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CONVERGENCE OF NONCONFORMING V-CYCLE AND F-CYCLE MULTIGRID ALGORITHMS FOR SECOND ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. The convergence of V-cycle and F-cycle multigrid algorithms with a sufficiently large number of smoothing steps is established for nonconforming finite element methods for second order elliptic boundary value problems.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Consider the variational problem of finding $u \in H_0^1(\Omega)$ such that

$$a(u,v) = F(v) \qquad \forall\, v \in H_0^1(\Omega)\,, \tag{1.1}$$

where $F \in H^{-1}(\Omega)$ and

$$a(v,w) = \int_\Omega \Big[\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + r(x) v w \Big]\, dx\,. \tag{1.2}$$

We assume that $a_{ij}, r \in C^1(\bar\Omega)$, $a_{12} = a_{21}$, $r \geq 0$ on $\bar\Omega$, and

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq c(\xi_1^2 + \xi_2^2) \qquad \forall\, x \in \bar\Omega\,, \xi_1, \xi_2 \in \mathbb{R}\,, \tag{1.3}$$

where c is a positive constant. Under these conditions the bilinear form $a(\cdot,\cdot)$ on $H_0^1(\Omega) \times H_0^1(\Omega)$ is bounded and coercive, and (1.1) has a unique solution.

It is well known (cf. §5.C and §14.A of [32]) that there exists $\alpha \in (\frac{1}{2}, 1]$ such that the solution u of (1.1) belongs to $H^{1+\alpha}(\Omega) \cap H_0^1(\Omega)$ whenever $F \in H^{-1+\alpha}(\Omega)$ and

$$\|u\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|F\|_{H^{-1+\alpha}(\Omega)}\,. \tag{1.4}$$

Approximate solutions of the variational problem (1.1) can be obtained by the finite element method (cf. [30, 25]). The resulting symmetric positive definite systems are sparse and can be solved efficiently by multigrid algorithms (cf. [36], [39], [7], [16], [49]). For the symmetric V-cycle algorithm with equal numbers of pre-smoothing and post-smoothing steps, the classical result by Braess and Hackbusch

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(cf. [35], [5], [3], [39], [9]) states that, in the case where $\alpha = 1$ (i.e., when Ω is convex),

$$\gamma_k \leq \frac{C}{C+m} \qquad \text{for } m = 1, 2, \dots ,$$

where γ_k is the contraction number of the k-th level V-cycle algorithm in the norm

$$\| \cdot \|_a = \sqrt{a(\cdot, \cdot)} \,, \tag{1.5}$$

m is the number of pre-smoothing and post-smoothing steps, and C (with or without subscripts) henceforth denotes a generic positive constant which is independent of k and m.

The case where $\frac{1}{2} < \alpha < 1$ (i.e., when Ω has re-entrant corners) is more subtle. It was not until the early nineties, after a multiplicative theory for multilevel methods (cf. [13, 12]) had been developed, that the following result was established (cf. [57, 53, 10, 56, 11, 34, 41]):

$$\gamma_k \leq \delta \,, \tag{1.6}$$

where $\delta \in (0, 1)$ is independent of k. The asymptotic behavior of γ_k with respect to m was studied in [22] by an additive convergence theory. It was shown that, for the P_1 finite element and with the Richardson relaxation scheme as smoother,

$$\gamma_k \leq \frac{C}{m^\alpha} \qquad \text{for } m \geq m_0 \,, \tag{1.7}$$

where the positive integer m_0 is independent of k. It then follows easily from (1.6) and (1.7) that

$$\gamma_k \leq \frac{C}{C + m^\alpha} \qquad \text{for } m = 1, 2, \dots .$$

In other words, a complete generalization of the result of Braess and Hackbusch to the case of less than full elliptic regularity has been obtained. The results in [22] were generalized to include other smoothers in [23].

In this paper we extend the theory in [22] to nonconforming finite elements and establish the same estimate (1.7). Since the V-cycle algorithm for nonconforming finite elements in general does not converge uniformly for $m = 1$, the estimate (1.7) is the best possible general estimate for such methods. As a by-product we also obtain similar estimates for nonconforming F-cycle algorithms (cf. [17], [40], [51], [49]). As far as we know, this is the first convergence result for nonconforming F-cycle algorithms, even though it has been known for some time (cf. [50]) that these algorithms perform very well in practice. The theory developed in this paper can also be applied to fourth order problems (cf. [58]).

We also note in passing that most of the general convergence results for nonconforming multigrid algorithms were obtained for the W-cycle algorithm and the variable V-cycle preconditioner (cf. [18], [14], [21] and the references therein, and also [44], [47] for the cascadic multigrid algorithm). The only existent nonconforming V-cycle convergence results involve either special elements (cf. [54], [45], [46]), conforming finite element spaces on coarse grids (cf. [52], [55]), or the suboptimal Galerkin approach (cf. [29]).

The rest of the paper is organized as follows. In Section 2 we set up V-cycle and F-cycle algorithms for nonconforming finite elements in an abstract setting. The assumptions for the convergence theory are stated in Section 3. We then prove a strengthened Cauchy-Schwarz inequality in Section 4. The estimate (1.7) for V-cycle and F-cycle algorithms is established in Section 5. In Section 6 we show

that the assumptions in Section 3 can be verified within an abstract framework for nonconforming multigrid methods. Applications to concrete nonconforming finite elements are then given in Section 7.

For future reference we state here two simple inequalities:

$$2ab \le (\theta a)^2 + (\theta^{-1} b)^2 \qquad \forall\, a, b \in \mathbb{R}\,,\ \theta \in (0,1)\,, \tag{1.8a}$$

$$(a+b)^2 \le (1+\theta^2)a^2 + (1+\theta^{-2})b^2 \qquad \forall\, a, b \in \mathbb{R}\,,\ \theta \in (0,1)\,. \tag{1.8b}$$

2. Nonconforming V-cycle and F-cycle multigrid algorithms

Let $\mathcal{T}_1$ be a triangulation of Ω and let the triangulations $\mathcal{T}_k$, for $k = 2, 3, \ldots$, be obtained by successive regular subdivisions. The mesh size h_k of $\mathcal{T}_k$ therefore satisfies the relation

$$h_k = 2h_{k+1} \qquad \text{for } k = 1, 2, \ldots\,. \tag{2.1}$$

The discontinuous energy space $H^1(\mathcal{T}_k)$ associated with $\mathcal{T}_k$ is defined by

$$H^1(\mathcal{T}_k) = \{v \in L_2(\Omega) :\ v\big|_T \in H^1(T) \quad \forall\, T \in \mathcal{T}_k\}\,. \tag{2.2}$$

We define the nonconforming variational form $a_k(\cdot,\cdot)$ on $H^1(\mathcal{T}_k)$ by

$$a_k(v,w) = \sum_{T\in\mathcal{T}_k} \int_T \Big[\sum_{i.j=1}^{2} a_{ij}(x) \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} + r(x) v w \Big]\, dx \qquad \forall\, v, w \in H^1(\mathcal{T}_k)\,, \tag{2.3}$$

and the corresponding nonconforming energy (semi-)norm $\|\cdot\|_{a_k}$ by

$$\|v\|_{a_k} = \sqrt{a_k(v,v)} \qquad \forall\, v \in H^1(\mathcal{T}_k)\,. \tag{2.4}$$

Note that $H^1(\mathcal{T}_1) \subset H^1(\mathcal{T}_2) \subset \cdots$ and

$$\|v\|_{a_{k-1}} = \|v\|_{a_k} \qquad \forall\, v \in H^1(\mathcal{T}_{k-1})\,. \tag{2.5}$$

Moreover, it follows from the boundedness and coercivity of $a(\cdot,\cdot)$ and the Poincaré inequality that

$$\|\zeta\|_{a_k} = \|\zeta\|_a \approx \|\zeta\|_{H^1(\Omega)} \approx |\zeta|_{H^1(\Omega)} \qquad \forall\, \zeta \in H^1_0(\Omega)\,. \tag{2.6}$$

Let $V_k \subset H^1(\mathcal{T}_k)$ be a nonconforming finite element space associated with $\mathcal{T}_k$ such that $a_k(\cdot,\cdot)$ is positive definite on V_k, and let $(\cdot,\cdot)_k$ be a discrete inner product on V_k. We can then represent $a_k(\cdot,\cdot)$ by the operator $A_k : V_k \longrightarrow V_k$ defined by

$$(A_k v_1, v_2)_k = a_k(v_1, v_2) \qquad \forall\, v_1, v_2 \in V_k\,. \tag{2.7}$$

Note that A_k is symmetric positive definite with respect to $(\cdot,\cdot)_k$ and the following relation holds:

$$a_k(A_k^s v_1, v_2) = a_k(v_1, A_k^s v_2) \qquad \forall\, v_1, v_2 \in V_k\,, s \in \mathbb{R}\,. \tag{2.8}$$

2.1. V-cycle and F-cycle multigrid algorithms.

The k-th level multigrid V-cycle and F-cycle algorithms are multilevel iterative methods for the equation

$$A_k z = g\,. \tag{2.9}$$

We assume that the finite element space V_{k-1} is connected to V_k by the (linear) intergrid transfer operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$, and denote its transpose with respect to the discrete inner products by $I_k^{k-1} : V_k \longrightarrow V_{k-1}$, i.e.,

$$(I_k^{k-1} v, w)_{k-1} = (v, I_{k-1}^k w)_k \qquad \forall\, v \in V_k\,, w \in V_{k-1}\,. \tag{2.10}$$

The operator $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the transpose of I_{k-1}^k with respect to the nonconforming variational forms, i.e.,

$$a_{k-1}(P_k^{k-1}v, w) = a_k(v, I_{k-1}^k w) \qquad \forall\, v \in V_k\,, w \in V_{k-1}\,. \tag{2.11}$$

These operators satisfy the well-known relation

$$I_k^{k-1} A_k = A_{k-1} P_k^{k-1}\,. \tag{2.12}$$

Finally we take Λ_k to be a number dominating the spectral radius $\rho(A_k)$ of A_k.

Algorithm 2.1 (The V-cycle algorithm)**.**

The k-th level symmetric V-cycle algorithm produces $MG_{\mathcal{V}}(k, g, z_0, m)$ as an approximate solution for (2.9) with initial guess z_0, where m denotes the number of pre-smoothing and post-smoothing steps.

For $k = 1$ we define

$$MG_{\mathcal{V}}(1, g, z_0, m) = A_1^{-1} g\,.$$

For $k \geq 2$ the approximate solution $MG_{\mathcal{V}}(k, g, z_0, m)$ is computed recursively in three steps:

Pre-smoothing. For $j = 1, \ldots, m$, compute z_j by

$$z_j = z_{j-1} + \Lambda_k^{-1}(g - A_k z_{j-1})\,.$$

Coarse grid correction. Let $r_{k-1} = I_k^{k-1}(g - A_k z_m)$ and compute z_{m+1} by

$$z_{m+1} = z_m + I_{k-1}^k MG_{\mathcal{V}}(k-1, r_{k-1}, 0, m)\,.$$

Post-smoothing. For $j = m+2, \ldots, 2m+1$, compute z_j by

$$z_j = z_{j-1} + \Lambda_k^{-1}(g - A_k z_{j-1})\,.$$

We then define $MG_{\mathcal{V}}(k, g, z_0, m) = z_{2m+1}$.

Algorithm 2.2. (The F-cycle algorithm)

The k-th level F-cycle algorithm (associated with the symmetric V-cycle algorithm) produces $MG_{\mathcal{F}}(k, g, z_0, m)$ as an approximate solution for (2.9).

For $k = 1$, we define

$$MG_{\mathcal{F}}(1, g, z_0, m) = A_1^{-1} g\,.$$

For $k \geq 2$, we define $MG_{\mathcal{F}}(k, g, z_0, m)$ recursively in three steps:

Pre-smoothing. For $j = 1, \ldots, m$, compute z_j by

$$z_j = z_{j-1} + \Lambda_k^{-1}(g - A_k z_{j-1})\,.$$

Coarse grid correction. Let $r_{k-1} = I_k^{k-1}(g - A_k z_m)$ and compute z_{m+1} by

$$z_{m+\frac{1}{2}} = MG_{\mathcal{F}}(k-1, r_{k-1}, 0, m)\,,$$
$$z_{m+1} = z_m + I_{k-1}^k MG_{\mathcal{V}}(k-1, r_{k-1}, z_{m+\frac{1}{2}}, m)\,.$$

Post-smoothing. For $j = m+1, \ldots, 2m+1$, compute z_j by

$$z_j = z_{j-1} + \Lambda_k^{-1}(g - A_k z_{j-1})\,.$$

We then define $MG_{\mathcal{F}}(k, g, z_0, m) = z_{2m+1}$.

Remark 2.3. We use the Richardson relaxation scheme as the smoother for simplicity. The theory in this paper can be applied to other smoothers if the definition of the mesh-dependent norms in Section 3 is modified appropriately (cf. [23]).

2.2. **Error representations.** Let $\mathbb{E}_{k,m} : V_k \longrightarrow V_k$ be the operator relating the initial error and the final error of the multigrid V-cycle algorithm applied to the equation (2.9), i.e.,

$$\mathbb{E}_{k,m}(z - z_0) = z - MG_{\mathcal{V}}(k, g, z_0, m)\,.$$

The operator $\mathbb{E}_{k,m}$ can be described in terms of the operators I^j_{j-1} and P^{j-1}_j $(2 \le j \le k)$ and the operators $R_j : V_j \longrightarrow V_j$ defined by

$$R_j = Id_j - \Lambda_j^{-1} A_j\,, \tag{2.13}$$

where $Id_j : V_j \longrightarrow V_j$ is the identity operator. Clearly we have

$$a_j(R_j v, w) = a_j(v, R_j w) \qquad \forall\, v, w \in V_j\,. \tag{2.14}$$

The following relations (cf. [36], [7]) are well known:

$$\mathbb{E}_{k,m} = R^m_k \left[\left(Id_k - I^k_{k-1}P^{k-1}_k\right) + I^k_{k-1}\mathbb{E}_{k-1,m}P^{k-1}_k\right] R^m_k \quad \text{for } k \ge 2\,, \tag{2.15}$$

$$\mathbb{E}_{1,m} = 0\,. \tag{2.16}$$

Using (2.15) and (2.16), we obtain an *additive* expression for $\mathbb{E}_{k,m}$:

$$\begin{aligned}
\mathbb{E}_{k,m} &= R^m_k \left[\left(Id_k - I^k_{k-1}P^{k-1}_k\right) + I^k_{k-1}\mathbb{E}_{k-1,m}P^{k-1}_k\right] R^m_k \\
&= R^m_k (Id_k - I^k_{k-1}P^{k-1}_k) R^m_k \\
&\qquad + R^m_k I^k_{k-1} R^m_{k-1} \Big[\left(Id_{k-1} - I^{k-1}_{k-2}P^{k-2}_{k-1}\right) \\
&\qquad\qquad + I^{k-1}_{k-2}\mathbb{E}_{k-2,m}P^{k-2}_{k-1}\Big] R^m_{k-1}P^{k-1}_k R^m_k \\
&= \sum_{j=2}^{k} T_{k,j,m} R^m_j (Id_j - I^j_{j-1}P^{j-1}_j) R^m_j T_{j,k,m}\,,
\end{aligned} \tag{2.17}$$

where

$$T_{k,k,m} = Id_k\,, \tag{2.18}$$

and for $j < k$, $T_{j,k,m} : V_k \longrightarrow V_j$ and $T_{k,j,m} : V_j \longrightarrow V_k$ are defined by

$$T_{j,k,m} = P^j_{j+1} R^m_{j+1} \cdots P^{k-1}_k R^m_k\,, \tag{2.19}$$

$$T_{k,j,m} = R^m_k I^k_{k-1} \cdots R^m_{j+1} I^{j+1}_j\,. \tag{2.20}$$

Note that for $1 \le j \le k \le \ell$ the following relations are valid:

$$T_{j,\ell,m} = T_{j,k,m} T_{k,\ell,m} \quad \text{and} \quad T_{\ell,j,m} = T_{\ell,k,m} T_{k,j,m}\,. \tag{2.21}$$

Also (2.11) and (2.14) imply

$$a_j(T_{j,k,m} v, w) = a_k(v, T_{k,j,m} w) \qquad \forall\, v \in V_k\,,\ w \in V_j\,. \tag{2.22}$$

Let $\tilde{\mathbb{E}}_{k,m} : V_k \longrightarrow V_k$ be the operator relating the initial error and the final error of the F-cycle algorithm applied to the equation (2.9), i.e.,

$$\tilde{\mathbb{E}}_{k,m}(z - z_0) = z - MG_{\mathcal{F}}(k, g, z_0, m)\,.$$

The following relations (cf. [49]) are also well known:

$$\tilde{\mathbb{E}}_{1,m} = 0\,, \tag{2.23}$$

$$\tilde{\mathbb{E}}_{k,m} = R^m_k \left[(Id_k - I^k_{k-1}P^{k-1}_k) + I^k_{k-1}\mathbb{E}_{k-1,m}\tilde{\mathbb{E}}_{k-1,m}P^{k-1}_k\right] R^m_k\,,\ k \ge 2\,. \tag{2.24}$$

3. Assumptions

In this section we state the assumptions for the convergence theory and derive some of their immediate consequences.

First we introduce a scale of mesh-dependent norms (cf. [4]) on each V_k:

$$\|v\|_{s,k} = \sqrt{(A_k^s v, v)_k} \qquad \forall\, s \in \mathbb{R}\,,\ v \in V_k\,. \tag{3.1}$$

It is clear from (2.4), (2.7) and (3.1) that

$$\|v\|_{0,k} = \sqrt{(v,v)_k} \qquad \forall\, v \in V_k\,, \tag{3.2}$$

$$\|v\|_{1,k} = \sqrt{a_k(v,v)} = \|v\|_{a_k} \qquad \forall\, v \in V_k\,, \tag{3.3}$$

$$\|A_k^s v\|_{t,k} = \|v\|_{t+2s,k} \qquad \forall\, v \in V_k\,,\ s,t \in \mathbb{R}\,. \tag{3.4}$$

Moreover the Cauchy-Schwarz inequality implies

$$\|v\|_{1+t,k} = \sup_{w \in V_k \setminus \{0\}} \frac{a_k(v,w)}{\|w\|_{1-t,k}} \qquad \forall\, t \in \mathbb{R}\,,\ v \in V_k\,. \tag{3.5}$$

To avoid the proliferation of constants we henceforth use the notation $A \lesssim B$ to represent the inequality $A \le (\text{constant}) \times B$, where the constant is independent of both the mesh (i.e., independent of the mesh size and the mesh level) and the number of smoothing steps. The statement $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

Assumptions on V_k. We assume that

$$(v,v)_k \approx \|v\|^2_{L_2(\Omega)} \qquad \forall\, v \in V_k\,, \tag{3.6}$$

$$\|v\|_{a_k} \lesssim h_k^{-1} \|v\|_{L_2(\Omega)} \qquad \forall\, v \in V_k\,. \tag{3.7}$$

Assumptions on I_{k-1}^k and P_k^{k-1}. We assume that I_{k-1}^k and P_k^{k-1} have the following properties:

$$\|I_{k-1}^k v\|^2_{1,k} \le (1+\theta^2)\|v\|^2_{1,k-1} + C_1 \theta^{-2} h_k^{2\alpha} \|v\|^2_{1+\alpha,k-1} \qquad \forall\, v \in V_{k-1}\,,\ \theta \in (0,1)\,, \tag{3.8}$$

$$\|I_{k-1}^k v\|^2_{1-\alpha,k} \le (1+\theta^2)\|v\|^2_{1-\alpha,k-1} + C_2 \theta^{-2} h_k^{2\alpha} \|v\|^2_{1,k-1} \qquad \forall\, v \in V_{k-1}\,,\ \theta \in (0,1)\,, \tag{3.9}$$

$$\|P_k^{k-1} v\|^2_{1-\alpha,k-1} \le (1+\theta^2)\|v\|^2_{1-\alpha,k} + C_3 \theta^{-2} h_k^{2\alpha} \|v\|^2_{1,k} \qquad \forall\, v \in V_k\,,\ \theta \in (0,1)\,, \tag{3.10}$$

where α is the index of elliptic regularity in (1.4) and the positive constants C_1, C_2 and C_3 are mesh-independent.

Remark 3.1. The estimates in [22] corresponding to (3.9) and (3.10) involve an index $\beta \in (0, \frac{1}{2})$ instead of α. Using the tools developed in Section 6 of this paper, one can also replace β by α in [22].

Assumptions on $I_{k-1}^k P_k^{k-1}$ and $P_k^{k-1} I_{k-1}^k$. We assume the nonconforming finite element spaces have the following approximation properties:

$$\|\big(Id_k - I_{k-1}^k P_k^{k-1}\big)v\|_{1-\alpha,k} \lesssim h_k^{2\alpha}\|v\|_{1+\alpha,k} \qquad \forall\, v \in V_k\,, \tag{3.11}$$

$$\|\big(Id_{k-1} - P_k^{k-1} I_{k-1}^k\big)v\|_{1-\alpha,k-1} \lesssim h_k^{\alpha}\|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}\,. \tag{3.12}$$

We now derive some simple consequences of the assumptions above.

First we note that (3.6) and (3.7) imply

$$\rho(A_k) \lesssim h_k^{-2}\,. \tag{3.13}$$

It follows easily (cf. [4], [36]) from (2.13), (3.1) and (3.13) that

$$\|v\|_{s,k} \lesssim h_k^{t-s}\|v\|_{t,k} \qquad \forall\, v \in V_k\,,\ 0 \le t \le s \le 2\,, \tag{3.14}$$

$$\|R_k v\|_{s,k} \le \|v\|_{s,k} \qquad \forall\, v \in V_k\,,\ s \in \mathbb{R}\,, \tag{3.15}$$

$$\|R_k^m v\|_{s,k} \lesssim h_k^{t-s} m^{(t-s)/2}\|v\|_{t,k} \qquad \forall\, v \in V_k\,,\ 0 \le t \le s \le 2\,,\ m \ge 1\,. \tag{3.16}$$

The estimate (3.8) and (3.12) imply through (2.11), (3.5) and (3.14) three additional estimates:

$$\|I_{k-1}^k v\|_{1,k} \lesssim \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}\,, \tag{3.17}$$

$$\|P_k^{k-1} v\|_{1,k-1} \lesssim \|v\|_{1,k} \qquad \forall\, v \in V_k\,, \tag{3.18}$$

$$\|\big(Id_{k-1} - P_k^{k-1} I_{k-1}^k\big)v\|_{1,k-1} \lesssim h_k^{\alpha}\|v\|_{1+\alpha,k-1} \qquad \forall\, v \in V_{k-1}\,. \tag{3.19}$$

Again, to avoid the proliferation of constants, we henceforth say that an estimate holds for m sufficiently large if it is valid for $m \ge m_*$, where the positive integer m_* is mesh-independent.

Lemma 3.2. *Given any number $\omega \in (0,1)$, the following estimates hold for m sufficiently large* :

$$\|I_{k-1}^k R_{k-1}^m v\|_{1,k} \le (1+\omega)\|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}\,, \tag{3.20}$$

$$\|P_k^{k-1} R_k^m v\|_{1-\alpha,k-1} \le (1+\omega)\|v\|_{1-\alpha,k} \qquad \forall\, v \in V_k\,, \tag{3.21}$$

$$\|R_k^m I_{k-1}^k v\|_{1+\alpha,k} \le (1+\omega)\|v\|_{1+\alpha,k-1} \qquad \forall\, v \in V_{k-1}\,. \tag{3.22}$$

Proof. From (2.1), (3.8), (3.15) and (3.16) we have

$$\begin{aligned}\|I_{k-1}^k R_{k-1}^m v\|_{1,k}^2 &\le (1+\omega^2)\|R_{k-1}^m v\|_{1,k-1}^2 + C_1\omega^{-2}h_k^{2\alpha}\|R_{k-1}^m v\|_{1+\alpha,k-1}^2\\ &\le (1+\omega^2)\|v\|_{1,k-1}^2 + C_1'\omega^{-2}m^{-\alpha}\|v\|_{1,k-1}^2\,,\end{aligned}$$

where the positive constant C_1' is independent of the mesh and the number of smoothing steps. The estimate (3.20) then follows if

$$m^{\alpha} \ge \frac{C_1'\omega^{-3}}{2}\,.$$

Similarly we obtain (3.21) using (3.10), (3.15) and (3.16). The estimate (3.22) then follows from (2.11), (2.14), (3.5) and (3.21). □

4. A strengthened Cauchy-Schwarz inequality

In this section we derive a strengthened Cauchy-Schwarz inequality which takes into account the effect of smoothing. We begin by estimating the bounds of the operator $T_{j,k,m}$ with respect to various mesh-dependent norms. For brevity we will sometimes suppress the parameter m and write $T_{j,k}$ instead of $T_{j,k,m}$.

From (2.18)–(2.20), (3.21) and (3.22) we immediately have the following lemma.

Lemma 4.1. *Let $j \le k$. Given any $\omega \in (0,1)$, the following estimates hold for m sufficiently large* :

$$\|T_{j,k,m}v\|_{1-\alpha,j} \le (1+\omega)^{k-j}\|v\|_{1-\alpha,k} \qquad \forall\, v \in V_k\,, \tag{4.1}$$

$$\|T_{k,j,m}v\|_{1+\alpha,k} \le (1+\omega)^{k-j}\|v\|_{1+\alpha,j} \qquad \forall\, v \in V_j\,. \tag{4.2}$$

The next three lemmas and one corollary are preparatory for the crucial estimate involving $T_{K,k,m}T_{k,K,m}$ for $k \le K$.

Lemma 4.2. *Let $k \le K$. Then the estimate*

$$\|T_{K,k,m}v\|_{1,K} \lesssim \|v\|_{1,k} \qquad \forall\, v \in V_k \tag{4.3}$$

holds for m sufficiently large.

Proof. Let $v \in V_k$ be arbitrary. Given any $\omega \in (0,1)$, we have, from (1.8a), (2.11), (2.18), (2.20), (3.1), (3.3), (3.15), (3.19), (4.2) and the Cauchy-Schwarz inequality, that

$$\begin{aligned}
\|T_{K,k}v\|_{1,K}^2 &= a_K\big(T_{K,k}v, T_{K,k}v\big)\\
&= a_K\big(R_K^m I_{K-1}^K T_{K-1,k}v, R_K^m I_{K-1}^K T_{K-1,k}v\big)\\
&\le a_K\big(I_{K-1}^K T_{K-1,k}v, I_{K-1}^K T_{K-1,k}v\big)\\
&= a_{K-1}\big(P_K^{K-1} I_{K-1}^K T_{K-1,k}v, T_{K-1,k}v\big)\\
&= a_{K-1}\big(T_{K-1,k}v, T_{K-1,k}v\big)\\
&\qquad + a_{K-1}\big([P_K^{K-1} I_{K-1}^K - Id_{K-1}]T_{K-1,k}v, T_{K-1,k}v\big)\\
&\le \|T_{K-1,k}v\|_{1,K-1}^2\\
&\qquad + \|(P_K^{K-1} I_{K-1}^K - Id_{K-1})T_{K-1,k}v\|_{1,K-1}\|T_{K-1,k}v\|_{1,K-1}\\
&\le (1+\theta_K^2)\|T_{K-1,k}v\|_{1,K-1}^2 + C_\diamond \theta_K^{-2} h_K^{2\alpha}\|T_{K-1,k}v\|_{1+\alpha,K-1}^2\\
&\le (1+\theta_K^2)\|T_{K-1,k}v\|_{1,K-1}^2 + C_\sharp \theta_K^{-2}(1+\omega)^{2(K-k)} h_K^{2\alpha}\|v\|_{1+\alpha,k}^2
\end{aligned} \tag{4.4}$$

for all $v \in V_k$, provided m is sufficiently large. Note that $\theta_K \in (0,1)$ is arbitrary, and the positive constant $C_\sharp$ is independent of the mesh, the number of smoothing steps and the parameters ω and θ_K.

Iterating (4.4), we find

$$\begin{aligned}
\|T_{K,k}v\|_{1,K}^2 \le{}& \Big[\prod_{k+1\le q\le K}(1+\theta_q^2)\Big]\|v\|_{1,k}^2\\
&+ C_\sharp\Big[\sum_{k+1\le p\le K}\Big(\prod_{p+1\le q\le K}(1+\theta_q^2)\Big)\theta_p^{-2}(1+\omega)^{2(p-k)}h_p^{2\alpha}\Big]\|v\|_{1+\alpha,k}^2\,,
\end{aligned} \tag{4.5}$$

where $\theta_{k+1},\dots,\theta_K$ are arbitrary numbers in $(0,1)$.

Note that $h_p = 2^{k-p} h_k$ by (2.1). We now choose

$$\omega = \left(\frac{4}{3}\right)^{\alpha/2} - 1$$

so that

$$(1+\omega)^{2(p-k)} h_p^{2\alpha} = \left[(1+\omega)^2 4^{-\alpha}\right]^{p-k} h_k^{2\alpha} = 3^{-\alpha(p-k)} h_k^{2\alpha}\,, \tag{4.6}$$

and then we take

$$\theta_q = \left(\frac{2}{3}\right)^{\alpha(q-k)/2} \qquad \text{for } k+1 \le q \le K\,. \tag{4.7}$$

Combining (4.5)–(4.7), we have

$$|\!|\!| T_{K,k} v |\!|\!|_{1,K}^2 \le \rho_1 |\!|\!| v |\!|\!|_{1,k}^2 + C_\sharp \rho_1 \rho_2 h_k^{2\alpha} |\!|\!| v |\!|\!|_{1+\alpha,k}^2\,, \tag{4.8}$$

where

$$\rho_1 = \prod_{j=1}^{\infty} \left[1 + \left(\frac{2}{3}\right)^{\alpha\cdot j}\right] < \infty \quad \text{and} \quad \rho_2 = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{\alpha\cdot j} < \infty\,.$$

The estimate (4.3) follows from (3.14) and (4.8). □

Lemma 4.2, (2.22) and (3.5) immediately imply the following corollary.

Corollary 4.3. *Let $k \le K$. Then the estimate*

$$|\!|\!| T_{k,K,m} v |\!|\!|_{1,k} \lesssim |\!|\!| v |\!|\!|_{1,K} \qquad \forall\, v \in V_K \tag{4.9}$$

holds for m sufficiently large.

Lemma 4.4. *Let $k \le K$. Then the estimate*

$$|\!|\!| T_{K,k,m} v |\!|\!|_{1-\alpha,K} \lesssim |\!|\!| v |\!|\!|_{1-\alpha,k} \qquad \forall\, v \in V_k \tag{4.10}$$

holds for m sufficiently large.

Proof. Let $v \in V_k$ be arbitrary. It follows from (2.18), (2.20), (3.9), (3.15) and (4.3) that, for m sufficiently large,

$$\begin{aligned} |\!|\!| T_{K,k} v |\!|\!|_{1-\alpha,K}^2 &= |\!|\!| R_K^m I_{K-1}^K T_{K-1,k} v |\!|\!|_{1-\alpha,K}^2 \\ &\le |\!|\!| I_{K-1}^K T_{K-1,k} v |\!|\!|_{1-\alpha,K}^2 \\ &\le (1+\theta_K^2) |\!|\!| T_{K-1,k} v |\!|\!|_{1-\alpha,K-1}^2 + C_2 \theta_K^{-2} h_K^{2\alpha} |\!|\!| T_{K-1,k} v |\!|\!|_{1,K-1}^2 \\ &\le (1+\theta_K^2) |\!|\!| T_{K-1,k} v |\!|\!|_{1-\alpha,K-1}^2 + C_\dagger \theta_K^{-2} h_K^{2\alpha} |\!|\!| v |\!|\!|_{1,k}^2 \end{aligned} \tag{4.11}$$

for any $\theta_K \in (0,1)$, where the positive constant $C_\dagger$ is independent of the mesh, the number of smoothing steps and the parameter θ_K. Iterating (4.11), we find

$$\begin{aligned} |\!|\!| T_{K,k} v |\!|\!|_{1-\alpha,K}^2 \le{}& \left[\prod_{k+1\le q\le K} (1+\theta_q^2)\right] |\!|\!| v |\!|\!|_{1-\alpha,k}^2 \\ &+ C_\dagger \left[\sum_{k+1\le p\le K} \left(\prod_{p+1\le q\le K} (1+\theta_q^2)\right) \theta_p^{-2} h_p^{2\alpha}\right] |\!|\!| v |\!|\!|_{1,k}^2\,, \end{aligned} \tag{4.12}$$

where $\theta_{k+1}, \ldots, \theta_K$ are arbitrary numbers in $(0,1)$.

As in the proof of Lemma 4.2, by choosing

$$\theta_q = \left(\frac{1}{2}\right)^{\alpha(q-k)/2} \qquad \text{for } k+1 \le q \le K\,, \tag{4.13}$$

we can deduce from (2.1) and (4.12) that

$$\|T_{K,k}v\|_{1-\alpha,K}^2 \lesssim \|v\|_{1-\alpha,k}^2 + h_k^{2\alpha}\|v\|_{1,k}^2 \,. \tag{4.14}$$

The estimate (4.10) now follows from (3.14) and (4.14). □

Lemma 4.5. *Let $k \le K$. Then the estimate*

$$\|T_{k,K,m}v\|_{1-\alpha,k}^2 \lesssim \|v\|_{1-\alpha,K}^2 + h_k^{2\alpha}\|v\|_{1,K}^2 \qquad \forall\, v \in V_K \tag{4.15}$$

holds for m sufficiently large.

Proof. Let $v \in V_K$ be arbitrary. From (2.19), (3.10), (3.15) and (4.9) we have, for m sufficiently large,

$$\begin{aligned} \|T_{k,K}v\|_{1-\alpha,k}^2 &= \|P_{k+1}^k R_{k+1}^m T_{k+1,K}v\|_{1-\alpha,k}^2 \\ &\le (1+\theta_{k+1}^2)\|T_{k+1,K}v\|_{1-\alpha,k+1}^2 + C_3\theta_{k+1}^{-2}h_{k+1}^{2\alpha}\|T_{k+1,K}v\|_{1,k+1}^2 \\ &\le (1+\theta_{k+1}^2)\|T_{k+1,K}v\|_{1-\alpha,k+1}^2 + C_\flat\theta_{k+1}^{-2}h_{k+1}^{2\alpha}\|v\|_{1,K}^2 \,, \end{aligned} \tag{4.16}$$

where $\theta_{k+1} \in (0,1)$ is arbitrary and the positive constant $C_\flat$ is independent of the mesh, the number of smoothing steps and the parameter θ_{k+1}.

Iterating (4.16), we find

$$\begin{aligned} \|T_{k,K}v\|_{1-\alpha,k}^2 \le &\Bigg[\prod_{k+1\le q\le K}(1+\theta_q^2)\Bigg]\|v\|_{1-\alpha,K}^2 \\ &+ C_\flat\Bigg[\sum_{k+1\le p\le K}\Bigg(\prod_{k+1\le q\le p-1}(1+\theta_q^2)\Bigg)\theta_p^{-2}h_p^{2\alpha}\Bigg]\|v\|_{1,K}^2 \,. \end{aligned} \tag{4.17}$$

Choosing θ_q by formula (4.13), we deduce (4.15) from (2.1) and (4.17) as in the proof of Lemma 4.2. □

We can now establish a crucial estimate.

Lemma 4.6. *Let $k \le K$. Then the estimate*

$$\|T_{k,K,m}T_{K,k,m}v\|_{1-\alpha,k} \lesssim \|v\|_{1-\alpha,k} \qquad \forall\, v \in V_k \tag{4.18}$$

holds for m sufficiently large.

Proof. It follows from (3.14), Lemma 4.2, Lemma 4.4 and Lemma 4.5 that

$$\begin{aligned} \|T_{k,K,m}T_{K,k,m}v\|_{1-\alpha,k}^2 &\lesssim \|T_{K,k,m}v\|_{1-\alpha,K}^2 + h_k^{2\alpha}\|T_{K,k,m}v\|_{1,K}^2 \\ &\lesssim \|v\|_{1-\alpha,k}^2 + h_k^{2\alpha}\|v\|_{1,k}^2 \lesssim \|v\|_{1-\alpha,k}^2 \,, \end{aligned}$$

provided m is sufficiently large. □

The following proposition is the main result of this section.

Proposition 4.7 (Strengthened Cauchy-Schwarz inequality with smoothing)**.** *Let $1 \le j \le k \le K$. Given any $\omega \in (0,1)$, the estimate*

$$\begin{aligned} &a_K\big(T_{K,j,m}R_j^q v_j, T_{K,k,m}R_k^q v_k\big) \\ &\qquad \lesssim q^{-\alpha}\left(\frac{1+\omega}{2^\alpha}\right)^{k-j}\big(h_j^{-\alpha}\|v_j\|_{1-\alpha,j}\big)\big(h_k^{-\alpha}\|v_k\|_{1-\alpha,k}\big) \end{aligned} \tag{4.19}$$

holds for all $v_j \in V_j$ and $v_k \in V_k$, provided m is sufficiently large.

Proof. Given any $\omega \in (0,1)$, from (2.1), (2.21), (2.22), (3.5), (3.15), (3.16), (4.1) and Lemma 4.6, we obtain

$$\begin{aligned} a_K\big(T_{K,j}R_j^q v_j, T_{K,k}R_k^q v_k\big) &= a_j\big(R_j^q v_j, T_{j,k}T_{k,K}T_{K,k}R_k^q v_k\big) \\ &\le \|R_j^q v_j\|_{1+\alpha,j}\|T_{j,k}T_{k,K}T_{K,k}R_k^q v_k\|_{1-\alpha,j} \\ &\lesssim \frac{h_j^{-2\alpha}}{q^\alpha}\|v_j\|_{1-\alpha,j}(1+\omega)^{k-j}\|v_k\|_{1-\alpha,k} \\ &= \frac{(1+\omega)^{k-j}}{q^\alpha}\left(\frac{h_k}{h_j}\right)^\alpha \big(h_j^{-\alpha}\|v_j\|_{1-\alpha,j}\big)\big(h_k^{-\alpha}\|v_k\|_{1-\alpha,k}\big) \\ &= q^{-\alpha}\left(\frac{1+\omega}{2^\alpha}\right)^{k-j} \big(h_j^{-\alpha}\|v_j\|_{1-\alpha,j}\big)\big(h_k^{-\alpha}\|v_k\|_{1-\alpha,k}\big)\,, \end{aligned}$$

provided m is sufficiently large. □

Corollary 4.8. *Let $v_k \in V_k$ for $1 \le k \le K$. Then the estimate*

$$a_K\Big(\sum_{k=1}^K T_{K,k,m}R_k^q v_k, \sum_{k=1}^K T_{K,k,m}R_k^q v_k\Big) \lesssim q^{-\alpha}\sum_{k=1}^K h_k^{-2\alpha}\|v_k\|_{1-\alpha,k}^2 \tag{4.20}$$

holds for m sufficiently large.

Proof. It follows from Proposition 4.7 that, given any $\omega \in (0,1)$,

$$\begin{aligned} a_K\Big(\sum_{k=1}^K T_{K,k}R_k^q v_k, \sum_{k=1}^K T_{K,k}R_k^q v_k\Big) &= \sum_{j,k=1}^K a_K\big(T_{K,j}R_j^q v_j, T_{K,k}R_k^q v_k\big) \\ &\lesssim q^{-\alpha}\sum_{j,k=1}^K \left(\frac{1+\omega}{2^\alpha}\right)^{|k-j|} \big(h_j^{-\alpha}\|v_j\|_{1-\alpha,j}\big)\big(h_k^{-\alpha}\|v_k\|_{1-\alpha,k}\big) \end{aligned} \tag{4.21}$$

for m sufficiently large. We now choose ω so that $(1+\omega)2^{-\alpha} < 1$. The estimate (4.20) then follows from (4.21) and a discrete Young's inequality (cf. [37]). □

5. Convergence of V-cycle and F-cycle algorithms

In this section we establish the asymptotic behavior of the contraction numbers of $\mathbb{E}_{K,m}$ and $\tilde{\mathbb{E}}_{K,m}$ $(K \ge 2)$ with respect to m.

The analysis in [22] for conforming V-cycle multigrid methods uses the fact that

$$[Id_k - I_{k-1}^k P_k^{k-1}]^2 = Id_k - I_{k-1}^k P_k^{k-1} \tag{5.1}$$

in the case $V_{k-1} \subset V_k \subset H^1(\Omega)$. Since (5.1) does not hold for nonconforming finite element spaces, we need to consider first the contraction property of an auxiliary operator $\mathscr{E}_{K,m} : V_K \longrightarrow V_K$ defined by

$$\begin{aligned} \mathscr{E}_{K,m} = \sum_{k=2}^K T_{K,k,m}\big[R_k^m(Id_k - I_{k-1}^k P_k^{k-1})h_k^{-2\alpha}A_k^{-\alpha} \\ \times (Id_k - I_{k-1}^k P_k^{k-1})R_k^m\big]T_{k,K,m}\,. \end{aligned} \tag{5.2}$$

Lemma 5.1. *The estimate*

$$\|\mathscr{E}_{K,m}v\|_{1,K} \lesssim m^{-\alpha}\|v\|_{1,K} \qquad \forall\, v \in V_K \tag{5.3}$$

holds for m sufficiently large.

Proof. Let $v \in V_K$ be arbitrary and

$$v_k = (Id_k - I_{k-1}^k P_k^{k-1}) h_k^{-2\alpha} A_k^{-\alpha} (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v \,. \tag{5.4}$$

We have, by (5.2), (5.4) and Corollary 4.8,

$$\begin{aligned} a_K(\mathscr{E}_{K,m} v, \mathscr{E}_{k,m} v) &= a_K\Big(\sum_{k=2}^K T_{K,k,m} R_k^m v_k, \sum_{k=2}^K T_{K,k,m} R_k^m v_k\Big) \\ &\lesssim m^{-\alpha} \sum_{k=2}^K h_k^{-2\alpha} \|v_k\|_{1-\alpha,k}^2 \end{aligned} \tag{5.5}$$

for m sufficiently large. Moreover, (3.4), (3.11) and (5.4) imply

$$\begin{aligned} \|v_k\|_{1-\alpha,k} &\lesssim \|A_k^{-\alpha}(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v\|_{1+\alpha,k} \\ &= \|A_k^{-\alpha/2}(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v\|_{1,k} \,. \end{aligned} \tag{5.6}$$

Let $w_k = (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v$. It follows from (2.8), (2.11), (2.14), (2.22), (3.1) and (5.2) that

$$\begin{aligned} &\sum_{k=2}^K h_k^{-2\alpha} \|A_k^{-\alpha/2}(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v\|_{1,k}^2 \\ &\quad = \sum_{k=2}^K h_k^{-2\alpha} a_k\big(A_k^{-\alpha/2} w_k, A_k^{-\alpha/2}(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v\big) \\ &\quad = \sum_{k=2}^K a_K\big(T_{K,k,m} R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) h_k^{-2\alpha} A_k^{-\alpha} w_k, v\big) \\ &\quad = a_K(\mathscr{E}_{K,m} v, v) \,. \end{aligned} \tag{5.7}$$

Combining (3.3), (5.5)–(5.7), and the Cauchy-Schwarz inequality, we find

$$\|\mathscr{E}_{K,m} v\|_{1,K}^2 = a_K\big(\mathscr{E}_{K,m} v, \mathscr{E}_{K,m} v\big) \lesssim m^{-\alpha} a_K(\mathscr{E}_{K,m} v, v) \le m^{-\alpha} \|\mathscr{E}_{k,m} v\|_{1,K} \, \|v\|_{1,K}$$

and (5.3) follows. □

We can now prove the convergence of the symmetric V-cycle algorithm.

Theorem 5.2 (Convergence of the symmetric V-cycle algorithm)**.** *There exist positive mesh-independent constants C and m_* such that*

$$\|\mathbb{E}_{K,m} v\|_{1,K} \le \frac{C}{m^\alpha} \|v\|_{1,K} \qquad \forall\, v \in V_K \,,\ K \ge 1 \,,\ m \ge m_* \,. \tag{5.8}$$

Proof. The case where $K = 1$ is trivial. Let $v \in V_K$ $(K \ge 2)$ be arbitrary and

$$v_k = (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v \,. \tag{5.9}$$

Then (2.17), (5.9) and Corollary 4.8 imply that

$$\begin{aligned} a_K(\mathbb{E}_{K,m} v, \mathbb{E}_{K,m} v) &= a_K\Big(\sum_{k=2}^K T_{K,k,m} R_k^m v_k, \sum_{k=2}^K T_{K,k,m} R_k^m v_k\Big) \\ &\lesssim m^{-\alpha} \sum_{k=2}^K h_k^{-2\alpha} \|v_k\|_{1-\alpha,k}^2 \end{aligned} \tag{5.10}$$

for m sufficiently large.

From (3.4) and (5.9) we have

$$\|v_k\|_{1-\alpha,k} = \|A_k^{-\alpha/2}(Id_k - I_{k-1}^k P_k^{k-1})R_k^m T_{k,K,m} v\|_{1,k}\,. \tag{5.11}$$

Combining (3.3), (5.7), (5.10)–(5.11), Lemma 5.1 and the Cauchy-Schwarz inequality, we find

$$\begin{aligned}\|\mathbb{E}_{K,m}v\|_{1,K}^2 &= a_K(\mathbb{E}_{K,m}v, \mathbb{E}_{K,m}v)\\ &\lesssim m^{-\alpha} a_K(\mathscr{E}_{K,m}v, v) \le m^{-\alpha}\|\mathscr{E}_{k,m}v\|_{1,K}\|v\|_{1,K} \le m^{-2\alpha}\|v\|_{1,K}^2\end{aligned}$$

and (5.8) follows. □

Finally we prove the convergence of the F-cycle algorithm.

Theorem 5.3 (Convergence of the F-cycle algorithm)**.** *There exist positive mesh-independent constants C and m_* such that*

$$\|\tilde{\mathbb{E}}_{K,m}v\|_{1,K} \le \frac{C}{m^\alpha}\|v\|_{1,K} \qquad \forall\, v \in V_K\,,\ K \ge 1\,,\ m \ge m_*\,. \tag{5.12}$$

Proof. Suppose that

$$\|\tilde{\mathbb{E}}_{k-1,m}v\|_{1,k-1} \le \eta_{k-1}\|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}\,. \tag{5.13}$$

From (2.24), (3.11), (3.15)–(3.18), (5.13) and Theorem 5.2, we have, for m sufficiently large,

$$\begin{aligned}\|\tilde{\mathbb{E}}_{k,m}v\|_{1,k} &\le \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{1,k} + \|I_{k-1}^k \mathbb{E}_{k-1,m}\tilde{\mathbb{E}}_{k-1,m}P_k^{k-1}R_k^m v\|_{1,k}\\ &\lesssim m^{-\alpha/2}h_k^{-\alpha}\|(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{1-\alpha,k} + m^{-\alpha}\|\tilde{\mathbb{E}}_{k-1,m}P_k^{k-1}R_k^m v\|_{1,k-1}\\ &\lesssim m^{-\alpha/2}h_k^{\alpha}\|R_k^m v\|_{1+\alpha,k} + m^{-\alpha}\eta_{k-1}\|P_k^{k-1}R_k^m v\|_{1,k-1}\\ &\lesssim m^{-\alpha}(1+\eta_{k-1})\|v\|_{1,k} \qquad \forall\, v \in V_k\,.\end{aligned}$$

In other words we have,

$$\|\tilde{\mathbb{E}}_{k,m}v\|_{1,k} \le \eta_k\|v\|_{1,k} \qquad \forall\, v \in V_k\,,\ m \ge m_\dagger\,, \tag{5.14}$$

where

$$\eta_k = C_\ddagger m^{-\alpha}(1+\eta_{k-1}) \tag{5.15}$$

and the positive constants $m_\dagger$ and $C_\ddagger$ are mesh-independent.

In view of (2.23) and (5.13)–(5.15), we can obtain by mathematical induction

$$\|\tilde{\mathbb{E}}_{k,m}v\|_{1,k} \le \frac{C_\ddagger}{m^\alpha - C_\ddagger}\|v\|_{1,k} \qquad \forall\, v \in V_k\,,\ k \ge 1\,,\ m \ge m_*\,, \tag{5.16}$$

provided $m_* \ge \max(m_\dagger, C_\ddagger^{1/\alpha})$. The estimate (5.12) follows immediately from (5.16). □

6. An abstract framework for nonconforming multigrid methods

In order to apply the convergence results in Section 5 to a specific nonconforming multigrid method, one must verify the assumptions (3.6)–(3.12). This can be accomplished through the framework developed in [21]. Indeed the standard discrete estimate (3.6) and inverse estimate (3.7) (cf. [30], [25]) are the assumptions (P) and (I) in [21], and (3.11) is established in Lemma 4.2 there.

The truly new ingredients among the assumptions in Section 3 are therefore the estimates (3.8)–(3.10) and (3.12). We will show in this section that they can be derived using the framework in [21] for second order problems and four additional

conditions (cf. (6.10)–(6.13) below). In the following discussion we will rely heavily on the results in [21].

6.1. **Results from** [21] **and new conditions.** A key ingredient of the theory in [21] is the relation between the nonconforming finite element space V_k and a conforming finite element space $\tilde{V}_k \subset H_0^1(\Omega)$ (referred to as a conforming relative of V_k in [21]). These spaces are connected by the linear maps $E_k : V_k \longrightarrow \tilde{V}_k$ and $F_k : \tilde{V}_k \longrightarrow V_k$. Two of the properties of these maps are (cf. (FE) and Lemma 3.1 in [21]):

$$F_k \circ E_k = Id_k \,, \tag{6.1}$$

$$\|F_k\tilde{v}\|_{L_2(\Omega)} \lesssim \|\tilde{v}\|_{L_2(\Omega)} \quad \text{and} \quad \|F_k\tilde{v}\|_{a_k} \lesssim \|\tilde{v}\|_{H^1(\Omega)} \qquad \forall\, \tilde{v} \in \tilde{V}_k \,. \tag{6.2}$$

Let $\zeta \in H^{1+\alpha}(\Omega) \cap H_0^1(\Omega)$, $\zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ be related by

$$a(\zeta, E_k v) = a_k(\zeta_k, v) \qquad \forall\, v \in V_k \,,$$
$$a(\zeta, E_{k-1} v) = a_{k-1}(\zeta_{k-1}, v) \qquad \forall\, v \in V_{k-1} \,.$$

Then the following estimates are valid within the framework in [21] (cf. Theorem 3.5 and Lemma 3.7 there):

$$\|\zeta - \zeta_k\|_{a_k} \lesssim h_k^{\alpha} \|\zeta\|_{H^{1+\alpha}(\Omega)} \,, \tag{6.3}$$

$$|\!|\!|\Pi_k\zeta - \zeta_k|\!|\!|_{1-\alpha,k} \lesssim h_k^{2\alpha} \|\zeta\|_{H^{1+\alpha}(\Omega)} \,, \tag{6.4}$$

$$|\!|\!|\zeta_{k-1} - P_k^{k-1}\zeta_k|\!|\!|_{1-\alpha,k-1} \lesssim h_k^{2\alpha} \|\zeta\|_{H^{1+\alpha}(\Omega)} \,, \tag{6.5}$$

where $\Pi_k : H_0^1(\Omega) \longrightarrow V_k$ is an interpolation operator.

The following estimates concerning I_{k-1}^k and Π_k are also established within the framework in [21] (cf. Lemma 3.3, (Π-1), (Π-2) and (I-2) there):

$$|\!|\!|I_{k-1}^k v|\!|\!|_{s,k} \lesssim |\!|\!|v|\!|\!|_{s,k-1} \qquad \forall\, v \in V_{k-1} \,,\ 0 \le s \le 1 \,, \tag{6.6}$$

$$\|\zeta - \Pi_k\zeta\|_{L_2(\Omega)} \lesssim h_k |\zeta|_{H^1(\Omega)} \qquad \forall\, \zeta \in H_0^1(\Omega) \,, \tag{6.7}$$

$$\|\zeta - \Pi_k\zeta\|_{a_k} \lesssim h_k^{\alpha} \|\zeta\|_{H^{1+\alpha}(\Omega)} \qquad \forall\, \zeta \in H^{1+\alpha}(\Omega) \cap H_0^1(\Omega) \,, \tag{6.8}$$

$$|\!|\!|\Pi_k\zeta - I_{k-1}^k \Pi_{k-1}\zeta|\!|\!|_{1-\alpha,k} \lesssim h_k^{2\alpha} \|\zeta\|_{H^{1+\alpha}(\Omega)} \qquad \forall\, \zeta \in H^{1+\alpha}(\Omega) \cap H_0^1(\Omega) \,. \tag{6.9}$$

We will derive the estimates (3.8)–(3.10) and (3.12) using (6.1)–(6.9) and four additional conditions imposed on the intergrid transfer operator I_{k-1}^k and the interpolation operator Π_k.

Four additional conditions. We assume that, in addition to the conditions (I-1) and (I-2) in [21],

$$|\!|\!|I_{k-1}^k v|\!|\!|_{0,k}^2 \le (1+\theta^2)|\!|\!|v|\!|\!|_{0,k-1}^2 + C_0\theta^{-2} h_k^{2\alpha} |\!|\!|v|\!|\!|_{\alpha,k-1}^2 \tag{6.10}$$

for all $v \in V_{k-1}$ and $\theta \in (0,1)$, where the positive constant C_0 is mesh-independent.

Furthermore, the interpolation operator $\Pi_{k-1} : H_0^1(\Omega) \longrightarrow V_{k-1}$ actually maps the larger space $H_0^1(\Omega) + V_k$ into V_{k-1} and satisfies, in addition to (Π-1) and (Π-2) in [21], the estimates

$$\|\Pi_{k-1} v\|_{a_{k-1}} \lesssim \|v\|_{a_k} \qquad \forall\, v \in H_0^1(\Omega) + V_k \,, \tag{6.11}$$

$$\|\Pi_{k-1} v - v\|_{L_2(\Omega)} \lesssim h_k \|v\|_{a_k} \qquad \forall\, v \in V_k \,, \tag{6.12}$$

$$|\!|\!|\Pi_{k-1} v|\!|\!|_{0,k-1}^2 \le (1+\theta^2)|\!|\!|v|\!|\!|_{0,k}^2 + C_0'\theta^{-2} h_k^{2\alpha} |\!|\!|v|\!|\!|_{\alpha,k}^2 \qquad \forall\, v \in V_k \,,\ \theta \in (0,1) \,, \tag{6.13}$$

where the positive constant C_0' is mesh-independent.

6.2. **Derivation of (3.8)–(3.10) and (3.12).** We begin by introducing an operator J_k. Let $Q_k : L_2(\Omega) \longrightarrow \tilde{V}_k$ be the L_2-orthogonal projection operator. Then we define

$$J_k\phi = F_k Q_k \phi \qquad \forall\, \phi \in L_2(\Omega)\,. \tag{6.14}$$

Lemma 6.1. *The operator J_k satisfies*

$$J_k E_k v = v \qquad \forall\, v \in V_k\,, \tag{6.15}$$

$$|\!|\!| J_k\zeta |\!|\!|_{1-\alpha,k} \lesssim \|\zeta\|_{H^{1-\alpha}(\Omega)} \qquad \forall\, \zeta \in H_0^{1-\alpha}(\Omega)\,. \tag{6.16}$$

Proof. The relation (6.15) follows immediately from (6.1) and (6.14).

From (3.2), (3.3), (3.6), (6.2) and standard properties of Q_k (cf. [15]) we have

$$|\!|\!| J_k\zeta |\!|\!|_{0,k} \approx \|F_k Q_k\zeta\|_{L_2(\Omega)} \lesssim \|Q_k\zeta\|_{L_2(\Omega)} \lesssim \|\zeta\|_{L_2(\Omega)} \qquad \forall\, \zeta \in L_2(\Omega)\,, \tag{6.17}$$

$$|\!|\!| J_k\zeta |\!|\!|_{1,k} = \|F_k Q_k\zeta\|_{a_k} \lesssim \|Q_k\zeta\|_{H^1(\Omega)} \lesssim \|\zeta\|_{H^1(\Omega)} \qquad \forall\, \zeta \in H_0^1(\Omega)\,. \tag{6.18}$$

The estimate (6.16) follows from (6.17), (6.18) and interpolation between Hilbert scales (cf. [48], [38], [16]). □

Lemma 6.2. *Given $\zeta_k \in V_k$, let $\zeta \in H_0^1(\Omega)$ be defined by*

$$a(\zeta,\phi) = a_k(\zeta_k, J_k\phi) \qquad \forall\, \phi \in H_0^1(\Omega)\,. \tag{6.19}$$

Then we have

$$a(\zeta, E_k v) = a_k(\zeta_k, v) \qquad \forall\, v \in V_k\,, \tag{6.20}$$

$$|\zeta|_{H^1(\Omega)} \lesssim |\!|\!| \zeta_k |\!|\!|_{1,k}\,, \tag{6.21}$$

$$\|\zeta\|_{H^{1+\alpha}(\Omega)} \lesssim |\!|\!| \zeta_k |\!|\!|_{1+\alpha,k}\,, \tag{6.22}$$

$$\|\Pi_k\zeta\|_{a_k} \lesssim |\!|\!| \zeta_k |\!|\!|_{1,k}\,. \tag{6.23}$$

Proof. The relation (6.20) follows immediately from (6.15) and (6.19).

From (3.3), (6.18), (6.19) and the Cauchy-Schwarz inequality we have

$$a(\zeta,\zeta) = a_k(\zeta_k, J_k\zeta) \le \|\zeta_k\|_{a_k}\|J_k\zeta\|_{a_k} \lesssim |\!|\!| \zeta_k |\!|\!|_{1,k}\|\zeta\|_{H^1(\Omega)}\,.$$

which implies (6.21) in view of (2.6). The estimate (6.23) then follows immediately from (2.6), (6.11) and (6.21).

Using (3.5), (6.16) and (6.19), we find

$$a_k(\zeta_k, J_k\phi) \le |\!|\!| \zeta_k |\!|\!|_{1+\alpha,k} |\!|\!| J_k\phi |\!|\!|_{1-\alpha,k} \lesssim |\!|\!| \zeta_k |\!|\!|_{1+\alpha,k}\|\phi\|_{H^{1-\alpha}(\Omega)} \qquad \forall\, \phi \in H_0^1(\Omega)\,.$$

Thus the right-hand side of (6.19) defines a linear functional F on $H_0^1(\Omega)$ which actually belongs to $H^{-1+\alpha}(\Omega)$ and

$$\|F\|_{H^{-1+\alpha}(\Omega)} \lesssim |\!|\!| \zeta_k |\!|\!|_{1+\alpha,k}\,. \tag{6.24}$$

The estimate (6.22) follows from (1.4) and (6.24). □

We are now ready to derive the estimates (3.8)–(3.10) and (3.12). In the following derivations we use C to denote a generic mesh-independent positive constant that is also independent of the parameter θ.

Lemma 6.3. *The estimate (3.8) holds.*

Proof. Let $\zeta_{k-1} \in V_{k-1}$ be arbitrary and $\zeta \in H_0^1(\Omega)$ be defined by

$$a(\zeta, \phi) = a_{k-1}(\zeta_{k-1}, J_{k-1}\phi) \qquad \forall\, \phi \in H_0^1(\Omega)\,. \tag{6.25}$$

Then it follows from (1.8b), (2.1), (2.5), (3.3), (3.14), (6.3), (6.6), (6.8), (6.9), (6.25) and Lemma 6.2 that

$$\begin{aligned}
|\!|\!| I_{k-1}^k \zeta_{k-1} |\!|\!|_{1,k}^2 &\le \big(\|\zeta_{k-1}\|_{a_k} + \|\zeta_{k-1} - I_{k-1}^k \zeta_{k-1}\|_{a_k}\big)^2 \\
&\le (1+\theta^2)\|\zeta_{k-1}\|_{a_{k-1}}^2 + C\theta^{-2}\big(\|\zeta_{k-1} - \zeta\|_{a_k}^2 + \|\zeta - \Pi_k\zeta\|_{a_k}^2 \\
&\qquad + \|\Pi_k\zeta - I_{k-1}^k \Pi_{k-1}\zeta\|_{a_k}^2 + \|I_{k-1}^k(\Pi_{k-1}\zeta - \zeta_{k-1})\|_{a_k}^2\big) \\
&\le (1+\theta^2)|\!|\!|\zeta_{k-1}|\!|\!|_{1,k-1}^2 + C\theta^{-2}h_k^{2\alpha}|\!|\!|\zeta_{k-1}|\!|\!|_{1+\alpha,k-1}^2\,.
\end{aligned}$$

□

Lemma 6.4. *The estimate (3.9) holds.*

Proof. Let C_* be a constant that is greater than or equal to the constants C_0 and C_1 in (6.10) and (3.8), and define, for any $\theta \in (0,1)$,

$$\langle v_1, v_2\rangle_{k-1,\theta} = (1+\theta^2)(v_1, v_2)_{k-1} + C_*\theta^{-2}h_k^{2\alpha}(A_{k-1}^{\alpha}v_1, v_2)_{k-1} \tag{6.26}$$

for all $v_1, v_2 \in V_{k-1}$. Note that A_{k-1} is symmetric positive definite with respect to the inner product $\langle\cdot,\cdot\rangle_{k-1,\theta}$.

It follows from (3.1), (3.8), (6.10) and (6.26) that

$$\begin{aligned}
|\!|\!| I_{k-1}^k v |\!|\!|_{0,k}^2 &\le \langle A_{k-1}^0 v, v\rangle_{k-1,\theta} \qquad \forall\, v \in V_{k-1}\,, \\
|\!|\!| I_{k-1}^k v |\!|\!|_{1,k}^2 &\le \langle A_{k-1} v, v\rangle_{k-1,\theta} \qquad \forall\, v \in V_{k-1}\,,
\end{aligned}$$

which imply through (3.1) and interpolation between Hilbert scales

$$|\!|\!| I_{k-1}^k v |\!|\!|_{1-\alpha,k}^2 \le \langle A_{k-1}^{1-\alpha} v, v\rangle_{k-1,\theta} = (1+\theta^2)|\!|\!| v |\!|\!|_{1-\alpha,k-1}^2 + C_*\theta^{-2}h_k^{2\alpha}|\!|\!| v |\!|\!|_{1,k-1}^2\,.$$

□

Lemma 6.5. *It holds that*

$$|\!|\!| \Pi_{k-1} v |\!|\!|_{1,k-1}^2 \le (1+\theta^2)|\!|\!| v |\!|\!|_{1,k}^2 + C_\natural\theta^{-2}h_k^{2\alpha}|\!|\!| v |\!|\!|_{1+\alpha,k}^2 \tag{6.27}$$

for all $v \in V_k$ and $\theta \in (0,1)$, where the positive constant $C_\natural$ is mesh-independent.

Proof. Let $\zeta_k \in V_k$ be arbitrary and define $\zeta \in H_0^1(\Omega)$ by (6.19).

From (1.8b), (2.1), (2.5), (3.1), (6.3), (6.8), (6.11) and Lemma 6.2 we have

$$\begin{aligned}
|\!|\!| \Pi_{k-1}\zeta_k |\!|\!|_{1,k-1}^2 &\le \big(\|\zeta_k\|_{a_k} + \|\zeta_k - \Pi_{k-1}\zeta_k\|_{a_k}\big)^2 \\
&\le (1+\theta^2)|\!|\!|\zeta_k|\!|\!|_{1,k}^2 + C\theta^{-2}\big(\|\zeta_k - \zeta\|_{a_k}^2 + \|\zeta - \Pi_{k-1}\zeta\|_{a_k}^2 \\
&\qquad + \|\Pi_{k-1}(\zeta - \zeta_k)\|_{a_k}^2\big) \\
&\le (1+\theta^2)|\!|\!|\zeta_k|\!|\!|_{1,k}^2 + C\theta^{-2}h_k^{2\alpha}|\!|\!|\zeta_k|\!|\!|_{1+\alpha,k}^2\,.
\end{aligned}$$

□

The next lemma follows from (6.13), (6.27) and interpolation between Hilbert scales, as in the proof of Lemma 6.4.

Lemma 6.6. *It holds that*

$$|\!|\!| \Pi_{k-1} v |\!|\!|_{1-\alpha,k-1}^2 \le (1+\theta^2)|\!|\!| v |\!|\!|_{1-\alpha,k}^2 + C_\diamond\theta^{-2}h_k^{2\alpha}|\!|\!| v |\!|\!|_{1,k}^2 \tag{6.28}$$

for all $v \in V_k$ and $\theta \in (0,1)$, where the positive constant $C_\diamond$ is mesh-independent.

We need one more estimate for the derivation of (3.10).

Lemma 6.7. *The following estimate holds*:

$$\|\Pi_{k-1}\zeta - \Pi_{k-1}\Pi_k\zeta\|_{L_2(\Omega)} \lesssim h_k|\zeta|_{H^1(\Omega)} \qquad \forall\, \zeta \in H_0^1(\Omega)\,. \tag{6.29}$$

Proof. From (2.1), (2.6), (6.7), (6.11) and (6.12) we have

$$\begin{aligned}\|\Pi_{k-1}\zeta - \Pi_{k-1}\Pi_k\zeta\|_{L_2(\Omega)} &\le \|\Pi_{k-1}\zeta - \zeta\|_{L_2(\Omega)} + \|\zeta - \Pi_k\zeta\|_{L_2(\Omega)} \\ &\qquad + \|\Pi_k\zeta - \Pi_{k-1}\Pi_k\zeta\|_{L_2(\Omega)} \\ &\lesssim h_k|\zeta|_{H^1(\Omega)} + h_k\|\Pi_k\zeta\|_{a_k} \lesssim h_k|\zeta|_{H^1(\Omega)}\,.\end{aligned}$$

□

Lemma 6.8. *The estimate* (*3.10*) *holds.*

Proof. Let $\zeta_k \in V_k$ be arbitrary. Define $\zeta \in H_0^1(\Omega)$ by (6.19) and $\zeta_{k-1} \in V_{k-1}$ by

$$a_{k-1}(\zeta_{k-1}, v) = a(\zeta, E_{k-1}v) \qquad \forall\, v \in V_{k-1}\,.$$

We have, by (1.8b),

$$\begin{aligned}&\|P_k^{k-1}\zeta_k\|_{1-\alpha,k-1}^2 \\ &\quad\le \big(\|\Pi_{k-1}\Pi_k\zeta\|_{1-\alpha,k-1} + \|\Pi_{k-1}\Pi_k\zeta - P_k^{k-1}\zeta_k\|_{1-\alpha,k-1}\big)^2 \\ &\quad\le (1+\theta^2)\|\Pi_{k-1}\Pi_k\zeta\|_{1-\alpha,k-1}^2 + C\theta^{-2}\|\Pi_{k-1}\Pi_k\zeta - P_k^{k-1}\zeta_k\|_{1-\alpha,k-1}^2\,.\end{aligned} \tag{6.30}$$

The first term on the right-hand side of (6.30) can be estimated using (1.8b), (3.3), (3.14), (6.4), (6.28) and Lemma 6.2:

$$\begin{aligned}\|\Pi_{k-1}\Pi_k\zeta\|_{1-\alpha,k-1}^2 &\le (1+\theta^2)\|\Pi_k\zeta\|_{1-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\Pi_k\zeta\|_{1,k}^2 \\ &\le (1+\theta^2)\big(\|\zeta_k\|_{1-\alpha,k} + \|\Pi_k\zeta - \zeta_k\|_{1-\alpha,k}\big)^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{1,k}^2 \\ &\le (1+\theta^2)^2\|\zeta_k\|_{1-\alpha,k}^2 + C\theta^{-2}h_k^{4\alpha}\|\zeta\|_{H^{1+\alpha}(\Omega)}^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{1,k}^2 \\ &\le (1+\theta^2)^2\|\zeta_k\|_{1-\alpha,k}^2 + C\theta^{-2}h_k^{4\alpha}\|\zeta_k\|_{1+\alpha,k} + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{1,k}^2 \\ &\le (1+\theta^2)^2\|\zeta_k\|_{1-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{1,k}^2\,.\end{aligned} \tag{6.31}$$

Similarly the second term on the right-hand side of (6.30) can be estimated using (2.1), (3.2), (3.6), (3.14), (6.4), (6.5), (6.29) and Lemma 6.2:

$$\begin{aligned}&\|\Pi_{k-1}\Pi_k\zeta - P_k^{k-1}\zeta_k\|_{1-\alpha,k-1}^2 \\ &\quad\le \big(\|\Pi_{k-1}(\Pi_k\zeta - \zeta)\|_{1-\alpha,k-1} + \|\Pi_{k-1}\zeta - \zeta_{k-1}\|_{1-\alpha,k-1} \\ &\qquad\qquad + \|\zeta_{k-1} - P_k^{k-1}\zeta_k\|_{1-\alpha,k-1}\big)^2 \\ &\quad\le C\big(h_k^{2(\alpha-1)}\|\Pi_{k-1}(\Pi_k\zeta - \zeta)\|_{0,k-1}^2 + h_k^{2\alpha}\|\zeta_k\|_{1,k}^2\big) \\ &\quad\le Ch_k^{2\alpha}\|\zeta_k\|_{1,k}^2\,.\end{aligned} \tag{6.32}$$

Combining (6.30)–(6.32), we have

$$\|P_k^{k-1}\zeta_k\|_{1-\alpha,k-1}^2 \le (1+\theta^2)^3\|\zeta_k\|_{1-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{1,k}^2\,,$$

which implies (3.10) since $\theta \in (0,1)$ is arbitrary. □

We now turn to the derivation of (3.12) by first noting that the estimate

$$\|P_k^{k-1}v\|_{1-\alpha,k-1} \lesssim \|v\|_{1-\alpha,k} \qquad \forall\, v \in V_k \tag{6.33}$$

follows from (3.10) and (3.14).

Lemma 6.9. *The estimate (3.12) holds.*

Proof. Let $\zeta_{k-1} \in V_{k-1}$ be arbitrary and define $\zeta \in H_0^1(\Omega)$ and $\zeta_k \in V_k$ by (6.25) and (6.20), respectively. Then it follows from (2.1), (3.1), (3.5), (3.14), (6.4)–(6.6), (6.9), (6.33) and Lemma 6.2 that

$$\begin{aligned}
&a_{k-1}\big((Id_{k-1} - P_k^{k-1} I_{k-1}^k)\zeta_{k-1}, w\big) \\
&\qquad = a_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k, w) - a_{k-1}\big(P_k^{k-1}(I_{k-1}^k\zeta_{k-1} - \zeta_k), w\big) \\
&\qquad \le |\!|\!|\zeta_{k-1} - P_k^{k-1}\zeta_k|\!|\!|_{1-\alpha,k-1}|\!|\!|w|\!|\!|_{1+\alpha,k-1} \\
&\qquad\qquad + |\!|\!|P_k^{k-1}(I_{k-1}^k\zeta_{k-1} - \zeta_k)|\!|\!|_{1-\alpha,k-1}|\!|\!|w|\!|\!|_{1+\alpha,k-1} \\
&\qquad \lesssim h_k^{2\alpha}\|\zeta\|_{H^{1+\alpha}(\Omega)}|\!|\!|w|\!|\!|_{1+\alpha,k-1} + \big(|\!|\!|I_{k-1}^k(\zeta_{k-1} - \Pi_{k-1}\zeta)|\!|\!|_{1-\alpha,k} \\
&\qquad\qquad + |\!|\!|I_{k-1}^k\Pi_{k-1}\zeta - \Pi_k\zeta|\!|\!|_{1-\alpha,k} + |\!|\!|\Pi_k\zeta - \zeta_k|\!|\!|_{1-\alpha,k}\big)|\!|\!|w|\!|\!|_{1+\alpha,k-1} \\
&\qquad \lesssim h_k^{2\alpha}\|\zeta\|_{H^{1+\alpha}(\Omega)}|\!|\!|w|\!|\!|_{1+\alpha,k-1} \\
&\qquad \lesssim h_k^{2\alpha}|\!|\!|\zeta_{k-1}|\!|\!|_{1+\alpha,k-1}|\!|\!|w|\!|\!|_{1+\alpha,k-1} \\
&\qquad \lesssim h_k^{\alpha}|\!|\!|\zeta_{k-1}|\!|\!|_{1,k-1}|\!|\!|w|\!|\!|_{1+\alpha,k-1}
\end{aligned}$$

for all $w \in V_{k-1}$, which implies (3.12) because of (3.5). □

Remark 6.10. The proof of Lemma 6.9 actually establishes the stronger estimate

$$|\!|\!|(Id_{k-1} - P_k^{k-1} I_{k-1}^k)v|\!|\!|_{1-\alpha,k-1} \lesssim h_k^{2\alpha}|\!|\!|v|\!|\!|_{1+\alpha,k-1} \qquad \forall\, v \in V_{k-1}\,.$$

7. Applications

In this section we apply the theory developed in Sections 3–6 to two nonconforming finite element methods for the variational problem (1.1). As demonstrated in [21], both of these methods satisfy the assumptions of the framework developed in that paper. Therefore, from the discussion in Section 6, it only remains for us to check the additional conditions (6.10)–(6.13) for these examples.

7.1. The nonconforming P_1 finite element method.

Let e be (a segment of) an edge of $T \in \mathcal{T}_k$. We define

$$M_{e,T}(v) = \frac{1}{|e|}\int_e v\big|_T\, ds \qquad \text{for } v \in H^1(\mathcal{T}_\ell)\,,\ \ell \ge 1\,. \tag{7.1}$$

Remark 7.1. In the case where $\ell > k$ and e is an edge of T, the integral on the right-hand side of (7.1) should be interpreted as

$$\int_e v\big|_T\, ds = \sum_{j=1}^{2^{\ell-k}} \int_{e_j} v\big|_{T_j}\, ds$$

where $e_1, \ldots, e_{2^{\ell-k}}$ are the edges from $\mathcal{T}_\ell$ that form a partition of e and $T_j \subset T$ is a triangle in $\mathcal{T}_\ell$ with e_j as an edge (cf. Figure 7.1 for the case $\ell - k = 2$).

The nonconforming P_1 finite element space V_k is defined by (cf. [31])

$$\begin{aligned}
V_k = \{v \in L_2(\Omega) :\ & v\big|_T \in \langle 1, x_1, x_2\rangle\ \forall\, T \in \mathcal{T}_k\,,\ M_{e,T}(v) = M_{e,T'}(v) \text{ if } e \\
& \text{is the common edge of } T \text{ and } T' \text{ in } \mathcal{T}_k, \text{ and } M_{e,T}(v) = 0 \\
& \text{if } e \subset \partial\Omega \text{ is an edge of } T\}\,.
\end{aligned} \tag{7.2}$$

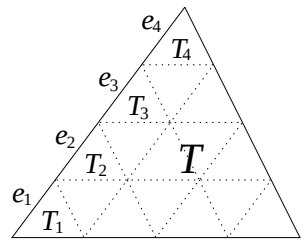

FIGURE 7.1. Interpretation of (7.1) when $\ell - k = 2$

Henceforth we will denote $M_{e,T}(v)$ by $M_e(v)$ if there is no ambiguity about T or if v has an unambiguous mean value on e.

Let $\mathcal{E}_k$ be the set of the internal edges of $\mathcal{T}_k$. We define $(\cdot,\cdot)_k$ by

$$(v_1, v_2)_k = h_k^2 \sum_{e\in\mathcal{E}_k} M_e(v_1)M_e(v_2) \qquad \forall\, v_1, v_2 \in V_k\,. \tag{7.3}$$

Let $\sum_{\ell=1}^{\infty} H^1(\mathcal{T}_\ell)$ be the subspace of $L_2(\Omega)$ whose members are finite sums of functions from the spaces $H^1(\mathcal{T}_1), H^1(\mathcal{T}_2), \ldots$. The weak interpolation operator $\Pi_k : \sum_{\ell=1}^{\infty} H^1(\mathcal{T}_\ell) \longrightarrow V_k$ is defined by

$$M_e\big(\Pi_k v\big) = \frac{1}{2} \sum_{e\subset\partial T, T\in\mathcal{T}_k} M_{e,T}(v) \qquad \forall\, e \in \mathcal{E}_k\,. \tag{7.4}$$

Note that there are exactly two triangles in the sum and the mean value of $\Pi_k v$ on e is just the average of the mean values of v on e from these two triangles.

The restriction of Π_k to $H_0^1(\Omega)$ is precisely the weak interpolation operator used in [21] (cf. also [19], [6]), and the intergrid transfer operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ in [21] is just the restriction of Π_k to V_{k-1}.

The following lemma, a simple consequence of the Bramble-Hilbert lemma (cf. [8]) and scaling, is the key to the estimates (6.10)–(6.13).

Lemma 7.2. *Let e_1, e_2 be two of the edges of $T \in \mathcal{T}_k$. Then we have*

$$|M_{e_1}(v) - M_{e_2}(v)| \lesssim |v|_{H^1(T)} \qquad \forall\, v \in H^1(T)\,.$$

Verification of (6.11). Let $e \in \mathcal{E}_{k-1}$ be the common edge of two triangles T and T' in $\mathcal{T}_{k-1}$, and let e_1 (resp. e_2) be the half of the edge e neighboring the two triangles T_1 and T_2 (resp. T_3 and T_4) in $\mathcal{T}_k$ (cf. Figure 7.2).

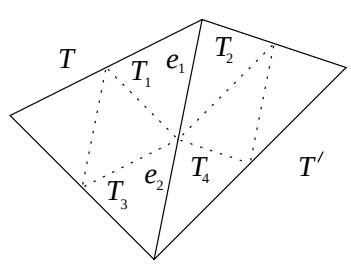

FIGURE 7.2. Two neighboring triangles in $\mathcal{T}_{k-1}$

Then for $v \in H_0^1(\Omega) + V_k$, we have

$$M_{e,T}(v) = \frac{1}{2}\big[M_{e_1}(v) + M_{e_2}(v)\big] = M_{e,T'}(v)\,. \tag{7.5}$$

In view of (7.4) and (7.5), the interpolation operator $\Pi_{k-1} : H_0^1(\Omega) + V_k \longrightarrow V_{k-1}$ can be analyzed locally on each triangle of $\mathcal{T}_{k-1}$.

Let T be a triangle in $\mathcal{T}_{k-1}$ subdivided into four triangles $T_1, \ldots, T_4$ and let e_1, $\ldots, e_6$ be the edges from $\mathcal{T}_k$ that are on ∂T (cf. Figure 7.3). For any $v \in H_0^1(\Omega)+V_k$ we obtain from (7.4), (7.5) and scaling

$$|\Pi_{k-1}v|^2_{H^1(T)} \lesssim \sum_{i,j=1}^{6} |M_{e_i}(v) - M_{e_j}(v)|^2\,. \tag{7.6}$$

Note that Lemma 7.2 implies

$$|M_{e_i}(v) - M_{e_j}(v)|^2 \lesssim \sum_{\ell=1}^{4} |v|^2_{H^1(T_\ell)} \qquad \forall\, v \in H^1(T) + V_k\,,\ 1 \le i,j \le 9. \tag{7.7}$$

For example, we have (cf. Figure 7.3)

$$\begin{aligned} |M_{e_1}(v) - M_{e_4}(v)|^2 &\lesssim |M_{e_1}(v) - M_{e_7}(v)|^2 + |M_{e_7}(v) - M_{e_9}(v)|^2 \\ &\qquad + |M_{e_9}(v) - M_{e_4}(v)|^2 \\ &\lesssim |v|^2_{H^1(T_1)} + |v|^2_{H^1(T_4)} + |v|^2_{H^1(T_3)}\,. \end{aligned}$$

It follows from (7.6) and (7.7) that

$$|\Pi_{k-1}v|^2_{H^1(T)} \lesssim \sum_{\ell=1}^{4} |v|^2_{H^1(T_\ell)} \qquad \forall\, v \in H^1(T) + V_k\,. \tag{7.8}$$

We obtain by summing (7.8) over all the triangles in $\mathcal{T}_{k-1}$

$$\sum_{T\in\mathcal{T}_{k-1}} \int_T |\nabla(\Pi_{k-1}v)|^2\,dx \lesssim \sum_{T\in\mathcal{T}_k} \int_T |\nabla v|^2\,dx \qquad \forall\, v \in H_0^1(\Omega) + V_k\,. \tag{7.9}$$

The estimate (6.11) then follows from (1.3), (2.3), (2.4), (7.9) and the Poincaré inequality for nonconforming P_1 finite element functions (cf. [24] and the references therein). □

Let T be a generic triangle in $\mathcal{T}_{k-1}$ subdivided into four triangles $T_1, \ldots, T_4 \in \mathcal{T}_k$ (cf. Figure 7.3). Henceforth we will denote by $e_1, \ldots, e_9$ the edges of $T_1, \ldots, T_4$ (cf.

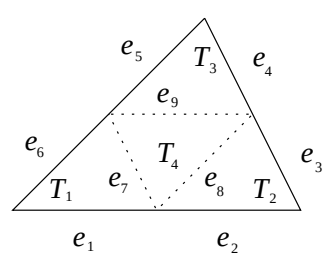

FIGURE 7.3. A divided triangle in $\mathcal{T}_{k-1}$

Figure 7.3) and denote by $\tilde{e}_j$ $(1 \le j \le 3)$ the edge of T that contains the edges e_{2j-1} and e_{2j} from $\mathcal{T}_k$.

Verification of (6.12). Let T be a generic triangle in $\mathcal{T}_{k-1}$. For any $v \in V_k$, we have by (7.4) and (7.5)

$$M_{\tilde{e}_j}(\Pi_{k-1}v) = \frac{1}{2}\big[M_{e_{2j-1}}(v) + M_{e_{2j}}(v)\big]\,. \tag{7.10}$$

Observe also that

$$M_{\tilde{e}_j}(w) = \frac{1}{2}\big[M_{e_{2j-1}}(w) + M_{e_{2j}}(w)\big] \qquad \forall\, w \in H^1(T)\,,$$

which together with (7.7) implies

$$|M_{\tilde{e}_j}(w) - M_{e_i}(w)| \lesssim |w|_{H^1(T)} \quad \forall\, w \in H^1(T)\,,\ 1 \le j \le 3\,,\ 1 \le i \le 9\,. \tag{7.11}$$

We have by scaling

$$\|\Pi_{k-1}v - v\|^2_{L_2(T)} \lesssim h_k^2 \sum_{j=1}^{9} \big[M_{e_j}(\Pi_{k-1}v) - M_{e_j}(v)\big]^2\,, \tag{7.12}$$

and it follows from (7.7), (7.8), (7.10) and (7.11) that

$$\begin{aligned} \sum_{j=1}^{9} \big[M_{e_j}(\Pi_{k-1}v) - M_{e_j}(v)\big]^2 &\lesssim \sum_{\ell=1}^{4} |v|^2_{H^1(T_\ell)} + |\Pi_{k-1}v|^2_{H^1(T)} \\ &\lesssim \sum_{\ell=1}^{4} |v|^2_{H^1(T_\ell)}\,. \end{aligned} \tag{7.13}$$

Combining (7.12) and (7.13), we find

$$\|\Pi_{k-1}v - v\|^2_{L_2(T)} \lesssim h_k^2 \sum_{\ell=1}^{4} |v|^2_{H^1(T_\ell)}\,.$$

In view of (1.3), (2.3) and (2.4), we obtain the estimate (6.12) by summing up the preceding estimate over all the triangles in $\mathcal{T}_{k-1}$. □

Verification of (6.13). Denote by $\mathcal{E}_T$ the set of the three edges of T. Then we have by (7.2) and (7.3)

$$(v,v)_k = \frac{h_k^2}{2} \sum_{T \in \mathcal{T}_k} \sum_{e \in \mathcal{E}_T} [M_e(v)]^2 \qquad \forall\, v \in V_k\,. \tag{7.14}$$

The relation (7.14) allows us to focus on a generic $T \in \mathcal{T}_{k-1}$ in the derivation of (6.13). From (1.8), (7.7) and (7.10) we have

$$\begin{aligned}
\sum_{j=1}^{3}[M_{\tilde{e}_j}(\Pi_{k-1}v)]^2 &\leq \frac{1}{2}\sum_{i=1}^{6}[M_{e_i}(v)]^2 \\
&= \frac{1}{4}\sum_{i=1}^{6}[M_{e_i}(v)]^2 + \frac{1}{4}\sum_{j=1}^{3}\Big\{\big(M_{e_{j+6}}(v) - [M_{e_{j+6}}(v) - M_{e_{2j-1}}(v)]\big)^2 \\
&\qquad\qquad + \big(M_{e_{j+6}}(v) - [M_{e_{j+6}}(v) - M_{e_{2j}}(v)]\big)^2\Big\} \\
&= \frac{1}{4}\sum_{i=1}^{6}[M_{e_i}(v)]^2 + \frac{(1+\theta^2)}{2}\sum_{i=7}^{9}[M_{e_i}(v)]^2 + C\theta^{-2}\sum_{\ell=1}^{4}|v|^2_{H^1(T_\ell)}
\end{aligned} \tag{7.15}$$

for all $v \in V_k$ and $\theta \in (0,1)$, where the positive constant C is mesh-independent.

Summing up the estimate (7.15) over all the triangles in $\mathcal{T}_{k-1}$ and using (1.3), (2.1), (2.3), (2.4), (3.2), (3.3), (7.2), (7.3) and (7.14), we arrive at

$$\begin{aligned}
|\!|\!|\Pi_{k-1}v|\!|\!|^2_{0,k-1} &= \frac{h_{k-1}^2}{2}\sum_{T\in\mathcal{T}_{k-1}}\sum_{\tilde{e}\in\mathcal{E}_T}[M_{\tilde{e}}(v)]^2 \\
&\leq (1+\theta^2)\frac{h_{k-1}^2}{4}\sum_{e\in\mathcal{E}_k}[M_e(v)]^2 + C\theta^{-2}h_k^2\sum_{T\in\mathcal{T}_k}|v|^2_{H^1(T)} \\
&\leq (1+\theta^2)(v,v)_k + C\theta^{-2}h_k^2\|v\|^2_{a_k} \\
&= (1+\theta^2)|\!|\!|v|\!|\!|^2_{0,k} + C\theta^{-2}h_k^2|\!|\!|v|\!|\!|^2_{1,k}\,.
\end{aligned} \tag{7.16}$$

The estimate (6.13) follows from (3.14) and (7.16). □

Verification of (6.10). Let e be an edge of $\mathcal{T}_k$ which is on the common boundary of $T, T' \in \mathcal{T}_{k-1}$ (cf. Figure 7.2). Then we have by (1.8a) and (7.4)

$$[M_e(I_{k-1}^k v)]^2 = \frac{1}{4}\big[M_{e,T}(v) + M_{e,T'}(v)\big]^2 \leq \frac{1}{2}[M_{e,T}(v)]^2 + \frac{1}{2}[M_{e,T'}(v)]^2\,. \tag{7.17}$$

For $T \in \mathcal{T}_{k-1}$, let $\mathcal{E}_T^b$ (resp. $\mathcal{E}_T^i$) denote the set of the edges from $\mathcal{T}_k$ that are on ∂T (resp. interior to T). We obtain from (7.3) and (7.17) the estimate

$$(I_{k-1}^k v, I_{k-1}^k v)_k \leq h_k^2\sum_{T\in\mathcal{T}_{k-1}}\Big(\frac{1}{2}\sum_{e\in\mathcal{E}_T^b}[M_{e,T}(v)]^2 + \sum_{e\in\mathcal{E}_T^i}[M_e(v)]^2\Big) \tag{7.18}$$

for all $v \in V_{k-1}$.

Again the estimate (7.18) allows us to focus on a generic $T \in \mathcal{T}_{k-1}$. It follows from (1.8b) and (7.11) that

$$\frac{1}{2}\sum_{i=1}^{6}[M_{e_i}(v)]^2 + \sum_{i=7}^{9}[M_{e_i}(v)]^2 \leq 2(1+\theta^2)\sum_{j=1}^{3}[M_{\tilde{e}_j}(v)]^2 + C\theta^{-2}|v|^2_{H^1(T)} \tag{7.19}$$

for all $v \in H^1(T)$ and $\theta \in (0,1)$, where the positive constant C is mesh-independent.

Summing up the estimate (7.19) over all the triangles in $\mathcal{T}_{k-1}$, we find by (1.3), (2.1), (2.3), (2.4), (3.2), (3.3), (7.14) and (7.18)

$$\begin{aligned}
\|\!| I_{k-1}^k v \|\!|_{0,k}^2 &= (I_{k-1}^k v, I_{k-1}^k v)_k \\
&\le h_k^2 \sum_{T\in\mathcal{T}_{k-1}} \Big(\frac{1}{2} \sum_{e\in\mathcal{E}_T^b} [M_{e,T}(v)]^2 + \sum_{e\in\mathcal{E}_T^i} [M_e(v)]^2 \Big) \\
&\le (1+\theta^2)\frac{h_{k-1}^2}{2} \sum_{T\in\mathcal{T}_{k-1}} \sum_{\tilde{e}\in\mathcal{E}_T} [M_{\tilde{e}}(v)]^2 + C\theta^{-2} h_k^2 \sum_{T\in\mathcal{T}_{k-1}} |v|_{H^1(T)}^2 \\
&\le (1+\theta^2)\|\!| v \|\!|_{0,k-1}^2 + C\theta^{-2} h_k^2 \|\!| v \|\!|_{1,k-1}^2
\end{aligned} \tag{7.20}$$

for all $v \in V_{k-1}$ and $\theta \in (0,1)$.

The estimate (6.10) follows from (3.14) and (7.20). □

7.2. **The rotated Q_1 finite element method.** We shall adopt the notation in subsection 7.1 with obvious modifications. For simplicity we assume that $\mathcal{T}_1$ (and hence any $\mathcal{T}_k$) is a triangulation of Ω by rectangles whose sides are parallel to the coordinate axes. The rotated Q_1 finite element space (cf. [42]) is defined by

$$\begin{aligned}
V_k = \{ v \in L_2(\Omega) : v\big|_R \in \langle 1, x_1, x_2, x_1^2 - x_2^2 \rangle \quad \forall\, R \in \mathcal{T}_k\,,\ M_{e,R}(v) = \\
M_{e,R'}(v) \text{ if } e \text{ is the common edge of } R \text{ and } R' \text{ in } \mathcal{T}_k, \text{ and } M_{e,R}(v) \\
= 0 \text{ if } e \subset \partial\Omega \text{ is an edge of } R \}\,.
\end{aligned} \tag{7.21}$$

We define the inner product $(\cdot,\cdot)_k$ by

$$(v_1, v_2)_k = h_k^2 \sum_{e\in\mathcal{E}_k} M_e(v_1) M_e(v_2) \qquad \forall\, v \in V_k\,. \tag{7.22}$$

The weak interpolation operator $\Pi_k : \sum_{\ell=1}^\infty H^1(\mathcal{T}_\ell) \longrightarrow V_k$ is defined by

$$M_e(\Pi_k v) = \frac{1}{2} \sum_{e\subset\partial R, R\in\mathcal{T}_k} M_e(v) \qquad \forall\, e \in \mathcal{E}_k\,. \tag{7.23}$$

The restriction of Π_k to $H_0^1(\Omega)$ is precisely the weak interpolation operator used in [21], and the intergrid transfer operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ used in [21] is just the restriction of Π_k to V_{k-1}.

The following analog of Lemma 7.2 is again a simple consequence of the Bramble-Hilbert lemma and scaling.

Lemma 7.3. *Let e_1 and e_2 be two edges of the rectangle $R \in \mathcal{T}_k$. Then we have*

$$|M_{e_1}(v) - M_{e_2}(v)| \lesssim |v|_{H^1(R)} \qquad \forall\, v \in H^1(R)\,.$$

Using Lemma 7.3, the verification of (6.10)–(6.13) proceeds as in subsection 7.1 and we therefore omit the details.

Remark 7.4. The results of this paper can also be applied to other nonconforming quadrilateral elements (cf. [33], [27], [28], [26]).

Remark 7.5. The nonconforming P_1 and the rotated Q_1 finite elements are equivalent to the lowest order triangular and rectangular Raviart-Thomas mixed finite elements (cf. [43], [2], [1]). The results of this paper can therefore be applied to multigrid methods for the lowest order triangular and rectangular Raviart-Thomas mixed methods (cf. [20], [1]).

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