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ASYMPTOTICS OF RECURRENCE COEFFICIENTS FOR ORTHONORMAL POLYNOMIALS ON THE LINE—MAGNUS'S METHOD REVISITED

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Abstract. We use Freud equations to obtain the main term in the asymptotic expansion of the recurrence coefficients associated with orthonormal polynomials $p_n(w^2)$ for weights $w = W\exp(-Q)$ on the real line where Q is an even polynomial of fixed degree with nonnegative coefficients or where $Q(x) = \exp(x^{2m})$, $m \geq 1$. Here $W(x) = |x|^\rho$ for some real $\rho > -1$.

1. Introduction

The idea of this article arose out of two papers by Magnus in 1986 (see [11] and [12]), who used Freud equations to study the asymptotic behaviour of recurrence coefficients for orthonormal polynomials $p_n(w^2)$ on the real line, where $w = W(x)\exp(-x^{2m})$, $m \in \mathbb{N}$ and $W(x) = |x|^\rho$ for some real fixed $\rho > -1$. Using these equations, Magnus established an important special case of Freud's conjecture. We refer the reader to [11], [13], the references cited therein and our discussion below for a comprehensive history of this conjecture. Freud difference equations are closely related to discrete Painlevé equations and have found many applications in approximation theory and more recently in mathematical physics; see [3] and the many references cited therein. Our main objective in this paper will be to show that the methods of Magnus in the above papers may be improved to obtain the main term in the asymptotic expansion of the recurrence coefficients associated with orthonormal polynomials $p_n(w^2)$ for weights $w = W\exp(-Q)$ on the real line, where Q is an even polynomial of fixed degree with nonnegative coefficients and where $Q(x) = \exp(x^{2m})$, $m \geq 1$. Here $W(x) = |x|^\rho$ for some real $\rho > -1$. The polynomial case (which we will denote by Q_P) provides an alternative proof to an earlier case studied by Bauldry, Máté and Nevai; see [2]. In this case, sharper results than ours have been obtained by Bleher and Its (see [3]) and Deift, Kriecherbauer and McLaughlin in [4] when $\rho = 0$. In these later papers, the analyticity of w is exploited together with Lax-Levermore theory and Riemann Hilbert techniques. The exponential case (which we will denote by Q_E) is new even in the case $\rho = 0$. One of the main objectives in this paper is to show that our method of proof allows us to treat even weights with varying rates of smooth decay at infinity and with possible points of nonanalyticity. The noneven analogues of our results require further ideas

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and will appear in a forthcoming paper. We refer the reader to [3], [4], [10], [13], [17] and the many references cited therein for a detailed account of the history of this subject. We finish this section with a short description of the structure of this paper and some needed notation. Because Q_P and Q_E are of different natures, we choose to structure the paper in such a way that they are delt with separately. This paper is thus organised as follows: In Section 2, we state our main results. In Section 3, we illustrate that the recurrence coefficients for Q_P satisfy a difference equation which admits unique solutions which may be well approximated and we carefully approximate these solutions by sequences involving the scaled endpoints of the equilibrium measure for $\exp(-2Q_P)$ using an improved technique which is of independent interest. In Section 4, we prove our main result for Q_P. In Section 5, we show that the recurrence coefficients for Q_E are a solution (for $\rho > -1$) and a unique solution (for $\rho = 0$) of a Freud difference equation and in Section 6, we indicate how to adapt the techniques used for Q_P to show that these later solutions may be well approximated. In this section, we also present the proof of our main result for Q_E.

For any two sequences (b_n) and (c_n) of nonnegative real numbers, we shall write

$$b_n = O(c_n),$$

if there exists a constant $C > 0$ such that for all n

$$b_n \le Cc_n,$$

and

$$b_n \sim c_n$$

if there exist two constants $0 < C_1 \le C_2 < \infty$ such that for all n

$$C_1 \le b_n/c_n \le C_2.$$

Similar notation will be used for functions and sequences of functions.

2. Main result

In this section, we state our main result.

Given a weight w as above, we may define a unique system of orthonormal polynomials (see [8])

$$p_n(w^2, x) := \gamma_n x^n + \cdots, \qquad \gamma_n := \gamma_n(w^2) > 0$$

satisfying

$$\int_{\mathbb{R}} p_n(w^2, x) p_n(w^2, x) w^2(x) dx = \delta_{m,n}, \qquad m, n \ge 0,$$

and denote by

$$-\infty < x_{n,n}(w^2) < x_{n-1,n}(w^2) < \cdots < x_{2,n}(w^2) < x_{1,n}(w^2) < \infty$$

the n simple zeros of $p_n(w^2)$. We write the three term recurrence of $p_n(w^2)$ in the form

$$x p_n(w^2, x) = \alpha_{n+1} p_{n+1}(w^2, x) + \alpha_n p_{n-1}(w^2, x), \qquad n \ge 0.$$

Here, $p_{-1}(w^2) = 0$, $p_0(w^2) = \left(\int w(x)\,dx\right)^{-1/2}$,

$$\alpha_n(w^2) = \gamma_{n-1}/\gamma_n > 0$$

and $\alpha_0 = 0$ are the associated recurrence coefficients. Let us put $\alpha := \alpha(w) = (\alpha_n)_{n=1}^{\infty}$ and let

$$J(\alpha) = \begin{pmatrix} 0 & \alpha_1 & & & & \\ \alpha_1 & 0 & \alpha_2 & & & \\ & \alpha_2 & 0 & \alpha_3 & & \\ & & \alpha_3 & 0 & \alpha_4 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

be the infinite Jacobi matrix associated with the recurrence coefficients α. The asymptotic behaviour of the recurrence coefficients is expressed in terms of the scaled endpoints of the equilibrium measure for $\exp(-2Q)$, $a_n = (a_n)_{n=1}^{\infty}$, where $a_u = a_u(\exp(-2Q))$ is the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}\, dt.$$

The number a_u is, as a real-valued function of u, uniquely defined and strictly increasing in $(0, \infty)$; see [9] and [17].

Following are our main results:

Theorem 2.1a. *Let $w = W \exp(-Q_P)$, where Q_P is an even polynomial of fixed degree and with nonnegative coefficients. Then*

$$\frac{\alpha_n}{a_n} = 1/2 \left[1 + O\left(\frac{1}{n}\right)\right], \qquad n \to \infty \tag{2.1}$$

and

$$\frac{\alpha_{n+1}}{\alpha_n} = 1 + O\left(\frac{1}{n}\right), \qquad n \to \infty. \tag{2.2}$$

Theorem 2.1b. *Let $w = W \exp(-Q_E)$, where $Q_E(x) = \exp(x^{2m})$, $m \geq 1$. Then*

$$\frac{\alpha_n}{a_n} = 1/2 \left[1 + O\left(\frac{\log n}{n}\right)\right], \qquad n \to \infty \tag{2.3}$$

and

$$\frac{\alpha_{n+1}}{\alpha_n} = 1 + O\left(\frac{\log n}{n}\right), \qquad n \to \infty. \tag{2.4}$$

3. The Freud equation and its approximation for $Q := Q_P$

In this section we will show that the recurrence coefficients generated by w satisfy a Fr eud equation whose solutions may be well approximated. Let us write

$$Q(x) = \sum_{j=1}^{M} x^{2mj} c_j, \qquad m \geq 1,\ x \in \mathbb{R},$$

for some nonnegative sequence $\{c_j\}$ and fixed M so that

$$Q'(x) = 2m \sum_{j=1}^{M} x^{2mj-1} j c_j.$$

Formally applying Q' to the operator $J(\alpha)$, we learn that

$$(Q'(J(\alpha)))_{i,j} = \int_{\mathbb{R}} Q'(x) p_i(x) p_j(x) w(x)\, dx.$$

The right-hand side of the latter equation above is easily interpreted as the spectral representation of the left-hand side.

Note that $Q'(J(\alpha))$ is a matrix. Throughout, $(J(\alpha))_{i,j}$ will denote the element of the $(i+1)$th row and $(j+1)$th column of the matrix $J(\alpha)$ and $(Q'(J(\alpha)))_{i,j}$ will denote the element of the $(i+1)$th row and $(j+1)$th column of the matrix $Q'(J(\alpha))$. Similar conventions will be used in the sequel.

Motivated by the discussion above, we define a sequence of operators F_n $(n \in \mathbb{N})$ acting on sequences $x = (x_n)_{n=1}^{\infty}$ of positive real numbers by

$$F_n(x) := x_n(Q'(J(x)))_{n,n-1}, \qquad n = 1, 2, 3, \ldots. \tag{3.1}$$

The Freud equations for obtaining the sequence α of recurrence coefficients for the orthonormal polynomials with weight $w = W\exp(-Q)$ from the operators defined above is then given by

$$F_n(\alpha) = n + \mathrm{odd}(n), \qquad n = 1, 2, 3, \ldots, \tag{3.2}$$

where $\mathrm{odd}(n)$ is 1 if n is odd and 0 if n is even.

It follows then quite easily using the method of [12, p. 367] and [11, Lemma 2.3] that the following holds:

Lemma 3.1. *The recurrence coefficients $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ are a unique positive solution of the Freud equations* (3.2) *for the weight w.*

We now approximate $F_n(\alpha)$ by a sequence of operators related to $F_n(a/2)$, where $a = (a_n/2)_{n=1}^{\infty}$ are the unique scaled endpoints of the equilibrium measure for $\exp(-2Q)$. The idea is similar to the notion of a consistent approximant solution first used in [11] but the generalization presented here works for even weights of polynomial and of faster than polynomial decay at infinity and allows for improved estimates. Let us define polynomials $p_n^* := x^n + \cdots$ of degree n, $n \geq 0$, by the recurrence

$$xp_n^*(x) := \frac{a_{n+1}}{2}p_{n+1}^*(x) + \frac{a_n}{2}p_{n-1}^*(x), \qquad n = 0, 1, \ldots,$$

where we may take $p_0^* = 1$ and $p_{-1}^* = 0$. By Favard's theorem ([8], Theorem 2.1.5), there exists a positive measure μ^* with all moments finite such that these polynomials are orthogonal with respect to μ^*. Without loss of generality, we may assume that the p_n^* are suitably scaled so that they are orthonormal with respect to μ^* and satisfy the recurrence above. We note that the p_n^* depend on a_n, the scaled endpoints of the support of the equilibrium measure for $\exp(-2Q)$. Let

$$J(a/2) = \begin{pmatrix} 0 & a_1/2 & & & & \\ a_1/2 & 0 & a_2/2 & & & \\ & a_2/2 & 0 & a_3/2 & & \\ & & a_3/2 & 0 & a_4/2 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

be the infinite Jacobi matrix associated with the recurrence coefficients $(a_n/2)_{n=0}^{\infty}$. Then the Freud operator applied to the sequence $(a_1/2, a_2/2, a_3/2, \ldots)$ gives

$$\begin{aligned} F_n(a/2) &= \frac{a_n}{2}(Q'(J(a/2)))_{n,n-1} \\ &= a_n/2\int_{\mathbb{R}} Q'(x)p_n^*(x)p_{n-1}^*(x)d\mu^*(x), \qquad n \geq 1. \end{aligned} \tag{3.3}$$

We shall prove

Theorem 3.2. *For $n \geq 1$, we have*

$$|F_n(\alpha) - F_n(a/2)| = O(1). \tag{3.4}$$

For the proof of Theorem 3.2 we need some preparatory material and lemmas. The Jacobi matrix for the Chebyshev polynomials of the second kind

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta,$$

which are orthonormal with respect to the weight $\frac{2}{\pi}\sqrt{1-x^2}$ on $[-1,1]$, is given by

$$J_0 = J(1/2) = \begin{pmatrix} 0 & 1/2 & & & & \\ 1/2 & 0 & 1/2 & & & \\ & 1/2 & 0 & 1/2 & & \\ & & 1/2 & 1/2 & 1/2 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.5}$$

We recall that

$$\frac{2}{\pi}\int_{-1}^{1} x^k U_n(x) U_{n-1}(x)\sqrt{1-x^2}\,dx = (J_0^k)_{n,n-1}, \qquad k \geq 1. \tag{3.6}$$

Let $\overline{a_n/2}$ be the constant sequence $(a_n/2, a_n/2, a_n/2, \ldots)$, which should not be confused with the sequence $a/2 = (a_1/2, a_2/2, a_3/2, \ldots)$. Then the Jacobi matrix for $\overline{a_n/2}$ is given by

$$J(\overline{a_n/2}) = a_n J_0, \tag{3.7}$$

and

$$F_s(\overline{a_n/2}) = \frac{a_n}{2}\left(Q'(a_n J_0)\right)_{s,s-1}. \tag{3.8}$$

Our main idea for proving Theorem 3.2 is to approximate the sequence $F_s(a/2)$ by $F_s(\overline{a_n/2})$ when s is close to n. This is achieved by the following lemma.

Lemma 3.3. *We have for $n \geq 1$*

$$|F_n(a/2) - F_n(\overline{a_n/2})| = O\left(a_n \sum_{j=1}^{M} j c_j a_n^{2mj-1} R_{2mj-1,n}\right), \tag{3.9}$$

where for positive integers k and n

$$\begin{aligned} R_{k,n} &= \frac{1}{a_n^k}\int_{\mathbb{R}} x^k p_n^*(x) p_{n-1}^*(x)\,d\mu^*(x) \\ &\quad - 2/\pi \int_{-1}^{1} x^k U_n(x) U_{n-1}(x)\sqrt{1-x^2}\,dx. \end{aligned} \tag{3.10}$$

Proof. Observe that

$$F_n(a/2) = \frac{a_n}{2} 2mj \sum_{j=1}^{M} \int_{\mathbb{R}} x^{2mj-1} p_n^*(x) p_{n-1}^*(x)\,d\mu^*(x)$$

and

$$F_n(\overline{a_n/2}) = \frac{a_n}{2}\frac{2}{\pi}\int_{-1}^{1} Q'(a_n x) U_n(x) U_{n-1}(x)\sqrt{1-x^2}\,dx.$$

Then the lemma follows. □

We now proceed to estimate $R_{k,n}$ for positive integers k and n and to this end we make use of an idea of [18]. The author thanks Walter Van Assche in this regard.

Lemma 3.4. *For positive integers k and $n \geq C$*

$$\begin{aligned}|R_{k,n}| &\leq (2k-1)\\ &\quad\times \max_{-k\leq j\leq k}\left(\frac{1}{2}\left|\frac{a_{n+j}}{a_n}-1\right|+\frac{1}{2}\left|\frac{a_{n+j-1}}{a_n}-1\right|\right)\sum_{j=0}^{k-1}\left(\frac{x_{1,n+k}}{a_n}\right)^{k-j-1},\end{aligned}\tag{3.11}$$

where $x_{1,n+k}$ is the largest zero of p^*_{n+k}.

Proof. First observe that

$$R_{k,n} = \frac{1}{a_n^k}(J(a/2)^k)_{n,n-1} - (J_0^k)_{n,n-1}.$$

Hence we will estimate the difference between the operators $J(a/2)^k/a_n^k$ and J_0^k. Recalling the identity

$$A^k - B^k = \sum_{j=0}^{k-1} B^j(A-B)A^{k-j-1},$$

and letting $e_n = (0,0,0,\cdots,0,1,0,0,\cdots)^*$ be the nth unit basis vector, we write

$$\begin{aligned}&\frac{1}{a_n^k}(J(a/2)^k)_{n,n-1} - (J_0^k)_{n,n-1}\\ &\qquad= e_n^*\left[\frac{1}{(a_n)^k}J(a/2)^k - J_0^k\right]e_{n-1}\\ &\qquad= \sum_{j=0}^{k-1} e_n^* J_0^j\left[\frac{1}{a_n}J(a/2) - J_0\right]\left(\frac{1}{a_n}J(a/2)\right)^{k-j-1} e_{n-1}.\end{aligned}$$

Here $e_n^*J_0^j$ is the nth row of J_0^j which, by using (3.6), has the form

$$\begin{aligned}&\begin{pmatrix}(J_0^j)_{n,0} & (J_0^j)_{n,1} & (J_0^j)_{n,2} & \cdots & (J_0^j)_{n,n+j} & \cdots\end{pmatrix}\\ &\quad= \Bigg(\frac{2}{\pi}\int_{-1}^{1}x^jU_nU_0\sqrt{1-x^2}\,dx \quad \frac{2}{\pi}\int_{-1}^{1}x^jU_nU_1\sqrt{1-x^2}\,dx\\ &\qquad\quad \frac{2}{\pi}\int_{-1}^{1}x^jU_nU_2\sqrt{1-x^2}\,dx \;\cdots\; \frac{2}{\pi}\int_{-1}^{1}x^jU_nU_{n+j}\sqrt{1-x^2}\,dx \quad \cdots\Bigg)\\ &\quad= \Bigg(0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad \frac{2}{\pi}\int_{-1}^{1}x^jU_n(x)U_{n-j}(x)\sqrt{1-x^2}\,dx\\ &\qquad\quad \cdots \quad \frac{2}{\pi}\int_{-1}^{1}x^jU_n(x)U_{n+j}(x)\sqrt{1-x^2}\,dx \quad 0 \quad \cdots\Bigg),\end{aligned}$$

where we used orthogonality. Thus $e_n^*J_0^j \in \operatorname{span}\{e^*_{n-j},\ldots,e^*_{n+j}\}$ which is a subset of $\operatorname{span}\{e^*_{n-k},\ldots,e^*_{n+k}\}$ whenever $j \leq k$. Similarly $(J(a/2)/a_n)^{k-j-1}e_{n-1}$ gives the $(n-1)$th column of the matrix $(J(a/2)/a_n)^{k-j-1}$, and it has the form

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{(a_n)^{k-j-1}} \int_{-\infty}^{\infty} x^{k-j-1} p_{n-1}^*(x) p_{n-k+j}^*(x) d\mu^*(x) \\ \vdots \\ \frac{1}{(a_n)^{k-j-1}} \int_{-\infty}^{\infty} x^{k-j-1} p_{n-1}^*(x) p_{n+k-j-2}^*(x) d\mu^*(x) \\ 0 \\ \vdots \end{pmatrix}$$

so that it belongs to $\operatorname{span}\{e_{n-k+j}, \ldots, e_{n+k-j-2}\}$ which for every $j \geq 0$ is a subset of $\operatorname{span}\{e_{n-k}, \ldots, e_{n+k}\}$. Let $P_{n,k}$ be the projection matrix

$$P_{n,k} = \begin{pmatrix} 0_{n-k} & 0 & 0 \\ 0 & I_{2k+1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it follows that

$$\begin{aligned} &\frac{1}{a_n^k}(J(a/2)^k)_{n,n-1} - (J_0^k)_{n,n-1} \\ &\quad = \sum_{j=0}^{k-1} e_n^* J_0^j P_{n,k} \left(\frac{1}{a_n} J(a/2) - J_0 \right) P_{n,k} (J(a/2)/a_n)^{k-j-1} e_{n-1}. \end{aligned}$$

Observe that the matrix $P_{n,k}(J(a/2)/a_n - J_0)P_{n,k}$ converges to 0 for every fixed k as $n \to \infty$ in any norm. Now, by the Cauchy-Schwarz inequality and the relation

$$\|Ax\|_2 \leq \|A\|_2 \|x\|_2,$$

we see that

$$\begin{aligned} &\left| \sum_{j=0}^{k-1} e_n^* J_0^j P_{n,k} \left(\frac{1}{a_n} J(a/2) - J_0 \right) P_{n,k} (J(a/2)/a_n)^{k-j-1} e_{n-1} \right| \\ &\quad \leq \|P_{n,k}(J(a/2)/a_n - J_0)P_{n,k}\|_2 \sum_{j=0}^{k-1} \|e_n^* J_0^j\|_2 \, \|(J(a/2)/a_n)^{k-j-1} e_{n-1}\|_2. \end{aligned}$$

We begin with the estimate of

$$\|e_n^* J_0^j\|_2 = \sqrt{\sum_{i=-j}^{j} |x_i|^2},$$

where

$$x_i = \frac{2}{\pi} \int_{-1}^{1} x^j U_n(x) U_{n+i}(x) \sqrt{1-x^2}\, dx.$$

Here, for each $-j \leq i \leq j$

$$|x_i| \leq \frac{2}{\pi} \int_{-1}^{1} |U_n(x) U_{n+i}(x)| \sqrt{1-x^2}\, dx \leq 1,$$

by Cauchy-Schwarz. Hence

$$\|e_n^* J_0^j\|_2 \le \sqrt{2j+1} \le \sqrt{2k-1}.$$

Next we estimate

$$\|(J(a/2)/a_n)^{k-j-1} e_{n-1}\|_2 = \sqrt{\sum_{i=-k+j+1}^{k-j-1} |y_i|^2},$$

where

$$y_i = \frac{1}{(a_n)^{k-j-1}} \int_{-\infty}^{\infty} x^{k-j-1} p_{n-1}^*(x) p_{n-1+i}^*(x) d\mu^*(x).$$

We recall the Gauss-Jacobi quadrature [8]

$$\int_{-\infty}^{\infty} P(x)\, d\mu^*(x) = \sum_{\ell=1}^{n} \lambda_{\ell,n} P(x_{\ell,n}),$$

for every polynomial P of degree at most $2n-1$, with

$$\lambda_{l,n} = \frac{2}{a_n p_{n-1}^*(x_{l,n}) p_n'^*(x_{l,n})} > 0, \qquad 1 \le l \le n.$$

Then, since $x^{k-j-1} p_{n-1}^*(x) p_{n+i-1}^*(x)$ is a polynomial of degree $k-j+2n+i-3 \le 2(k+n)-4$,

$$\begin{aligned} y_i &= \frac{1}{(a_n)^{k-j-1}} \sum_{l=1}^{n+k} \lambda_{l,n+k} x_{l,n+k}^{k-j-1} p_{n-1}^*(x_{l,n+k}) p_{n+i-1}^*(x_{l,n+k}) \\ &\le \left(\frac{x_{1,n+k}}{a_n}\right)^{k-j-1} \sqrt{\sum_{l=1}^{n+k} \lambda_{l,n+k} p_{n-1}^{*2}(x_{l,n+k})} \\ &\quad \times \sqrt{\sum_{l=1}^{n+k} \lambda_{l,n+k} p_{n+i-1}^{*2}(x_{l,n+k})} \\ &\le \left(\frac{x_{1,n+k}}{a_n}\right)^{k-j-1}, \end{aligned}$$

where we used Cauchy-Schwarz and Gauss-Jacobi again. Thus

$$\begin{aligned} \|(J(a/2)/a_n)^{k-j-1} e_{n-1}\|_2 &\le \sqrt{2k-2j-1} \left(\frac{x_{1,n+k}}{a_n}\right)^{k-j-1} \\ &\le \sqrt{2k-1} \left(\frac{x_{1,n+k}}{a_n}\right)^{k-j-1}. \end{aligned}$$

Finally, we estimate

$$\begin{aligned} \|P_{n,k}(J(a/2)/a_n - J_0)P_{n,k}\|_2 &\le \|P_{n,k}(J(a/2)/a_n - J_0)P_{n,k}\|_\infty \\ &\le \max_{-k\le j\le k} \left(\frac{1}{2}\left|\frac{a_{n+j}}{a_n} - 1\right| + \frac{1}{2}\left|\frac{a_{n+j-1}}{a_n} - 1\right| \right). \end{aligned}$$

Then, combining our estimates, we have (3.11) and the lemma is proved. □

We now use Lemma 3.4 to estimate (3.9).

Lemma 3.5. *For $n \geq C$ we have*

$$|F_n(a/2) - F_n(\overline{a_n/2})| = O(1). \tag{3.12}$$

Proof. First observe that for a given positive integer k

$$\sum_{j=0}^{k-1} \left(\frac{x_{1,n+k}}{a_n}\right)^{k-j-1} = \sum_{j=0}^{k-1} \left(\frac{x_{1,n+k}}{a_{n+k}}\right)^{k-j-1} \left(\frac{a_{n+k}}{a_n}\right)^{k-j-1}.$$

Next we observe that [6, Theorem 7] implies that

$$\frac{x_{1,n+k}}{a_{n+k}/2} \leq 2.$$

It is important at this point to mention that Freud's result holds for the special system of orthogonal polynomials generated by the measure μ^* and with recurrence coefficients $(a_n/2)_{n=1}^{\infty}$.

Thus we have, using the strict monotonicity of $(a_n/2)_{n=1}^{\infty}$,

$$\sum_{j=0}^{k-1} \left(\frac{x_{1,n+k}}{a_n}\right)^{k-j-1} \leq \sum_{j=0}^{k-1} \left(\frac{a_{n+k}}{a_n}\right)^{k-j-1} \leq k \left(\frac{a_{n+k}}{a_n}\right)^{k-1}.$$

Inserting this estimate into (3.11) and using (3.9) then gives for $n \geq C$

$$\begin{aligned} &|F_n(a/2) - F_n(\overline{a_n/2})| \\ &\quad = O\left(a_n \sum_{j=1}^{\infty} j^2 c_j a_n^{2mj-1} \left(\frac{a_{n+2mj-1}}{a_n}\right)^{2mj-2} \max_{|i| \leq 2mj-1} \left|\frac{a_{n+i}}{a_n} - 1\right|\right). \end{aligned} \tag{3.13}$$

We proceed to estimate (3.13). We make heavy use of the following estimates which are well known; see for example [9].

(a) Uniformly for $u \in [v/2, 2v]$ and $v \geq C$

$$\left|\frac{a_u}{a_v} - 1\right| \sim \left|\frac{u}{v} - 1\right|. \tag{3.14}$$

(b) For all $v \geq C$ and $u \geq 1$,

$$\frac{a_{uv}}{a_v} = O(u). \tag{3.15}$$

(c) For $u \geq C$,

$$Q^{(j)}(a_u) \sim \frac{u}{a_u^j}, \qquad j = 0, 1, 2, 3. \tag{3.16}$$

Now for $n \geq C$,

$$
\begin{aligned}
(3.17)\ &|F_n(a/2) - F_n(\overline{a_n/2})| \\
&= O\left(a_n \sum_{j=1}^{M} j^2 c_j a_n^{2mj-1} \left(\frac{a_{n+2j-1}}{a_n}\right)^{2mj-2} \max_{|i|\leq 2mj-1} \left|\frac{a_{n+i}}{a_n} - 1\right|\right) \\
&= O\left(\frac{a_n}{n} \sum_{j=1}^{M} j^3 c_j a_n^{2mj-1} \left(1 + \frac{2mj-1}{n}\right)^{2mj-2}\right) \\
&= O\left(\frac{a_n^3}{n} \sum_{j=1}^{M} j^3 c_j a_n^{2mj-3}\right) \\
&= O\left(\frac{a_n^3}{n} Q^{(3)}(a_n)\right) \\
&= O(1).
\end{aligned}
$$

This establishes (3.12) and the lemma. □

We are now in the position to prove Theorem 3.2.

Proof. Set for $x \in \mathbb{R}$

$$M_n(x) := n \log x - 1/2\pi \int_0^{\pi} Q(2x\cos(\theta))d\theta.$$

Then (see for example [11, Section 4]) we always have

$$
\begin{aligned}
n - F_n(\overline{a_n/2}) &= n - (a_n/2)Q'(J(\overline{a_n/2}))_{n,n-1} \\
&= x\frac{\partial}{\partial x} M_n(x)|_{x=a_n/2} \\
&= 0.
\end{aligned}
$$

Thus

$$n - F_n(\overline{a_n/2}) = 0. \tag{3.18}$$

Now recalling that the recurrence coefficients are a unique solution of (3.2), we may apply Lemma 3.5, (3.18) and the triangle inequality to deduce the result. □

4. The Proof of Theorem 2.1a

In this section, we present the proof of Theorem 2.1a. We make use of ideas in [11].

Proof. Let us write

$$Q'(x) := 2m \sum_{j=1}^{M} j c_j x^{2mj-1},\ x \in \mathbb{R}.$$

Choose $\varepsilon > 0$, a positive integer m and a real positive sequence $\gamma = (\gamma_n)_{n=1}^{\infty}$ with the property that there exist positive constants C_1 and C_2 such that uniformly for $n \geq 1$

$$C_1 \leq \frac{\gamma_n}{a_n} \leq C_2. \tag{4.1}$$

Moreover let

$$\sigma := \{\sigma_1, \sigma_2, \ldots\} \tag{4.2}$$

be any real sequence having only finitely many nonzero terms.

Using Favard's theorem, we use the sequence γ_n to generate orthonormal polynomials $p_n^{'}$ with respect to some nonnegative even mass distribution $d\Delta$. Then if $J(\gamma)$ is the Jacobi matrix corresponding to the recurrence coefficients γ_n, we know from Section 3 that if j is a natural number fixed for the moment, then we have for every $n \geq 1$

$$\int_{\mathbb{R}} x^{2mj-1} p_n(x) p_{n-1}(x) d\Delta(x) = ((J(\gamma))^{2mj-1})_{n,n-1}. \tag{4.3}$$

Thus we may define the associated Freud equations by

$$G_{n,j}(\gamma) = \gamma_n((J(\gamma))^{2mj-1})_{n,n-1}, \qquad n = 1, 2, \ldots. \tag{4.4}$$

We observe for later use the fact that this sequence depends only on the finite number of terms

$$\{\gamma_{n-mj+1}, \ldots, \gamma_{n+mj-1}\}.$$

Now with this sequence of operators we define the Fréchet derivative by

$$H_j(\gamma) := \begin{pmatrix} \frac{\partial G_{1,j}(\gamma)}{\partial \log \gamma_1^2} & \cdots & \frac{\partial G_{n,j}(\gamma)}{\partial \log \gamma_1^2} & \cdots \\ \vdots & \cdots & \vdots & \\ \frac{\partial G_{1,j}(\gamma)}{\partial \log \gamma_n^2} & \cdots & \frac{\partial G_{n,j}(\gamma)}{\partial \log \gamma_n^2} & \cdots \\ \vdots & \cdots & \vdots & \ddots \end{pmatrix}. \tag{4.5}$$

We denote the quadratic form associated with

$$\{\sigma_1, \sigma_2, \ldots\}$$

by

$$(H_j(\gamma), \sigma) := \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_n \sigma_k \frac{\partial G_{n,j}(\gamma)}{\partial \log \gamma_k^2}. \tag{4.6}$$

First using (3.14) we know that

$$\frac{a_{n+1}}{a_n} \to 1, \qquad n \to \infty.$$

Thus it follows from [11, Theorem 5.3] that the matrix (4.5) is symmetric and that, subsequently, there exists C depending only on C_1 and C_2 in (4.1) such that

$$\sum_{n=1}^{\infty} a_n^{2mj} \sigma_n^2 \leq C(H_j(\gamma), \sigma). \tag{4.7}$$

Here it is crucial that the sequence

$$\{\sigma_1, \sigma_2, \ldots\}$$

contains finitely many nonzero terms.

The Fréchet derivative is used in the appreciation of the distance between two sequences a' and a'' in terms of $F_n(a')$ and $F_n(a'')$. We join them by a rectilinear path

$$\log \gamma_n(t) = \log a_n' + t\left(\log a_n'' - \log a_n'\right), \qquad 0 \leq t \leq 1, \ n = 1, 2, \ldots,$$

and integrate along the matrix (4.5). More precisely, we proceed as follows.

We specialize our choice of sequences defined above in the following way. First set

$$\overline{\sigma}_n := \log \alpha_n^2 - \log(a_n/2)^2, \qquad n = 1, 2, 3, \ldots, \tag{4.8}$$

and put

$$\sigma_n := (0, 0, \ldots, 0, \{\overline{\sigma}_n\}_{n=v+1-mj}^{N-1+mj}, 0, 0, \ldots), \tag{4.9}$$

where N and v are positive integers with $N - v$ finite. Thus σ_n is a real sequence with finitely many nonzero terms.

Next we set for $0 \leq t \leq 1$

$$\log \gamma_n^2(t) = \log(a_n/2)^2 + t\overline{\sigma}_n, \qquad n = 1, 2, \ldots. \tag{4.10}$$

Observe that $\gamma_n(t) > 0$ for every $t \in (0, 1)$ and moreover ([10, (2.10)]), ensure that (4.1) holds uniformly along the path (4.10). Thus uniformly along this path, (4.7) will hold with the choice of γ_n and the sequence σ_n.

Next observe that using well-known results concerning functions of several variables (see for example [5, Chapter 3]) and recalling the definition of the path (4.10), we may write

$$\int_0^1 \sum_{k=1}^{\infty} \overline{\sigma}_k \frac{\partial G_{n,j}(\gamma)}{\partial \log \gamma_k^2}\, dt = G_{n,j}(\alpha) - G_{n,j}(a/2). \tag{4.11}$$

To see this, it is useful to recall that

$$G_{n,j}(\alpha) = \alpha_n((J(\alpha))^{2mj-1})_{n,n-1}$$

and

$$G_{n,j}(a/2) = a_n/2((J(a/2))^{2mj-1})_{n,n-1}$$

are the same polynomials depending only on $2mj - 2$ terms in different variables α_n and $a_n/2$. Moreover if we wish to write either of the above formulae using their spectral representation, they become

$$G_{n,j}(\alpha) = \alpha_n \int_{\mathbb{R}} x^{2mj-1} p_n(x) p_{n-1}(x) w(x) dx$$

and

$$G_{n,j}(a/2) = a_n/2 \int_{\mathbb{R}} x^{2mj-1} p_n^*(x) p_{n-1}^*(x) d\mu^*(x).$$

Thus

$$2m \sum_{j=1}^{M} j c_j G_{n,j}(\alpha) = F_n(\alpha)$$

and

$$2m \sum_{j=1}^{M} j c_j G_{n,j}(a/2) = F_n(a/2).$$

Now we multiply (4.11) by $\overline{\sigma}_n$ for $v \leq n \leq N$ and so we obtain

$$\int_0^1 \sum_{n=v}^{N} \sum_{k=n-mj+1}^{n+mj-1} \overline{\sigma}_n \overline{\sigma}_k \frac{\partial G_{n,j}(\gamma)}{\partial \log \gamma_k^2} dt = \sum_{n=v}^{N} \overline{\sigma}_n (G_{n,j}(\alpha) - G_{n,j}(a/2)) \tag{4.12}$$

which holds uniformly along the path given by (4.10). Thus we have that

$$\int_0^1 \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \sigma_n\sigma_k \frac{\partial G_{n,j}(\gamma)}{\partial \log \gamma_k^2} dt = \sum_{n=v}^{N} \overline{\sigma}_n (G_{n,j}(\alpha) - G_{n,j}(a/2)). \tag{4.13}$$

Now we apply the operator $2m\sum_{j=1}^{M} jc_j$ to (4.13). This gives, recalling that $N-v$ is finite,

$$2m\sum_{j=1}^{M} jc_j \int_0^1 \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \sigma_n\sigma_k \frac{\partial G_{n,j}(\gamma)}{\partial \log \gamma_k^2} dt = O\left(\sum_{n=v}^{N} |\overline{\sigma}_n||F_n(\alpha) - F_n(a/2)|\right).$$

Now recalling (4.7) which holds for each fixed j and the fact that for each fixed j

$$\sum_{n=v}^{N} \overline{\sigma}_n{}^2 a_n^{2mj} \le \sum_{n=v+1-mj}^{N-1+mj} \overline{\sigma}_n{}^2 a_n^{2mj}$$

gives

$$\begin{aligned} 2m\sum_{j=1}^{M} jc_j \sum_{n=v}^{N} \overline{\sigma}_n{}^2 a_n^{2mj} &= O\left(\sum_{n=v}^{N} |\overline{\sigma}_n|\right) \\ &= O\left(\left(\sum_{n=v}^{N} \overline{\sigma}_n{}^2\right)^{1/2} \left(\sum_{n=v}^{N} 1\right)^{1/2}\right). \end{aligned} \tag{4.14}$$

We may now set $v = n$ and $N = n+1$ so that (4.14) becomes

$$2m\sum_{j=1}^{M} jc_j a_{n+1}^{2mj} \overline{\sigma}_{n+1}{}^2 = O\left(|\overline{\sigma}_{n+1}|\right).$$

Thus it follows from the above, (3.14) and (3.16) that

$$\begin{aligned} |\overline{\sigma}_n| &= O\left(\frac{1}{a_n Q'(a_n)}\right) \\ &= O\left(\frac{1}{n}\right). \end{aligned}$$

Recalling, as we may (see [10]), that

$$\left|\log\left(\frac{\alpha_n}{a_n/2}\right)\right| \sim \left|1 - \frac{\alpha_n}{a_n/2}\right|$$

finally gives (2.1). Equation (2.2) follows from (2.1) and (3.14). $\square$

5. The Freud equation and its approximation for $Q := Q_E$

In this section we will show that the recurrence coefficients generated by w also satisfy a Freud equation whose solutions may be well approximated. Let us write

$$Q'(x) = 2m\sum_{j=1}^{\infty} x^{2mj-1} jc_j, \qquad x \in \mathbb{R},$$

where $c_j := \frac{1}{j!}$. Applying this to the operator $J(\alpha)$ we obtain the formula

$$Q'(J(\alpha))_{i,j} = \int_{\mathbb{R}} Q'(x) p_i(x) p_j(x) w(x)\, dx.$$

Set

$$F_n(x) := x_n(Q'(J(x)))_{n,n-1}, \qquad n = 1, 2, 3, \ldots, \tag{5.1}$$

The Freud equations for obtaining the sequence α of recurrence coefficients for the orthonormal polynomials with weight $w = W \exp(-Q)$ from the operators defined above is then again given by

$$F_n(\alpha) = n + \mathrm{odd}(n), \qquad n = 1, 2, 3, \ldots, \tag{5.2}$$

where $\mathrm{odd}(n)$ is 1 if n is odd and 0 if n is even.

We prove

Theorem 5.1. *The recurrence coefficients $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ are a unique positive solution of the Freud equations* (5.2) *for the weight w with $\rho = 0$ and are a positive solution for all $\rho > -1$.*

Our proof will be broken down into several steps. We begin with Lemma 5.2 which follows as in [12, Lemma 1].

Lemma 5.2. *For $\rho > -1$, the recurrence coefficients α are a solution of the Freud equations* (5.2).

Proof. We observe that by orthogonality we obtain

$$\int_{\mathbb{R}} (p_n p_{n-1})'(x) w(x)\, dx = \int_{\mathbb{R}} p_n'(x) p_{n-1}(x) w(x)\, dx.$$

Now we write $p_n'(x) = n\gamma_n/\gamma_{n-1} p_{n-1}(x) + \pi_{n-2}(x)$, where π_{n-2} is some polynomial of degree at most $n-2$. Then it is easy to see that

$$\int_{\mathbb{R}} (p_n p_{n-1})'(x) w(x)\, dx = n/\alpha_n.$$

On the other hand integration by parts gives

$$\begin{aligned}\int_{\mathbb{R}} &(p_n p_{n-1})'(x) w(x)\, dx \\ &= \int_{\mathbb{R}} Q'(x) p_n(x) p_{n-1}(x) w(x)\, dx - \frac{\rho}{\alpha_n}\mathrm{odd}(n) \\ &= Q'(J(\alpha))_{n,n-1} - \frac{\rho}{\alpha_n}\mathrm{odd}(n)\end{aligned}$$

which shows that $F_n(\alpha) = n + \mathrm{odd}(n)$ for $n \geq 1$. □

We now show that α for $\exp(-Q)$ is the only positive solution of (5.2). This requires an extension of Theorem 2.2 in [11] where the proof there relies heavily on the polynomial decay of the underlying weight. The following lemma is essential. An earlier special case is actually due to Nevai [15].

Lemma 5.3. *Let μ be a monotone increasing function on $(-\infty, \infty)$ with all moments ν_j $(j = 0, 1, 2, \ldots)$ finite, and assume*

$$\int_{-\infty}^{\infty} P(x) Q'(x)\, d\mu(x) = \int_{-\infty}^{\infty} P'(x)\, d\mu(x), \tag{5.3}$$

for all polynomials P. Further assume that μ is normalized so that $\mu(-\infty) = 0$ and $\mu(+\infty) = \int_{-\infty}^{\infty} w(x)\, dx$. Then μ is absolutely continuous on $(-\infty, \infty)$ and $\mu'(x) = \exp(-Q(x))$ for all $x \in \mathbb{R}$.

Proof. Define q_n by

$$q_n Q'(q_n) = n, \qquad n \geq 1.$$

It is easy to see that for every fixed positive $k > 0$ and uniformly for $n \geq 1$

$$q_{kn} \sim a_n \sim (\log n)^{1/2m}.$$

We first show that for n even

$$\nu_n = \int_{\mathbb{R}} x^n \, d\mu(x) = O(q_{2n+2}^n). \tag{5.4}$$

Indeed, we write

$$\nu_n = \int_{|x| \leq q_{2n+2}} x^n \, d\mu(x) + \int_{|x| > q_{2n+2}} x^n \, d\mu(x) = I_1 + I_2.$$

Then

$$I_1 \leq q_{2n+2}^n \int_{-\infty}^{\infty} d\mu(x) = O(q_{2n+2}^n)$$

since by assumption all the moments of μ are finite. Next, observe that for $|x| \geq q_{2n+2}$ the monotonicity of $xQ'(x)$ on $(0, \infty)$ gives

$$\begin{aligned} xQ'(x) - (n+1) &\geq q_{2n+2} Q'(q_{2n+2}) - (n+1) \\ &= (2n+2) - (n+1) = n+1. \end{aligned} \tag{5.5}$$

Now (5.3) implies

$$\int_{-\infty}^{\infty} x^{n+1} Q'(x) \, d\mu(x) = (n+1)\nu_n,$$

so that

$$\int_{-\infty}^{\infty} x^n [xQ'(x) - (n+1)] \, d\mu(x) = 0, \tag{5.6}$$

for all $n \geq 0$. Thus by (5.5) and (5.6)

$$\begin{aligned} I_2 &\leq \frac{1}{n+1} \int_{|x| \geq q_{2n+2}} x^n [xQ'(x) - (n+1)] \, d\mu(x) \\ &= -\frac{1}{n+1} \int_{|x| < q_{2n+2}} x^n [xQ'(x) - (n+1)] \, d\mu(x) \\ &= \int_{|x| < q_{2n+2}} x^n \, d\mu(x) - \frac{1}{n+1} \int_{|x| < q_{2n+2}} x^{n+1} Q'(x) \, d\mu(x) \\ &\leq \int_{|x| < q_{2n+2}} x^n \, d\mu(x) = O(q_{2n+2}^n). \end{aligned}$$

Equation (5.4) then follows.

Next we show that (5.3) and (5.4) imply that

$$\int_{-\infty}^{\infty} e^{itx} Q'(x) \, d\mu(x) = it \int_{-\infty}^{\infty} e^{itx} \, d\mu(x), \tag{5.7}$$

for every fixed $t \in \mathbb{R}$. The idea follows ([11, (2.8)]) where we approximate e^{itx} by its partial sums $\sum_{j=0}^{n} (itx)^j / j!$ and show that

$$\lim_{n \to \infty,\, n \text{ even}} \int_{\mathbb{R}} \left| e^{itx} - \sum_{j=0}^{n} (itx)^j / j! \right| |Q'(x)| \, d\mu(x) = 0 \tag{5.8}$$

and

$$\lim_{n\to\infty,\, n\,\mathrm{even}} \int_{\mathbb{R}} \left| e^{itx} - \sum_{j=0}^{n-1} (itx)^j/j! \right| d\mu(x) = 0. \tag{5.9}$$

We prove (5.8). The proof of (5.9) is similar and easier. Recalling that

$$Q'(x) = 2m \sum_{j=1}^{\infty} x^{2jm-1} j c_j \tag{5.10}$$

and using (5.4) and the inequality

$$|e^{iu} - \sum_{j=0}^{n} (iu)^j/j!| \le \frac{|u|^{n+1}}{(n+1)!}, \qquad u \in \mathbb{R},$$

we may write for any $0 < \delta < 1/2$

$$\begin{aligned}
&\int_{\mathbb{R}} |e^{itx} - \sum_{j=0}^{n} (itx)^j/j!||Q'(x)|\, d\mu(x) \\
&= O\left(\frac{|t|^{n+1}}{(n+1)!} \sum_{j=1}^{\infty} j c_j \int_{\mathbb{R}} |x|^{2mj+n} d\mu(x) \right) \\
&= O\left(\frac{|t|^{n+1}}{(n+1)!} \sum_{j=1}^{\infty} j c_j \nu_{2mj+n}] \right) \\
&= O\left(\frac{|t|^{n+1}}{(n+1)!} \sum_{j=1}^{n+1} j c_j q_{4mj+2n+2}^{2mj+n} \right) + \\
&+ O\left(\frac{|t|^{n+1}}{(n+1)!} \sum_{j=n+2}^{\infty} j c_j q_{4mj+2n+2}^{2mj+n} \right) \\
&= O\left(\frac{n^{\delta n}|t|^{n+1}}{(n+1)!} \sum_{j=1}^{\infty} j c_j \right) + O\left(\frac{|t|^{n+1}}{(n+1)!} \sum_{j=n+2}^{\infty} \frac{q_{4mj+2n+2}^{2mj+n}}{(j-1)!} \right) \\
&= O\left(\frac{n^{\delta n}|t|^{n+1}}{(n+1)!} \right) + O\left(\frac{|t|^{n+1}}{(n+1)!} \right) \\
&= O(1), \qquad n \to \infty.
\end{aligned} \tag{5.11}$$

Thus (5.8) is established and so (5.7) then follows easily. Lemma 5.3 then follows quite easily using the method of [11, Theorem 2.2] and the fact that Q is entire. □

6. Estimates for the Freud operators and the proof of Theorem 2.1b

In this section, we prove Theorem 2.1b. The idea following Theorem 2.1a is to estimate $F_n(\alpha)$ by a sequence of operators acting on $a/2 := \{a_n/2\}$. Because Q grows faster than a polynomial, we need some modifications in the methods of Theorem 2.1a. We will now indicate these modifications below.

Step 1. First, as in Theorem 2.1a, define as before polynomials p_n^* of degree n, $n \geq -1$, by

$$2xp_n^*(x) := a_{n+1}p_{n+1}^*(x) + a_n p_{n-1}^*(x).$$

Then by Favard's theorem, there exists a positive measure μ^* with all moments finite such that these polynomials are orthonormal with respect to μ^*. Let

$$J(a/2) = \begin{pmatrix} 0 & a_1/2 & & & & \\ a_1/2 & 0 & a_2/2 & & & \\ & a_2/2 & 0 & a_3/2 & & \\ & & a_3/2 & 0 & a_4/2 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

be the infinite Jacobi matrix associated with the recurrence coefficients $(a_n/2)_{n=0}^{\infty}$. Then the Freud operator applied to the sequence $(a_1/2, a_2/2, a_3/2, \ldots)$ gives

$$\begin{aligned} F_n(a/2) &= \frac{a_n}{2}(Q'(J(a/2)))_{n,n-1} \\ &= a_n/2 \int_{\mathbb{R}} Q'(x)p_n^*(x)p_{n-1}^*(x)d\mu^*(x), \qquad n \geq 1. \end{aligned} \tag{6.1}$$

We now need a related sequence of "cousins" defined by

$$F_n^*(a/2) =: (a_n/2)2m\sum_{j=1}^{\infty} jc_j \int_{\mathbb{R}} x^{2mj-1}p_n^*(x)p_{n-1}^*(x)d\mu^*(x), \qquad n \geq 1. \tag{6.2}$$

We shall prove

Theorem 6.1. *For $n \geq C$, we have*

$$|F_n(\alpha) - F_n^*(a/2)| = O((\log n)^{3/2}). \tag{6.3}$$

Step 2. To prove Theorem 6.1, we follow the method of Lemmas 3.3–3.5 and deduce that for $n \geq C$

$$\begin{aligned} &|F_n^*(a/2) - F_n(\overline{a_n/2})| \\ &\quad = O\left(a_n \sum_{j=1}^{\infty} j^2 c_j a_n^{2mj-1}\left(\frac{a_{n+2mj-1}}{a_n}\right)^{2mj-2} \max_{|i|\leq 2mj-1}\left|\frac{a_{n+i}}{a_n} - 1\right|\right), \end{aligned} \tag{6.4}$$

where $\overline{a_n/2}$ is the constant sequence $(a_n/2, a_n/2, \ldots)$. We proceed to estimate (6.4) and here we now use that

$$a_u \sim (\log u)^{1/2m}$$

and the following easily proven identities (see for example [9]):

(a) Uniformly for $u \in [v/2, 2v]$ and $v \geq C$

$$\left|\frac{a_u}{a_v} - 1\right| \sim \left|\frac{u}{v} - 1\right| \frac{1}{\log v}. \tag{6.5}$$

(b) For all $v \geq C$ and $u \geq 1$

$$\frac{a_{uv}}{a_v} = O(u^{\frac{C}{\log v}}). \tag{6.6}$$

(c) For $u \geq C$

$$Q^{(j)}(a_u) \sim \frac{u(\log u)^{j-1/2}}{a_u^j}, \qquad j = 0, 1, 2, 3. \tag{6.7}$$

Now applying the estimates (a)–(c) above, we see that for $n \geq C$,

$$
\begin{aligned}
(6.8)\quad & |F_n^*(a/2) - F_n(\overline{a_n/2})| \\
&= O\left(a_n \sum_{j=1}^{\infty} j^2 c_j a_n^{2mj-1} \left(\frac{a_{n+2mj-1}}{a_n}\right)^{2mj-2} \max_{|i|\leq 2mj-1} \left|\frac{a_{n+i}}{a_n} - 1\right|\right) \\
&= O\left(\frac{a_n}{n\log n} \sum_{j=1}^{[\sqrt{n}]-1} j^3 c_j a_n^{2mj-1} \left(1 + \frac{2mj-1}{n}\right)^{\frac{Cj}{\log n}}\right) \\
&\quad + O\left(a_n \sum_{j=[\sqrt{n}]}^{n-1} j^2 c_j (\log n)^{Cn+1}\right) \\
&\quad + O\left(a_n \sum_{j=n}^{\infty} j^2 c_j (\log j)^{Cj+1}\right) \\
&= O\left(\frac{a_n^3}{n\log n} \sum_{j=1}^{[\sqrt{n}]-1} j^3 c_j a_n^{2j-3}\right) + O(1) \\
&= O\left(\frac{a_n^3}{n\log n} Q^{(3)}(a_n)\right) + O(1) \\
&= O((\log n)^{3/2}).
\end{aligned}
$$

Equation (6.8) together with the method of Theorem 2.1a gives Theorem 6.1. □

Step 3. Put

$$\overline{\sigma}_n := \log \alpha_n^2 - \log(a_n/2)^2, \qquad n = 1, 2, 3, \ldots, \tag{6.9}$$

Then using the definition of F_n^*, the fact that Q is entire, Theorem 6.1 and the method of Theorem 2.1a, we obtain that

$$2m \sum_{j=1}^{\infty} j c_j a_{n+1}^{2mj} \overline{\sigma}_{n+1}{}^2 = O\left(|\overline{\sigma}_{n+1}|(\log(n+1))^{3/2}\right).$$

Thus it follows from the above and (6.7) that

$$
\begin{aligned}
|\overline{\sigma}_n| &= O\left(\frac{(\log n)^{3/2}}{a_n Q'(a_n)}\right) \\
&= O\left(\frac{\log n}{n}\right).
\end{aligned}
$$

Using that

$$\left|\log\left(\frac{\alpha_n}{a_n/2}\right)\right| \sim \left|1 - \frac{\alpha_n}{a_n/2}\right|$$

finally gives (2.3). Equation (2.4) follows from (2.3) and (6.5). □

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