MATHEMATICS OF COMPUTATION
Volume 73, Number 245, Pages 181–190
S 0025-5718(03)01557-6
Article electronically published on June 6, 2003

LAGRANGE INTERPOLATION ON SUBGRIDS OF TENSOR PRODUCT GRIDS

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Dedicated to Mariano Gasca, without whom the field of polynomial interpolation would be very much depleted, on the occasion of his 60th birthday.

Abstract. This note shows that a wide class of algebraically motivated constructions for Lagrange interpolation polynomials always yields a tensor product interpolation space as long as the nodes form a tensor product grid or a lower subset thereof.

1. Introduction

Although polynomial interpolation in one variable is a classical topic—according to [1], one of the oldest surveys on (mainly polynomial) interpolation, the name "interpolation" itself dates back to Wallis in 1655—and covered by almost any textbook on numerical analysis, the theory of polynomial interpolation in several variables is much more recent. As pointed out in [11], most of the early work on (essentially bivariate) interpolation in more than one variable about 1900 considered either "rectangular" tensor product grids or their "triangular" subgrids, and so it is no surprise that most of the results are obtained quite directly from the univariate theory. It should be mentioned, however, that the oldest publication in multivariate polynomial interpolation, namely, the paper by Kronecker [14], considers a much more general configuration of points, cf. [11].

Let $\mathbb{K}$ be a field of characteristic zero, for example $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, the rational, real and complex numbers, respectively. By $\Pi = \mathbb{K}[x_1, \dots, x_d]$ we denote the algebra of all polynomials with coefficients in $\mathbb{K}$, that is, all finite linear combinations of monomials of the form

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^d} f_\alpha \, x^\alpha, \qquad \# \left\{ \alpha \in \mathbb{N}_0^d \;:\; f_\alpha \neq 0 \right\} < \infty.$$

Here we use standard multi-index notation, that is, we write $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$. For two multi-indices $\alpha, \beta \in \mathbb{N}_0^d$ we write $\alpha \leq \beta$ if all components of α are majorized by those of β, i.e., $\beta \in \alpha + \mathbb{N}_0^d$ or, equivalently, $\alpha \in \beta - \mathbb{N}_0^d$.

Given a finite set $\mathcal{X} \subset \mathbb{K}^d$ of points, the *Lagrange interpolation* problem consists of finding, for a given data vector $y = (y_x \,:\, x \in \mathcal{X}) \in \mathbb{K}^{\mathcal{X}}$, a polynomial $f = f_y \in \Pi$ such that

$$f(\mathcal{X}) = y, \qquad \text{that is,} \qquad f(x) = y_x, \quad x \in \mathcal{X}. \tag{1.1}$$

Received by the editor January 31, 2001 and, in revised form, June 8, 2002.
2000 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.
Key words and phrases. Interpolation, Gröbner bases, H-bases, tensor products.

Clearly, equation (1.1) has infinitely many solutions: if f^* is one such solution, then the totality of all solutions can be written as $f^* + I_{\mathcal{X}}$, where $I_{\mathcal{X}} \subset \Pi$ is the *ideal* of all polynomials vanishing at $\mathcal{X}$, i.e., $I_{\mathcal{X}} = \{f \in \Pi \,:\, f(\mathcal{X}) = 0\}$. In order to obtain a *unique* solution of the Lagrange interpolation problem, one therefore has to determine a finite dimensional linear subspace $\mathcal{P}$ of Π which forms an *interpolation space* with respect to $\mathcal{X}$, that is, such that for any $y \in \mathbb{K}^{\mathcal{X}}$ there exists a *unique* polynomial $f \in \mathcal{P}$ which satisfies (1.1). It is well-known that in contrast to the univariate case there is no universal space of polynomials which admits unique Lagrange interpolation for all point sets of a given cardinality, and so the interpolation space will depend on the set $\mathcal{X}$ of interpolation points. Constructions of such interpolation spaces have been considered first by Buchberger and Möller [8] in the context of Gröbner bases, and later, but independently, by de Boor et al. [3], [5] under the name of *least interpolation*; see [19] for some remarks on the connection between these approaches, in particular, how the approach by Buchberger and Möller can be interpreted as a "least" approach based on term orders. The algebraic interpretation of least interpolation has also been pointed out in [12].

The simplest point configurations for multivariate interpolation are doubtless those which are either *tensor product grids* or properly structured subgrids thereof. A tensor product grid is defined as follows. Let

$$\mathcal{X}_j = \{x_{j,k} \;:\; k \in \mathbb{N}_0\} \subset \mathbb{K}, \qquad j = 1, \ldots, d,$$

be the coordinate projections on the abscissae; then we define for $\alpha \in \mathbb{N}_0^d$ the interpolation node $x_\alpha \in \mathbb{K}^d$ as

$$x_\alpha = (x_{1,\alpha_1}, \ldots, x_{d,\alpha_d}),$$

and for $\beta \in \mathbb{N}_0^d$ the *tensor product grid* $\mathcal{X}_\beta$ as

$$\mathcal{X}_\beta := (\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_d)^\beta := \bigcup_{\alpha \le \beta} \{x_\alpha\}.$$

The simplicity of these grids stems from the well-known fact that the *tensor product* space

$$\mathcal{P}_\beta = \operatorname{span}_{\mathbb{K}} \{x^\alpha \;:\; \alpha \le \beta\}$$

is an interpolation space for the interpolation points $\mathcal{X}_\beta$, though there obviously also exist other interpolation spaces for these point configurations.

However, it is an interesting and important observation that in the case of a tensor product grid both constructions of interpolation spaces, the one by Buchberger and Möller and the least interpolation (as specifically pointed out in [4]), always lead to the tensor product space, while non-tensor-product spaces always appear as unnatural and artificial constructs in that case. Thus it seems reasonable that, regardless of the method for constructing the interpolation space, tensor product data is best handled by tensor product spaces.

It is the purpose of this note to substantiate this argument by showing that within a large class of algebraically "natural" construction methods for interpolation spaces, namely, *normal form interpolations*, which include the constructions by Buchberger and Möller as well as the least interpolant, the tensor product space is always the unique space to be chosen for tensor product grids.

The necessary algebraic background for normal interpolation, a concept generalizing the idea of implicit interpolation via division with remainder, will be provided in Section 2, while the main result of this paper, Theorem 3.3, will be stated, proven and commented upon in Section 3.

2. Normal form interpolation

To illustrate the simple idea behind normal form interpolation, let us have a brief look at univariate Lagrange interpolation first. To that end, let $x_0, \dots, x_n \in \mathbb{K}$ be distinct points and let $g \in \Pi$ be any univariate polynomial. The Lagrange interpolation polynomial $Lg \in \Pi$ of g with respect to $\mathcal{X}$ is then a polynomial of degree $\le n$, and since $(g - Lg)(\mathcal{X}) = 0$, we observe that there exists a polynomial $q \in \Pi$ such that

$$g(x) - Lg(x) = q(x) \underbrace{(x - x_0) \cdots (x - x_n)}_{=:f(x)} \qquad \text{or} \qquad g(x) = q(x)\, f(x) + Lg(x).$$

Since $\deg f = n + 1$ and $\deg Lg \le n$, the representation on the right-hand side is nothing but *division with remainder* of g by f. Also note that f depends directly on the interpolation points, as it is the unique polynomial with monomial leading term which vanishes precisely at the interpolation nodes. In other words, f is the normalized generator of the *principal ideal* of all polynomials vanishing at $x_0, \dots, x_n$. Moreover, since $\deg qf \le \deg g$, it also follows that $\deg Lg < \deg g$ if $q \ne 0$ and, trivially, $\deg Lg = \deg g$ if $q = 0$. Consequently, the interpolation operator is *degree-reducing*, that is, $\deg Lg \le \deg g$ for any $g \in \Pi$.

Algebraically, the univariate Lagrange interpolation is thus based on a degree-reducing remainder of dividing g by a generator f of the ideal of all polynomials that vanish at the node set $\mathcal{X}$. To carry over this approach to the multivariate setting, we first recall how the notion of degree of a polynomial is extended to several variables by means of a *graded ring*, cf. [10, p. 30]. A monoid Γ, that is, an (additive) semigroup with neutral (zero) element, equipped with a well-ordering "$<$", defines a *grading* of Π if there exists a direct sum decomposition

$$\Pi = \bigoplus_{\gamma \in \Gamma} \Pi_\gamma \tag{2.1}$$

into $\mathbb{K}$-vector spaces (originally additive Abelian subgroups) Π_γ such that

$$\Pi_\gamma\, \Pi_{\gamma'} \subseteq \Pi_{\gamma+\gamma'}, \qquad \gamma, \gamma' \in \Gamma. \tag{2.2}$$

The subspaces Π_γ, $\gamma \in \Gamma$, are called the *homogenous subspaces* of Π. This notion coincides with that of classical homogenous subspaces if one uses the *H-grading* related to the monoid $\Gamma = \mathbb{N}_0$ equipped with its canonical order. It follows from (2.2) and $\mathbb{K} \cdot \mathbb{K} = \mathbb{K}$ that $\mathbb{K} \subset \Pi_0$—the constant polynomials always have degree zero. The grading will be called *strict* if the constants are the only polynomials of degree zero, that is, if $\Pi_0 = \mathbb{K}$, and it is called *monomial* if each of the vector spaces Π_γ, $\gamma \in \Gamma$, is spanned by monomials.

By (2.1), any polynomial $f \in \Pi$ can be written uniquely as a sum $f = \sum_{\gamma \in \Gamma} f_\gamma$ of its homogeneous components with only finitely many nonzero terms, and therefore there exists an index

$$\delta(f) = \max_{<} \{\gamma \;:\; f_\gamma \ne 0\},$$

which will be called the Γ-*degree* of f. The maximal part $\lambda(f) = f_{\delta(f)}$ is called the Γ-*leading term* of f.

For a finite set $\mathcal{F} \subset \Pi$ of polynomials, we denote by

$$\langle \mathcal{F} \rangle = \left\{ \sum_{f \in \mathcal{F}} q_f \, f \ : \ q_f \in \Pi \right\}$$

the ideal generated by $\mathcal{F}$, and we say that $\mathcal{F}$ is a *basis* for an ideal I if $I = \langle \mathcal{F} \rangle$. By Hilbert's Basis Theorem, often referred to by its German name "Basissatz" even in the English literature, every ideal has a finite basis. In theory as well as in applications, however, one is interested in particular bases which can also be handled algorithmically. Important special cases are H-bases, considered by Macaulay [15] as early as 1916, and Buchberger's Gröbner bases, introduced in 1965 [6], [7]. Both are special cases of Γ-*bases*, obtained by choosing an appropriate grading monoid. Specifically, Gröbner bases are obtained by choosing $\Gamma = \mathbb{N}_0^d$, equipped with a *term order* $\prec$, and setting

$$\Pi_\alpha := \mathbb{K} \cdot x^\alpha, \qquad \alpha \in \Gamma,$$

while H-bases result from the grading by total degree or *H-grading* which is obtained with $\Gamma = \mathbb{N}_0$ and

$$\Pi_k := \sum_{|\alpha| = k} \mathbb{K} \cdot x^\alpha, \qquad k \in \Gamma,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha \in \mathbb{N}_0^d$. Note that if the term order is subordinate to total degree, that is, $|\alpha| > |\beta|$ implies $\alpha \succ \beta$, then any Gröbner basis with respect to that term order is an H-basis as well. A (finite) set $\mathcal{G} \subset \Pi$ is called a Γ-basis for an ideal I if

$$f \in I \qquad \Longleftrightarrow \qquad f = \sum_{g \in \mathcal{G}} q_g \, g, \qquad \delta(f) \ge \delta\left(q_g \, g\right), \quad g \in \mathcal{G}, \tag{2.3}$$

or, equivalently, if the homogeneous ideal generated by the leading terms of $\mathcal{G}$ covers all the leading terms of all polynomials in the ideal, that is,

$$\lambda(I) \cap \Pi_\gamma = V_\gamma(\mathcal{G}), \qquad \gamma \in \Gamma, \tag{2.4}$$

where

$$V_\gamma(\mathcal{F}) = \left\{ \sum_{f \in \mathcal{F}} q_f \, \lambda(f) \ : \ q_f \in \Pi_{\gamma - \delta(f)} \right\} \subseteq \Pi_\gamma, \tag{2.5}$$

is the intersection of Π_γ with the homogeneous ideal $\langle \lambda(\mathcal{F}) \rangle_h$ generated by $\lambda(\mathcal{F})$. Observe that a Γ-basis is distinguished from a "general" basis by the degree constraints on the right-hand side of (2.3). It is possible to show that for any ideal there exists a finite Γ-basis and, connecting us back to the univariate case, that for any polynomial $f \in \Pi$, any ideal I and any Γ-basis $\mathcal{G}$ of I there exists a *unique* polynomial $\nu_{\mathcal{G}} f$, called the *normal form* of f with respect to $\mathcal{G}$, such that

$$f = \nu_{\mathcal{G}} f + \sum_{g \in \mathcal{G}} q_g \, g, \qquad \delta(f) \ge \delta\left(q_g \, g\right), \quad g \in \mathcal{G}, \tag{2.6}$$

$\delta\left(\nu_{\mathcal{G}} f\right) \le \delta(f)$, and that the remainder $\nu_{\mathcal{G}} f$ is reduced with respect to $\mathcal{G}$. Choosing an *arbitrary* inner product $(\cdot, \cdot) : \Pi \times \Pi \to \mathbb{K}$, a polynomial $g = \sum_{\gamma \in \Gamma} g_\gamma$ is called *reduced* with respect to a finite set $\mathcal{F} \subset \Pi$ if

$$\left(g_\gamma, V_\gamma(\mathcal{F})\right) = 0, \qquad \gamma \in \Gamma. \tag{2.7}$$

Moreover, the normal form can be constructed algorithmically by means of the orthogonal reduction process introduced in [20], see also [16]. This algorithm, which computes the remainder $\nu_{\mathcal{G}} f$ as well as the Γ*–representation* $\sum_{g \in \mathcal{G}} q_g \, g$ of the ideal part $f - \nu_{\mathcal{G}} f \in \langle \mathcal{G} \rangle$, has two parameters: the grading Γ and an inner product which can be chosen in addition and allows for further degrees of freedom. Though in [16], [20] only H-bases are considered, at least in detail, the methods developed there can be transferred almost literally to arbitrary grading monoids. It is worthwhile to mention that in the case of Gröbner bases the inner product is irrelevant, while it becomes a crucial ingredient as soon as the homogeneous spaces Π_γ, $\gamma \in \Gamma$, are of dimension > 1.

It may appear that the normal forms depend not only on the specific ideal I, the grading and the inner product, but also on the choice of the Γ-basis $\mathcal{G}$, which would cause a serious problem: while at least the reduced Gröbner basis with respect to a given term order is unique for any ideal $I \subset \Pi$, this is no longer true even for reduced H-bases. For details see [16, Theorem 6.5], which characterizes *all* reduced H-bases for a given ideal. Fortunately, though the Γ-basis may not be unique for a given ideal, the normal form is.

Lemma 2.1. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two Γ-bases for the ideal I. Then for any $f \in \Pi$ we have $\nu_{\mathcal{G}} f = \nu_{\mathcal{G}'} f$.*

Proof. Since both $\mathcal{G}$ and $\mathcal{G}'$ are Γ-bases for I, we conclude from (2.4) that

$$V_\gamma(\mathcal{G}) = \lambda(I) \cap \Pi_\gamma = V_\gamma(\mathcal{G}'), \qquad \gamma \in \Gamma,$$

and thus the normal form $\nu_{\mathcal{G}} f$ with respect to $\mathcal{G}$ satisfies $f - \nu_{\mathcal{G}} f \in I$ and is reduced with respect to $\mathcal{G}$. Thus, we obtain by means of (2.7) that

$$0 = \Big((\nu_{\mathcal{G}} f)_\gamma, V_\gamma(\mathcal{G}) \Big) = \Big((\nu_{\mathcal{G}} f)_\gamma, V_\gamma(\mathcal{G}') \Big), \qquad \gamma \in \Gamma,$$

that is, $\nu_{\mathcal{G}} f$ is also reduced with respect to $\mathcal{G}'$. Consequently, since the reduced remainder with respect to a Γ-basis is unique, it follows that $\nu_{\mathcal{G}} f = \nu_{\mathcal{G}'} f$. ☐

These observations on normal forms finally bring us back to the Lagrange interpolation problem. For a finite set $\mathcal{X} \subset \mathbb{K}^d$ of nodes, we let $\mathcal{G}$ be any Γ-basis of the ideal $I_{\mathcal{X}}$—such bases can even be computed efficiently, as long as the homogeneous spaces Π_γ have finite dimension, by a modification of the H-bases algorithm presented in [17]. By Lemma 2.1, the normal form $\nu(I_{\mathcal{X}}, f) := \nu_{\mathcal{G}} f$ essentially depends on $I_{\mathcal{X}}$ only, which enables us to speak about *the* normal form of f with respect to $I_{\mathcal{X}}$, but keep in mind that there is still a dependence on the underlying grading and inner product. Since $f - \nu(I_{\mathcal{X}}, f) \in I_{\mathcal{X}}$, hence $(f - \nu(I_{\mathcal{X}} f))(\mathcal{X}) = 0$, the two polynomials f and $\nu_{I_{\mathcal{X}}} f$ coincide on $\mathcal{X}$. Consequently, the normal form $\nu(I_{\mathcal{X}}, f)$ is a *degree-reducing* interpolation polynomial and the finite dimensional vector space $\mathcal{P} = \nu(I_{\mathcal{X}}, \Pi)$ is a degree-reducing interpolation space. This is the concept of normal form interpolation. Note that for a given node set $\mathcal{X}$ there usually exists a variety of different normal form interpolation spaces, depending on the grading monoid and, as pointed out in [20], on the choice of the inner product on Π as well.

There are several choices of normal form interpolation spaces known. First, choosing $\Gamma = \mathbb{N}_0^d$ equipped with a term order (the inner product then becomes irrelevant), the normal form spaces are the interpolation spaces introduced in [8], see also [18]. Combined with the inner product

$$(f,g) = \sum_{\alpha\in\mathbb{N}_0^d} f_\alpha\, g_\alpha, \qquad f(x) = \sum_{\alpha\in\mathbb{N}_0^d} f_\alpha\, x^\alpha, \quad f(x) = \sum_{\alpha\in\mathbb{N}_0^d} f_\alpha\, x^\alpha,$$

the H-grading gives Macaulay's *inverse systems*, see [13], the oldest normal form interpolation space, while the least interpolant is obtained from using the H-grading together with the inner product

$$(f,g) = (f(D)\,g)\,(0) = \sum_{\alpha\in\mathbb{N}_0^d} \frac{f_\alpha\, g_\alpha}{\alpha!},$$

which is responsible for many of the striking properties [3], [4], [5] of the least interpolant which are not available for Macaulay's inverse systems, cf. [20].

3. Interpolation on lower sets of tensor product grids

It is obvious that the tensor product space $\mathcal{P}_\beta$ is a suitable interpolation space for the tensor product node set $\mathcal{X}_\beta$, since the *Lagrange fundamental polynomials*

$$\ell_\alpha(x) := \left(\prod_{j=1}^d \prod_{k_j=0}^{\beta_j} \left(x_j - x_{j,k_j}\right)\right) \Big/ \left(\prod_{j=1}^d \left(x_j - x_{j,\alpha_j}\right)\right), \qquad \alpha \le \beta,$$

belong to $\mathcal{P}_\beta$ and vanish at $\mathcal{X}_\beta \setminus \{x_\alpha\}$ but not at x_α. Therefore, $\mathcal{P}_\beta$ is a natural interpolation space for the grid $\mathcal{X}_\beta$, but, as we will see, in terms of normal form interpolation it is even *the* natural interpolation space.

Theorem 3.1. *Suppose that Γ induces a strict monomial grading. For any normal form interpolation space $\mathcal{P} = \nu\left(I_\mathcal{X}, \Pi\right)$ for the tensor product grid $\mathcal{X} = \mathcal{X}_\beta$ we have that $\mathcal{P} = \mathcal{P}_\beta$.*

We will prove the theorem for a more general class of node configurations. To that end, recall that besides the tensor product grids the *triangular* ones, formed by the points

$$\mathcal{X}_k = \left\{(x_{1,\alpha_1},\dots,x_{d,\alpha_d}) \;:\; |\alpha| := \alpha_1 + \cdots + \alpha_d \le k\right\}, \qquad k \in \mathbb{N}_0,$$

were among the earliest to be studied systematically, beginning with Biermann [2] in 1903 in a context of cubature. Nevertheless, the methods on the triangular grid are still almost the same as in the tensor product setting; a more systematic investigation of grids which admit tensor-product-like methods has been performed by Werner [21]. This approach leads to the notion of a *lower set*, which means a set $L \subset \mathbb{N}_0^d$ with the property that

$$L = \left(L - \mathbb{N}_0^d\right) \cap \mathbb{N}_0^d.$$

Note that the complement $\mathbb{N}_0^d \setminus L$ of a lower set L is an *upper set* $U \subset \mathbb{N}_0^d$ which satisfies $U = U + \mathbb{N}_0^d$. Any nontrivial upper set is infinite but nevertheless *finitely generated*.

Lemma 3.2. *Let U be an upper set. Then there exists a finite set $\mathcal{A} \subset \mathbb{N}_0^d$ such that*

$$U = \bigcup_{\alpha\in\mathcal{A}} \left(\alpha + \mathbb{N}_0^d\right). \tag{3.1}$$

Proof. By Dickson's Lemma, cf. [9, p. 69], the monomial ideal $\{x^\alpha \,:\, \alpha \in U\}$ has a finite basis whose exponents we collect into $\mathcal{A}$, which implies that (in the sense of monomial ideals)

$$\{x^\alpha \,:\, \alpha \in U\} = \langle x^\alpha \,:\, \alpha \in \mathcal{A}\rangle ,$$

which is equivalent to (3.1). □

For a *finite* lower set L we define the point set $\mathcal{X}_L := \{x_\alpha \,:\, \alpha \in L\}$ and the polynomial subspace $\mathcal{P}_L = \operatorname{span}_{\mathbb{K}} \{x^\alpha \,:\, \alpha \in L\}$. Obviously, $\dim \mathcal{P}_L = \# \mathcal{X}_L$. Now we are in a position to state and prove the main result of this paper, from which Theorem 3.1 follows immediately.

Theorem 3.3. *Suppose that* Γ *induces a strict monomial grading. For any finite lower set* L *and any normal form interpolation space with respect to* $\mathcal{X}_L$ *we have that* $\nu\left(I_{\mathcal{X}_L}, \Pi\right) = \mathcal{P}_L$.

Proof. Let $U = \mathbb{N}_0^d \backslash L$ be the complimentary upper set for L and let $\mathcal{A}$ be a *minimal* generating set for U, that is,

$$\left(\mathcal{A} + \left(\mathbb{N}_0^d \setminus \{0\}\right)\right) \cap \mathcal{A} = \emptyset. \tag{3.2}$$

The main point of the proof is the fact that the polynomials

$$f_\alpha = \prod_{j=1}^{d} \prod_{k_j=0}^{\alpha_j - 1} \left(x_j - x_{j,\alpha_j}\right) = x^\alpha + \cdots , \qquad \alpha \in \mathcal{A}, \tag{3.3}$$

are well-defined and form a *universal* Γ-basis for the ideal $I_{\mathcal{X}_L}$, that is, a Γ-basis for any strict monomial grading. In particular, they form a universal Gröbner and H-basis.

To see that $f_\alpha \in \mathcal{P}_L$, we first note that for any $\alpha \in \mathcal{A}$ and any $\beta \le \alpha$ such that $\beta \ne \alpha$ it follows that $\beta \in L$, as otherwise we could write $\alpha = \beta + \eta$ for some $\eta \in \mathbb{N}_0^d \setminus \{0\}$ as well as $\beta = \alpha' + \eta'$, $\alpha' \in \mathcal{A}$, $\eta' \in \mathbb{N}_0^d$, since $\mathcal{A}$ is a generating set for U. But this way we would obtain that $\alpha = \alpha' + \eta + \eta'$, where $\eta + \eta' \ne 0$, which contradicts (3.2).

The second step is to show that $f_\alpha \in I_{\mathcal{X}}$, $\alpha \in \mathcal{A}$, that is, that $f_\alpha\left(\mathcal{X}_L\right) = 0$. For that end, fix $\alpha \in \mathcal{A}$ and pick any $\beta \in L$. Since $U = \mathcal{A} + \mathbb{N}_0^d$ is an upper set, there must exist at least one index $j \in \{1, \dots, d\}$ such that $\beta_j < \alpha_j$, that is, $\beta_j \in \{0, \dots, \alpha_j - 1\}$ and therefore

$$\prod_{k=0}^{\alpha_j - 1} \left(\left(x_\beta\right)_j - x_{j,k}\right) = \prod_{k=0}^{\alpha_j - 1} \left(x_{j,\beta_j} - x_{j,k}\right) = 0,$$

which yields that $f_\alpha\left(x_\beta\right) = 0$. Since α and β were arbitrary, this implies that $f_\alpha \in I_{\mathcal{X}}$ and, in terms of ideals, that

$$F := \langle f_\alpha \,:\, \alpha \in \mathcal{A}\rangle \subseteq I_{\mathcal{X}}. \tag{3.4}$$

Next, we show that the polynomials f_α, $\alpha \in \mathcal{A}$, form a universal Gröbner basis for the ideal they generate. Since any monomial grading can be refined into a term order by using the lexicographical ordering $\prec_l$ to define the term order $\prec_\delta$, subordinate to Γ, as

$$\alpha \prec_\delta \beta \qquad \Longleftrightarrow \qquad \delta\left(x^\alpha\right) < \delta\left(x^\beta\right) \quad \text{or} \quad \delta\left(x^\alpha\right) = \delta\left(x^\beta\right) \text{ and } \alpha \prec_l \beta,$$

this also implies that the set $\{f_\alpha \,:\, \alpha \in \mathcal{A}\}$ is a universal Γ-basis for any monomial term order. To check the Gröbner basis property, we pick an arbitrary term order

$\prec$ and incorporate the well-known characterization of Gröbner bases which says that a finite set of polynomials is a Gröbner basis for the ideal they generate if and only if all syzygies of leading terms reduce to zero, cf. [9]. Since the grading is strict, the leading term of a polynomial of the form $x_j - x_{j,\alpha_j}$ is x_j; and since the leading term of a product of polynomials is the product of their leading terms, it follows from (3.3) that the leading term of any f_α with respect to $\prec$ takes the form

$$\lambda_\prec(f_\alpha) = \prod_{j=1}^{d} x_j^{\alpha_j} = x^\alpha,$$

independently of $\prec$. Of course, instead of checking all syzygies of leading terms, one checks only a finite generating set of this module, namely the so-called S-*polynomials*. For that end, choose $\alpha, \alpha' \in \mathcal{A}$ and let $\eta \in \mathbb{N}_0^d$ be the generator of the intersection of the two cones $\alpha + \mathbb{N}_0^d$ and $\alpha' + \mathbb{N}_0^d$, that is,

$$\left(\alpha + \mathbb{N}_0^d\right) \cap \left(\alpha' + \mathbb{N}_0^d\right) = \eta + \mathbb{N}_0^d.$$

Then the S-polynomial is

$$S(f_\alpha, f_{\alpha'}) = x^{\eta-\alpha} f_\alpha - x^{\eta-\alpha'} f_{\alpha'}.$$

Set

$$g_\alpha(x) := \prod_{j=1}^{d} \prod_{k_j=\alpha_j}^{\eta_j} \left(x_j - x_{j,k_j}\right) \qquad \text{and} \qquad g_{\alpha'}(x) := \prod_{j=1}^{d} \prod_{k_j=\alpha'_j}^{\eta_j} \left(x_j - x_{j,k_j}\right),$$

and note that $\lambda_\prec(g_\alpha) = x^{\eta-\alpha}$ as well as $\lambda_\prec(g_{\alpha'}) = x^{\eta-\alpha'}$, and that

$$g_\alpha(x)\, f_\alpha(x) = g_{\alpha'}(x)\, f_{\alpha'}(x) = \prod_{j=1}^{d} \prod_{k_j=0}^{\eta_j} \left(x_j - x_{j,k_j}\right).$$

Consequently,

$$\begin{aligned} S(f_\alpha, f_{\alpha'}) &= \left(x^{\eta-\alpha} - g_\alpha\right) f_\alpha - \left(x^{\eta-\alpha'} - g_{\alpha'}\right) f_{\alpha'} \\ &= \left(\lambda(g_\alpha) - g_\alpha\right) f_\alpha - \left(\lambda(g_{\alpha'}) - g_{\alpha'}\right) f_{\alpha'}, \end{aligned}$$

which confirms that any S-polynomial can be written as a combination of f_α and $f_{\alpha'}$, where the coefficient polynomials $\lambda(g_\alpha) - g_\alpha$ and $\lambda(g_{\alpha'}) - g_{\alpha'}$ are of sufficiently small degree. In other words, the S-polynomial reduces to zero modulo $\mathcal{F} := \{f_\alpha \;:\; \alpha \in \mathcal{A}\}$.

To finish the proof we have to show that $F = I_\mathcal{X}$, which follows immediately from the fact that $\nu_\mathcal{F}(\Pi)$, the vector space of all normal spaces modulo the Gröbner basis $\mathcal{F}$, takes the form

$$\nu_\mathcal{F}(\Pi) = \bigcap_{\alpha\in\mathcal{A}} \ker \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \left\{x^\alpha \;:\; \alpha \in \mathbb{N}_0^d \setminus U\right\} = \left\{x^\alpha \;:\; \alpha \in L\right\},$$

cf. [19]. Thus,

$$\dim \Pi/F = \dim \nu_\mathcal{F}(\Pi) = \#L = \dim \Pi/I_\mathcal{X},$$

which implies together with $F \subseteq I_\mathcal{X}$ that $I_\mathcal{X} = F = \langle\mathcal{F}\rangle$. Now, since $\mathcal{F}$ is a Gröbner basis with respect to a grading that refines the grading induced by Γ, it follows that $\mathcal{F}$ is also a Γ-basis for $I_\mathcal{X}$, and Lemma 2.1 yields that

$$\nu(I_\mathcal{X}, \Pi) = \nu(\mathcal{F}, \Pi) = \mathcal{P}_L,$$

as was claimed. □

We end the paper with some remarks on Theorem 3.3.

(1) The fact that tensor product configurations boil down to tensor spaces has already been emphasized as an important property of the least interpolation space by de Boor et al in [3]. Of course, the proof there is more intuitive, as it deals with a particular situation.

(2) In a sloppy way, Theorem 3.3 could be formulated as

The only reasonable interpolation space for tensor product grids is the space of tensor product polynomials.

This fact also has practical consequences for the *computation* of interpolation spaces, for example by the algorithms given in [8], [4], [18]. In many cases, in particular when the grading is subordinate to the one by total degree, tensor product configurations become *singular*, and the interpolation space changes dramatically as soon as the grid $\mathcal{X}_L$ is subject to any arbitrarily small perturbation. Clearly, this behavior results in instability of the algorithms, which leads to the following "rule of thumb":

When interpolating on a lower subset of a tensor product grid, it is reasonable to use a tensor product space from the beginning.

(3) The assumption that Γ induces a *monomial* grading is crucial in Theorem 3.3. For example, if $V \in (\mathbb{K} \setminus \{0\})^{d \times d}$ is a nonsingular matrix, then the non-monomial grading

$$\Pi_\alpha = \operatorname{span}_{\mathbb{K}} \left\{ (Vx)^\alpha \right\}, \quad \alpha \in \mathbb{N}_0^d,$$

where $\Gamma = \mathbb{N}_0^d$, equipped with a term order, leads to the "generic" interpolation spaces

$$\operatorname{span}_{\mathbb{K}} \left\{ (Vx)^\alpha \ : \ \alpha \in L \right\}$$

for the grid $\mathcal{X}_L$.

(4) Also, the property that tensor product points yield tensor product spaces does not hold true for arbitrary degree-reducing interpolation spaces. A simple example is to consider the points $(\pm 1, \pm 1) \in \mathbb{R}^2$ and the valid interpolation space spanned by $\left\{1, x, y, x^2 + xy + y^2\right\}$, which reduces the total degree but obviously is not a tensor product space.

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