3.39), and we denote by U^n the approximate solution at time t_n. We split (3.40) into its LHS and RHS. The LHS is a linear first-order system which can be written in the form

$$
\partial_t U + \sum_{d=1}^{D} C^{(d)}\partial_{x_d} U = 0 \tag{3.41}
$$

where the $C^{(d)}$ are block matrices defined by

$$
C^{(d)} = \begin{pmatrix} \mathrm{Diag}(\lambda_{ld}) & 0 \\ D^{(d)} & \Sigma^{(d)} \end{pmatrix}. \tag{3.42}
$$

As we show below, system (3.41) is hyperbolic diagonalizable if and only if $\mu \neq 0$. As a consequence we should discretize it by any method for hyperbolic linear systems including finite volumes on unstructured meshes. Anyway, for the sake of simplicity, we shall continue in this paper to deal with cartesian grids and applying a stable high-order conservative method, we obtain $U^{n+1/2}$. In particular, this yields

$$
u_\alpha^{n+1/2} = u_\alpha^n - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d}\left(v_{d,\alpha+e_d/2}^n - v_{d,\alpha-e_d/2}^n\right). \tag{3.43}
$$

The numerical solution U^{n+1} is then an approximation of the limit when ϵ tends to zero of the solution of the system

$$
\left\{
\begin{array}{l}
\partial_t f_l^\epsilon = \displaystyle\frac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right), \quad 1 \le l \le N, \\
\partial_t u^\epsilon = 0, \\
\partial_t v_j^\epsilon = \displaystyle\frac{1}{\epsilon}\left(A_j - v_j - (\sqrt{\epsilon}2\mu\theta\sqrt{N'} + \theta^2 N')\sum_{d=1}^{D}\sum_{m=1}^{N'} \sigma_{mj}\sigma_{md}\partial_{x_d} f_{N+m}^\epsilon\right),
\end{array}
\right.
\tag{3.44}
$$

for the initial condition $U^{n+1/2}$. The following lemma is proven in subsections 3.2.1 and 3.2.2:

Lemma 3.3. *Denote by $U^\epsilon(t) = R^\epsilon(t, t_n, U^{n+1/2})$ the solution of* (3.44) *with the initial condition $U^\epsilon(t_n) = U^{n+1/2}$, and let*

$$
\overline{U^{n+1}} = \lim_{\epsilon \to 0} U^\epsilon(t_{n+1}).
$$

Then $\overline{U^{n+1}}$ is given by

$$\overline{u^{n+1}} = u^{n+1/2}, \quad \overline{f_l^{n+1}} = M_l(\overline{u^{n+1}}), \ \ 1 \le l \le N,$$

and

$$\overline{v_j^{n+1}} = A_j(\overline{u^{n+1}}) - \partial_{x_j} B(\overline{u^{n+1}}), \ \ 1 \le j \le D.$$

As a consequence, setting $U^{n+1} = \overline{U^{n+1}}$, we obtain

$$u_\alpha^{n+1} = u_\alpha^n - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d} \left[(A_d(u^n) - \partial_{x_d} B(u^n))_{\alpha+e_d/2} - (A_d(u^n) - \partial_{x_d} B(u^n))_{\alpha-e_d/2} \right]. \tag{3.45}$$

This equality insures that, if converging, the method leads to the right solution. In practice we have to define an approximate value for $\partial_{x_d} B(u^n)_\alpha$. Here we choose a standard central difference formula, like

$$\partial_{x_d} B(u^n)_\alpha = \frac{B(u^n_{\alpha+e_d}) - B(u^n_{\alpha-e_d})}{2\Delta x_d}. \tag{3.46}$$

Hence we set

$$\begin{cases} u_\alpha^{n+1} = u_\alpha^{n+1/2}, & \\ f_{\alpha,l}^{n+1} = M_l(u_\alpha^{n+1}), & 1 \le l \le N, \\ v_{\alpha,j}^{n+1} = A_j(u_\alpha^{n+1}) - \dfrac{B(u^{n+1}_{\alpha+e_d}) - B(u^{n+1}_{\alpha-e_d})}{2\Delta x_d}, & 1 \le j \le D. \end{cases} \tag{3.47}$$

It remains to prove the hyperbolicity of system (3.41). For all $\xi \in \mathbb{R}^D$, $|\xi| = 1$, and for all $\mu > 0$, let us set

$$C_\mu(\xi) = \sum_{d=1}^{D} \xi_d C^{(d)} = \begin{pmatrix} \Lambda(\xi) & 0 \\ D_\mu(\xi) & \Sigma_\mu(\xi) \end{pmatrix}. \tag{3.48}$$

The following two results are proved in subsection 3.2.1 for $D = 1$ and in subsection 3.2.2 for $D = 2$, respectively.

Lemma 3.4. *For all $\mu > 0$ and for all $\xi \in \mathbb{R}^D$ such that $|\xi| = 1$, $\Sigma_\mu(\xi)$ has real eigenvalues and is diagonalizable. Moreover there exists an orthogonal matrix $P(\xi)$ such that the matrix $\Pi(\mu, \xi)$ of right eigenvectors of $\Sigma_\mu(\xi)$ can be written, for $k = 2, \ldots, D+1$ and $j = 1, \ldots, D+1$, as*

$$\Pi_{1j}(\mu, \xi) = P_{1j}(\xi) \frac{\sqrt{D+1}}{\mu}, \qquad \Pi_{kj}(\mu, \xi) = P_{kj}(\xi). \tag{3.49}$$

Proposition 3.4. *For all $\mu > 0$, the system (3.41) is hyperbolic diagonalizable. For all $\xi \in \mathbb{R}^2$, with $|\xi| = 1$, the matrix of eigenvectors of $C_\mu(\xi)$ takes the form*

$$Q(\mu, \xi) = \begin{pmatrix} I & 0 \\ R & \Pi(\mu, \xi) \end{pmatrix} \tag{3.50}$$

where I is the identity matrix in $\mathbb{R}^N$, $\Pi(\mu, \xi)$ is defined in (3.49), and, for $1 \le l \le N$ and $1 \le d \le D$,

$$R_{1,l} = 1, \ \ R_{1+d,l} = \lambda_{ld}.$$

For $\mu = 0$, the system is not (strongly) hyperbolic.

3.2.1. *One space dimension.* In the case $D = 1$, the system (3.40) takes the simpler form

$$\left\{\begin{array}{ll} \partial_t f_l^\epsilon + & \lambda_l \partial_x f_l^\epsilon = \frac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right) \qquad 1 \le l \le N, \\ \partial_t u^\epsilon + & \partial_x v^\epsilon = 0, \\ \partial_t v^\epsilon + & \sum_{l=1}^{N} (\lambda_l^2 - \frac{\mu^2}{2}) \partial_x f_l^\epsilon + \frac{\mu^2}{2} \partial_x u \\ & = \frac{1}{\epsilon}\left(A - v - (\theta\mu\sqrt{2}\sqrt{\epsilon} + \theta^2)\partial_x (f_{N+1}^\epsilon + f_{N+2}^\epsilon)\right). \end{array}\right. \tag{3.51}$$

Let us first prove Lemma 3.3.

Proof of Lemma 3.3 for $D = 1$. We have to solve the system

$$\left\{\begin{array}{ll} \partial_t f_l^\epsilon & = \frac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right) \quad 1 \le l \le N, \\ \partial_t u^\epsilon & = 0, \\ \partial_t v^\epsilon & = \frac{1}{\epsilon}\left(A(u^\epsilon) - v^\epsilon - (\theta\mu\sqrt{2}\sqrt{\epsilon} + \theta^2)\partial_x (f_{N+1}^\epsilon + f_{N+2}^\epsilon)\right). \end{array}\right. \tag{3.52}$$

Since $u^\epsilon = u^{n+1/2}$, there is an analytical solution. □

A direct computation shows that, as far as $\mu \neq 0$, the left-hand side of system (3.51) is hyperbolic diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_l, -\mu/\sqrt{2}, \mu/\sqrt{2}$, while for $\mu = 0$ it is not diagonalizable. The matrix $P(\xi)$ is

$$P(\xi) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and the diagonal variables are

$$\left\{\begin{array}{l} f_1, \dots, f_N, \\ y = \frac{1}{2}\left[\mu\left(u - \sum_{l=1}^{N} f_l\right) - \sqrt{2}\left(v - \sum_{l=1}^{N} \lambda_l f_l\right)\right], \\ z = \frac{1}{2}\left[\mu\left(u - \sum_{l=1}^{N} f_l\right) + \sqrt{2}\left(v - \sum_{l=1}^{N} \lambda_l f_l\right)\right]. \end{array}\right.$$

On the other hand, we have

$$u = \sum_{l=1}^{N} f_l + (y+z)/\mu, \qquad v = \sum_{l=1}^{N} \lambda_l f_l + (-y+z)/\sqrt{2}. \tag{3.53}$$

Let us present now our numerical schemes in explicit form. We begin by applying a high-order conservative scheme on the first N diagonal variables and a first-order upwind scheme on y, z. Formulas (3.47) give

(3.54)
$$\begin{aligned} u_\alpha^{n+1} = \quad & u_\alpha^n - \frac{\Delta t}{\Delta x}\sum_{l=1}^N \lambda_l (f^n_{\alpha+1/2,l} - f^n_{\alpha-1/2,l}) + \frac{\mu \Delta t}{2\sqrt{2}\theta^2 \Delta x}(B^n_{\alpha+1} - 2B^n_\alpha + B^n_{\alpha-1}) \\ & + \frac{\Delta t}{4\Delta x^2}(B^n_{\alpha+2} - 2B^n_\alpha + B^n_{\alpha-2}). \end{aligned}$$

We retrieve the same hyperbolic numerical flux given by the kinetic splitting, namely $\mathcal{A}^n_{\alpha+1/2} = \sum_{l=1}^N \lambda_l f^n_{\alpha+1/2,l}$. To this flux, we have to add a supplementary numerical diffusion depending on μ (end of the first line). However, the only requirement on μ is its positivity, so that we can take it very small, for instance setting

$$\mu = 2\sqrt{2}\theta^2 \eta, \quad \eta << 1.$$

We remark that the time step restriction is to be given a posteriori. The resolution of the LHS requires only a hyperbolic CFL condition but it is clear that we must impose a parabolic condition. In the limit $\mu \to 0$, we retrieve some of the schemes obtained by the kinetic splitting, and provided that we keep first-order schemes on the unknown y, z, we can even put $\mu = 0$ in (3.54). We can also take a higher order scheme on these two variables. Hence, as also confirmed by numerical tests, the main advantage of this technique is the fact that it could be applied to noncartesian meshes. We intend to explore this possibility in a future paper.

3.2.2. *Two space dimensions.* Let us set

$$s_{ijk} = \sum_{m=1}^3 \sigma_{mi}\sigma_{mj}\sigma_{mk}$$

and

$$s_{ij}(\xi) = s_{1ij}\xi_1 + s_{2ij}\xi_2 \quad \text{for } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Proof of Lemma 3.3 for $D = 2$. We have to solve the 2×2 system of PDE for v given by

(3.55)
$$\begin{aligned} \partial_t v_j^\epsilon + \frac{a^\epsilon}{\sqrt{\epsilon}} \sum_{d=1}^2 \sum_{k=1}^2 s_{djk} \partial_{x_k} v_k^\epsilon \quad & = \tfrac{1}{\epsilon}\Big[A_j(u^{n+1/2}) - v_j^\epsilon - \partial_{x_j} B(u^{n+1/2}) \\ & \quad + \rho_j^\epsilon \sqrt{\epsilon} + e^{-(t-t_n)/\epsilon}(p_j^\epsilon + q_j^\epsilon \sqrt{\epsilon})\Big], \end{aligned}$$

for $j = 1, 2$. Here a^ϵ, $\rho_j^\epsilon(x)$, $p_j^\epsilon(x)$, $q_j^\epsilon(x)$ are uniformly bounded for $\epsilon \in [0, 1]$. By Fourier transform in the space variable we can write the solution and make $\epsilon \to 0$, so that we obtain the result. □

Proof of Proposition 3.4 for $D = 2$. The matrix $D_\mu(\xi)$ in (3.48) takes the form

$$\left\{\begin{array}{rl} D_{1l}(\xi) = & 0, \\ D_{2l}(\xi) = & \left(\lambda_{l1}^2 - \mu^2/3 - \mu(s_{111}\lambda_{l1} + s_{112}\lambda_{l2})\right)\xi_1 \\ & + (\lambda_{l1}\lambda_{l2} - \mu(s_{211}\lambda_{l1} + s_{212}\lambda_{l2}))\,\xi_2, \\ D_{3l}(\xi) = & (\lambda_{l1}\lambda_{l2} - \mu(s_{211}\lambda_{l1} + s_{212}\lambda_{l2}))\,\xi_1 \\ & + \left(\lambda_{l2}^2 - \mu^2/3 - \mu(s_{221}\lambda_{l1} + s_{222}\lambda_{l2})\right)\xi_2, \end{array}\right. \tag{3.56}$$

for $1 \le l \le N$. The matrix $\Sigma_\mu(\xi)$ in (3.48) is given by

$$\Sigma_\mu(\xi) = \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1\mu^2/3 & \mu s_{11}(\xi) & \mu s_{12}(\xi) \\ \xi_2\mu^2/3 & \mu s_{12}(\xi) & \mu s_{22}(\xi) \end{pmatrix}. \tag{3.57}$$

The eigenvalues of $C_\mu(\xi)$ are $\Lambda_l(\xi) = \xi_1\lambda_{l1} + \xi_2\lambda_{l2}$, $l = 1, \dots, N$, and the eigenvalues of $\Sigma_\mu(\xi)$. We can take as eigenvector for $\Lambda_l(\xi)$

$$(f, u, v_1, v_2) = (e_l, 1, \lambda_{l1}, \lambda_{l2}), \tag{3.58}$$

where e_l is the l^{th} canonical vector of $\mathbb{R}^N$. Consequently $C_\mu(\xi)$ is diagonalizable if and only if $\Sigma_\mu(\xi)$ is diagonalizable. To end the proof, it remains to prove Lemma 3.4. □

Proof of Lemma 3.4. We put $F(\xi, \mu, \beta) = \det(\Sigma_\mu(\xi) - \beta)$. By dividing each column of $F(\xi, \mu, \beta)$ by μ, we remark that there exists a function $G(\xi, x)$, not depending on μ such that $F(\xi, \mu, \beta) = \mu^3 G(\xi, x = \beta/\mu)$. For $\mu = \sqrt{3}$, the roots x of $G(\xi, x) = 0$ are the eigenvalues of a symmetric matrix. Therefore there exist three real roots $x_1(\xi), x_2(\xi), x_3(\xi)$ for this equation. This proves that $\Sigma_\mu(\xi)$ has three real eigenvalues $\mu x_1(\xi) = \beta_1(\xi), \mu x_2(\xi) = \beta_2(\xi), \mu x_3(\xi) = \beta_3(\xi)$.

Now for $\mu = \sqrt{3}$, $\Sigma_\mu(\xi)$ is diagonalizable and there exists an orthogonal matrix $P(\xi)$ whose columns are eigenvectors for it. It is easy to see that for any $\mu > 0$ the matrix $\Pi(\mu, \xi)$ defined by (3.49) is the regular matrix of right eigenvectors we are looking for. This ends the proof of the lemma. □

3.3. **Higher order in time, implicit schemes.** To construct approximations, which have a higher order in time, we remark that we have obtained some functions $\mathcal{A}_d$ and $\mathcal{B}_d$ such that the formula (3.3) may be viewed as a first-order discretization of the differential equation

$$u'_\alpha = u_\alpha - \sum_{d=1}^{D} \frac{\overline{\mathcal{A}}_{\alpha+e_d/2} - \overline{\mathcal{A}}_{\alpha-e_d/2}}{\Delta x_d} + \sum_{d=1}^{D} \frac{\mathcal{B}_{\alpha+e_d,d} - 2\mathcal{B}_{\alpha,d} + \mathcal{B}_{\alpha-e_d,d}}{\Delta x_d^2}\,. \tag{3.59}$$

Here $\overline{\mathcal{A}}_{\alpha+e_d/2}$ is the limit of $\mathcal{A}_d\left(u_{\alpha-a+e_d}, \dots, u_{\alpha+a}\right)$ when $\Delta t \to 0$ and Δx is fixed, and $\mathcal{B}_{\alpha,d} = \mathcal{B}_d\left(u_{\alpha-b}, \dots, u_{\alpha+b}\right)$ (if a first-order scheme is used on the first N equations of the kinetic model, then $\mathcal{A} = \overline{\mathcal{A}}$). Therefore, one may discretize (3.59) with any suitable scheme which is high-order in time, using for example Runge-Kutta methods [31, 13, 12], or one may make an implicit treatment of the parabolic part of the equation. The stability condition on the hyperbolic part remains identical. As usual the order in time has to be related to order in space. The TVD properties of the numerical solution are kept by dividing the CFL condition by 2. Remark that (1.3) and the numerical choices above can be viewed as a “flux function manufacture" furnishing $\mathcal{A}_d$ and $\mathcal{B}_d$, which can then be mixed with any (stable) discretization in time.

4. Stability and convergence theory in the scalar case

Diagonality and quasimonotonicity in the sense of [14] are two important features of models (1.3). These properties are essential to prove the convergence of the sequence u^ϵ in [5], as well as in the proof of numerical convergence for the hyperbolic case first given in [2]. Here the kinetic splitting method preserves both diagonality and monotonicity so that, roughly speaking, the scheme (3.3) obtained by this method owns the same stability properties as H^ϵ_Δ, including the cell entropy inequality. The semi-conservative splitting method destroys these features and therefore to show convergence seems to be more difficult. However, we have already pointed out that for a large class of discretizations, the scheme (3.43), (3.47) is close to the one obtained by the kinetic splitting method. Therefore, in this section we study the stability properties of the scheme obtained by kinetic splitting on the system (1.3) with the choice (3.36), namely the scheme (3.3) with (3.34), (3.13) and (3.37). As a consequence of our results, and thanks to a recent result by Bouchut [3], we can finally state a new and quite general convergence result for a class of monotone flux splitting schemes.

In the sequel, the parameter ϵ will depend on Δx through the relation (3.30) or using Proposition 3.3, and we suppress the superscript ϵ in the notation. We call DKS the discrete kinetic scheme given by (3.1), (3.3) with (3.34), (3.13), (3.37), and any of the models of the form (1.3), (3.36). According to the time step restriction (3.37), Δx and Δt tend to zero in such a way that for all d, $\xi_d := \Delta t/\Delta x_d^2$ is constant. We denote by $\mathcal{V}$ the volume of each cell, namely $\mathcal{V} = \Delta x_1 \dots \Delta x_D$.

Convergence to a weak solution will be proven under the following hypotheses:
(H1) $u_0 \in \mathrm{L}^\infty(\mathbb{R}^D) \cap L^1(\mathbb{R}^D) \cap \mathrm{BV}(\mathbb{R}^D)$ and M is a MMF on $I = [-\|u_0\|_\infty, \|u_0\|_\infty]$.
(H2) H_Δ preserves the extrema, namely

$$\forall f = (f_{\alpha,l})_{1\le l\le L, \alpha\in\mathbb{Z}^D},\ \ \forall l,\ \ \forall\alpha,\ \ \min_{\alpha'\in\mathbb{Z}^D} f_{\alpha',l} \le (H_\Delta(\Delta t) f)_{\alpha,l} \le \max_{\alpha'\in\mathbb{Z}^D} f_{\alpha',l}.$$

If the scheme H_Δ is monotone, then we have convergence to the entropy solution of the problem and it is only necessary to suppose that
(H1′) $u_0 \in \mathrm{L}^\infty(\mathbb{R}^D) \cap L^1(\mathbb{R}^D)$ and M is a MMF on $I = [-\|u_0\|_\infty, \|u_0\|_\infty]$.

The main result of the present section is given by the following convergence theorem.

Theorem 4.1. *(i) Suppose that (H1) and (H2) are satisfied and that H_Δ is TVD. Then, as $\Delta x \to 0$, while keeping constant the ratio $\Delta t/\Delta x_d^2$, the numerical solution (given by the DKS) u_Δ converges in $\mathrm{L}^\infty([0,T], \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^D))$ towards a weak solution $u \in C([0,T], \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^D)) \cap \mathrm{L}^\infty([0,T]\times\mathbb{R}^D)$ of (1.1), (1.2).*

(ii) Suppose that H_Δ is monotone and that (H1′) is satisfied. Then the result of (i) holds. Moreover the convergence takes place in $\mathrm{L}^\infty([0,T], L^1(\mathbb{R}^D))$, the limit function $u \in C([0,T], L^1(\mathbb{R}^D)) \cap \mathrm{L}^\infty([0,T]\times\mathbb{R}^D)$ and u is the unique weak entropy solution of the problem in the sense of Carrillo, which is: for any $\eta : \mathbb{R} \to \mathbb{R}$ convex Lipschitz, there holds

$$\partial_t[\eta(u)] + \mathrm{div}[A_\eta(u)] - \Delta[B_\eta(u)] \le 0 \quad \text{in } \mathbb{R}^D\times]0,\infty[, \tag{4.1}$$

with $A'_\eta = \eta' A'$, $B'_\eta = \eta' B'$, and

$$\nabla_x[B(u)] \in L^2(\mathbb{R}^D\times]0,\infty[). \tag{4.2}$$

This result follows from Propositions 4.1 to 4.5 below. Let us recall the following useful inequalities:

$$\|u^0_\Delta\|_\infty \le \|u_0\|_\infty,\ \ \|u^0_\Delta\|_1 \le \|u_0\|_1,\ \ \mathrm{TV}(u^0_\Delta) \le \mathrm{TV}(u_0), \tag{4.3}$$

and

$$\|u^0_\Delta - u_0\|_1 \le \|\Delta x\|\mathrm{TV}(u_0). \tag{4.4}$$

Proposition 4.1. *Under hypotheses (H1),(H2), the DKS is L^∞ stable:*

$$\|u^n_\Delta\|_\infty \le \|u_0\|_\infty. \tag{4.5}$$

Proof. The DKS may be summarized as

$$f^n_{\alpha,l} = M_l(u^n_\alpha),\ \ u^{n+1}_\alpha = \sum_{l=1}^{N+N'} f^{n+1/2}_{\alpha,l},\ \ n \ge 0 \tag{4.6}$$

where $f^{n+1/2}_{\alpha,l}$ is given by formula (3.18)–(3.21) or their multidimensional analogues. We know that (4.5) is true for $n = 0$. Take now $n \ge 1$ and suppose that $u^{n-1}_\alpha \in [-\|u_0\|_\infty, \|u_0\|_\infty]$. Condition (H1) and (4.6) yield

$$M_l(-\|u_0\|_\infty) \le f^{n-1}_{\alpha,l} \le M_l(\|u_0\|_\infty),$$

and by (H2), we have

$$M_l(-\|u_0\|_\infty) \le f^{n-1/2}_{\alpha,l} \le M_l(\|u_0\|_\infty). \tag{4.7}$$

Now, summing on l, we find (4.5). □

Remark that the previous proof also shows that $\|f^n_{\Delta,l}\|_\infty \le \|M_l(u_0)\|_\infty$. We now study the BV and L^1 stability of the scheme.

Proposition 4.2. *Under hypotheses (H1),(H2), if H_Δ is TVD, then the DKS is TVD and the two following estimates hold true:*

$$\forall t \in [0, +\infty[,\quad \|u_\Delta(t)\|_1 \le \|u_0\|_1 + C_1 t\mathrm{TV}(u_0); \tag{4.8}$$

for all $T \ge 0$ and all compact sets K, there exists a constant $C = C(T, K, u_0)$ such that

$$\forall t, t' \in [0, T],\ \ \|u_\Delta(t) - u_\Delta(t')\|_{L^1(K)} \le C(|t - t'| + \Delta t)^{1/3}. \tag{4.9}$$

Notice that, in the purely hyperbolic case, the Lipschitz continuity of the numerical flux functions or, for monotone schemes, the L^1-contraction property, is used to prove the following time stability estimate:

$$\|u^{n+1}_\Delta - u^n_\Delta\|_1 \le C\Delta t.$$

From that, one concludes with

$$\|u_\Delta(t') - u_\Delta(t)\|_1 \le C(|t - t'| + \Delta t).$$

In our case such a method only leads to

$$\|u^{n+1}_\Delta - u^n_\Delta\|_1 \le C\sqrt{\Delta t},$$

which is not enough to conclude. Instead we use a different technique, originally inspired by Kružkov [20], to prove (4.9).

Proof of Proposition 4.2. By (4.3), $\mathrm{TV}(u_\Delta^0) \le \mathrm{TV}(u_0)$. For $n \ge 1$, suppose that $\mathrm{TV}(u_\Delta^{n-1}) \le \mathrm{TV}(u_0)$. We have

$$\mathrm{TV}(u_\Delta^n) = \sum_{\alpha\in\mathbb{Z}^D} \sum_{d=1}^{D} \frac{\mathcal{V}}{\Delta x_d} \left| \sum_{l=1}^{N+N'} \left(f_{\alpha+e_d,l}^{n-1/2} - f_{\alpha,l}^{n-1/2} \right) \right|.$$

Setting $\mathrm{TV}(f_\Delta) = \sum_{l=1}^{N+N'} \mathrm{TV}(f_{l,\Delta})$ and using the fact that (HS) is TVD, we have the estimate

$$\mathrm{TV}(u_\Delta^n) \le \mathrm{TV}(f_\Delta^{n-1}) = \sum_{\alpha\in\mathbb{Z}^D} \sum_{d=1}^{D} \frac{\mathcal{V}}{\Delta x_d} \sum_{l=1}^{N+N'} \left| M_l(u_{\alpha+e_d}^{n-1}) - M_l(u_\alpha^{n-1}) \right|.$$

By (H1) and (2.1), we conclude that

$$\mathrm{TV}(u_\Delta^n) \le \mathrm{TV}(u_\Delta^{n-1}) \le \mathrm{TV}(u_0), \tag{4.10}$$

and the DKS is TVD. In order to prove (4.8), let us write u_α^{n+1} as

$$u_\alpha^{n+1} = \sum_{l=1}^{N} M_l(u_\alpha^n) - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d} \left(\mathcal{A}_{\alpha+e_d/2}^n - \mathcal{A}_{\alpha-e_d/2}^n \right) + \sum_{l=N+1}^{N+N'} f_{\alpha,l}^{n+1/2}.$$

The numerical flux function $\mathcal{A}$ is Lipschitz continuous. Moreover the scheme (HS) is monotone on the last N' equations, which gives

$$\|f_{\Delta,l}^{n+1/2}\|_1 \le \|f_{\Delta,l}^n\|_1 = \|M_l(u_\Delta^n)\|_1, \quad l \ge N+1.$$

Consequently, denoting by C_1 the Lipschitz constant of $\mathcal{A}$, it holds that

$$\|u_\Delta^{n+1}\|_1 \le \sum_{l=1}^{N+N'} \|M_l(u_\Delta^n)\|_1 + C_1 \Delta t \mathrm{TV}(u_0).$$

Hypothesis (H1), (2.1) and L^∞ stability also give

$$\|u_\Delta^{n+1}\|_1 \le \|u_\Delta^n\|_1 + C_1 \Delta t \mathrm{TV}(u_0),$$

which finally implies (4.8).

Let us now prove (4.9). We use here Kružkov's method; see Lemmas 1 and 5 in [20]. We first prove that for all integers $m \ge 0$, $p \ge 1$, with $(m+p)\Delta t \le T$, there exists a constant $C(T, u_0) > 0$ such that, for all $\phi \in \mathcal{C}_0^2(\mathbb{R}^D)$,

$$\left| \int_{\mathbb{R}^D} \left(u_\Delta^{m+p} - u_\Delta^m \right) \phi dx \right| \le C(T, u_0) \cdot p\Delta t \cdot \|\phi\|_{\mathcal{C}_0^2}. \tag{4.11}$$

For all $\phi \in \mathcal{C}_0^2(\mathbb{R}^D)$, let us set

$$\phi_\alpha = \frac{1}{\mathcal{V}} \int_{\mathrm{I}_\alpha} \phi dx.$$

We use the scheme the DKS in the form (3.3). We have

(4.12)

$$\begin{aligned}\left|\int_{\mathbb{R}^D}(u_\Delta^{m+p}-u_\Delta^m)\phi dx\right| &= \mathcal{V}\left|\sum_{\alpha\in\mathbb{Z}^D}(u_\alpha^{m+p}-u_\alpha^m)\phi_\alpha\right| \\ &\le \mathcal{V}\sum_{\alpha\in\mathbb{Z}^D}\sum_{n=m}^{m+p-1}\sum_{d=1}^{D}\frac{\Delta t}{\Delta x_d}\left|\left(\mathcal{A}^n_{\alpha+e_d/2}-\mathcal{A}^n_{\alpha-e_d/2}\right)\phi_\alpha\right| \\ &+\mathcal{V}\sum_{\alpha\in\mathbb{Z}^D}\sum_{n=m}^{m+p-1}\sum_{d=1}^{D}\frac{\Delta t}{\Delta x_d^2}\left|(\mathcal{B}^n_{\alpha+e_d,d}-2\mathcal{B}^n_{\alpha+e_d,d}+\mathcal{B}^n_{\alpha+e_d,d})\phi_\alpha\right| \\ &= H+P.\end{aligned}$$

The Lipschitz continuity of the numerical flux functions and of the MMF implies that $\mathcal{A}$ and $\mathcal{B}$ are Lipschitz continuous. Using Green's formula we obtain

$$\begin{aligned} H &\le \mathcal{V}\sum_{\alpha\in\mathbb{Z}^D}\sum_{n=m}^{m+p-1}\sum_{d=1}^{D}\frac{\Delta t}{\Delta x_d}\left|\mathcal{A}^n_{\alpha+e_d/2}(\phi_\alpha-\phi_{\alpha-e_d})\right| \\ &\le \Delta t\sum_{n=m}^{m+p-1}\sum_{d=1}^{D}\sup_\alpha\left|\frac{\phi_\alpha-\phi_{\alpha-e_d}}{\Delta x_d}\right|\cdot C\cdot\|u_\Delta^n\|_1 \\ &\le C(T,u_0)p\Delta t\|\phi\|_{\mathcal{C}_0^1}. \qquad (4.13)\end{aligned}$$

In the same way

$$P\le C(T,u_0)p\Delta t\|\phi\|_{\mathcal{C}_0^2}, \qquad (4.14)$$

which ends the proof of (4.11).

We follow now the proof of Lemma 5 in [20]. Let K be a compact set and take $X>0$ such that $K\subset B(0,X/2):=\{x\in\mathbb{R}^D,\ \ |x|<X/2\}$, $\rho\in]0,X/2[$. Put

$$\psi(x)=\begin{cases}\operatorname{sgn}(u_\Delta^{m+p}(x)-u_\Delta^m(x)) & \text{if } |x|\le X-\rho,\\ 0 & \text{otherwise.}\end{cases}$$

A smooth approximation of ψ is constructed by a standard smoothing function $\xi_\rho(x)=\rho^{-D}\xi(x/\rho)$, where $\xi\in\mathcal{C}_0^\infty(\mathbb{R}^D)$, $\xi\ge 0$, $\xi(x)=0$ for $|x|>1$ and $\int_{\mathbb{R}^D}\xi dx=1$. The function $\psi_\rho=\xi_\rho*\psi\in\mathcal{C}_0^\infty$ with support in $B(0,X)$ and

$$\|\psi_\rho\|_{\mathcal{C}_0^2(\mathbb{R}^D)}\le C_2(\rho^{-1}+\rho^{-2}). \qquad (4.15)$$

This yields

(4.16)

$$\begin{aligned}\textstyle\int_K|u_\Delta^{m+p}(x)-u_\Delta^m(x)|dx &\le \textstyle\int_{B(0,X-\rho)}(u_\Delta^{m+p}(x)-u_\Delta^m(x))\psi_\rho(x)dx \\ &+\textstyle\int_{B(0,X-\rho)}(u_\Delta^{m+p}(x)-u_\Delta^m(x))(\psi(x)-\psi_\rho(x))dx \\ &= I_1+I_2.\end{aligned}$$

Using (4.11) and (4.15), we obtain

$$|I_1|\le C(T,u_0)p\Delta t(\rho^{-1}+\rho^{-2})+2\|u_0\|_\infty X^{D-1}\rho.$$

To estimate I_2, we follow exactly the proof of Lemma 1 in [20] and we find

$$
\begin{aligned}
|I_2| &\le 2\sup_{|z|\le\rho}\int_{\mathbb{R}^D}|(u_\Delta^{m+p}(x)-u_\Delta^m(x))-(u_\Delta^{m+p}(x-z)-u_\Delta^m(x-z))|dx \\
&\le C_3\rho\,\mathrm{TV}(u_0).
\end{aligned} \tag{4.17}
$$

Choosing $\rho=(p\Delta t)^{1/3}$, which is always possible by taking X large enough, gives

$$\|u_\Delta^{m+p}-u_\Delta^m\|_{L^1(K)}\le C(T,X,u_0)(p\Delta t)^{1/3}$$

and (4.9) follows. □

Proposition 4.3. *Under the hypothesis (H1), if H_Δ is monotone, then the DKS is monotone, L^1-contracting, TVD and estimate* (4.9) *holds. Moreover the estimate* (4.8) *can be improved in*

$$\forall t\in[0,+\infty[,\quad \|u_\Delta(t)\|_1\le\|u_0\|_1. \tag{4.18}$$

Proof of Proposition 4.3. Monotonicity is clear. Let us consider $\overline{u}_0$, such that $\|\overline{u}_0\|_\infty\le\|u_0\|_\infty$. For all $n\ge 0$, we have

$$\|u_\Delta^{n+1}-\overline{u}_\Delta^{n+1}\|_1\le\sum_{l=1}^{N+N'}\|f_{l,\Delta}^{n+1/2}-\overline{f}_{l,\Delta}^{n+1/2}\|_1.$$

The scheme H_Δ is monotone and therefore L^1-contracting:

$$\sum_{l=1}^{N+N'}\|f_{l,\Delta}^{n+1/2}-\overline{f}_{l,\Delta}^{n+1/2}\|_1\le\sum_{l=1}^{N+N'}\|f_{l,\Delta}^{n}-\overline{f}_{l,\Delta}^{n}\|_1.$$

Condition (2.1), L^∞-stability and the fact that M is a MMF on $[-\|u_0\|_\infty,\|u_0\|_\infty]$ imply that

$$
\begin{aligned}
\sum_{l=1}^{N+N'}\|f_{l,\Delta}^{n}-\overline{f}_{l,\Delta}^{n}\|_1 &= \sum_{l=1}^{N+N'}\|M_l(u_\Delta^n)-M_l(\overline{u}_\Delta^n)\|_1 \\
&= \|\sum_{l=1}^{N+N'}M_l(u_\Delta^n)-M_l(\overline{u}_\Delta^n)\|_1=\|u_\Delta^n-\overline{u}_\Delta^n\|_1,
\end{aligned} \tag{4.19}
$$

which proves that the scheme is L^1-contracting. If H_Δ is monotone, then it is TVD so that Proposition 4.2 applies. □

We can deal now with the problem of the entropy inequality (4.1).

Proposition 4.4. *Suppose that the hypotheses (H1),(H2) are satisfied and that a discrete entropy inequality holds for each component of H_Δ; i.e., for all convex function S, for all $l=1,\dots,N$, there exists a Lipschitz continuous function R_l such that*

$$\frac{S(f_{\alpha,l}^{n+1/2})-S(f_{\alpha,l}^n)}{\Delta t}+\sum_{d=1}^{D}\lambda_{ld}\frac{R_{\alpha+e_d/2,l}^n-R_{\alpha-e_d/2,l}^n}{\Delta x_d}\le 0, \tag{4.20}$$

with

$$R_{\alpha+e_d/2,l}^n=R_{ld}\left(f_{\alpha-s_l+e_d,l}^n,\dots,f_{\alpha+s_l,l}^n\right),\quad R_{ld}\left(f_l,\dots,f_l\right)=S(f_l).$$

Then a discrete entropy inequality holds for DKS: for all convex entropy η with the related entropy functions defined by $A'_\eta = \eta' A'$, $B'_\eta = \eta' B'$, there exist two Lipschitz continuous functions Φ and Π such that

$$\frac{\eta(u_\alpha^{n+1}) - \eta(u_\alpha^n)}{\Delta t} + \sum_{d=1}^{D} \left(\frac{\Phi^n_{\alpha+e_d/2} - \Phi^n_{\alpha-e_d/2}}{\Delta x_d} - \frac{\Pi^n_{\alpha+e_d,d} - 2\Pi^n_{\alpha,d} + \Pi^n_{\alpha-e_d,d}}{\Delta x_d^2} \right) \leq 0, \tag{4.21}$$

with

$$\Phi^n_{\alpha+e_d/2} = \Phi_d\left(u^n_{\alpha-s+e_d}, \dots, u^n_{\alpha+s}\right), \quad \Phi_d\left(u, \dots, u\right) = A_{\eta,d}(u)$$

and

$$\Pi^n_{\alpha,d} = \Pi_d\left(u^n_{\alpha-e_d}, u^n_\alpha, u^n_{\alpha+e_d}\right), \quad \Pi_d\left(u, u, u\right) = B_\eta(u).$$

In order to prove this proposition, let us recall some results from [5]. Let η be a convex entropy for (1.1). If η is a Kružkov entropy ($\eta(u) = |u-k| - |k|$), then the kinetic entropies are defined by

$$S_{l,\eta}(f) = |f - M_l(k)| - |M_l(k)|, \quad l = 1, \dots, N+N'.$$

Let us define, for every convex $\mathcal{C}^2$-entropy η, the associated kinetic entropy

$$S_{l,\eta}(f) = \int_{\mathbb{R}} \frac{1}{2}\left(|f - M_l(k)| - |M_l(k)|\right)\eta''(k)dk + \frac{1}{2} f\left(\eta'(-\infty) + \eta'(+\infty)\right). \tag{4.22}$$

The function $S_{l,\eta}$ is convex. Moreover, if M is strictly monotone on I and if $\eta \in \mathcal{C}^2(\mathbb{R})$, then $S_{l,\eta} \in \mathcal{C}^2([M_l(-\|u_0\|_\infty), M_l(\|u_0\|_\infty)])$ and we have, for all $w \in I$,

$$S'_{l,\eta}(M_l(w)) = \eta'(w), \tag{4.23}$$

$$S''_{l,\eta}(M_l(w)) = \eta''(w)\, M'_l(w)^{-1}. \tag{4.24}$$

Otherwise, the identity (4.23) holds in the sense of subdifferentials: for all $f \in [M_l(-\|u_0\|_\infty), M_l(\|u_0\|_\infty)]$, and for all $w \in I$,

$$S_{l,\eta}(M_l(w)) - S_{l,\eta}(f) - \eta'(w)(M_l(w) - f) \leq 0. \tag{4.25}$$

We have also

$$\sum_{l=1}^{N+N'} S_{l,\eta}(M_l(w)) = \eta(w) - \eta(0), \tag{4.26}$$

$$\sum_{l=1}^{N+N'} \gamma_{ld} S_{l,\eta}(M_l(w)) = A_{\eta,d}(w), \tag{4.27}$$

$$\sum_{l=1}^{N+N'} \theta_{ld}\theta_{lj} S_{l,\eta}(M_l(w)) = \delta_{jd} B_\eta(w), \tag{4.28}$$

where the notation of (1.3) and of Theorem 4.1 has been used. Due to the particular form of the present kinetic model, the last two identities can be reformulated as

$$\sum_{l=1}^{N} \lambda_{ld} S_{l,\eta}(M_l(w)) = A_{\eta,d}(w), \;\; S_{N+m,\eta}(M_{N+m}(w)) = \frac{B_\eta(w)}{2D\theta^2}, \;\; m = 1, \dots, 2D. \tag{4.29}$$

Proof of Proposition 4.4. Let η be a convex entropy. Set

$$E^n_{\alpha,l} = S_{l,\eta}(f^{n+1}_{\alpha,l}) - S_{l,\eta}(f^n_{\alpha,l}) + \Delta t \sum_{d=1}^{D} \gamma_{ld} \frac{R^n_{\alpha+e_d/2,l} - R^n_{\alpha-e_d/2,l}}{\Delta x_d}, \quad l = 1, \dots, N+N'.$$

We apply (4.20) with $S = S_{l,\eta}$ for each l:

$$\begin{aligned} E^n_{\alpha,l} &= E^n_{\alpha,l} + S_{l,\eta}(f^{n+1/2}_{\alpha,l}) - S_{l,\eta}(f^{n+1/2}_{\alpha,l}) \\ &\le S_{l,\eta}(f^{n+1}_{\alpha,l}) - S_{l,\eta}(f^{n+1/2}_{\alpha,l}) = S_{l,\eta}(M_l(u^{n+1}_\alpha)) - S_{l,\eta}(f^{n+1/2}_{\alpha,l}). \end{aligned} \tag{4.30}$$

Moreover, by (4.23) (or (4.25)), (4.6), (2.1) and by the convexity of $S_{l,\eta}$, we have

$$\begin{aligned} \textstyle\sum_{l=1}^{N+N'} E^n_{\alpha,l} &\le \textstyle\sum_{l=1}^{N+N'} \Big[S_{l,\eta}(M_l(u^{n+1}_\alpha)) - S_{l,\eta}(f^{n+1/2}_{\alpha,l}) \\ &\qquad - \eta'(u^{n+1}_\alpha)(M_l(u^{n+1}_\alpha) - f^{n+1/2}_{\alpha,l}) \Big] \le 0. \end{aligned} \tag{4.31}$$

Using (4.26), we obtain

$$\sum_{l=1}^{N+N'} E^n_{\alpha,l} = \eta(u^{n+1}_\alpha) - \eta(u^n_\alpha) + \Delta t \sum_{d=1}^{D} \sum_{l=1}^{N+N'} \gamma_{ld} \frac{R^n_{\alpha+e_d/2,l} - R^n_{\alpha-e_d/2,l}}{\Delta x_d}.$$

Moreover

$$\begin{aligned} \textstyle\sum_{l=1}^{N} \gamma_{ld} R^n_{\alpha+e_d/2,l} &= \textstyle\sum_{l=1}^{N} \lambda_{ld} R_{ld}(M_l(u^n_{\alpha-s_l+e_d}), \dots, M_l(u^n_{\alpha+s_l})) \\ &= \Phi_d(u^n_{\alpha-s_l+e_d}, \dots, u^n_{\alpha+s_l}), \end{aligned}$$

and

$$\Phi_d(u, \dots, u) = \sum_{l=1}^{N} \lambda_{ld} S_{l,\eta}(M_l(u)) = A_{\eta,d}(u).$$

For $l = N+m$, Lemma 3.1 and (4.29) yield

$$\begin{aligned} \sum_{l=N+1}^{N+N'} \gamma_{ld} R^n_{\alpha+e_d/2,l} &= \sum_{m=1}^{2D} \gamma^\epsilon \sigma_{md} \sum_{k=-1}^{2} a^m_{kd} S_{N+m,\eta}(M_{N+m}(u^n_{\alpha+ke_d})) \\ &= \frac{\gamma^\epsilon}{2D\theta^2\sqrt{2}} \sum_{k=-1}^{2} (a_{kd} - a_{2-k-1,d}) B_\eta(u^n_{\alpha+ke_d}), \end{aligned}$$

with $\gamma^\epsilon = \mu + \frac{\theta\sqrt{2D}}{\sqrt{\epsilon}}$. Consequently, by arguing as in subsection 3.1, we find

$$\Pi^n_{\alpha,d} = m_d B_\eta(u^n_{\alpha-e_d}) + (1-2m_d) B_\eta(u^n_\alpha) + m_d B_\eta(u^n_{\alpha+e_d}), \tag{4.32}$$

which concludes the proof. $\square$

Let us now obtain some estimates, which in turn will imply (4.2).

Proposition 4.5. *We make the same assumptions as in Proposition 4.4. There exist a constant $C > 0$ and real numbers $a_1, \dots, a_D$, such that for all $N \ge 0$*

$$\begin{aligned} \mathcal{V} \Delta t \textstyle\sum_{\alpha\in\mathbb{Z}^D} \sum_{n=0}^{N} \sum_{d=1}^{D} & \left| 2a_d \frac{B^n_{\alpha+2e_d} - B^n_{\alpha-2e_d}}{4\Delta x_d} + (1-2a_d) \frac{B^n_{\alpha+e_d} - B^n_{\alpha-e_d}}{2\Delta x_d} \right|^2 \\ &\le C \sup_{0\le n\le N} \|u^n_\Delta\|_1. \end{aligned} \tag{4.33}$$

Let us recall that we have an estimate of the solution in L^1 by (4.8) for the general case and by Proposition 4.3 for the monotone schemes.

Proof. We choose here the entropy $\eta(u) = u^2/2$. As in the proof of Proposition 4.4, we have again the inequality (4.31), but here we have to be more precise. Remark that each term of the sum in (4.31) is nonpositive. Hence

$$\begin{aligned}\sum_{l=1}^{N+N'} E_{\alpha,l}^n \quad &\le \sum_{l=N+1}^{N+N'} \Big[S_{l,\eta}(M_l(u_\alpha^{n+1})) - S_{l,\eta}(f_{\alpha,l}^{n+1/2}) \\ &\qquad - \eta'(u_\alpha^{n+1})(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2}) \Big] .\end{aligned} \tag{4.34}$$

We recall that $N' = 2D$, and $M_l(u) = \frac{B(u)}{2D\theta^2}$ for $l = N+1, \dots, N+N'$. Moreover, for all l, the inequality (4.7) and the continuity of M_l insure that $[M_l(u_\alpha^{n+1}), f_{\alpha,l}^{n+1/2}] \subset M_l(I)$, for $I = [-\|u_0\|_\infty, \|u_0\|_\infty]$. Let us first suppose that $B' > 0$ on I. This yields

$$\begin{aligned}\sum_{l=1}^{N+N'} E_{\alpha,l}^n \quad &\le -\tfrac{1}{2} \sum_{l=N+1}^{N+N'} \Big(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2} \Big)^2 \\ &\qquad \times \min \Big\{ S''_{l,\eta}(g), \quad g \in [M_l(u_\alpha^{n+1}), f_{\alpha,l}^{n+1/2}] \Big\} .\end{aligned}$$

Applying (4.24), we obtain

$$\sum_{l=1}^{N+N'} E_{\alpha,l}^n \le -\frac{1}{2} \sum_{l=N+1}^{N+N'} \Big(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2} \Big)^2 . \tag{4.35}$$

Let us integrate this inequality and divide by Δt

$$\begin{aligned}\frac{1}{2}\mathcal{V}\Delta t \sum_{n=0}^{N} \sum_{\alpha\in\mathbb{Z}^D} \sum_{l=N+1}^{N+N'} \left| \frac{M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2}}{\sqrt{\Delta t}} \right|^2 \quad &\le \quad -\mathcal{V} \sum_{\alpha\in\mathbb{Z}^D} \big(\eta(u_\alpha^N) - \eta(u_\alpha^0) \big) \\ &\le \quad C \sup_{0\le n\le N} \|u_\Delta^n\|_1 .\end{aligned} \tag{4.36}$$

On the other hand, we remark that for all l, it holds that

$$\frac{f_{\alpha,l}^{n+1} - f_{\alpha,l}^n}{\Delta t} + \sum_{d=1}^{D} \gamma_{ld} \frac{f_{\alpha+e_d/2,l}^n - f_{\alpha-e_d/2,l}^n}{\Delta x_d} = \frac{1}{\Delta t} \Big(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2} \Big) , \tag{4.37}$$

with $f_{\alpha,l}^n = M_l(u_\alpha^n)$. Consequently, substracting equations $N+2d-1$ and $N+2d$ yields

$$\begin{aligned}&\frac{\gamma^\epsilon}{2D\theta^2\sqrt{2}\Delta x_d} \big(a_d(B_{\alpha+2e_d}^n - B_{\alpha-2e_d}^n) + (1-2a_d)(B_{\alpha+e_d}^n - B_{\alpha-e_d}^n) \big) \\ &= \frac{1}{\Delta t} \Big((M_{N+2d-1}(u_\alpha^{n+1}) - f_{\alpha,N+2d-1}^{n+1/2}) - (M_{N+2d}(u_\alpha^{n+1}) - f_{\alpha,N+2d}^{n+1/2}) \Big)\end{aligned}$$

where we have put $a_d = a_{-1,d} + a_{2,d}$ and used (3.8). Now we put $\rho_d = \frac{\gamma^\epsilon \Delta x_d}{2D\theta^2\sqrt{2}}$. Using the properties of the scheme, it is easy to prove that ρ_d is a constant not depending on Δx. Using (4.36), it follows that

$$\begin{aligned}\mathcal{V}\Delta t \rho_d^2 \xi_d \sum_{\alpha\in\mathbb{Z}^D} \sum_{n=0}^{N} \quad &|2a_d \tfrac{B_{\alpha+2e_d}^n - B_{\alpha-2e_d}^n}{4\Delta x_d} + (1-2a_d) \tfrac{B_{\alpha+e_d}^n - B_{\alpha-e_d}^n}{2\Delta x_d}|^2 \\ &\le 2C \sup_{0\le n\le N} \|u_\Delta^n\|_1 .\end{aligned}$$

Here $\xi_d = \Delta t/\Delta x_d^2$ is constant.

Let us consider now the degenerate case, where possibly $B' = 0$ on a subset of $M_l^{-1}([M_l(u_\alpha^{n+1}), f_{\alpha,l}^{n+1/2}])$, for some $l \geq N+1$. There exists an integer J, not depending on Δx, α and n, such that for all $l \geq N+1$, there exists $k_l \leq J$ and

$$[M_l(u_\alpha^{n+1}), f_{\alpha,l}^{n+1/2}] = \bigcup_{1\leq j\leq k_l} [M_l(w_{l,j}^-), M_l(w_{l,j}^+)],$$

with $M_l' > 0$ on $]w_{l,j}^-, w_{l,j}^+[$, $w_{l,j}^- < w_{l,j}^+ < w_{l,j+1}^-$, and $M_l(w_{l,j}^+) = M_l(w_{l,j+1}^-)$.

Let us now fix $l \geq N+1$ and suppose that $M_l(u_\alpha^{n+1}) < f_{\alpha,l}^{n+1/2}$. Then,

$$\begin{aligned} & S_{l,\eta}(M_l(u_\alpha^{n+1})) - S_{l,\eta}(f_{\alpha,l}^{n+1/2}) - \eta'(u_\alpha^{n+1})(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2}) \\ = \; & \sum_{j=1}^{k_l} \left[S_{l,\eta}(M_l(w_{l,j}^-)) - S_{l,\eta}(M_l(w_{l,j}^+)) - \eta'(u_\alpha^{n+1})(M_l(w_{l,j}^-) - M_l(w_{l,j}^+)) \right]. \end{aligned}$$

Since η' and M_l are nondecreasing, it holds that

$$\eta'(w_{l,j}^-) \geq \eta'(u_\alpha^{n+1}).$$

Therefore, there exists a constant $C > 0$ such that

$$\begin{aligned} S_{l,\eta}(M_l(u_\alpha^{n+1})) & - S_{l,\eta}(f_{\alpha,l}^{n+1/2}) - \eta'(u_\alpha^{n+1})(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2}) \\ & \leq -\tfrac{1}{2} \sum_{j=1}^{k_l} \left(M_l(w_{l,j}^-) - M_l(w_{l,j}^+) \right)^2 \\ & \leq -C \left(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2} \right)^2. \end{aligned} \tag{4.38}$$

Hence we recover (4.35), up to a constant. The case $M_l(u_\alpha^{n+1}) > f_{\alpha,l}^{n+1/2}$ is similar. The end of the proof is the same as in the nondegenerate case. □

We have an important corollary which completes this result.

Theorem 4.2. *Take $K = D = 1$. Consider a monotone scheme for problem* (1.1), (1.2) *which is written in the form* (3.3), *with $\mathcal{B}_\alpha^n$ given by a linear formula* (3.13) *and let*

$$\mathcal{A}_{\alpha+1/2}^n = \mathcal{A}_\alpha^{n,+} - \mathcal{A}_{\alpha+1}^{n,-}, \tag{4.39}$$

for some functions $\mathcal{A}_\alpha^{n,\pm} = \mathcal{A}^\pm(u_\alpha^n)$, with $\mathcal{A}^{\pm\prime} \geq 0$ for all $u \in I$. Suppose that $u_0 \in \mathrm{L}^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then u_α^n converges in $\mathrm{L}^\infty([0,T], L^1(\mathbb{R}))$ towards the (unique) entropy weak solution $u \in C([0,T], L^1(\mathbb{R})) \cap \mathrm{L}^\infty([0,T] \times \mathbb{R})$ of (1.1), (1.2).

Proof. Thanks to a recent result by F. Bouchut [3], obtained in the purely hyperbolic case, this scheme can be derived by applying the kinetic splitting method to the model defined by the following choices:

$$\left\{ \begin{aligned} & \gamma^\epsilon = \left(\lambda, 0, -\lambda, -\frac{\mu}{\sqrt{2}} - \frac{\theta}{\sqrt{\epsilon}}, \frac{\mu}{\sqrt{2}} + \frac{\theta}{\sqrt{\epsilon}} \right)^T, \\ & M = \left(\frac{\mathcal{A}^+}{\lambda}, u - \frac{\mathcal{A}^+ + \mathcal{A}^-}{\lambda} - \frac{B}{\theta^2}, \frac{\mathcal{A}^-}{\lambda}, \frac{B}{2\theta^2}, \frac{B}{2\theta^2} \right)^T. \end{aligned} \right. \tag{4.40}$$

Now, the function M is a MMF if

$$\sup_{u\in I} \frac{\mathcal{A}^{+\prime}(u)+\mathcal{A}^{-\prime}(u)}{\lambda}+\frac{B'(u)}{\theta^2}\leq 1. \tag{4.41}$$

It is always possible to find such λ and θ. It remains to choose (H_Δ). In the last two equations, it is given by Proposition 3.2. In the first three, we take the upwind scheme

$$\mathcal{A}^n_{\alpha+1/2}=\lambda\left(f^n_{\alpha+1/2,1}-f^n_{\alpha+1/2,3}\right)=\mathcal{A}^{n,+}_\alpha-\mathcal{A}^{n,-}_\alpha.$$

This concludes the proof. □

5. Numerical results

In this section, we shall present some numerical results performed by using our diffusive BGK approximations. We also make detailed comparison between our schemes and the existing SD2 scheme by A. Kurganov and E. Tadmor [21]. The section ends with some further numerical simulations with the high-order PPM method and a 2D example.

5.1. **Comparison tests.** To fix a diffusive kinetic scheme for the equations (1.1), one needs to specify

- a diffusive BGK model,
- a splitting method for this model (kinetic or semi-conservative),
- a scalar hyperbolic problem solver,
- a time integration method.

In all our 1D simulations, the BGK model applied is the DRM1, the DRM2 or the FDM, introduced in Section 2 and in [2]. Both kinetic and semi-conservative splitting are considered. A specific choice yields a system (3.3), or in its semi-discrete form (3.59).

This system is then solved by a second-order Runge-Kutta (mid-point) method. That is, we combine two first-order (in time) Euler forward steps into a second-order time step. Each Euler forward step contains two substeps, namely, a convection step for the homogeneous linear hyperbolic system (3.16) or (3.41) and a projection step for the source terms. Each equation of the linear hyperbolic system is solved by a second-order MUSCL type scheme with minmod limiter, or first-order upwind scheme, as described in the previous sections. In the projection step, on the other hand, a "relaxed" scheme is always used as we directly project f_l^ϵ to the corresponding equilibrium state. Though we could solve explicitly the ODE (3.23) or the PDE (3.52) for nonzero ϵ, it turns out that this "relaxed" scheme always provides fewer numerical smearing effects.

More specifically, we shall perform numerical simulations with three particular numerical schemes based on diffusive BGK models. Depending on the splitting method adopted, they are

- KS1: the kinetic splitting method (3.15) with $a_0=1, a_1=0, m=0$, and ϵ determined by (3.30). The first N homogeneous equations are solved by MUSCL method, whereas the last two are solved by upwind method.
- KS2: the kinetic splitting method (3.15) with $a_0=1, a_1=0, m=0$, and ϵ determined by (3.30) but we add also a second-order correction on the last two equations, so that all homogeneous equations are solved by MUSCL method.

- SCS: the semi-conservative splitting method (3.51). All homogeneous equations are solved by MUSCL method.

Three other schemes are also used for comparison.

- Naive: a direct decoupling of the hyperbolic part and diffusion part for the original equation(s) (1.1). The quasilinear hyperbolic part is solved by MUSCL type scheme with minmod limiter, and the parabolic part by central difference.
- JPT: namely, the original splitting technique devised by S. Jin, L. Pareschi and G. Toscani [17].
- SD2: the second-order central semi-discretized scheme proposed by A. Kurganov and E. Tadmor [21]. Note that our SD2 code adopts a mid-point scheme for time integration, rather than DUMKA3 devised by A. Medovikov in the original paper [21].

We use $\theta = \lambda$ in all cases, and $\mu = \sqrt{2}\lambda$ unless otherwise mentioned. The MMF condition is not strictly satisfied in our codes. In fact, we take $\lambda = \max|A'(u)|$ unless otherwise mentioned. We obtain satisfactory numerical results, though a more rigorous study of this point is yet to be made. We remark here that this indifference to the violation of the MMF condition has also been reported for purely hyperbolic systems [2].

First we demonstrate the numerical solutions at time $t = 0.7$ with these schemes for fine grids $\triangle x = 0.01$ on a 1D Burgers type equation

$$u_t + (u^2)_x = 0.1(\nu(u)u_x)_x, \tag{5.1}$$

with the strongly degenerate diffusion

$$\nu(u) = \begin{cases} 0, & |u| \le 0.25, \\ 1, & |u| > 0.25, \end{cases}$$

and the initial condition

$$u(x,0) = \begin{cases} 1, & -\frac{1}{\sqrt{2}} - 0.4 < x < -\frac{1}{\sqrt{2}} - 0.4, \\ -1, & \frac{1}{\sqrt{2}} - 0.4 < x < \frac{1}{\sqrt{2}} - 0.4, \\ 0, & \text{otherwise}. \end{cases} \tag{5.2}$$

We use the flux decomposition model. Thanks to the symmetry of the solution, we only depict the right branch of the solution in Figure 1. As remarked before, the kinetic splitting with MUSCL method yields a solution with wrong speed. Other schemes, on the other hand, reach essentially the same solution. In fact, the solution with SCS is the same as that with the JPT scheme. We note that our semi-conservative splitting method is a natural extension of the JPT technique. They coincide for the simple case here. We also observe that the solution given by KS1 is identical to the one given by the naive splitting scheme. This fact may be proved by writing explicitly the numerical flux for the last two equations with upwind scheme. The functions M_1 and M_3 do not depend on $B(u)$, and M_2 gives no effect as f_2 is stationary. This leads to the independence of the scheme to the parameter θ.

In these numerical tests, our CFL condition (3.22) amounts to $\triangle t \le 125 \times 10^{-4}$. Yet numerical experiments are satisfactory at much bigger time step size. We use $\triangle t = 0.00025$ for the two kinetic splitting schemes and the naive splitting scheme, and $\triangle t = 0.0001$ for SD2. The schemes SCS and JPT are used with even bigger time step size $\triangle t = 0.0005$. We notice that small wiggles are still observed around

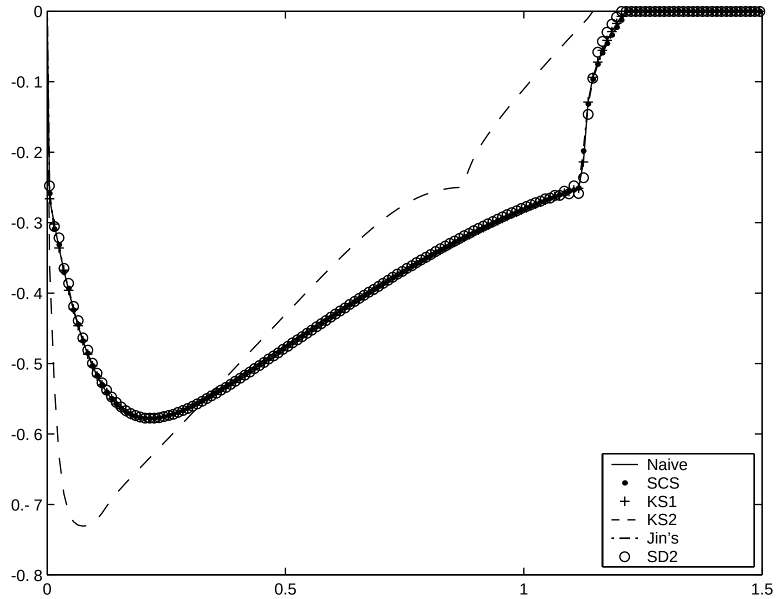

FIGURE 1. Comparison among different splitting methods on Pb. (5.1)–(5.2) , with $\triangle x = 0.01$ and $\triangle t = 0.00025$ for Naive, KS1 and KS2; $\triangle t = 0.0001$ for SD2; $\triangle t = 0.0005$ for SCS and JPT.

the degenerate points for SD2, though the time step size is already the smallest one.

Therefore, for coarse grids, we make a comparison among the KS1, JPT, SCS and SD2. We take the numerical solution by the SCS with fine grid as the "exact" solution. The results are shown in Figures 2 to 3.

The semi-conservative splitting again gives the same result as that obtained by the JPT scheme. Around the border of degeneracy, SD2 produces some oscillations. We summarize the L^1 errors in Table 1. It turns out that all these schemes are comparable in accuracy.

TABLE 1. The L^1 errors ($\times 10^{-2}$).

	SCS	KS1	JPT	SD2
$\triangle x = 0.02, \triangle t = 0.0005$	1.8394	1.7158	1.8394	2.4556
$\triangle x = 0.04, \triangle t = 0.001$	2.7105×10^2	2.5528	2.7105	3.4870
$\triangle x = 0.08, \triangle t = 0.005$	5.6389	5.1421	5.6389	7.0031

Thirdly, since the diffusive BGK model includes the parameter μ, we examine its effect on the accuracy for kinetic and semi-conservative splitting schemes. We denote $\mu = \sqrt{2}\alpha\lambda$ and fix discretization parameters $\triangle x = 0.04, \triangle t = 0.001$. When using KS1, the numerical experiments indicate that the solution is virtually not influenced by μ. The L^1 error remains at 2.5528×10^{-2} over $\alpha \in [0.001, 8]$. A

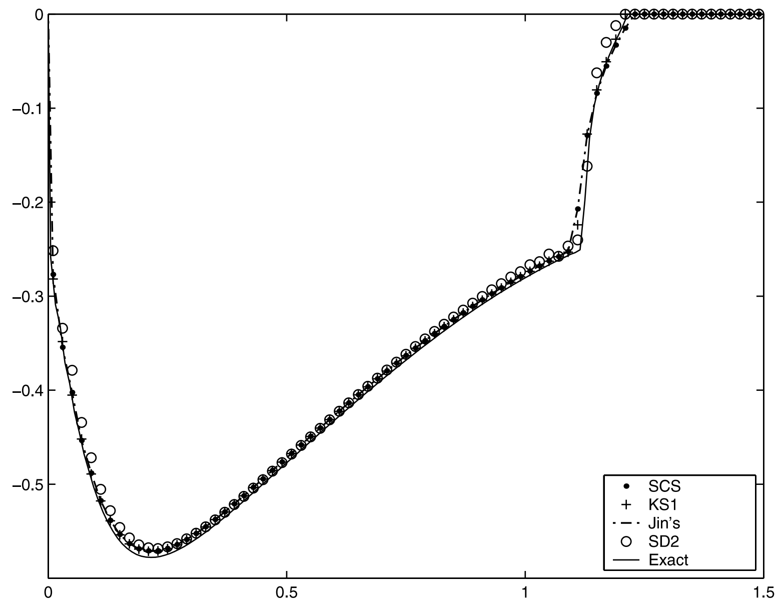

FIGURE 2. Comparison among different splitting methods on Pb. (5.1)–(5.2), with $\triangle x = 0.02, \triangle t = 0.0005$.

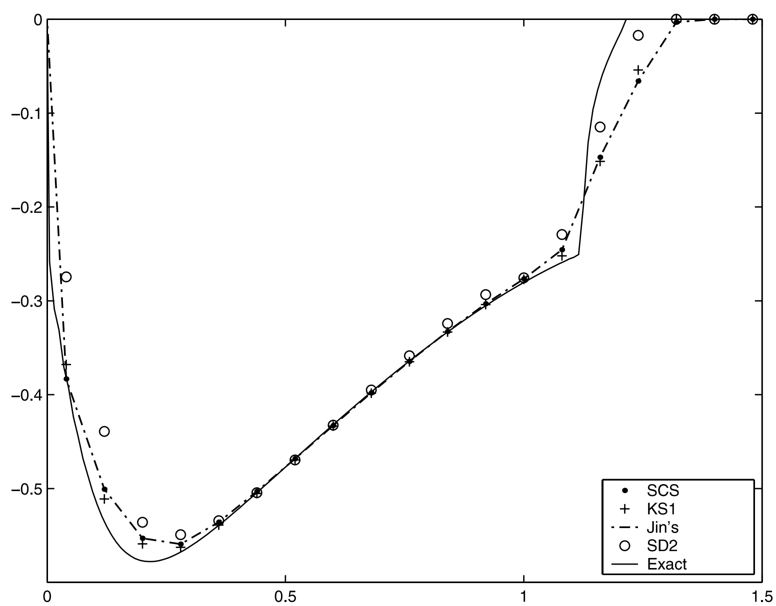

FIGURE 3. Comparison among different splitting methods on Pb. (5.1)–(5.2), with $\triangle x = 0.08, \triangle t = 0.005$.

detailed analysis shows that this is a result of the choice

$$\left(\frac{\mu}{\sqrt{2}} + \frac{\theta}{\sqrt{\epsilon}}\right) = \frac{2\theta^2}{\triangle x}.$$

On the other hand, for the SCS scheme, the L^1 error varies from 2.7035×10^{-2} to 2.7950×10^{-2} over $\alpha \in [0.001, 8]$. Smaller μ improves, yet with limitation when the error is mainly caused by discretization factors. Table 2 gives a quantitative illustration of the L^1 errors.

TABLE 2. The L^1 errors ($\times 10^{-2}$) for different α.

α	8	4	2	1	0.5
Error($\times 10^2$)	2.7590	2.7425	2.7186	2.7105	2.7071
α	0.25	0.125	0.0625	0.03125	0.001
Error($\times 10^2$)	2.7052	2.7043	2.7039	2.7037	2.7035

5.2. **A three phase flow model.** This example is taken from [19], and we refer to that paper for comparison with our results. We deal with a one dimensional 2×2 system modeling a multiphase flow in a porous medium:

$$\left\{ \begin{array}{l} \partial_t u + \partial_x f(u) = \epsilon \partial_{xx} d(u), \\ \partial_t v + \partial_x g(u,v) = \epsilon \partial_{xx} d(u), \end{array} \right. \tag{5.3}$$

with

$$(u_0(x), v_0(x)) = (0.4, 0.6) \text{ if } x < 1, \quad (0,0) \text{ otherwise.}$$

The unknown functions u and v are the phase saturations (gas and water) and then take values in $[0,1]$. The flux function $A(u,v) = (f(u), g(u,v))$ is defined by

$$\left\{ \begin{array}{l} f(u) = \dfrac{u^2}{u^2 + (1-u)^2/10}, \\ \\ g(u,v) = \dfrac{(1-u)^2 + u^2/10}{10u^2 + (1-u)^2} f(v) = \phi(u) f(v), \end{array} \right.$$

and the diffusion function $B(u,v) = (\epsilon d(u), \epsilon d(v))$ is such that

$$d'(w) = 4w(1-w).$$

In the following we take $\epsilon = 0.1$. For all (u,v), $A'(u,v)$ is a triangular matrix, while $B'(u,v)$ is diagonal, and we require monotonicity on the Maxwellian functions. First, we may remark that for all $(u,v) \in [0,1]^2$, the eigenvalues of $A'(u,v)$ are positive. Therefore, for the DRM2, M has positive eigenvalues if for all $(u,v) \in I$, the parameters λ_m, λ_p and θ satisfy

$$\left\{ \begin{array}{l} 1 - \dfrac{d'(u)}{\theta^2} \geq \alpha, \\ \\ 0 \leq \lambda_m \leq \dfrac{f'(u)}{1 - \dfrac{d'(u)}{\theta^2}} \leq \lambda_p, \end{array} \right. \qquad \left\{ \begin{array}{l} 1 - \dfrac{d'(v)}{\theta^2} \geq \alpha, \\ 0 \leq \lambda_m \leq \dfrac{\phi(u) f'(v)}{1 - \dfrac{d'(v)}{\theta^2}} \leq \lambda_p, \end{array} \right. \tag{5.4}$$

where $\alpha \in]0,1[$ is fixed. These parameters are evaluated at each time step. As for the scalar case, our experiments show that the accuracy is higher when α is close to 1, so that we have taken $\alpha = 0.9$. The solution is computed by the kinetic splitting method, with a second-order discretization in space and a first-order one in time. The value of the parameter m in the approximation of $\partial_{xx} d$ is $1/4$.

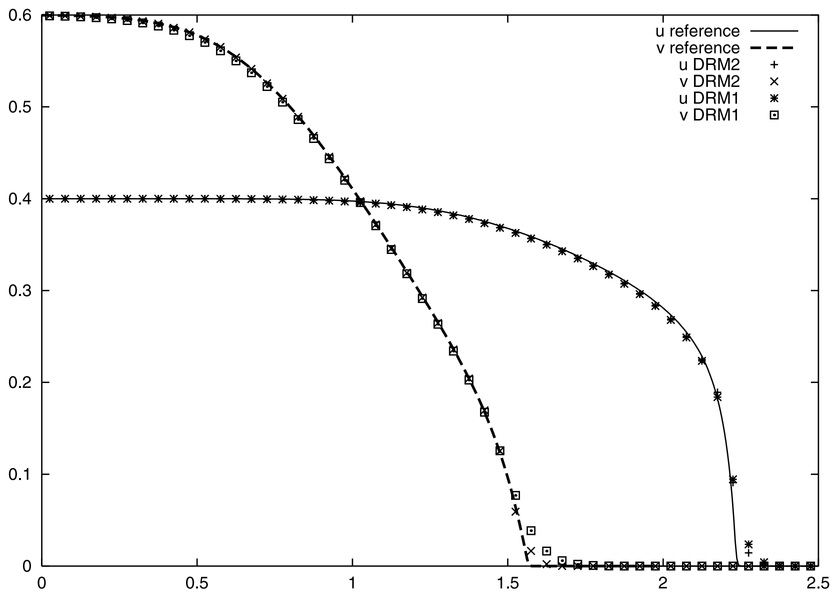

FIGURE 4. Approximation to the 2×2 system (5.3): $\Delta x = 0.05$.

As far as we deal with first-order (in space) schemes, the DRM2 and the FDM coincide. It is not the case, *a priori*, for second-order schemes. However, in this particular case the numerical results are the same. We present the results obtained with both DRM1 and DRM2 (equivalently FDM) models, a time $t = 0.5$. Figure 4 represents the comparison between the two models for a space step $\Delta x = 0.05$. As expected, the DRM2 is better than the DRM1 where the characteristic velocities are $-\lambda$, λ instead of $0 \leq \lambda_m \leq \lambda_p$.

5.3. **Further tests.** Let us mention that higher resolution methods such as PPM can be successfully incorporated to the diffusive BGK models. The diffusive BGK model has also been promptly applied to higher space dimensions. For instance, Figure 5 shows the numerical solution at $t = 0.5$ for the 2D Burgers type equation

$$u_t + (u^2)_x + (u^2)_y = 0.1(\nu(u)u_x)_x + 0.1(\nu(u)u_y)_y, \tag{5.5}$$

with the strongly degenerate diffusion

$$\nu(u) = \begin{cases} 0, & |u| \leq 0.25, \\ 1, & |u| > 0.25, \end{cases}$$

and the initial condition

$$u(x, y, 0) = \begin{cases} -1, & (x - 0.5)^2 + (y - 0.5)^2 \leq 0.16, \\ 1, & (x + 0.5)^2 + (y + 0.5)^2 \leq 0.16, \\ 0, & \text{otherwise.} \end{cases}$$

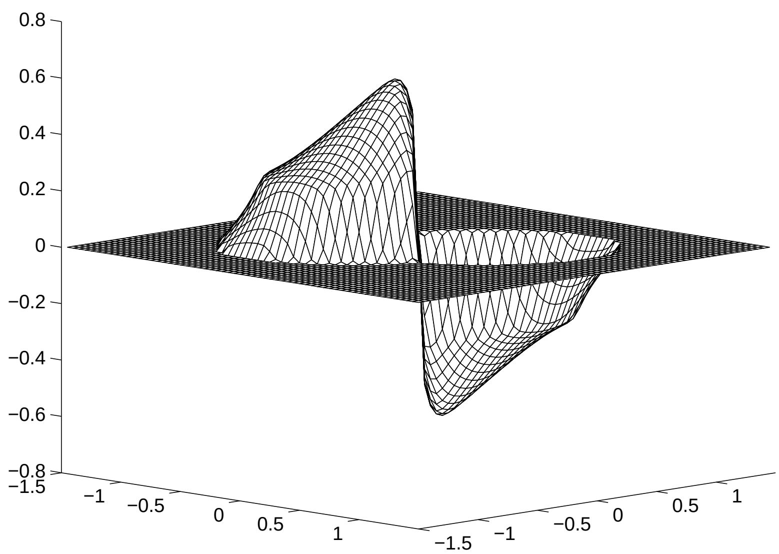

FIGURE 5. Approximation to the 2D Burgers type equation (5.5).

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