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ALL NUMBERS WHOSE POSITIVE DIVISORS HAVE INTEGRAL HARMONIC MEAN UP TO 300

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Abstract. A positive integer n is said to be *harmonic* when the harmonic mean $H(n)$ of its positive divisors is an integer. Ore proved that every perfect number is harmonic. No nontrivial odd harmonic numbers are known. In this article, the list of all harmonic numbers n with $H(n) \leq 300$ is given. In particular, such harmonic numbers are all even except 1.

1. Introduction

A positive integer n is said to be *perfect* if $\sigma(n) = 2n$, where $\sigma(n)$ denotes the sum of the positive divisors of n. It is an open problem whether or not an odd perfect number exists. In this connection, Ore [8] introduced the concept of *harmonic numbers*. A positive integer n is said to be harmonic if the harmonic mean of its positive divisors

$$H(n) = \frac{n\tau(n)}{\sigma(n)} \tag{1}$$

is an integer, where $\tau(n)$ denotes the number of the positive divisors of n. Ore proved the following fact which represents the relationship between perfect numbers and harmonic numbers.

Theorem 1.1 ([8]). *Every perfect number is harmonic.*

The converse of this theorem does not hold. For example, 140 is not perfect, but $H(140) = 5$. Ore listed all harmonic numbers up to 10^4 and this list was extended by Garcia [5] to 10^7 and by Cohen [2] to $2 \cdot 10^9$. No nontrivial odd harmonic numbers have been discovered. Ore conjectured the following statement. If this conjecture holds, it follows that odd perfect numbers do not exist.

Conjecture. *All harmonic numbers other than* 1 *must be even.*

Kanold [7] showed the following fact.

Theorem 1.2 ([7]). *For any positive integer c, there exist only finitely many numbers n satisfying $H(n) = c$.*

In [6, B2], Guy wrote: “Which values does the harmonic mean take? Presumably not $4, 12, 16, 18, 20, 22, \ldots$; does it take the value 23?” Cohen [2] settled the first of these questions for the first two values.

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Theorem 1.3 ([2]). *Let n be harmonic and $H(n) \leq 13$. Then n is one of the 13 numbers listed in Table 4 (at the end of this paper). In particular, the numbers n with $H(n) = 4$ or 12 do not exist.*

This result is extended to $H(n) \leq 300$. For this, some propositions and a computer were used. The program is written by Mathematica®, and it takes about three months to get the result. For $H(n) \leq 200$, we need only about two days.

Main result. *Let n be harmonic and $H(n) \leq 300$. Then n is one of the 280 numbers listed in Table 4 (at the end of this paper). In particular, such harmonic numbers are all even except 1. The table includes the nonexistence of numbers n with $H(n)$ being equal to one of the numbers that Guy listed. In the table, the values of $H(n) \leq 300$ are omitted when there is no corresponding value of n.*

For example, we solve the equation $H(n) = 14$ in §5. In §2, we recall the known facts about harmonic numbers. Then the general method of the search for the harmonic numbers n with $H(n) = c$ is explained in §3. In this search, the following proposition proved in §4 is useful.

Proposition 1.4. *Let p be prime. If $H(n) = 2p$, then $2p \mid n$. If $H(n) = 3p$, then $p \mid n$.*

This is analogous to the following fact due to Cohen.

Proposition 1.5 ([2]). *Let p be prime. If $H(n) = p$, then either $p \mid n$ or n is an even perfect number.*

Cohen and Sorli [3] introduced the concept of harmonic seeds.

Definition. Let d be a divisor of an integer n. We call d a *unitary divisor* of n and n a *unitary multiple* of d if $(d, n/d) = 1$. A harmonic number is called a *harmonic seed* if it does not have a smaller proper unitary divisor which is harmonic (we call a unitary divisor d *proper* if $d > 1$).

Every harmonic number is a unitary multiple of a certain harmonic seed. It is conjectured that such a harmonic seed is unique. Table 4 includes the harmonic seed of all harmonic numbers listed, and in all of these cases the seed is unique.

2. Known facts

In this section, we recall the known facts about harmonic numbers. The following lemma is a fundamental property of H, and we often use it without special mention.

Lemma 2.1. *Let n, m, e, f be positive integers and p, q primes.*

- *H is multiplicative, i.e., $H(nm) = H(n)H(m)$ if $(n, m) = 1$.*
- *H is monotone, i.e., $H(p^e) < H(p^f) < H(q^f)$ if $e < f$ and $p < q$.*

Proof. The first statement is clear from the definition (1). The second statement is also clear from the fact that H is averaging of positive numbers (this statement is a special case of Lemma 7 in Cohen and Deng [4]). $\square$

We denote the number of the distinct primes dividing n by $\omega(n)$.

Theorem 2.2 ([8], [10]). *Let n be a harmonic number and $\omega(n) \leq 2$. Then n is an even perfect number.*

Ore [8] proved the nonexistence of harmonic numbers n with $\omega(n) = 1$. In 1973, Pomerance proved that a harmonic number n with $\omega(n) = 2$ must be an even perfect number (cf. [10]), and Callan [1] rediscovered the proof of the same fact in 1992.

Theorem 2.3 ([8]). *Let n be a harmonic number greater than 6. Then n is not squarefree.*

Theorem 2.4 ([3]). *For any integer n,*

$$H(n) > \frac{2^{\omega(n)+1}}{\omega(n)+1},$$

with the following exceptions (in which p denotes a prime): $n = p$, $n = 2p$, $n = 6p$ $(p \neq 3)$, $n = 30p$ $(7 \leq p \leq 23)$, $n = 1, 15, 21, 70$. By Theorems 2.2 and 2.3, a harmonic number n greater than 6 satisfies the inequality.

Theorem 2.5 ([5]). *Let n be an odd harmonic number and $p^e \parallel n$. Then $p^e \equiv 1 \pmod 4$.*

In order to prove Theorem 2.5, we provide the following lemma, which is often used later. Let p be a prime and let Q be a rational number. Suppose that $Q = p^e m/n$ with $p \nmid mn$. Then we denote by $\mathrm{ord}_p(Q)$ the exponent e in this paper.

Lemma 2.6. *Let p be an integer (not necessarily prime). If $p \equiv 1 \pmod 4$, then*

$$\mathrm{ord}_2(1 + p + \cdots + p^e) = \mathrm{ord}_2(e + 1).$$

If $p \equiv 3 \pmod 4$ and $\mathrm{ord}_2(p + 1) = m$, then

$$\mathrm{ord}_2(1 + p + \cdots + p^e) = \begin{cases} \mathrm{ord}_2(e + 1) + m - 1, & \text{if } e \text{ is odd,} \\ 0, & \text{if } e \text{ is even.} \end{cases}$$

Proof of Theorem 2.5. Let p be a prime and e a positive integer. If $p^e \equiv 1 \pmod 4$. Then $\mathrm{ord}_2(H(p^e)) = 0$ by Lemma 2.6. If $p^e \equiv 3 \pmod 4$, then $\mathrm{ord}_2(H(p^e)) < 0$. Hence $H(n)$ cannot be integral if $p^e \equiv 3 \pmod 4$, $p^e \parallel n$ and $2 \nmid n$. □

Lemma 2.6 is a standard fact. For example, see [9]. The following simple proof is due to Koichi Tanaka, an undergraduate student of Kyushu University.

Proof of Lemma 2.6. Let $e + 1 = 2^k l$, where k is an integer and l is odd. The statement is clear when $k = 0$. Suppose that $k \geq 1$. Then

$$\begin{aligned} 1 + p + \cdots + p^e &= (p^{e+1} - 1)/(p - 1) = (p^{2^k l} - 1)/(p - 1) \\ &= (p^{2^{(k-1)} l} + 1)(p^{2^{(k-2)} l} + 1) \cdots (p^{2l} + 1)(p^l + 1)(p^l - 1)/(p - 1). \end{aligned}$$

The last part of this expression $(p^l - 1)/(p - 1) = p^{l-1} + \cdots + p + 1$ is odd, and the rest of the parts are all even. In particular, $p^{2^{(k-i)} l} + 1 \equiv 2 \pmod 4$ for $1 \leq i \leq k-1$. Since $p^l + 1 = (p + 1)(p^{l-1} - \cdots + 1)$,

$$\mathrm{ord}_2(p^l + 1) = \mathrm{ord}_2(p + 1) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 4, \\ m, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Now, the proof is complete. □

3. General algorithm

In this section, we give the general algorithm of searching all integers n satisfying $H(n) = c$ for a fixed integer c. Roughly speaking, it has three steps.

(1) List the possibilities of $\omega(n)$, the number of distinct primes dividing n.
(2) For each value of $\omega(n)$, list the possibilities of the *types of exponents* in the factorization of n.
(3) For each type of exponents, list the possibilities of primes dividing n.

Then check whether or not $H(n) = c$ for the finite possibilities of n. Practically, the existence of this algorithm verifies Theorem 1.2.

This algorithm finishes in finite time, but not always in reasonable time. We can make this time shorter, using Propositions 1.4 and 1.5, Lemma 4.1, and some methods explained in §5.

3.1. **Possibilities of the numbers of distinct primes.** Recall that we denote the number of distinct primes dividing n by $\omega(n)$. Suppose that n is harmonic. Then either n is an even perfect number or $\omega(n) \geq 3$ by Theorem 2.2. And Theorem 2.4 gives an upper bound of $\omega(n)$. But in order to get the upper bound, we may use the following method.

For example, suppose that $\omega(n) = 5$. Then $H(n) \geq H(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)$. For harmonic number n, $H(n) \geq H(2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) = 13.75$ because such n must not be squarefree by Theorem 2.3 (see also Lemma 4.2). Since $H(n)$ is integral, we have $H(n) \geq 14$. Thus, we get Table 1.

Note that Cohen [2] gives a more precise version of such a table.

Example. Suppose that $H(n) = 5$. Then $\omega(n)$ must be 2 or 3. Next, suppose that $H(n) = 14$. Then n cannot be an even perfect number, so $\omega(n)$ must be $3, 4$ or 5.

3.2. **Possibilities of the types of exponents.** We denote by $(e_1, \ldots, e_r)$ the type of exponents of $n = p_1^{e_1} \cdots p_r^{e_r}$ with $e_1 \geq \cdots \geq e_r$. Suppose a harmonic number n has the type $(e_1, \ldots, e_r)$. Since $H(p^e) < e+1$, we have $H(n) \leq (e_1+1) \cdots (e_r+1) - 1$. We can take the round-up of the value $H(2^{e_1} \cdot 3^{e_2} \cdots q_r^{e_r})$ as the lower bound of $H(n)$, where q_r is the rth prime.

Example. Suppose that $H(n) = 5$ and n is not an even perfect number. Then $\omega(n)$ must be 3. The type of exponents is not $(1, 1, 1)$ by Theorem 2.3. When it is $(2, 1, 1)$, the lower bound of H is 5 since $H(2^2 \cdot 3 \cdot 5) = 4.28 \cdots$. The upper bound is $3 \cdot 2 \cdot 2 - 1 = 11$. If the type is $(2, 2, 1)$ or $(3, 1, 1)$, then $H(n) \geq 6$. Hence the only possibility of the type of exponents is $(2, 1, 1)$.

Table 1. The lower bound of integral $H(n)$

$\omega(n)$	Theorem 2.4	this method
3	4	5
4	7	8
5	11	14
6	19	26
7	32	49
8	57	92
9	103	176
10	187	340

Cohen and Deng [4] have already given the inequality $H(n) \leq \tau(n) - 1$ for harmonic numbers n. Using Theorem 1.3, they also showed $H(n) \leq \tau(n) - 8$ when n is an even harmonic number and $n \neq 6, 28, 140, 496, 8128$. An improved result is possible using the main result of this paper.

3.3. **Possibilities of primes.** We define $S(n) = \sigma(n)/n$. Let p, q be primes and e, f positive integers. If $p < q$ and $e < f$, then it is easily verified that

$$1 < S(q^e) < S(q^f) < S(p^e) < S(p^f) < 2$$

and $S(p^e) \to p/(p-1)$ as $e \to \infty$. Suppose that the type of exponents of n is $(e_1, \ldots, e_r)$. Then $\tau(n) = (e_1 + 1) \cdots (e_r + 1)$. Since $H(n) = c$, which is fixed, $S(n)$ must be equal to $\tau(n)/H(n) = (e_1 + 1) \cdots (e_r + 1)/c$. If the smallest prime dividing n is p, then $S(n) < S(p^{e_1}) \cdots S(p^{e_r}) < (p/(p-1))^r$, so it follows that $p < S(n)^{1/r} \cdot (S(n)^{1/r} - 1)^{-1}$. In this way, we have the finite possibilities of the second smallest prime dividing n, third smallest prime, and so on.

Example. Suppose that $H(n) = 5$ and n is not an even perfect number. Then the type of the exponents of n must be $(2, 1, 1)$. So we have $\tau(n) = 3 \cdot 2 \cdot 2 = 12$ and $S(n) = \tau(n)/H(n) = 12/5$. If n were odd, then $S(n) \leq S(3^2 \cdot 5 \cdot 7) = 208/105 < 12/5$, a contradiction. Therefore it follows that $2 \mid n$. Next, if the second smallest prime dividing n were greater than 5, then $S(n) \leq S(2^2 \cdot 7 \cdot 11) = 24/11 < 12/5$, a contradiction. Therefore it follows that the second smallest prime dividing n is 3 or 5. By Proposition 1.5, we have $5 \mid n$. If $3 \mid n$, then the possibilities of n are $2^2 \cdot 3 \cdot 5$, $2 \cdot 3^2 \cdot 5$ and $2 \cdot 3 \cdot 5^2$, but $H(n)$ is not integral in these cases. Let p be a prime greater than 5. If $n = 2^2 \cdot 5 \cdot p$, then $H(p) = 5/H(2^2 \cdot 5) = 7/4$, so $p = 7$. We get a solution $n = 2^2 \cdot 5 \cdot 7 = 140$. If $n = 2 \cdot 5^2 \cdot p$, then $H(p) = 5/H(2 \cdot 5^2) = 31/20$, but this does not have a solution. Similarly, there does not exist a solution in the case that $n = 2 \cdot 5 \cdot p^2$. Hence all the solutions of $H(n) = 5$ are 140 and an even perfect number 496. We remark that Cohen and Sorli [3] have given a simpler proof of this fact, but their proof is not as algorithmically viable.

4. Proof of Proposition 1.4

In this section, we give the proof of Proposition 1.4.

Lemma 4.1. *If $H(n)$ is even, then n is even.*

Proof. Let p be an odd prime and e a positive integer. Then $\mathrm{ord}_2(H(p^e)) \leq 0$ by Lemma 2.6. Hence $H(n)$ cannot be even for an odd integer n. □

The following lemma is a special case of Lemma 5 in Cohen and Deng [4].

Lemma 4.2. *Let e, f be nonnegative integers and p, q primes. If $e < f$ and $p < q$, then $H(p^e q^f) > H(p^f q^e)$.*

Proof of Proposition 1.4. We give only the proof of the first statement. The second statement can be proved similarly. Let $H(n) = 2p$. In view of Lemma 4.1, it is sufficient to show that $p \mid n$. Assume that $p \nmid n$. Then $p \mid \tau(n)$ since $n\tau(n) = 2p\,\sigma(n)$. We put $n = q^{kp-1}m$ with a prime q and positive integers k and m. We can assume without loss of generality that $q \nmid m$. Since $\omega(n) \geq 3$ by Theorem 2.2, it follows that $\omega(m) \geq 2$. So we have $H(n) = H(q^{kp-1}m) \geq H(2^{kp-1})H(6) > kp$. Hence $k = 1$ and $n = q^{p-1}m$. Since

$$\frac{q-1}{q}p < H(q^{p-1}) < p$$

and $H(n) = 2p$, we have

$$2 < H(m) < \frac{2q}{q-1}.$$

Therefore it follows that

$$\frac{20}{9} = H(10) < \frac{2q}{q-1},$$

so the possibilities of q are only $2, 3, 5$ and 7.

(i) Assume that $n = 7^{p-1}m$ with $7 \nmid m$. Since $6p/7 < H(7^{p-1}) < p$, we have $2 < H(m) < 7/3$. Therefore the only possibility of m is 10. From the equation $H(7^{p-1} \cdot 10) = 2p$, we have $7^{p-1} = 3$, a contradiction.

(ii) Assume that $n = 5^{p-1}m$ with $5 \nmid m$. Since $4p/5 < H(5^{p-1}) < p$, we have $2 < H(m) < 5/2$. But we give a better estimation below. When $p = 2, 3, 5$, the statement holds by Theorem 1.3. From now on, we assume that $p \geq 7$. Then it follows that

$$H(5^{p-1}) = \frac{4 \cdot 5^{p-1}p}{5^p - 1} \leq \frac{4 \cdot 5^6}{5^7 - 1}p.$$

Hence $H(m) \geq (5^7 - 1)/(2 \cdot 5^6) > 2.4999$.

Claim. There does not exist an integer m satisfying $2.4999 < H(m) < 2.5$.

Suppose that $2.4999 < H(m) < 2.5$. If $\omega(m) \geq 3$, then $H(m) \geq H(2 \cdot 3 \cdot 5) > 2.5$. So we have $\omega(m) \leq 2$.

Assume that $\omega(m) = 2$ and let $m = p^e q^f$ be the factorization of m. If $\max(e, f) \geq 2$, then $H(m) \geq H(2^2 \cdot 3) > 2.5$ by Lemmas 2.1 and 4.2. So we have $f = e = 1$. If m is odd, then $H(m) \geq H(3 \cdot 5) = 2.5$, a contradiction. Assume that m is even. Since $H(2 \cdot 13) = 2.47\cdots$ and $H(2 \cdot 17) = 2.51\cdots$, it is impossible that $2.4999 < H(m) < 2.5$. Similarly, we can deal with the case that $\omega(m) = 1$.

(iii) We can deal with the case $n = 3^{p-1}m$ with $3 \nmid m$ similarly. In fact, there does not exist an integer m satisfying $2.998 < H(m) < 3$, $\omega(m) \geq 2$ and $3 \nmid m$.

(iv) Assume that $n = 2^{p-1}m$ with $2 \nmid m$. Since

$$\frac{1}{2}p < H(2^{p-1}) = \frac{2^{p-1}p}{2^p - 1} \leq \frac{2^6}{2^7 - 1}p,$$

we have $3.96875 < H(m) < 4$. Such integers m satisfying $\omega(m) \geq 2$ and $2 \nmid m$ are only $3^2 \cdot 23$ and products of distinct two odd primes. First, put $H(2^{p-1}3^2 \cdot 23) = 2p$. Then we have $2^{p-1} = 104$, a contradiction.

Next, put $H(2^{p-1}p_1p_2) = 2p$, where p_1 and p_2 are distinct odd primes. Then it follows that

$$\frac{2^{p-1}p}{2^p - 1} \cdot \frac{p_1}{\frac{p_1+1}{2}} \cdot \frac{p_2}{\frac{p_2+1}{2}} = 2p.$$

Therefore we have

$$2^{p-2}p_1p_2 = (2^p - 1) \cdot \frac{p_1 + 1}{2} \cdot \frac{p_2 + 1}{2}.$$

Hence the odd integer $2^p - 1$ is equal to either one of p_1, p_2 or the product p_1p_2.

If $2^p - 1 = p_1$, then

$$2^{p-2}p_2 = \frac{p_1 + 1}{2} \cdot \frac{p_2 + 1}{2} = 2^{p-1} \cdot \frac{p_2 + 1}{2},$$

which has no solution. If $2^p - 1 = p_1p_2$, then

$$2^{p-2} = \frac{p_1 + 1}{2} \cdot \frac{p_2 + 1}{2}.$$

Hence we have $p_1p_2+1=(p_1+1)(p_2+1)$, a contradiction. Now, all the possibilities of $p \nmid n$ are denied, so the proof is complete. □

5. Only solution of $H(n) = 14$

In this section, we show that $n = 18620$ is the only solution of $H(n) = 14$ using Proposition 1.4. For such an integer n, it follows that $\omega(n) \geq 3$ and $14 \mid n$ by Theorem 2.2 and Proposition 1.4. Let

$$\mathrm{ord}_2(n) = s, \qquad \mathrm{ord}_7(n) = t.$$

We have $1 \leq s \leq 9$ since $H(2^{10} \cdot 7 \cdot 3) > 14$. Similarly we have $1 \leq t \leq 7$. Table 2 is the table of $H(2^s)$.

Suppose that $s = 9$ and $n = 2^9 \cdot 7^t m$ with $(m, 14) = 1$. Since $H(n)$ is an integer, it is necessary that $31 \mid \tau(n)$ or $31 \mid m$. Assume that $31 \mid \tau(n)$. Then n has a prime raised to 30th power or higher as a factor. In this case, $H(n) > H(2^9 \cdot 3^{30}) > 14$, a contradiction. In the case that $31 \mid m$, we also have $H(n) \geq H(2^9 \cdot 7 \cdot 31) > 14$. In this way, the possibilities of $s = 7, 8, 9$ are denied.

In the case that $s = 6$, it is necessary that $127 \mid n$. We put $n = 2^6 \cdot 127^u m$, where u is a positive integer and $(m, 254) = 1$. If u is odd, then $\mathrm{ord}_2(H(n)) \leq 0$ by Lemma 2.6, hence it is impossible that $H(n) = 14$. If u is even, then $H(n) \geq H(2^6 \cdot 127^2 \cdot 3) > 14$. A similar argument denies the possibility of $s = 4$. Put $n = 2^4 \cdot 31^u m$ with $(m, 62) = 1$. Then u must be even and it is necessary that $\mathrm{ord}_2(H(m)) = -3$ by Lemma 2.6. But $H(n) \geq H(2^4 \cdot 31^2 \cdot 47) > 14$ in this case.

From now on, the following clear fact is often used.

Lemma 5.1. *Let m be a positive integer. The smallest values of $H(m)$ are as follows:*

$$H(1) = 1,\ H(2) = 4/3 = 1.33\cdots,\ H(3) = 3/2 = 1.5,\ H(5) = 5/3 = 1.66\cdots.$$

In other cases, $H(m) > 1.7$.

Suppose that $s = 5$ and $n = 2^5 \cdot 7^t m$ with $(m, 14) = 1$. If $t \geq 3$, then $H(n) \geq H(2^5 \cdot 7^3) > 14$, a contradiction. Assume that $t = 2$. Since $H(2^5 \cdot 7^2) = (2^6 \cdot 7)/(3 \cdot 19)$, it is necessary that $19 \mid n$. But $H(n) \geq H(2^5 \cdot 7^2 \cdot 19) > 14$. Assume that $t = 1$. Then $H(m) = 14/H(2^5 \cdot 7) = 1.31\cdots$, a contradiction to Lemma 5.1. Therefore the possibility of $s = 5$ is denied.

Next, we give Table 3, the table of $H(7^t)$.

The possibilities of $4 \leq t \leq 7$ are denied by an argument similar to that of the cases of $7 \leq s \leq 9$. Assume that $t = 1$ and $n = 7m$ with $7 \nmid m$. Then $H(m) = H(n)/H(7) = 8$. By Theorem 1.3, we have $m = 2^5 \cdot 3 \cdot 7$. But this is a contradiction to $7 \nmid m$. Hence $t = 2$ or 3. When $t = 3$, it is necessary that $s = 3$ since $\mathrm{ord}_2(H(7^3)) = -2$. The remaining possibilities are $s = 3$, $t = 2, 3$, or $s = 1, 2$, $t = 2$.

Table 2.

s	$H(2^s)$	s	$H(2^s)$	s	$H(2^s)$
1	$2^2/3$	4	$(2^4 \cdot 5)/31$	7	$2^{10}/(3 \cdot 5 \cdot 17)$
2	$(2^2 \cdot 3)/7$	5	$2^6/(3 \cdot 7)$	8	$(2^8 \cdot 3^2)/(7 \cdot 73)$
3	$2^5/(3 \cdot 5)$	6	$(2^6 \cdot 7)/127$	9	$(2^{10} \cdot 5)/(3 \cdot 11 \cdot 31)$

Table 3.

t	$H(7^t)$	t	$H(7^t)$	t	$H(7^t)$
1	$7/2^2$	4	$(5\cdot 7^4)/2801$	7	$7^7/(2^2\cdot 5^2\cdot 1201)$
2	$7^2/19$	5	$7^5/(2^2\cdot 19\cdot 43)$		
3	$7^3/(2^2\cdot 5^2)$	6	$7^7/(29\cdot 4733)$		

Suppose that $s=3$ and $n=2^3\cdot 7^t m$, where $(m,14)=1$ and $t=2$ or 3. First, assume that $t=3$. Since $H(2^3\cdot 7^3)=(2^3\cdot 7^3)/(3\cdot 5^3)$, it is necessary that $5\mid n$. If $\mathrm{ord}_5(n)\geq 2$, then $H(n)\geq H(2^3\cdot 7^3\cdot 5^2)>14$. If $n=2^3\cdot 7^3\cdot 5m$ with $(m,70)=1$, then $H(m)=14/H(2^3\cdot 7^3\cdot 5)=1.14\cdots$, a contradiction to Lemma 5.1. Secondly, assume that $t=2$. Since $H(2^3\cdot 7^2)=(2^5\cdot 7^2)/(3\cdot 5\cdot 19)$, it is necessary that $19\mid n$. But we deduce a contradiction by the above argument.

Suppose that $s=2$ and $t=2$. Since $H(2^2\cdot 7^2)=(2^2\cdot 3\cdot 7)/19$, it is necessary that $19\mid n$. If $19^3\mid n$, then $H(n)\geq H(2^2\cdot 7^2\cdot 19^3)>14$. Assume that $19^2\parallel n$ and put $n=2^2\cdot 7^2\cdot 19^2 m$ with $(m,266)=1$. Then $H(m)=14/H(2^2\cdot 7^2\cdot 19^2)=1.11\cdots$. Hence the only possibility is $19\parallel n$. Since $H(2^2\cdot 7^2\cdot 19)=(2\cdot 3\cdot 7)/5$, it is necessary that $5\mid n$. In fact, $H(2^2\cdot 7^2\cdot 19\cdot 5)=14$, so we get the solution $n=18620$.

Suppose that $s=1$ and $t=2$. Since $H(2\cdot 7^2)=(2^2\cdot 7^2)/(3\cdot 19)$, it is necessary that $19\mid n$. If $19^4\mid n$, then $H(n)\geq H(2\cdot 7^2\cdot 19^4)>14$. If $n=2\cdot 7^2\cdot 19^3 m$ with $(m,266)=1$, then $H(m)=14/H(2\cdot 7^2\cdot 19^3)=1.07\cdots$. If $n=2\cdot 7^2\cdot 19^2 m$ with $(m,266)=1$, then $H(m)=14/H(2\cdot 7^2\cdot 19^2)=1.43\cdots$. Hence we have $19\parallel n$ and $n=2\cdot 7^2\cdot 19m$ with $(m,266)=1$. Since $H(2\cdot 7^2\cdot 19)=(2\cdot 7^2)/(3\cdot 5)$, it is necessary that $5\mid m$. If $5^2\mid m$, then $H(n)\geq H(2\cdot 7^2\cdot 19\cdot 5^2)>14$. If $n=2\cdot 7^2\cdot 19\cdot 5m'$ with $(m',1330)=1$, then $H(m')=14/H(2\cdot 7^2\cdot 19\cdot 5)=1.28\cdots$, a contradiction.

We have checked all possibilities; hence 18620 is the only solution of $H(n)=14$.

6. Open problems

The problems in this section are proposed by the pioneers or the authors.

Problem 1. Does a nontrivial odd harmonic number exist?

Ore conjectured that the answer is “no”. If the conjecture is true, then odd perfect numbers do not exist.

Problem 2. Are there infinitely many harmonic numbers? How about harmonic seeds?

It seems that the answer to this problem is “yes”, but it is not clear. On this topic, the authors’ question is as follows.

Problem 3. Are there infinitely many harmonic seeds n with $\omega(n)=3$? If not, find all such n. Does an odd one exist?

All such numbers which the authors know are $n=270$ with $H(n)=6$, $n=672$ with $H(n)=8$, and $n=6200$ with $H(n)=10$. How about the same problem with $\omega(n)=4,5,\dots$? Note that there exist only finitely many harmonic numbers with a fixed type of exponents, because of the inequality $H(n)<\tau(n)$ and Theorem 1.2.

Cohen and Sorli [3] conjectured that a harmonic seed of a harmonic number is always unique.

Problem 4. Does every harmonic number have a unique harmonic seed?

We say that n is *powerful* if $p \mid n$ implies $p^2 \mid n$, where p is prime. Cohen and Sorli [3] implied that nontrivial harmonic numbers are not powerful.

Problem 5. Does a nontrivial powerful harmonic number exist?

It is showed that there are no nontrivial powerful harmonic numbers less than 10^{12} in [3]. Euler showed that the factorization of an odd perfect number must have the form $p^e p_1^{2e_1} \cdots p_r^{2e_r}$ with $p \equiv e \equiv 1 \pmod 4$. If odd powerful harmonic numbers other than 1 do not exist, the form of an odd perfect number must be $pp_1^{2e_1} \cdots p_r^{2e_r}$ with $p \equiv 1 \pmod 4$.

Nontrivial harmonic numbers listed in Table 4 are perfect numbers or abundant numbers. In other words, if $H(n)$ is integral and $1 < H(n) \leq 300$, then $S(n) = \sigma(n)/n \geq 2$.

Problem 6. Does a nontrivial deficient harmonic number exist?

A harmonic number n is deficient if and only if $H(n) > \tau(n)/2$. Cohen and Deng [4] remarked that $H(n) < 2\tau(n)/3$ for an even harmonic number n.

A positive integer n is said to be *arithmetic* if the arithmetic mean of its positive divisors $A(n) = \sigma(n)/\tau(n)$ is an integer. For example, odd primes are arithmetic. Ore observed that almost all (small) harmonic numbers n with $\omega(n) \geq 3$ are arithmetic and conjectured that all such numbers are arithmetic. But he soon found the counterexample 950976. Such counterexamples are marked with an asterisk in Table 4. On this topic, the following facts hold.

Proposition 6.1. *Let n be harmonic. Then n is arithmetic if and only if $H(n) \mid n$. In particular, even perfect numbers are not arithmetic.*

Proof. The first statement is clear from the equation $H(n)A(n) = n$. Since $H(n) = p$ for an even perfect number $n = 2^{p-1}(2^p-1)$, the second statement is also clear. $\square$

In view of Proposition 6.1, Proposition 1.5 says: "If $H(n)$ is a prime and n is not an even perfect number, then n is arithmetic." And the first statement of Proposition 1.4 says: "If $H(n)$ is a double of a prime, then n is arithmetic."

Problem 7. Assume that $H(n)$ is a triple of a prime. Is n arithmetic?

If $H(n)$ is a triple of a prime and less than 300, then n is arithmetic. But it is not clear whether or not 3 divides n when $H(n) > 300$.

Table 4. All harmonic numbers with $H(n) \le 300$

$H(n)$	n	factorization of n	seed
1	1		
2	6	$2 \cdot 3$	seed
3	28	$2^2 \cdot 7$	seed
5	140	$2^2 \cdot 5 \cdot 7$	$2^2 \cdot 7$
	496	$2^4 \cdot 31$	seed
6	270	$2 \cdot 3^3 \cdot 5$	seed
7	8128	$2^6 \cdot 127$	seed
8	672	$2^5 \cdot 3 \cdot 7$	seed
9	1638	$2 \cdot 3^2 \cdot 7 \cdot 13$	seed
10	6200	$2^3 \cdot 5^2 \cdot 31$	seed
11	2970	$2 \cdot 3^3 \cdot 5 \cdot 11$	$2 \cdot 3^3 \cdot 5$
13	105664	$2^6 \cdot 13 \cdot 127$	$2^6 \cdot 127$
	33550336	$2^{12} \cdot 8191$	seed
14	18620	$2^2 \cdot 5 \cdot 7^2 \cdot 19$	seed
15	8190	$2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$2 \cdot 3^2 \cdot 7 \cdot 13$
	18600	$2^3 \cdot 3 \cdot 5^2 \cdot 31$	$2^3 \cdot 5^2 \cdot 31$
17	27846	$2 \cdot 3^2 \cdot 7 \cdot 13 \cdot 17$	$2 \cdot 3^2 \cdot 7 \cdot 13$
	8589869056	$2^{16} \cdot 131071$	seed
19	117800	$2^3 \cdot 5^2 \cdot 19 \cdot 31$	$2^3 \cdot 5^2 \cdot 31$
	137438691328	$2^{18} \cdot 524287$	seed
21	55860	$2^2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 19$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
24	30240	$2^5 \cdot 3^3 \cdot 5 \cdot 7$	seed
	32760	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	seed
25	173600	$2^5 \cdot 5^2 \cdot 7 \cdot 31$	seed
26	242060	$2^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
27	167400	$2^3 \cdot 3^3 \cdot 5^2 \cdot 31$	$2^3 \cdot 5^2 \cdot 31$
	^{*950976}	$2^6 \cdot 3^2 \cdot 13 \cdot 127$	$2^6 \cdot 127$
	^{*301953024}	$2^{12} \cdot 3^2 \cdot 8191$	$2^{12} \cdot 8191$
29	237510	$2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 29$	$2 \cdot 3^2 \cdot 7 \cdot 13$
	539400	$2^3 \cdot 3 \cdot 5^2 \cdot 29 \cdot 31$	$2^3 \cdot 5^2 \cdot 31$
31	$23 \cdots 8139952128$	$2^{30} \cdot (2^{31} - 1)$	seed
35	2229500	$2^2 \cdot 5^3 \cdot 7^3 \cdot 13$	seed
37	4358600	$2^3 \cdot 5^2 \cdot 19 \cdot 31 \cdot 37$	$2^3 \cdot 5^2 \cdot 31$
	5085231579136	$2^{18} \cdot 37 \cdot 524287$	$2^{18} \cdot 524287$
39	726180	$2^2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
41	2290260	$2^2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 19 \cdot 41$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
42	1089270	$2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	seed
44	332640	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	$2^5 \cdot 3^3 \cdot 5 \cdot 7$
	360360	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$
45	4754880	$2^6 \cdot 3^2 \cdot 5 \cdot 13 \cdot 127$	$2^6 \cdot 127$
	1509765120	$2^{12} \cdot 3^2 \cdot 5 \cdot 8191$	$2^{12} \cdot 8191$
46	695520	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 23$	$2^5 \cdot 3^3 \cdot 5 \cdot 7$

TABLE 4. (continued)

$H(n)$	n	factorization of n	seed
	753480	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 23$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$
47	1421280	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 47$	$2^5 \cdot 3^3 \cdot 5 \cdot 7$
	1539720	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 47$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$
48	4713984	$2^9 \cdot 3^3 \cdot 11 \cdot 31$	seed
49	5772200	$2^3 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$	$2^3 \cdot 5^2 \cdot 31$
	8506400	$2^5 \cdot 5^2 \cdot 7^3 \cdot 31$	seed
	6734495875072	$2^{18} \cdot 7^2 \cdot 524287$	$2^{18} \cdot 524287$
50	6051500	$2^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$	seed
51	2845800	$2^3 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31$	$2^3 \cdot 5^2 \cdot 31$
	16166592	$2^6 \cdot 3^2 \cdot 13 \cdot 17 \cdot 127$	$2^6 \cdot 127$
	5133201408	$2^{12} \cdot 3^2 \cdot 17 \cdot 8191$	$2^{12} \cdot 8191$
53	8872200	$2^3 \cdot 3^3 \cdot 5^2 \cdot 31 \cdot 53$	$2^3 \cdot 5^2 \cdot 31$
	50401728	$2^6 \cdot 3^2 \cdot 13 \cdot 53 \cdot 127$	$2^6 \cdot 127$
	16003510272	$2^{12} \cdot 3^2 \cdot 53 \cdot 8191$	$2^{12} \cdot 8191$
54	*2178540	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
60	2457000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$	seed
61	$14 \cdots 6537079808$	$2^{30} \cdot 61 \cdot (2^{31} - 1)$	$2^{30} \cdot (2^{31} - 1)$
	$26 \cdots 5953842176$	$2^{60} \cdot (2^{61} - 1)$	seed
70	23088800	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$	seed
73	318177800	$2^3 \cdot 5^2 \cdot 19 \cdot 31 \cdot 37 \cdot 73$	$2^3 \cdot 5^2 \cdot 31$
	371221905276928	$2^{18} \cdot 37 \cdot 73 \cdot 524287$	$2^{18} \cdot 524287$
75	18154500	$2^2 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$	$2^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$
	$^*57 \cdots 3498803200$	$2^{30} \cdot 5^2 \cdot (2^{31} - 1)$	$2^{30} \cdot (2^{31} - 1)$
77	11981970	$2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19$	$2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$
78	115048440	$2^3 \cdot 3^2 \cdot 5 \cdot 13^2 \cdot 31 \cdot 61$	seed
80	23569920	$2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 31$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$
81	29410290	$2 \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	seed
82	44660070	$2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 41$	$2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$
83	90409410	$2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 83$	$2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$
84	32997888	$2^9 \cdot 3^3 \cdot 7 \cdot 11 \cdot 31$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$
85	80832960	$2^6 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 127$	$2^6 \cdot 127$
	25666007040	$2^{12} \cdot 3^2 \cdot 5 \cdot 17 \cdot 8191$	$2^{12} \cdot 8191$
86	14303520	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 43$	$2^5 \cdot 3^3 \cdot 5 \cdot 7$
	15495480	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 43$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$
87	137891520	$2^6 \cdot 3^2 \cdot 5 \cdot 13 \cdot 29 \cdot 127$	$2^6 \cdot 127$
	43783188480	$2^{12} \cdot 3^2 \cdot 5 \cdot 29 \cdot 8191$	$2^{12} \cdot 8191$
88	255428096	$2^9 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31$	seed
89	423184320	$2^6 \cdot 3^2 \cdot 5 \cdot 13 \cdot 89 \cdot 127$	$2^6 \cdot 127$
	134369095680	$2^{12} \cdot 3^2 \cdot 5 \cdot 89 \cdot 8191$	$2^{12} \cdot 8191$
	$19 \cdots 1548169216$	$2^{88} \cdot (2^{89} - 1)$	seed
91	75038600	$2^3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	$2^3 \cdot 5^2 \cdot 31$
	110583200	$2^5 \cdot 5^2 \cdot 7^3 \cdot 13 \cdot 31$	$2^5 \cdot 5^2 \cdot 7^3 \cdot 31$

Table 4. (continued)

$H(n)$	n	factorization of n	seed
	87548446375936	$2^{18} \cdot 7^2 \cdot 13 \cdot 524287$	$2^{18} \cdot 524287$
92	108421632	$2^9 \cdot 3^3 \cdot 11 \cdot 23 \cdot 31$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$
94	221557248	$2^9 \cdot 3^3 \cdot 11 \cdot 31 \cdot 47$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$
96	17428320	$2^5 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	seed
	45532800	$2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31$	seed
	*459818240	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	seed
	*10200236032	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$	seed
97	559903400	$2^3 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31 \cdot 97$	$2^3 \cdot 5^2 \cdot 31$
	825120800	$2^5 \cdot 5^2 \cdot 7^3 \cdot 31 \cdot 97$	$2^5 \cdot 5^2 \cdot 7^3 \cdot 31$
	653246099881984	$2^{18} \cdot 7^2 \cdot 97 \cdot 524287$	$2^{18} \cdot 524287$
99	23963940	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
	1630964808	$2^3 \cdot 3^4 \cdot 11^3 \cdot 31 \cdot 61$	seed
101	287425800	$2^3 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31 \cdot 101$	$2^3 \cdot 5^2 \cdot 31$
	1632825792	$2^6 \cdot 3^2 \cdot 13 \cdot 17 \cdot 101 \cdot 127$	$2^6 \cdot 127$
	518453342208	$2^{12} \cdot 3^2 \cdot 17 \cdot 101 \cdot 8191$	$2^{12} \cdot 8191$
102	37035180	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
105	69266400	$2^5 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$
	81695250	$2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$	seed
106	115462620	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 53$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
107	233103780	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 107$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
	$13 \cdots 7783728128$	$2^{106} \cdot (2^{107} - 1)$	seed
108	52141320	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19$	seed
110	27027000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
114	46683000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 19$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
115	56511000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 23$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
116	71253000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
117	644271264	$2^5 \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 31 \cdot 61$	seed
118	144963000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 59$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
120	*142990848	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31$	seed
121	8698459616	$2^5 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 127$	seed
125	*73924348400	$2^4 \cdot 5^2 \cdot 7 \cdot 31^2 \cdot 83 \cdot 331$	seed
127	$14 \cdots 1199152128$	$2^{126} \cdot (2^{127} - 1)$	seed
128	1867650048	$2^{10} \cdot 3^4 \cdot 11 \cdot 23 \cdot 89$	seed
130	300154400	$2^5 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$
132	766284288	$2^9 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31$	$2^9 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31$
135	163390500	$2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$	$2^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$
139	3209343200	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31 \cdot 139$	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$
140	164989440	$2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 31$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$
143	1265532840	$2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13^2 \cdot 31 \cdot 61$	$2^3 \cdot 3^2 \cdot 5 \cdot 13^2 \cdot 31 \cdot 61$
144	*1379454720	$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
	*30600708096	$2^{14} \cdot 3 \cdot 7 \cdot 19 \cdot 31 \cdot 151$	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$
145	526480500	$2^2 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 29$	$2^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$

Table 4. (continued)

$H(n)$	n	factorization of n	seed
	$16\cdots1465292800$	$2^{30}\cdot5^2\cdot29\cdot(2^{31}-1)$	$2^{30}\cdot(2^{31}-1)$
147	4409499089268	$2^2\cdot3^3\cdot7^4\cdot13\cdot467\cdot2801$	seed
149	2705020500	$2^2\cdot3\cdot5^3\cdot7^2\cdot13\cdot19\cdot149$	$2^2\cdot5^3\cdot7^2\cdot13\cdot19$
	$85\cdots1321676800$	$2^{30}\cdot5^2\cdot149\cdot(2^{31}-1)$	$2^{30}\cdot(2^{31}-1)$
150	2876211000	$2^3\cdot3^2\cdot5^3\cdot13^2\cdot31\cdot61$	seed
152	447828480	$2^9\cdot3^3\cdot5\cdot11\cdot19\cdot31$	$2^9\cdot3^3\cdot11\cdot31$
153	499974930	$2\cdot3^5\cdot5\cdot7^2\cdot13\cdot17\cdot19$	$2\cdot3^5\cdot5\cdot7^2\cdot13\cdot19$
155	110886522600	$2^3\cdot3\cdot5^2\cdot7\cdot31^2\cdot83\cdot331$	seed
156	428972544	$2^9\cdot3^3\cdot7\cdot11\cdot13\cdot31$	$2^9\cdot3^3\cdot11\cdot31$
158	1862023680	$2^9\cdot3^3\cdot5\cdot11\cdot31\cdot79$	$2^9\cdot3^3\cdot11\cdot31$
159	1558745370	$2\cdot3^5\cdot5\cdot7^2\cdot13\cdot19\cdot53$	$2\cdot3^5\cdot5\cdot7^2\cdot13\cdot19$
160	51001180160	$2^{14}\cdot5\cdot7\cdot19\cdot31\cdot151$	$2^{14}\cdot7\cdot19\cdot31\cdot151$
161	758951424	$2^9\cdot3^3\cdot7\cdot11\cdot23\cdot31$	$2^9\cdot3^3\cdot11\cdot31$
163	7279591410	$2\cdot3^2\cdot5\cdot7^2\cdot13\cdot19\cdot41\cdot163$	$2\cdot3^2\cdot5\cdot7^2\cdot13\cdot19$
164	1352913408	$2^9\cdot3^3\cdot7\cdot11\cdot31\cdot41$	$2^9\cdot3^3\cdot11\cdot31$
165	8154824040	$2^3\cdot3^4\cdot5\cdot11^3\cdot31\cdot61$	$2^3\cdot3^4\cdot11^3\cdot31\cdot61$
166	2738824704	$2^9\cdot3^3\cdot7\cdot11\cdot31\cdot83$	$2^9\cdot3^3\cdot11\cdot31$
167	5510647296	$2^9\cdot3^3\cdot7\cdot11\cdot31\cdot167$	$2^9\cdot3^3\cdot11\cdot31$
168	318729600	$2^7\cdot3^3\cdot5^2\cdot7\cdot17\cdot31$	$2^7\cdot3^3\cdot5^2\cdot17\cdot31$
	326781000	$2^3\cdot3^3\cdot5^3\cdot7^2\cdot13\cdot19$	seed
	481572000	$2^5\cdot3^3\cdot5^3\cdot7^3\cdot13$	seed
169	13660770240	$2^6\cdot3^2\cdot5\cdot13^3\cdot17\cdot127$	$2^6\cdot127$
	$23\cdots4766487552$	$2^{30}\cdot13^2\cdot61\cdot(2^{31}-1)$	$2^{30}\cdot(2^{31}-1)$
	$44\cdots6199327744$	$2^{60}\cdot13^2\cdot(2^{61}-1)$	$2^{60}\cdot(2^{61}-1)$
171	8410907232	$2^5\cdot3^2\cdot7^2\cdot13\cdot19^2\cdot127$	seed
	^{*221908282624}	$2^8\cdot7\cdot19^2\cdot37\cdot73\cdot127$	seed
172	10983408128	$2^9\cdot7\cdot11^2\cdot19\cdot31\cdot43$	$2^9\cdot7\cdot11^2\cdot19\cdot31$
173	23855232960	$2^6\cdot3^2\cdot5\cdot13\cdot29\cdot127\cdot173$	$2^6\cdot127$
	7574491607040	$2^{12}\cdot3^2\cdot5\cdot29\cdot173\cdot8191$	$2^{12}\cdot8191$
176	191711520	$2^5\cdot3^2\cdot5\cdot7^2\cdot11\cdot13\cdot19$	$2^5\cdot3^2\cdot5\cdot7^2\cdot13\cdot19$
	500860800	$2^7\cdot3^3\cdot5^2\cdot11\cdot17\cdot31$	$2^7\cdot3^3\cdot5^2\cdot17\cdot31$
	5058000640	$2^8\cdot5\cdot7\cdot11\cdot19\cdot37\cdot73$	$2^8\cdot5\cdot7\cdot19\cdot37\cdot73$
	112202596352	$2^{14}\cdot7\cdot11\cdot19\cdot31\cdot151$	$2^{14}\cdot7\cdot19\cdot31\cdot151$
181	13581986600	$2^3\cdot5^2\cdot7^2\cdot13\cdot19\cdot31\cdot181$	$2^3\cdot5^2\cdot31$
	20015559200	$2^5\cdot5^2\cdot7^3\cdot13\cdot31\cdot181$	$2^5\cdot5^2\cdot7^3\cdot31$
	$15\cdots8794044416$	$2^{18}\cdot7^2\cdot13\cdot181\cdot524287$	$2^{18}\cdot524287$
184	400851360	$2^5\cdot3^2\cdot5\cdot7^2\cdot13\cdot19\cdot23$	$2^5\cdot3^2\cdot5\cdot7^2\cdot13\cdot19$
	1047254400	$2^7\cdot3^3\cdot5^2\cdot17\cdot23\cdot31$	$2^7\cdot3^3\cdot5^2\cdot17\cdot31$
	10575819520	$2^8\cdot5\cdot7\cdot19\cdot23\cdot37\cdot73$	$2^8\cdot5\cdot7\cdot19\cdot37\cdot73$
	234605428736	$2^{14}\cdot7\cdot19\cdot23\cdot31\cdot151$	$2^{14}\cdot7\cdot19\cdot31\cdot151$
186	540277920	$2^5\cdot3^2\cdot5\cdot7^2\cdot13\cdot19\cdot31$	$2^5\cdot3^2\cdot5\cdot7^2\cdot13\cdot19$
	^{*14254365440}	$2^8\cdot5\cdot7\cdot19\cdot31\cdot37\cdot73$	$2^8\cdot5\cdot7\cdot19\cdot37\cdot73$

TABLE 4. (continued)

$H(n)$	n	factorization of n	seed
187	407386980	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
	27726401736	$2^3 \cdot 3^4 \cdot 11^3 \cdot 17 \cdot 31 \cdot 61$	$2^3 \cdot 3^4 \cdot 11^3 \cdot 31 \cdot 61$
188	819131040	$2^5 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 47$	$2^5 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$
	2140041600	$2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31 \cdot 47$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31$
	21611457280	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73$	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
	479411093504	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 47 \cdot 151$	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$
189	623397600	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$
	*675347400	$2^3 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	$2^3 \cdot 5^2 \cdot 31$
	*995248800	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7^3 \cdot 13 \cdot 31$	$2^5 \cdot 5^2 \cdot 7^3 \cdot 31$
	*787936017383424	$2^{18} \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 524287$	$2^{18} \cdot 524287$
191	3328809120	$2^5 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 191$	$2^5 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$
	8696764800	$2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31 \cdot 191$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31$
	87825283840	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 \cdot 191$	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
	1948245082112	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151 \cdot 191$	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$
192	57575890944	$2^{13} \cdot 3^2 \cdot 11 \cdot 13 \cdot 43 \cdot 127$	seed
193	108061356200	$2^3 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31 \cdot 97 \cdot 193$	$2^3 \cdot 5^2 \cdot 31$
	159248314400	$2^5 \cdot 5^2 \cdot 7^3 \cdot 31 \cdot 97 \cdot 193$	$2^5 \cdot 5^2 \cdot 7^3 \cdot 31$
	$12 \cdots 7277222912$	$2^{18} \cdot 7^2 \cdot 97 \cdot 193 \cdot 524287$	$2^{18} \cdot 524287$
195	900463200	$2^5 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$
	3221356320	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13^2 \cdot 31 \cdot 61$	$2^5 \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 31 \cdot 61$
	8628633000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 13^2 \cdot 31 \cdot 61$	seed
197	4720896180	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 197$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
	321300067176	$2^3 \cdot 3^4 \cdot 11^3 \cdot 31 \cdot 61 \cdot 197$	$2^3 \cdot 3^4 \cdot 11^3 \cdot 31 \cdot 61$
198	*22385029489560	$2^3 \cdot 3^{10} \cdot 5 \cdot 23 \cdot 107 \cdot 3851$	seed
200	*714954240	$2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31$	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31$
201	2481357060	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19 \cdot 67$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
202	3740553180	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19 \cdot 101$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
203	2008725600	$2^5 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 29 \cdot 31$	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$
	2369162250	$2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 29$	$2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$
204	886402440	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 17 \cdot 19$	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19$
205	2839922400	$2^5 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31 \cdot 41$	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$
	3349505250	$2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 41$	$2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$
207	1199250360	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19 \cdot 23$	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19$
209	513513000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 19$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
211	24362612820	$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 53 \cdot 211$	$2^2 \cdot 5 \cdot 7^2 \cdot 19$
212	2763489960	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19 \cdot 53$	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19$
213	3702033720	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19 \cdot 71$	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19$
214	5579121240	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19 \cdot 107$	$2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19$
215	1162161000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 43$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
216	43947421401888	$2^5 \cdot 3^6 \cdot 23 \cdot 137 \cdot 547 \cdot 1093$	seed
217	1179832600464	$2^4 \cdot 3 \cdot 7^2 \cdot 19 \cdot 31^2 \cdot 83 \cdot 331$	seed
218	2945943000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 109$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$

Table 4. (continued)

$H(n)$	n	factorization of n	seed
220	3831421440	$2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31$	$2^9 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31$
221	10952611488	$2^5 \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 17 \cdot 31 \cdot 61$	$2^5 \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 31 \cdot 61$
222	1727271000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
224	13073550336	$2^{10} \cdot 3^4 \cdot 7 \cdot 11 \cdot 23 \cdot 89$	$2^{10} \cdot 3^4 \cdot 11 \cdot 23 \cdot 89$
	*66433720320	$2^{13} \cdot 3^3 \cdot 5 \cdot 11 \cdot 43 \cdot 127$	seed
226	5275179000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 19 \cdot 113$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
227	10597041000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 19 \cdot 227$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
228	2716826112	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31$	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31$
229	12941019000	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 23 \cdot 229$	$2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$
230	*3288789504	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 31$	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31$
232	4146734592	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 29 \cdot 31$	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31$
233	150115204512	$2^5 \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 31 \cdot 61 \cdot 233$	$2^5 \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 31 \cdot 61$
235	*6720569856	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 47$	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31$
236	8436460032	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 59$	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31$
237	11296276992	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 79$	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31$
239	34174812672	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 239$	$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31$
240	1307124000	$2^5 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$	seed
	1381161600	$2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17 \cdot 31$	seed
	153003540480	$2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$
241	2096328767456	$2^5 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 127 \cdot 241$	$2^5 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 127$
245	3622293071600	$2^4 \cdot 5^2 \cdot 7^3 \cdot 31^2 \cdot 83 \cdot 331$	seed
	22047495446340	$2^2 \cdot 3^3 \cdot 5 \cdot 7^4 \cdot 13 \cdot 467 \cdot 2801$	$2^2 \cdot 3^3 \cdot 7^4 \cdot 13 \cdot 467 \cdot 2801$
248	57897151488	$2^{10} \cdot 3^4 \cdot 11 \cdot 23 \cdot 31 \cdot 89$	$2^{10} \cdot 3^4 \cdot 11 \cdot 23 \cdot 89$
252	*1553357978368	$2^8 \cdot 7^2 \cdot 19^2 \cdot 37 \cdot 73 \cdot 127$	seed
	*54934276752360	$2^3 \cdot 3^6 \cdot 5 \cdot 23 \cdot 137 \cdot 547 \cdot 1093$	seed
253	17624538624	$2^9 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19 \cdot 23 \cdot 31$	$2^9 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31$
254	237191556096	$2^{10} \cdot 3^4 \cdot 11 \cdot 23 \cdot 89 \cdot 127$	$2^{10} \cdot 3^4 \cdot 11 \cdot 23 \cdot 89$
255	2777638500	$2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19$	$2^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$
256	19209881600	$2^{11} \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	seed
258	32950224384	$2^9 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31 \cdot 43$	$2^9 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31$
260	2144862720	$2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$
261	4738324500	$2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 29$	$2^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$
262	100383241728	$2^9 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31 \cdot 131$	$2^9 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31$
263	201532767744	$2^9 \cdot 3 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31 \cdot 263$	$2^9 \cdot 7 \cdot 11^2 \cdot 19 \cdot 31$
264	15174001920	$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73$	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
	*43861478400	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 23 \cdot 31 \cdot 89$	seed
	336607789056	$2^{14} \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 31 \cdot 151$	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$
265	8659696500	$2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 53$	$2^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$
266	3134799360	$2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 31$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$
267	14541754500	$2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 89$	$2^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$
269	43952044500	$2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 269$	$2^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19$
270	*2701389600	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$

TABLE 4. (continued)

$H(n)$	n	factorization of n	seed
	*71271827200	$2^8 \cdot 5^2 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$	seed
272	23450730240	$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73$	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
	520212037632	$2^{14} \cdot 3 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 151$	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$
273	57648181500	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7^3 \cdot 13^3 \cdot 17$	seed
275	31638321000	$2^3 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13^2 \cdot 31 \cdot 61$	$2^3 \cdot 3^2 \cdot 5^3 \cdot 13^2 \cdot 31 \cdot 61$
276	31727458560	$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73$	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
	703816286208	$2^{14} \cdot 3 \cdot 7 \cdot 19 \cdot 23 \cdot 31 \cdot 151$	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$
277	888988066400	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31 \cdot 139 \cdot 277$	$2^5 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31$
278	22933532160	$2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 31 \cdot 139$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$
279	*42763096320	$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
	997978703400	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 31^2 \cdot 83 \cdot 331$	seed
282	64834371840	$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73$	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
	1438233280512	$2^{14} \cdot 3 \cdot 7 \cdot 19 \cdot 31 \cdot 47 \cdot 151$	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$
284	97941285120	$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 71 \cdot 73$	$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
	2172650274816	$2^{14} \cdot 3 \cdot 7 \cdot 19 \cdot 31 \cdot 71 \cdot 151$	$2^{14} \cdot 7 \cdot 19 \cdot 31 \cdot 151$
285	42054536160	$2^5 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19^2 \cdot 127$	$2^5 \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 19^2 \cdot 127$
	54648009000	$2^3 \cdot 3^2 \cdot 5^3 \cdot 13^2 \cdot 19 \cdot 31 \cdot 61$	$2^3 \cdot 3^2 \cdot 5^3 \cdot 13^2 \cdot 31 \cdot 61$
	*1109541413120	$2^8 \cdot 5 \cdot 7 \cdot 19^2 \cdot 37 \cdot 73 \cdot 127$	$2^8 \cdot 7 \cdot 19^2 \cdot 37 \cdot 73 \cdot 127$
	*24613169545216	$2^{14} \cdot 7 \cdot 19^2 \cdot 31 \cdot 127 \cdot 151$	seed
287	180789462659988	$2^2 \cdot 3^3 \cdot 7^4 \cdot 13 \cdot 41 \cdot 467 \cdot 2801$	$2^2 \cdot 3^3 \cdot 7^4 \cdot 13 \cdot 467 \cdot 2801$
290	83410119000	$2^3 \cdot 3^2 \cdot 5^3 \cdot 13^2 \cdot 29 \cdot 31 \cdot 61$	$2^3 \cdot 3^2 \cdot 5^3 \cdot 13^2 \cdot 31 \cdot 61$
291	427721411658996	$2^2 \cdot 3^3 \cdot 7^4 \cdot 13 \cdot 97 \cdot 467 \cdot 2801$	$2^2 \cdot 3^3 \cdot 7^4 \cdot 13 \cdot 467 \cdot 2801$
293	1291983233155524	$2^2 \cdot 3^3 \cdot 7^4 \cdot 13 \cdot 293 \cdot 467 \cdot 2801$	$2^2 \cdot 3^3 \cdot 7^4 \cdot 13 \cdot 467 \cdot 2801$
295	169696449000	$2^3 \cdot 3^2 \cdot 5^3 \cdot 13^2 \cdot 31 \cdot 59 \cdot 61$	$2^3 \cdot 3^2 \cdot 5^3 \cdot 13^2 \cdot 31 \cdot 61$
296	16569653760	$2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 19 \cdot 31 \cdot 37$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$
297	*125356165141536	$2^5 \cdot 3^{10} \cdot 7 \cdot 23 \cdot 107 \cdot 3851$	seed
298	428555439000	$2^3 \cdot 3^2 \cdot 5^3 \cdot 13^2 \cdot 31 \cdot 61 \cdot 149$	$2^3 \cdot 3^2 \cdot 5^3 \cdot 13^2 \cdot 31 \cdot 61$
299	9866368512	$2^9 \cdot 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 31$	$2^9 \cdot 3^3 \cdot 11 \cdot 31$

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