MATHEMATICS OF COMPUTATION
Volume 73, Number 248, Pages 1583–1600
S 0025-5718(04)01628-X
Article electronically published on March 9, 2004

CRITERIA FOR THE APPROXIMATION PROPERTY FOR MULTIGRID METHODS IN NONNESTED SPACES

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ABSTRACT. We extend the abstract frameworks for the multigrid analysis for nonconforming finite elements to the case where the assumptions of the second Strang lemma are violated. The consistency error is studied in detail for finite element discretizations on domains with curved boundaries. This is applied to prove the approximation property for conforming elements, stabilized Q_1/P_0-elements, and nonconforming elements for linear elasticity on nonpolygonal domains.

Proving the approximation property for the multigrid analysis for nonconforming finite element discretizations is formalized in [7, 4, 15] for many cases: it suffices to verify criteria on the approximation quality and the consistency error. In these papers, it is required that a continuous bilinear form can be extended to a nonconforming finite element space, which is not valid for many interesting applications.

The purpose of this paper is to establish a full set of criteria which guarantees the approximation property for a wide range of nonnested discretizations, where we do not assume that the discrete bilinear form coincides with the continuous bilinear form for all conforming functions. In the notation, we follow Bramble [5, Chap. 4], and our results can be applied directly to the multigrid theory described there. The results extend known results by Brenner [7] and Stevenson [15], and they provide a systematic and constructive way of studying nonnested multigrid algorithms for more general nonnested spaces and varying forms.

The paper is organized as follows. First, we introduce an abstract setting describing a multigrid hierarchy for nonconforming discretizations of an elliptic partial differential equation without full regularity. As usual, the multigrid approximation property is derived by comparison with the finite element approximation property, which we formulate using an interpolation operator and its adjoint with respect to the energy scalar product. In a second step (Section 1.9), we derive the approximation property from consistency assumptions on a conforming comparison space, similar to the approach in [7].

In Section 2, we consider the case of conforming finite elements on a polygonal approximation of the computational domain. Here, we choose a comparison space consisting of curved finite elements. After introducing a suitable interpolation operator, we use the equivalence of the operator norm scale (used throughout Section 1)

Received by the editor January 23, 2001 and, in revised form, March 21, 2003.
2000 *Mathematics Subject Classification.* Primary 65N55, 65F10.
Key words and phrases. Multigrid analysis, nonnested forms, approximation property, curved boundaries, stabilized finite elements.
This work was supported in part by the Deutsche Forschungsgemeinschaft.

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to the standard Sobolev norm scale; note that this is the only step where regularity of the continuous problem is required. Then, the consistency assumptions can be proved in the Sobolev norm scale; this is done in Sections 2.5, 2.6 and 2.7. In Section 2.8, we show that these estimates lead to improved a priori finite element estimates as well, which extends results from [6] to the case of Neumann boundary conditions.

In Section 3, the results are first applied to linear elasticity with conforming finite elements. Then we show that they carry over to the case in which the bilinear form is modified by a well-known stabilization technique. Finally, we combine our results with [7] to obtain multigrid convergence for nonconforming finite element approximations on curved domains as well.

1. The abstract setting

We consider an abstract setting, where we assume that the discrete problem is connected with the continuous problem by an interpolation operator π_j. In the first step, we show that the approximation property is a consequence of an approximation assumption on the adjoint interpolation. In the second step, we derive properties of the adjoint interpolation by comparison with a suitable conforming finite element space.

1.1. **The continuous problem.** Let $H_1 \subset H$ be separable real Hilbert spaces with inner products $a(\cdot,\cdot)$ and $(\cdot,\cdot)$, and let H_1 be dense in H with continuous injection. Following [11, Sect. 2.1], this defines an unbounded self-adjoint strictly positive operator A in H with domain

$$\begin{aligned}\mathrm{dom}(A) = \{u \in H_1 \quad | \quad & \text{the linear form } v \mapsto a(u,v) \text{ for } v \in H_1 \\ & \text{is continuous in the topology induced by } H\}\end{aligned}$$

by the relation

$$a(v,w) = (Av,w), \qquad v \in \mathrm{dom}(A),\ w \in H_1. \tag{1}$$

We have $H_1 = \mathrm{dom}(A^{1/2}) = [H, \mathrm{dom}(A)]_{1/2}$ [11, Sect. 2.4, Prop. 2.1]. Furthermore, $H_{2\alpha} := \mathrm{dom}(A^\alpha)$, $\alpha \geq 0$, are Hilbert spaces (equipped with the inner products $(v,w)_{2\alpha} = (A^\alpha v, A^\alpha w)$); for $\alpha \in [0,1]$ we have $H_{2\alpha} = [H, H_2]_\alpha$, $H_\alpha = [H, H_1]_\alpha$, and $H_{1+\alpha} = [H_1, H_2]_\alpha$ [16, Th. 1.15.3]. We denote the dual space by $H_{-\alpha} = H'_\alpha$.

Within the abstract setting, we fix a regularity parameter $\beta \in (0,1]$.

1.2. **The discrete problem.** Let M_j, $j = 0, ..., J$, be discrete spaces with inner products $(\,.\,,\,.\,)_j$, let

$$A_j \colon M_j \to M_j$$

be symmetric positive definite operators on M_j, and let

$$a_j(v_j, w_j) = (A_j v_j, w_j)_j, \qquad v_j, w_j \in M_j,$$

be the associated bilinear forms. For $\alpha \in [0,2]$, we define the discrete norms

$$\|v_j\|_{j,\alpha} = (A_j^{\alpha/2} v_j, A_j^{\alpha/2} v_j)_j^{1/2}, \qquad v_j \in M_j. \tag{2}$$

Finally, we set $\lambda_j = \|A_j\|_j = \sup\limits_{v_j \neq 0} \frac{\|A_j v_j\|_j}{\|v_j\|_j}$, and we require

$$\lambda_j \lesssim \lambda_{j-1}, \qquad j = 1, ..., J. \tag{S}$$

1.3. **Interpolation.** We assume that the continuous spaces and the discrete spaces are connected by surjective interpolation operators

$$\pi_j \colon H_{1-\beta} \to M_j, \qquad j = 0, ..., J.$$

For the interpolation, we require

$$(\Pi) \qquad \|\pi_j v\|_{j,1-\beta} \lesssim \|v\|_{1-\beta}, \qquad v \in H_{1-\beta}.$$

1.4. **The A-adjoint interpolation.** Let $\pi_j^* \colon M_j \to H_{1+\beta}$ be the A-adjoint interpolation; i.e.,

$$a(\pi_j^* v_j, w) = a_j(v_j, \pi_j w), \qquad v_j \in M_j,\ w \in H_{1-\beta}.$$

For the adjoint interpolation, we assume

$$(G) \qquad \|v_j - \pi_j \pi_j^* v_j\|_{j,1-\beta} \lesssim \lambda_j^{-\beta}\ \|v_j\|_{j,1+\beta}, \qquad v_j \in M_j.$$

1.5. **Prolongation and restriction.** We assume that the discrete spaces are connected by prolongation operators

$$I_j \colon M_{j-1} \to M_j, \qquad j = 1, ..., J.$$

We assume the compatibility of the prolongation I_j and the interpolation π_j

$$(P) \qquad \|\pi_j v - I_j \pi_{j-1} v\|_{j,1-\beta} \lesssim \lambda_j^{-\beta}\ \|v\|_{1+\beta}, \qquad v \in H_{1+\beta},$$

and the stability

$$(B) \qquad \|I_j v_{j-1}\|_{j,1-\beta} \lesssim \|v_{j-1}\|_{j-1,1-\beta}, \qquad v_{j-1} \in M_{j-1}.$$

The restriction $I_j^T \colon M_j \to M_{j-1}$, $j = 1, ..., J$, is given by

$$(I_j^T v_j, w_{j-1})_{j-1} = (v_j, I_j w_{j-1})_j, \qquad v_j \in M_j,\ w_{j-1} \in M_{j-1}.$$

1.6. **The A-adjoint prolongation.** Let $I_j^* = A_{j-1}^{-1} I_j^T A_j \colon M_j \to M_{j-1}$ be the A-adjoint prolongation; i.e.,

$$a_{j-1}(I_j^* v_j, w_{j-1}) = a_j(v_j, I_j w_{j-1}), \qquad v_j \in M_j,\ w_{j-1} \in M_{j-1}.$$

1.7. **Duality.** By duality with respect to the bilinear forms a and a_j we have

$$(3) \qquad \|v_j\|_{j,1-\alpha} = \sup_{w_j \in M_j} \frac{|a_j(v_j, w_j)|}{\|w_j\|_{j,1+\alpha}}, \quad \|v\|_{1-\alpha} = \sup_{w \in H_{1+\alpha}} \frac{|a(v, w)|}{\|w\|_{1+\alpha}}$$

for $\alpha \in \{-\beta, 0, \beta\}$. This implies the dual estimates for (Π)

$$(\Pi^*) \qquad \|\pi_j^* v_j\|_{1+\beta} \lesssim \|v_j\|_{j,1+\beta}, \qquad v_j \in M_j,$$

for (B)

$$(B^*) \qquad \|I_j^* v_j\|_{j-1,1+\beta} \lesssim \|v_j\|_{j,1+\beta}, \qquad v_j \in M_j,$$

and for (P)

$$(P^*) \qquad \|\pi_j^* u_j - \pi_{j-1}^* I_j^* u_j\|_{1-\beta} \lesssim \lambda_j^{-\beta}\ \|u_j\|_{j,1+\beta}.$$

1.8. **The approximation property.** Now we can derive the approximation property for the multigrid analysis in the form [5, Assumption A.10].

Theorem 1. *The approximation property*

$$(A) \qquad |a_j(u_j - I_j I_j^* u_j, u_j)| \lesssim \left(\frac{\|A_j u_j\|_j^2}{\lambda_j}\right)^\beta a_j(u_j,u_j)^{1-\beta}, \qquad u_j \in M_j,$$

follows from (S), (G), (Π), (P), *and* (B).

Proof. The approximation property (A) is a simple consequence of

$$\|u_j - I_j I_j^* u_j\|_{j,1-\beta} \lesssim \lambda_j^{-\beta}\, \|u_j\|_{j,1+\beta}, \qquad u_j \in M_j, \tag{4}$$

(cf. [7, Lem. 4.7]). Using

$$\begin{aligned} \mathrm{id}_{M_j} - I_j I_j^* &= \mathrm{id}_{M_j} - \pi_j\pi_j^* + \pi_j\pi_j^* - I_j I_j^* \\ &= \mathrm{id}_{M_j} - \pi_j\pi_j^* + (\pi_j - I_j\pi_{j-1})\pi_j^* \\ &\qquad + I_j\pi_{j-1}(\pi_j^* - \pi_{j-1}^* I_j^*) + I_j(\pi_{j-1}\pi_{j-1}^* - \mathrm{id}_{M_{j-1}})I_j^*, \end{aligned}$$

we obtain (4) by (G), (P), (Π^*), (B), (Π), (P^*), (B), (G), (S), and (B^*). □

1.9. **Consistency properties.** In this subsection, we present a sufficient criterion for (G) which does not involve the A-adjoint interpolation operator $\pi_j^* : M_j \to H_1$. To achieve this, we assume that an operator

$$\varphi_j \colon M_j \to H_1$$

exists such that φ_j is a stable right inverse of π_j, i.e., $\pi_j \circ \varphi_j = \mathrm{id}_{M_j}$, and

$$(\Phi) \qquad \|\varphi_j v_j\|_1 \lesssim \|v_j\|_{j,1}, \qquad v_j \in M_j.$$

Now we can derive a bound for the error of the adjoint interpolation in the discrete energy norm from the *first consistency and approximation assumption*

$$(C) \qquad |a_j(\pi_j v, \pi_j w) - a(v,w)| \lesssim \lambda_j^{-\beta/2}\, \|v\|_{1+\beta}\|w\|_1, \qquad v \in H_{1+\beta},\ w \in H_1.$$

Lemma 2. *Assume that* (Π), (Φ) *and* (C) *are satisfied. Then we have*

$$(E) \qquad \|v_j - \pi_j\pi_j^* v_j\|_{j,1} \lesssim \lambda_j^{-\beta/2}\, \|v_j\|_{j,1+\beta}, \qquad v_j \in M_j.$$

Proof. (C) can be written equivalently as

$$(C') \qquad \|(\mathrm{id}_{H_1} - \pi_j^*\pi_j)v\|_1 \lesssim \lambda_j^{-\beta/2}\|v\|_{1+\beta}, \qquad v \in H_{1+\beta},$$

and using (3), this is equivalent to

$$(C^*) \qquad \|(\mathrm{id}_{H_1} - \pi_j^*\pi_j)v\|_{1-\beta} \lesssim \lambda_j^{-\beta/2}\|v\|_1, \qquad v \in H_1.$$

Now inserting

$$\mathrm{id}_{M_j} - \pi_j\pi_j^* = (\mathrm{id}_{M_j} - \pi_j\pi_j^*)\pi_j\varphi_j = \pi_j(\mathrm{id}_{H_1} - \pi_j^*\pi_j)\varphi_j,$$

we get from (Π), (C^*), and (Φ)

$$(E^*) \qquad \|v_j - \pi_j\pi_j^* v_j\|_{j,1-\beta} \lesssim \lambda_j^{-\beta/2}\, \|v_j\|_{j,1}, \qquad v_j \in M_j.$$

Again by duality (3), (E^*) is equivalent to (E). □

To obtain (G), we need the *second consistency and approximation assumption*

$$(D) \qquad |a_j(\pi_j v, \pi_j w) - a(v,w)| \lesssim \lambda_j^{-\beta}\, \|v\|_{1+\beta}\|w\|_{1+\beta}, \qquad v, w \in H_{1+\beta}.$$

Lemma 3. *Assume that* (Π), (*E*), *and* (*D*) *are satisfied. Then we have* (*G*).

Proof. (*D*) can be written equivalently as

$$(D') \qquad \|(\mathrm{id}_{H_1} - \pi_j^*\pi_j)v\|_{1-\beta} \lesssim \lambda_j^{-\beta}\, \|v\|_{1+\beta}, \qquad v \in H_{1+\beta},$$

and the assertion follows directly from

$$\mathrm{id}_{M_j} - \pi_j\pi_j^* = (\mathrm{id}_{M_j} - \pi_j\pi_j^*)^2 + \pi_j(\mathrm{id}_{H_1} - \pi_j^*\pi_j)\pi_j^*$$

by applying (*E*), (*E**), (Π), (*D*′), and (Π*). □

2. Curved finite elements

As an application of our abstract theory, we consider the case of finite element approximations for elliptic problems on domains with curved boundaries. The discretization with Lagrange elements of lowest order on domains with curved boundaries will be done on a polygonal or polyhedral approximation of the domain. In this case, the approximating spaces are not contained in $L^2(\Omega)$, and the theory of [7] cannot be applied. By comparison with curved finite elements as they are introduced and analyzed for triangles by Zlámal [20] (and in a more general formulation in [1, 10, 12]), we derive a bound for the consistency error. For a different approach for analyzing curved boundaries, see [6].

2.1. **Local transformations.** Let $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, be a Lipschitz domain, and let $\Omega_j \subset \mathbf{R}^d$ be a polygonal approximation of Ω such that Ω_j can be decomposed into elements $E \in \mathcal{E}_j$. That is, $E \subset \Omega_j$ and

$$\bar{\Omega}_j = \bigcup_{E \in \mathcal{E}_j} \bar{E} \text{ and } E \cap E' = \emptyset \text{ for } E, E' \in \mathcal{E}_j,\ E \neq E'.$$

Every element $E \in \mathcal{E}_j$ is assumed to be the image of a reference element $\hat{E}$ (e.g., the unit triangle/quadrilateral for $d = 2$, or the unit tetrahedron/hexahedron for $d = 3$) under an affine (multi-) linear mapping $T_E \colon \hat{E} \to E$. We require quasi-uniformity with respect to a mesh parameter h_j; i.e., we assume that all affine mappings $T_E \colon \hat{E} \to E$ satisfy $\|T_E\| \simeq \|T_E^{-1}\|^{-1} \simeq h_j$.

We assume that for all element transformations, $\phi_E \colon \bar{E} \to \bar{E}^\phi \subset \bar{\Omega}$ exist such that $\phi_E \in C^2(\bar{E})^d$, $\det(\phi_E) > 0$, satisfying $\phi_E = \phi_{E'}$ on $\bar{E} \cap \bar{E}'$ and

$$(5) \qquad \|D^k\phi_E\|_\infty \approxeq 1, \qquad \|D^k\phi_E^{-1}\|_\infty \approxeq 1, \qquad k = 0, 1, 2.$$

This implies

$$(6) \qquad \|v\|_{k,E^\phi} \approxeq \|v \circ \phi_E\|_{k,E}, \qquad v \in H^k(E^\phi)^m,\ k = 0, 1, 2$$

in terms of standard Sobolev norms $\|\cdot\|_{k,\Omega} = \|\cdot\|_{H^k(\Omega)}$ (see, e.g., [19, Th. 4.1]). In addition, we need

$$(7) \qquad \phi_E(P) = P \text{ for all element vertices } P \in \bar{E},\ E \in \mathcal{E}_j.$$

Let $\phi_j = (\phi_E)_{E \in \mathcal{E}_j} \in C^{0,1}(\bar{\Omega}_j)^d$ denote the global map resulting from a combination of the ϕ_E. We require that $\phi_j : \Omega_j \to \Omega$ be bijective. From (5) and (7), we find that the approximation is improving by

$$(8) \qquad \|\,\mathrm{id} - \phi_j\|_\infty \lesssim h_j^2 \text{ and } \|\,\mathrm{id} - \phi_j^{-1}\|_\infty \lesssim h_j^2$$

and

$$(9) \qquad \|\,\mathbf{I} - D\phi_j\|_\infty \lesssim h_j \text{ and } \|\,\mathbf{I} - D\phi_j^{-1}\|_\infty \lesssim h_j.$$

2.2. **The model problem.** We consider the bilinear form

$$a(v,w) = \int_\Omega \underline{a}\ \nabla v \cdot \nabla w\, dx, \qquad v,w \in H^1(\Omega)^m,$$

where $\underline{a} \in C^{0,1}(\mathbf{R}^d)^{md\times md}$ is a symmetric and positive semidefinite matrix function such that the bilinear form a is uniformly elliptic in the space

$$H_1 = \{v \in H^1(\Omega)^m \mid v = 0 \text{ on } \Gamma\} \tag{10}$$

(equipped with the norm $\|\cdot\|_{1,\Omega}$), where $\Gamma \subset \partial\Omega$ has positive measure.

Applying the abstract setting from Section 1.1 to $H_1 \subset H := L^2(\Omega)^m$, this defines an unbounded self-adjoint strictly positive operator A and a scale of Hilbert spaces $H_{2\alpha} = \mathrm{dom}(A^\alpha)$.

From the assumptions on the bilinear form a and $H_1 = \mathrm{dom}(A^{1/2})$ the norm equivalence

$$\|v\|_1 = \sqrt{a(v,v)} \ \eqsim\ \|v\|_{1,\Omega}, \qquad v \in H_1,$$

follows and therefore for $\alpha \in [0,1]$ (by interpolation)

$$\|v\|_\alpha \ \eqsim\ \|v\|_{\alpha,\Omega}, \qquad v \in H_\alpha,$$

where the norm on the right-hand side denotes the norm in

$$H^\alpha(\Omega)^m := [L^2(\Omega)^m, H^1(\Omega)^m]_\alpha$$

(see, e.g., [8, Th. 12.2.3] for the equivalence to the classical definition).

We have to consider two different Hilbert scales: the operator-induced scale H_α which we use in the multigrid analysis and the Sobolev scale

$$H^{1+\alpha}(\Omega)^m := [H^1(\Omega)^m, H^2(\Omega)^m]_\alpha, \qquad \alpha \in [0,1],$$

in the analysis of the interpolation error and the consistency error. Thus, we assume in addition that for some regularity parameter $\beta \in (0,1]$ the relation

$$[H_1, H_2]_\alpha = [H_1, H^2(\Omega)^m \cap H_2]_\alpha, \qquad \alpha \in [0,\beta], \tag{R}$$

holds. In this form, the regularity is required in Corollary 7 below. Note that this is just another way for stating the usual regularity assumption for elliptic boundary problems.

2.3. **The finite element setting.** For the boundary, we assume additionally that $\Gamma_j := \phi_j^{-1}(\Gamma) \subset \partial\Omega_j$ can be represented as a union of element sides. Let

$$M_j \subset \{v_j \in H^1(\Omega_j)^m \mid v_j = 0 \text{ on } \Gamma_j\}$$

be a conforming finite element space on the polygonal domain Ω_j, so that—according to the mesh requirements—an inverse inequality

$$\|v_j\|_{1,E} \lesssim h_j^{-1}\ \|v_j\|_{0,E}, \qquad v_j \in M_j,\ E \in \mathcal{E}_j, \tag{I}$$

holds, and the nodal interpolation operator $\psi_j \colon C^0(\bar\Omega_j)^m \to H^1(\Omega_j)$ (obtained by pointwise evaluation at the nodal points) satisfies

$$\|v - \psi_j v\|_{2-k,E} \lesssim h_j^k\ \|v\|_{2,E}, \qquad v \in H^2(E)^m,\ k = 1,2,\ E \in \mathcal{E}_j, \tag{Q}$$

and $\psi_j(C^0(\Omega_j)^m \cap H_1) = M_j$. This applies, e.g., to all conforming finite elements of Lagrange type evaluated at their nodal points.

The approximated bilinear form on Ω_j is denoted by

$$a_j(v,w) = \int_{\Omega_j} \underline{a}\, \nabla v \cdot \nabla w \, dx, \qquad v,w \in H^1(\Omega_j)^m, \tag{11}$$

and we define the inner product on Ω_j by

$$(v,w)_j = \int_{\Omega_j} v \cdot w \, dx, \qquad v,w \in L^2(\Omega_j)^m .$$

This defines the discrete norm scale (2) which satisfies

$$\|v_j\|_{j,\alpha} \approx \|v_j\|_{\alpha,\Omega_j}, \qquad v_j \in M_j, \alpha \in [0,1]. \tag{12}$$

2.4. **Construction of the interpolation operator.** Based on the piecewise smooth mapping $\phi_j \in C^{0,1}(\bar{\Omega}_j)^d$ introduced in Section 2.1, we define the corresponding comparison mapping $\varphi_j \colon L^2(\Omega_j) \to L^2(\Omega)$ by $\varphi_j v_j = v_j \circ \phi_j^{-1}$; this yields a comparison space

$$M_j^{\varphi} = \{w \in H^1(\Omega)^m \mid w \circ \phi_j \in M_j\}$$

in the sense of [7]. The interpolation operator π_j (required for the application of the criteria in Section 1.9) is obtained by the following theorem.

Theorem 4. *An interpolation operator $\pi_j \colon L^2(\Omega)^m \to M_j$ exists satisfying the identity $\pi_j \circ \varphi_j = \mathrm{id}_{M_j}$ together with the following estimates:*

$$\|\pi_j w\|_{k,\Omega_j} \lesssim \|w\|_{k,\Omega}, \quad w \in H^k(\Omega)^m \cap H_1,\ k=0,1, \tag{13}$$

$$\|w \circ \phi_j - \pi_j w\|_{0,\Omega_j} \lesssim h_j \|w\|_{1,\Omega}, \quad w \in H_1, \tag{14}$$

$$\|w \circ \phi_j - \pi_j w\|_{2-k,\Omega_j} \lesssim h_j^k \|w\|_{2,\Omega}, \quad w \in H^2(\Omega)^m \cap H_1,\ k=1,2. \tag{15}$$

Proof. Let $Q_j \colon L^2(\Omega_j)^m \to M_j$ be the orthogonal projection onto M_j; i.e.,

$$(Q_j v, w_j)_{0,\Omega_j} = (v, w_j)_{0,\Omega_j}, \qquad v \in L^2(\Omega_j)^m, w_j \in M_j.$$

Defining the interpolation π_j by $\pi_j v = Q_j(v \circ \phi_j)$ for $v \in L^2(\Omega)^m$ gives by construction $(\pi_j \circ \varphi_j) v_j = v_j$ for all $v_j \in M_j$. We have $\|Q_j\|_{1,\Omega_j} \lesssim 1$ (cf. [9]), and together with $\|Q_j\|_{0,\Omega_j} = 1$ and (6) we obtain (13).

Now we prove (15). The case $k=2$ follows from (6) and (Q) by

$$\begin{aligned} \|w \circ \phi_j - \pi_j w\|_{0,\Omega_j}^2 &= \|w \circ \phi_j - Q_j(w \circ \phi_j)\|_{0,\Omega_j}^2 = \inf_{v_j \in M_j} \|w \circ \phi_j - v_j\|_{0,\Omega_j}^2 \\ &\le \|w \circ \phi_j - \psi_j(w \circ \phi_j)\|_{0,\Omega_j}^2 \\ &\lesssim h_j^4 \sum_{E \in \mathcal{E}_j} \|w \circ \phi_j\|_{2,E}^2 \approx h_j^4 \sum_{E \in \mathcal{E}_j} \|w\|_{2,E^\phi}^2, \end{aligned} \tag{16}$$

and the case $k = 1$ is obtained using (Q), (I) and (16) in

$$\begin{aligned}
\|w \circ \phi_j - \pi_j w\|_{1,\Omega_j} &\approx \|w \circ \phi_j - Q_j(w \circ \phi_j)\|_{1,\Omega_j} \\
&\lesssim \|w \circ \phi_j - \psi_j(w \circ \phi_j)\|_{1,\Omega_j} \\
&\qquad + \|\psi_j(w \circ \phi_j) - Q_j(w \circ \phi_j)\|_{1,\Omega_j} \\
&\lesssim \|w \circ \phi_j - \psi_j(w \circ \phi_j)\|_{1,\Omega_j} \\
&\qquad + h_j^{-1}\|\psi_j(w \circ \phi_j) - Q_j(w \circ \phi_j)\|_{0,\Omega_j} \\
&\lesssim \|w \circ \phi_j - \psi_j(w \circ \phi_j)\|_{1,\Omega_j} \\
&\qquad + h_j^{-1}\|\psi_j(w \circ \phi_j) - w \circ \phi_j\|_{0,\Omega_j} \\
&\qquad + h_j^{-1}\|w \circ \phi_j - Q_j(w \circ \phi_j)\|_{0,\Omega_j} \\
&\lesssim h_j\Big(\sum_{E \in \mathcal{E}_j} \|w \circ \phi_j\|_{2,E}^2\Big)^{1/2} \approx h_j\Big(\sum_{E \in \mathcal{E}_j} \|w\|_{2,E^\phi}^2\Big)^{1/2}.
\end{aligned}$$

In the same way, we obtain (14) from

$$\begin{aligned}
\|w \circ \phi_j - \pi_j w\|_{0,\Omega_j} &= \|w \circ \phi_j - Q_j(w \circ \phi_j)\|_{0,\Omega_j} \\
&\lesssim h_j \, \|w \circ \phi_j\|_{1,\Omega_j} \approx h_j \, \|w\|_{1,\Omega}. \quad \square
\end{aligned}$$

Combining Theorem 4 and (6), we directly obtain the following corollary.

Corollary 5. *For $\pi_j^\varphi : L^2(\Omega)^m \to M_j^\varphi$ defined by $\pi_j^\varphi v = \varphi_j \pi_j v$, we have*

$$\begin{aligned}
\|\pi_j^\varphi w\|_{1,\Omega} &\lesssim \|w\|_{1,\Omega}, && w \in H_1, \\
\|w - \pi_j^\varphi w\|_{0,\Omega} &\lesssim h_j \, \|w\|_{1,\Omega}, && w \in H_1, \\
\|w - \pi_j^\varphi w\|_{2-k,\Omega} &\lesssim h_j^k \, \|w\|_{2,\Omega}, && w \in H^2(\Omega)^m \cap H_1, \ k = 1, 2.
\end{aligned}$$

2.5. **Consistency error.** The main result of Section 2 is the following theorem which provides a bound for the consistency error.

Theorem 6. *We have for $v \in H_k \cap H^k(\Omega)^m$, $k = 1, 2$, and $w \in H_2 \cap H^2(\Omega)^m$*

$$|a(v, w) - a_j(\pi_j v, \pi_j w)| \lesssim h_j^k \, \|v\|_{k,\Omega} \|w\|_{2,\Omega}. \tag{17}$$

Before we prove the theorem, we formulate a direct consequence in fractional spaces because in applications without full regularity the consistency and approximation assumptions (C) and (D) in Section 1.9 are required for an intermediate space.

Corollary 7. *We have for $v \in H_{1+\alpha}$ and $w \in H_{1+\beta}$*

$$|a(v, w) - a_j(\pi_j v, \pi_j w)| \lesssim h_j^{\alpha+\beta} \, \|v\|_{1+\alpha,\Omega} \|w\|_{1+\beta,\Omega}, \qquad \alpha \in \{0, \beta\}, \tag{18}$$

where β is the regularity parameter.

Proof. We obtain the assertion in $[H_1, H^2(\Omega)^m \cap H_2]_\beta$ by interpolation of the bilinear form $a(v, w) - a_j(\pi_j v, \pi_j w)$; see [16, Section 1.19.5]. Thus, the assertion follows directly from the regularity assumption (R). $\square$

2.6. **The energy estimate.** The estimate (17) for $k = 1$ can be proved in two steps:

$$|a(v,w) - a_j(\pi_j v, \pi_j w)| \le |a(v,w) - a(\pi_j^\varphi v, \pi_j^\varphi w)| + |a(\pi_j^\varphi v, \pi_j^\varphi w) - a_j(\pi_j v, \pi_j w)|.$$

The bound for the first term is a simple consequence of Corollary 5.

Lemma 8. *We have for $v \in H_1$ and $w \in H_2 \cap H^2(\Omega)^m$*

$$|a(v,w) - a(\pi_j^\varphi v, \pi_j^\varphi w)| \lesssim h_j \ \|v\|_{1,\Omega}\|w\|_{2,\Omega}. \tag{19}$$

Proof. Integration by parts gives for $v \in H_1$ and $w \in H_2 \cap H^2(\Omega)^m$

$$|a(v,w)| \lesssim \|v\|_{0,\Omega}\|w\|_{2,\Omega} + \int_{\partial\Omega} v \cdot (\underline{a}\nabla w) \cdot n \, d\sigma \lesssim \|v\|_{0,\Omega}\|w\|_{2,\Omega} \tag{20}$$

due to the boundary conditions included into the spaces H_1 and H_2; from

$$\begin{aligned} |a(v,w) - a(\pi_j^\varphi v, \pi_j^\varphi w)| &= |a(v - \pi_j^\varphi v, w) + a(\pi_j^\varphi v, w - \pi_j^\varphi w)| \\ &\lesssim \|v - \pi_j^\varphi v\|_{0,\Omega}\|w\|_{2,\Omega} + \ \|\pi_j^\varphi v\|_{1,\Omega}\|w - \pi_j^\varphi w\|_{1,\Omega} \\ &\lesssim h_j \ \|v\|_{1,\Omega}\|w\|_{2,\Omega}, \end{aligned}$$

we obtain (19) by combining (20) and Corollary 5. □

Now we obtain the energy estimate by combining Lemma 8 with the following result; cf. [10, Lem. 8]. Note that this part of the proof does not require boundary conditions.

Lemma 9. *We have for $v, w \in H^1(\Omega)^m$*

$$|a(\pi_j^\varphi v, \pi_j^\varphi w) - a_j(\pi_j v, \pi_j w)| \lesssim h_j \ \|v\|_{1,\Omega}\|w\|_{1,\Omega}.$$

Proof. Let $v_j = \pi_j v$ and $w_j = \pi_j w$. On each element E, we apply the chain rule to obtain

$$\begin{aligned} &\int_{E^\phi} \underline{a}\nabla(v_j \circ \phi_E^{-1}) \cdot \nabla(w_j \circ \phi_E^{-1}) \, dy - \int_E \underline{a}\nabla v_j \cdot \nabla w_j \, dx \\ &\quad = \int_E (\nabla v_j) \cdot \Big(|\det(D\phi_E)|(D\phi_E^{-1} \circ \phi_E)^T \ \underline{a} \ D\phi_E^{-1} \circ \phi_E - \underline{a} \Big) \cdot (\nabla w_j) \, dx \\ &\quad \le \|\nabla v_j\|_{0,E} \ \| \, |\det(D\phi_E)|(D\phi_E^{-1} \circ \phi_E)^T \ \underline{a} \ D\phi_E^{-1} \circ \phi_E - \underline{a}\|_\infty \ \|\nabla w_j\|_{0,E}. \end{aligned}$$

Since (9) gives

$$\| \, |\det(D\phi_E)|(D\phi_E^{-1} \circ \phi_E)^T \ \underline{a} \ (D\phi_E^{-1} \circ \phi_E) - \underline{a}\|_{\infty,E} \lesssim h_E,$$

we obtain the assertion by summing over all elements and applying the Schwarz inequality together with (13). □

2.7. **The dual estimate.** Now we prove Theorem 6 in the case $k = 2$. For this purpose, we consider a linear extension operator

$$\eta \colon H^2(\Omega)^m \to H^2(\mathbf{R}^d)^m;$$

i.e., $(\eta v)_{|\Omega} = v$ and

$$\|\eta w\|_{k,\mathbf{R}^d} \lesssim \|w\|_{k,\Omega}, \qquad w \in H^k(\Omega)^m, \ k = 1, 2 \tag{21}$$

(cf. [14, Th. VI.3.5]). In particular, we have for the nodal interpolation operator

$$\psi_j(\eta w) = \psi_j(w \circ \phi_j), \qquad w \in H^2(\Omega)^m. \tag{22}$$

The estimate (17) for $k = 2$ can be proved in two steps:

$$|a(v,w) - a_j(\pi_j v, \pi_j w)| \le |a(v,w) - a_j(\eta v, \eta w)| + |a_j(\eta v, \eta w) - a_j(\pi_j v, \pi_j w)|$$

combining the following lemmata.

Lemma 10. *For $w \in H^2(\Omega)^m$ and $k = 1, 2$, we have*

$$\|\eta w - \pi_j w\|_{2-k,\Omega_j} \lesssim h_j^k \ \|w\|_{2,\Omega}. \tag{23}$$

Proof. Using (22) and (Q) gives for $w \in H^2(\Omega)^m$ and $k = 1, 2$

$$\begin{aligned} \|\eta w - w \circ \phi_j\|_{2-k,E} &\le \|\eta w - \psi_j(\eta w)\|_{2-k,E} + \|\psi_j(w \circ \phi_j) - w \circ \phi_j\|_{2-k,E} \\ &\lesssim h_E^k \ \|\eta w\|_{2,E} + h_E^k \ \|w \circ \phi_j\|_{2,E}. \end{aligned} \tag{24}$$

Summing up the elements and applying (6) and (21) yields

$$\|\eta w - w \circ \phi_j\|_{2-k,\Omega_j} \lesssim h_j^k \ \|w\|_{2,\Omega}.$$

Together with (15), this gives the assertion. □

In the next step, we estimate the error which is introduced by the domain approximation. Therefore, we define the boundary homotopy

$$G_j : [0,1] \times \partial\Omega_j \to \mathbf{R}^d, \qquad (t,x) \mapsto (1-t)x + t\phi_j(x).$$

Lemma 11. *We have for $v \in H^1(\mathbf{R}^d)^m$ and $G \subset G_j([0,1] \times \partial\Omega_j)$*

$$\|v\|_{0,G} \lesssim h_j \ \|v\|_{1,\mathbf{R}^d}.$$

Proof. We obtain from the transformation theorem and the trace theorem

$$\|v\|_{0,G}^2 \le \|v\|_{0,G_j([0,1]\times\partial\Omega_j)}^2 = \int_0^1 \int_{\partial\Omega_j} |\det DG_j| \ |v(G_j(t,x))|^2 \, dx \, dt \lesssim h_j^2 \ \|v\|_{1,\mathbf{R}^d}^2 ,$$

since (8) gives $|\det DG_j| \lesssim h_j^2$. □

Lemma 12. *We have for $v \in H^k(\Omega)^m$, $k = 1, 2$, and $w \in H^2(\Omega)^m$*

$$|a(v,w) - a_j(\eta v, \eta w)| \lesssim h_j^k \ \|v\|_{k,\Omega} \|w\|_{2,\Omega}.$$

Proof. Let $\Omega \triangle \Omega_j := (\Omega \setminus \Omega_j) \cup (\Omega_j \setminus \Omega) \subset G_j([0,1] \times \partial\Omega_j)$. We have

$$\begin{aligned} |a(v,w) - a_j(\eta v, \eta w)| &\le \int_{\Omega\triangle\Omega_j} \underline{a} \nabla(\eta v) \cdot \nabla(\eta w) \, dx \\ &\lesssim \|\nabla(\eta v)\|_{0,\Omega\triangle\Omega_j} \|\nabla(\eta w)\|_{0,\Omega\triangle\Omega_j} \\ &\lesssim h_j^{k-1} \ \|\nabla(\eta v)\|_{k-1,\mathbf{R}^d} \ h_j \ \|\nabla(\eta w)\|_{1,\mathbf{R}^d} \\ &\lesssim h_j^k \ \|v\|_{k,\Omega} \|w\|_{2,\Omega} \end{aligned}$$

by applying Lemma 11 and (21). □

Finally, we state the dual estimate corresponding to Lemma 8.

Lemma 13. *For $v, w \in H_2 \cap H^2(\Omega)^m$, we have*

$$|a_j(\eta v, \eta w) - a_j(\pi_j v, \pi_j w)| \lesssim h_j^2 \ \|v\|_{2,\Omega} \|w\|_{2,\Omega}. \tag{25}$$

Proof. We write

$$\begin{aligned} a_j(\pi_j v, \pi_j w) - a_j(\eta v, \eta w) &= a_j(\eta v - \pi_j v, \eta w - \pi_j w) \\ &\quad - a_j(\eta v - \pi_j v, \eta w) - a_j(\eta v, \eta w - \pi_j w) \end{aligned}$$

and estimate the terms separately. Using ellipticity and (23), we can estimate the first term by

$$|a_j(\eta v - \pi_j v, \eta w - \pi_j w)| \lesssim \|\eta v - \pi_j v\|_{1,\Omega_j} \|\eta w - \pi_j w\|_{1,\Omega_j} \lesssim h_j^2 \ \|v\|_{2,\Omega} \|w\|_{2,\Omega}.$$

The other two terms are of the same form, so that it is sufficient to estimate the second one. Here, integration by parts yields

$$|a_j(\eta v - \pi_j v, \eta w)| \lesssim \|\eta v - \pi_j v\|_{0,\Omega_j} \|\eta w\|_{2,\Omega_j} + \Big| \int_{\partial\Omega_j} (\eta v - \pi_j v) \cdot g \, d\sigma \Big|$$

with $g = (\underline{a} \nabla \eta w) \cdot n$. Because of (23), only the boundary integral remains to be estimated. We achieve this by splitting $\partial\Omega_j$ into the Dirichlet boundary part Γ_j and the Neumann boundary part $\partial\Omega_j \setminus \Gamma_j$.

Let $G = \bigcup_{t\in[0,1]} G_{j,t}(\Gamma_j)$ be the stripe containing both Γ and Γ_j. From (8) it follows that $|x - G_{j,1}(x)| \lesssim h_j^2$ for $x \in \Gamma_j$. Since v vanishes on Γ, we obtain the Poincaré estimate

$$\|\eta v\|_{0,G} \lesssim h_j^2 \ \|\nabla(\eta v)\|_{0,G} \lesssim h_j^2 \ \|v\|_{1,\Omega} \,. \tag{26}$$

Integrating the identity

$$(\eta v)^2(x) = -\int_0^1 \frac{d}{dt} (\eta v)^2 (G_j(t,x)) \, dt, \qquad x \in \Gamma_j,$$

along lines connecting Γ and Γ_j yields

$$\|\eta v\|_{0,\Gamma_j}^2 \lesssim \|\eta v\|_{1,G} \|\eta v\|_{0,G} \lesssim h_j^2 \ \|\eta v\|_{1,G}^2$$

using (26). This gives

$$\|\eta v\|_{0,\Gamma_j} \lesssim h_j \ \|\eta v\|_{1,G} \,, \tag{27}$$

and analogously (by extending g to Ω_j)

$$\|g\|_{0,\partial\Omega_j\setminus\Gamma_j} \lesssim h_j \ \|g\|_{1,\Omega} \,. \tag{28}$$

Thus, we have for the Dirichlet part

$$\begin{aligned} \int_{\Gamma_j} (\eta v - \pi_j v) \cdot g \, d\sigma &= \int_{\Gamma_j} (\eta v) \cdot g \, d\sigma \lesssim \|\eta v\|_{0,\Gamma_j} \|g\|_{0,\Gamma_j} \\ &\lesssim h_j \ \|\eta v\|_{1,G} \|g\|_{1,\Omega_j} \lesssim h_j^2 \ \|v\|_{2,\Omega} \|w\|_{2,\Omega} \end{aligned}$$

applying (27), Lemma 11, and (21). Finally, we have for the Neumann part

$$\int_{\partial\Omega_j\setminus\Gamma_j} (\eta v - \pi_j v) \cdot g \, d\sigma \lesssim \|\eta v - \pi_j v\|_{1,\Omega_j} \|g\|_{0,\partial\Omega_j\setminus\Gamma_j} \lesssim h_j \|v\|_{2,\Omega} \ h_j \|w\|_{2,\Omega} \,,$$

where we used the trace theorem for $\|\eta v - \pi_j v\|_{0,\partial\Omega_j\setminus\Gamma_j}$, (23), and (28). $\square$

2.8. **An optimal a priori estimate.** Before we proceed with the multigrid analysis, we comment on optimal a priori estimates for polygonal approximations in the case of full regularity (see also [6]).

Theorem 14. *For $f \in L^2(\Omega)^m$ let $u \in H_1$ be the solution of*

$$a(u,v) = (f,v)_{0,\Omega}, \qquad v \in H_1.$$

For $f_j \in M_j$ let $u_j \in M_j$ be the solution of

$$a_j(u_j,v_j) = (f_j,v_j)_j, \qquad v_j \in M_j.$$

If the consistency error of the right-hand side can be bounded by

$$|(f,v)_{0,\Omega} - (f_j,\pi_j v)_j| \lesssim h_j^k \, \|f\|_{0,\Omega}\|v\|_{k,\Omega}, \qquad v \in H^k(\Omega)^m \cap H_k, \tag{29}$$

for $k = 0,1,2$, we have in the case of full regularity ($\beta = 1$)

$$\|u - u_j\|_{2-k,\Omega_j\cap\Omega} \lesssim \|\eta u - u_j\|_{2-k,\Omega_j} \lesssim h_j^k \, \|f\|_{0,\Omega}, \qquad k = 1,2.$$

Proof. We denote $u^* = \pi_j^* u_j$ and consider the splitting

$$\eta u - u_j = \eta(u - u^*) + (\eta u^* - \pi_j u^*) + (\pi_j u^* - u_j).$$

The first term is estimated by $\|\eta(u^* - u)\|_{2-k,\Omega_j} \lesssim \|u^* - u\|_{2-k,\Omega}$ and

$$\begin{aligned} \|u^* - u\|_{2-k,\Omega} \quad &\eqsim \quad \sup_{v\in H^k(\Omega)^m\cap H_k} \frac{|a(u^* - u, v)|}{\|v\|_{k,\Omega}} \\ &\eqsim \quad \sup_{v\in H^k(\Omega)^m\cap H_k} \frac{|(f_j,\pi_j v)_j - (f,v)_{0,\Omega}|}{\|v\|_{k,\Omega}} \lesssim h_j^k \, \|f\|_{0,\Omega} \end{aligned}$$

using duality in the first equation and (29) for $k = 1,2$. The second term is estimated with (23), (R) and (Π^*):

$$\|\eta u^* - \pi_j u^*\|_{2-k,\Omega_j} \lesssim h_j^k \, \|u^*\|_{2,\Omega} \eqsim h_j^k \, \|u^*\|_2 \lesssim h_j^k \, \|u_j\|_{j,2}.$$

The last term is estimated by (E) for $k = 1$, and (G) for $k = 2$, which gives

$$\|\pi_j u^* - u_j\|_{2-k,\Omega_j} \lesssim h_j^k \, \|u_j\|_{j,2}.$$

Now, the assertion follows from $\|u_j\|_{j,2} = \|f_j\|_j$ and

$$\begin{aligned} \|f_j\|_j \quad &= \quad \sup_{v_j\in M_j} \frac{|(f_j,v_j)_j|}{\|v_j\|_j} \eqsim \sup_{v_j\in M_j} \frac{|(f_j,\pi_j\varphi_j v_j)_j|}{\|\varphi_j v_j\|_{0,\Omega}} \\ &\lesssim \quad \sup_{v_j\in M_j} \frac{|(f_j,\pi_j\varphi_j v_j)_j - (f,\varphi_j v_j)_{0,\Omega}|}{\|\varphi_j v_j\|_{0,\Omega}} + \sup_{v_j\in M_j} \frac{|(f,\varphi_j v_j)_{0,\Omega}|}{\|\varphi_j v_j\|_{0,\Omega}} \lesssim \|f\|_{0,\Omega} \end{aligned}$$

using (6) and (29) for $k = 0$. □

2.9. **Uniform refinement on domains with curved boundaries.** The standard uniform refinement procedure on polygonal domains has to be enhanced by an additional step for curved boundaries (for a realization in the software system *UG*, see [3]). In the first step, by uniform decomposition of all elements $E \in \mathcal{E}_{j-1}$ into 2^d elements, we obtain $\tilde{\mathcal{E}}_j$ with the corresponding finite element space $\tilde{M}_j \subset H^1(\Omega_{j-1})^m$. Then we obtain $\mathcal{E}_j$ by moving all element vertices $\tilde{P}$ in $\tilde{\mathcal{E}}_j$ with $\tilde{P} \in \partial\Omega_{j-1} \setminus \partial\Omega$ onto the boundary $\partial\Omega$ by $P = \phi_{j-1}(\tilde{P})$. This procedure transforms an element $\tilde{E} \in \tilde{\mathcal{E}}_j$ into an element $E \in \mathcal{E}_j$. Combining the corresponding transformations from the reference element $T_{\tilde{E}} \colon \hat{E} \to \tilde{E}$ and $T_E \colon \hat{E} \to E$

gives a mapping $S_E := T_E \circ T_{\tilde{E}}^{-1} \colon \tilde{E} \to E$ which can be combined to a piecewise smooth mapping $S_j \in C^{0,1}(\Omega_{j-1}, \Omega_j)$; this gives the finite element space $M_j = \{v \in H^1(\Omega_j)^m \mid v \circ S_j \in \tilde{M}_j\}$.

We require that the assumptions of Section 2.1 hold uniformly for all meshes which are generated by iterating this procedure, and we assume

$$\|v \circ S_j\|_{k,\tilde{E}} \approxeq \|v\|_{k,E}, \qquad v \in H^k(E)^m,\ E = S_j\tilde{E},\ \tilde{E} \in \tilde{\mathcal{E}}_j,\ k = 0,1,2. \tag{30}$$

Note that this implies a sufficiently fine resolution of the coarsest mesh Ω_0.

3. Application to linear elasticity

We apply the results of the previous section to linear elasticity. Then, we have $m = d$, and the bilinear form is defined by

$$a(u,v) = \int_\Omega \mathbf{C}\varepsilon\, v \cdot \varepsilon\, w\, dx,$$

where $\varepsilon\, v = \frac{1}{2}\big(\nabla v + (\nabla v)^T\big)$ and $\mathbf{C}\varepsilon\, v = 2\mu\varepsilon\, v + \lambda \operatorname{div} v\, \mathbf{I}$ (with positive Lamé constants λ, μ). Since the Dirichlet boundary Γ has nonzero measure, Korn's inequality yields

$$\|v\|_\alpha \approxeq \|v\|_{\alpha,\Omega}, \qquad v \in H_\alpha \subset H^\alpha(\Omega)^d,\ \alpha \in [0,1]. \tag{31}$$

Furthermore we require that the regularity assumption (R) holds for some $\beta \in (0,1]$, which implies the norm equivalence (31) for $\alpha \in (1, 1+\beta]$ as well.

3.1. **Conforming finite elements.** We apply the setting in Section 2 to the standard finite element space $M_j \subset H^1(\Omega_j)^d$ of piecewise (bi-/tri-) linear functions satisfying $v_j = 0$ on Γ_j, and we check all requirements for the approximation property in the case of uniform refinement as it is described in Section 2.9.

Scaling. The refinement procedure and (5) give $2h_{j-1} \approxeq h_j$, $j = 1, ..., J$. On the other hand, the ellipticity and boundedness of the bilinear form a together with the quasi-uniformity of the mesh give $\lambda_j \approxeq h_j^{-2}$; this implies (S).

Interpolation. The existence of an appropriate interpolation satisfying (Π) is shown in Theorem 4.

Prolongation. Let $\tilde{I}_j \colon M_{j-1} \to \tilde{M}_j$, $\tilde{I}_j v_{j-1} = v_{j-1}$, be the standard conforming prolongation; this defines the (nonnested) prolongation $I_j \colon M_{j-1} \to M_j$ by $I_j v_{j-1} = \tilde{I}_j v_{j-1} \circ S_j^{-1} = v_{j-1} \circ S_j^{-1}$.

Lemma 15. *We have for $k = 1, 2$*

$$\|\pi_j v - I_j \pi_{j-1} v\|_{2-k,\Omega_j} \lesssim h_{j-1}^k \|v\|_{2,\Omega}, \qquad v \in H^2(\Omega)^d.$$

Proof. We have the identity

$$\pi_j v - I_j \pi_{j-1} v = \big(\pi_j v - v \circ \phi_j\big) + \big(v \circ \phi_j - (\pi_{j-1} v) \circ S_j^{-1}\big).$$

The first term is estimated in (15). The second term is decomposed as

$$\begin{aligned} v \circ \phi_j \circ S_j - \pi_{j-1} v \quad = \quad & (v \circ \phi_j \circ S_j - \psi_{j-1}(v \circ \phi_j \circ S_j)) \\ & + (\psi_{j-1}(v \circ \phi_{j-1}) - v \circ \phi_{j-1}) + (v \circ \phi_{j-1} - \pi_{j-1} v)\,. \end{aligned}$$

Here, the first summand can be estimated as desired using (Q), (6) and (30), the second using (Q) and (6), and the third using (15). $\square$

From (13) and (30), we obtain L^2-stability of $\pi_j v - I_j \pi_{j-1}$. Thus, (P) follows by interpolation. Finally, (B) is a direct consequence of (6).

Consistency. For φ_j defined in Section 2.4, we obtain (Φ) from (6). Now, the consistency assumptions (C) and (D) follow from Corollary 7.

3.2. **Stabilized finite elements.** Now we apply our criteria to Q_1/P_0-elements [13, Chap. 4.4] which are commonly used in engineering applications for reducing locking effects (see, e.g., [2]). Although this discretization is not fully stable for the Stokes problem, it improves the quality of finite element solutions for problems in elasticity and plasticity; cf. [17].

By static condensation, the Q_1/P_0-discretization corresponds to using M_j with the stabilized bilinear form

$$\bar{a}_j(v,w) = \int_{\Omega_j} \mathbf{C}\bar{\varepsilon}\, v \cdot \bar{\varepsilon}\, w \, dx, \qquad v, w \in H^1(\Omega_j)^d,$$

where the so-called B-bar operator is defined by

$$\bar{\varepsilon}\, v = \varepsilon\, v - \frac{1}{3} \operatorname{div} v\, \mathbf{I} + \frac{1}{3} \overline{\operatorname{div}}\, v\, \mathbf{I}\,, \quad \overline{\operatorname{div}}\, v|_E = \frac{1}{|E|} \int_E \operatorname{div} v \, dx, \quad v \in H^1(\Omega_j)^d,$$

on every quadrilateral/hexahedron $E \in \mathcal{E}_j$.

From $\bar{a}(v,v) \eqsim a(v,v) \eqsim \|\varepsilon\, v\|^2_{0,\Omega_j} \eqsim \|v\|^2_{1,\Omega_j}$ in M_j (cf. [18]), we obtain the norm equivalence

$$\|\bar{A}_j^{\alpha/2} v_j\|_j \eqsim \|A_j^{\alpha/2} v_j\|_j \eqsim \|v_j\|_{\alpha,\Omega_j}, \qquad v_j \in M_j,\ \alpha \in [0,1]. \tag{32}$$

Thus, (Π), (P), (B), and (Φ) carry over from the previous section, and it remains to show (C) and (D). Following [18, Lem. 2.5 and 2.6], we have for $v, w \in H^1(\Omega)^d$

$$\begin{aligned}
&(\overline{\operatorname{div}}\, v, \overline{\operatorname{div}}\, w)_{0,\Omega} - (\operatorname{div} v, \operatorname{div} w)_{0,\Omega} \\
&\qquad = (\overline{\operatorname{div}}\, v, \overline{\operatorname{div}}\, w)_{0,\Omega} - (\operatorname{div} v, \operatorname{div} w)_{0,\Omega} + (\overline{\operatorname{div}}\, v, \operatorname{div} w - \overline{\operatorname{div}}\, w)_{0,\Omega} \\
&\qquad = (\overline{\operatorname{div}}\, v - \operatorname{div} v, \operatorname{div} w)_{0,\Omega} = (\overline{\operatorname{div}}\, v - \operatorname{div} v, \operatorname{div} w - \overline{\operatorname{div}}\, w)_{0,\Omega}.
\end{aligned}$$

This gives for $v \in H^k(\Omega)^d$, $w \in H^2(\Omega)^d$, $k = 1, 2$,

$$\begin{aligned}
|\bar{a}_j(\pi_j v, \pi_j w) - a_j(\pi_j v, \pi_j w)| &\eqsim |(\overline{\operatorname{div}}\, \pi_j v, \overline{\operatorname{div}}\, \pi_j w)_{0,\Omega} - (\operatorname{div} \pi_j v, \operatorname{div} \pi_j w)_{0,\Omega}| \\
&\lesssim h_j^k\, \|v\|_{k,\Omega_j} \|w\|_{2,\Omega_j}.
\end{aligned} \tag{33}$$

The consistency error can be estimated in the two steps

$$\begin{aligned}
|\bar{a}_j(\pi_j v, \pi_j w) - a(v,w)| \quad \le \quad & |\bar{a}_j(\pi_j v, \pi_j w) - a_j(\pi_j v, \pi_j w)| \\
& + |a_j(\pi_j v, \pi_j w) - a(v,w)|
\end{aligned}$$

which can be estimated by (33) and Corollary 7; this yields (C) and (D).

3.3. **Nonconforming finite elements.** Finally, we show how one can combine our results for curved boundaries with the analysis in [7] for nonconforming elements $\hat{M}_j \subset L^2(\Omega_j)^2$.

We consider nonconforming P_1-elements on triangles with the bilinear form

$$a_j(v,w) = \sum_{E\in\mathcal{E}_j} \int_E \mathbf{C}\varepsilon\, v \cdot \varepsilon\, w\, dx, \qquad v,w \in H_0^1(\Omega_j)^2 + \hat{M}_j,$$

and Dirichlet boundary conditions on $\Gamma_j = \partial\Omega_j$; cf. [8, Chap. 9.4]. Following [7, Sect. 5], one can construct $\hat{\pi}_j \colon H_0^1(\Omega_j)^2 \to \hat{M}_j$, $v \longmapsto \hat{v}_j$, by averaging with

$$\int_e \hat{v}_j(x)\, ds = \int_e v(x)\, ds \qquad \text{for every edge } e \subset \partial E,\ v \in H_0^1(\Omega)^2.$$

The arguments from [7, Sect. 5] then show for $k = 1, 2$

$$(\hat{\Pi}) \quad \|v - \hat{\pi}_j v\|_{0,\Omega_j} + h_j\, \|v - \hat{\pi}_j v\|_{1,\mathcal{E}_j} \lesssim h_j^k\, \|v\|_{k,\Omega_j}, \qquad v \in H^k(\Omega_j)^2 \cap H_0^1(\Omega_j)^2$$

(with the norm $\|v\|_{1,\mathcal{E}_j}^2 = \sum_{E\in\mathcal{E}_j} \|v\|_{1,E}^2$).

In our application, the consistency error [7, (N-1) and (N-2)] is required in a more general form.

Lemma 16. *We have for $v \in H^k(\mathbf{R}^2)^2 \cap H_0^1(\Omega)^2$ and $w \in H^2(\mathbf{R}^2)^2 \cap H_0^1(\Omega)^2$*

$$|a_j(v - \hat{\pi}_j v, w)| \lesssim h_j^k\, \|v\|_{k,\mathbf{R}^2} \|w\|_{2,\mathbf{R}^2}, \qquad k = 1, 2.$$

Proof. We have

$$a_j(v - \hat{\pi}_j v, w) = -\int_{\Omega_j} (v - \hat{\pi}_j v)\, \mathrm{div}\ \mathbf{C}\varepsilon w + \sum_{E\in\mathcal{E}_j} \sum_{e\subset\partial E} \int_e (v - \hat{\pi}_j v)(\,\mathbf{C}\varepsilon w \cdot n)\, ds$$

(where the last sum runs over all edges). This is decomposed into

$$\begin{aligned} \sum_{e\subset\partial E} \int_e (v - \hat{\pi}_j v)(\,\mathbf{C}\varepsilon w \cdot n)\, ds \quad &= \quad \sum_{e\not\subset\partial\Omega_j} \int_e [v - \hat{\pi}_j v](\,\mathbf{C}\varepsilon w \cdot n)\, ds \\ &\quad + \sum_{e\subset\partial\Omega_j} \int_e (v - \hat{\pi}_j v)(\,\mathbf{C}\varepsilon w \cdot n)\, ds, \end{aligned}$$

where $[v - \hat{\pi}_j v]$ denotes the jump of $v - \hat{\pi}_j v$ in the direction of n. Following [7, formulae (5.27) and (5.34)], we have

$$\sum_{e\not\subset\partial\Omega_j} \int_e [v - \hat{\pi}_j v](\,\mathbf{C}\varepsilon w \cdot n)\, ds \lesssim h_j^k\, \|v\|_{k,\mathbf{R}^2} \|w\|_{2,\mathbf{R}^2},$$

and, since $v - \hat{\pi} v$ has average zero on every edge $e \subset \partial\Omega_j$, we can insert constants c_e such that

$$\begin{aligned} \sum_{e\subset\partial\Omega_j} \int_e (v - \hat{\pi}_j v)(\,\mathbf{C}\varepsilon w \cdot n)\, ds \quad &= \quad \sum_{e\subset\partial\Omega_j} \int_e (v - \hat{\pi}_j v)(\,\mathbf{C}\varepsilon w \cdot n - c_e)\, ds \\ &\lesssim \quad h_j^{k-\frac{1}{2}}\, \|v\|_{k,\mathbf{R}^2}\, h_j^{\frac{1}{2}} \|w\|_{2,\mathbf{R}^2}. \end{aligned}$$

□

Now, we consider the extension to curved boundaries within two steps: first from $\hat{M}_j$ to a conforming space in $H_0^1(\Omega_j)^2$, and then to $H_0^1(\Omega)^2$. For the first step, let $M_j \subset H_0^1(\Omega_j)^2$ be the space of conforming quadratic finite elements. We define an operator $\hat{\varphi}_j \colon \hat{M}_j \to M_j$, $\hat{w}_j \longmapsto w_j$ by $w_j(x) = 0$ for the nodal points on the boundary, by averaging

$$w_j(x) = \frac{1}{|\mathcal{E}_j(x)|} \sum_{E \in \mathcal{E}_j(x)} \hat{w}_j|_E(x), \qquad \mathcal{E}_j(x) = \{E \in \mathcal{E}_j \mid x \in \bar{E}\},$$

for all corner points which are not on the boundary, and finally by

$$w_j(\tfrac{1}{2}(x+y)) = \frac{1}{4}\Big(6\hat{w}_j(\tfrac{1}{2}(x+y)) - w_j(x) - w_j(y)\Big)$$

for the edge midpoint $\frac{1}{2}(x+y)$ of two corner points $x, y \in \bar{E}$ in order to preserve the edge mean value. This satisfies the identity $\hat{\pi}_j \circ \hat{\varphi}_j = \mathrm{id}_{\hat{M}_j}$ by construction, and we have the stability $\|\hat{\varphi}_j \hat{w}_j\|_{1,\Omega_j} \lesssim \|\hat{w}_j\|_{1,\mathcal{E}_j}$.

For the second step, following Section 2.4, we define operators $\pi_j \colon L^2(\Omega)^2 \to M_j$ and $\varphi_j \colon M_j \to H_0^1(\Omega)^2$ with $\pi_j \circ \varphi_j = \mathrm{id}_{M_j}$. Then, we can combine the operators to $\hat{\pi}_j \circ \pi_j \colon L^2(\Omega)^2 \to \hat{M}_j$ and $\varphi_j \circ \hat{\varphi}_j \colon \hat{M}_j \to H_0^1(\Omega)^2$ satisfying the identity $(\hat{\pi}_j \circ \pi_j) \circ (\varphi_j \circ \hat{\varphi}_j) = \mathrm{id}_{\hat{M}_j}$ and the stability $\|\varphi_j \hat{\varphi}_j \hat{w}_j\|_{1,\Omega_j} \lesssim \|\hat{w}_j\|_{1,\mathcal{E}_j}$.

Lemma 17. *We have for $v, w \in H^2(\mathbf{R}^2)^2 \cap H_0^1(\Omega)^2$*

$$|a_j(\pi_j v - \hat{\pi}_j \pi_j v, \pi_j w)| \lesssim h_j^2 \, \|v\|_{2,\mathbf{R}^2} \|w\|_{2,\mathbf{R}^2}. \tag{34}$$

Proof. Let $N_j \subset M_j$ be the space of conforming linear elements. Then, we have for the nodal interpolation $\psi_j^N \colon C^0(\bar{\Omega}_j)^m \to N_j$ the identity $\psi_j^N w = \hat{\pi}_j \psi_j^N w$, which gives

$$w - \hat{\pi}_j \pi_j w = w - \psi_j^N w + \hat{\pi}_j(\psi_j^N w - w + w - \pi_j w).$$

From norm equivalence and direct scaling arguments we obtain

$$\|\hat{\pi}_j w_j\|_{0,\Omega_j} \lesssim \|w_j\|_{0,\Omega_j} \text{ and } \|\nabla \hat{\pi}_j w_j\|_{0,\mathcal{E}_j} \lesssim \|\nabla w_j\|_{0,\Omega_j} \tag{35}$$

for $w_j \in M_j$, and combining with Lemma 10 and ($\hat{\Pi}$), this yields

$$\|\pi_j w - \hat{\pi}_j \pi_j w\|_{0,\Omega_j} \le \|\pi_j w - w\|_{0,\Omega_j} + \|w - \hat{\pi}_j \pi_j w\|_{0,\Omega_j} \lesssim h_j^2 \, \|w\|_{2,\mathbf{R}^2} \tag{36}$$

and $\|\nabla(\pi_j w - \hat{\pi}_j \pi_j w)\|_{0,\mathcal{E}_j} \lesssim h_j \, \|w\|_{2,\mathbf{R}^2}$. Since we have

$$\begin{aligned} a_j(\pi_j v - \hat{\pi}_j \pi_j v, \pi_j w) \;=\; & a_j(\pi_j v - \hat{\pi}_j \pi_j v, \pi_j w - w) + a_j(v - \hat{\pi}_j v, w) \\ & + a_j((\pi_j v - v) - \hat{\pi}_j(\pi_j v - v), w), \end{aligned}$$

the assertion follows from Lemma 10, (36), and Lemma 16 for $k = 2$ and for $k = 1$ by inserting $\pi_j v - v$. □

Corollary 18. *We have for $v \in H^k(\Omega)^2 \cap H_0^1(\Omega)^2$ and $w \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2$*

$$|a(v,w) - a_j(\hat{\pi}_j \pi_j v, \hat{\pi}_j \pi_j w)| \lesssim h_j^k \, \|v\|_{k,\Omega} \|w\|_{2,\Omega}, \qquad k = 1, 2.$$

Proof. We consider

$$\begin{aligned} a(v,w) - a_j(\hat{\pi}_j \pi_j v, \hat{\pi}_j \pi_j w) \;=\; & a(v,w) - a_j(\pi_j v, \pi_j w) \\ & - a_j(\pi_j v - \hat{\pi}_j \pi_j v, \pi_j w - \hat{\pi}_j \pi_j w) \\ & + a_j(\pi_j v - \hat{\pi}_j \pi_j v, \pi_j w) \\ & + a_j(\pi_j v, \pi_j w - \hat{\pi}_j \pi_j w). \end{aligned}$$

This proves the assertion by applying Theorem 6, (36), and Lemma 17 for $k = 2$ and applying Lemma 16 for $k = 1$ by inserting $\pi_j v$ (where we replace v, w by the extensions $\eta v, \eta w$, using $\pi_j v = \pi_j \eta v$). □

The refinement of $\hat{M}_{j-1}$ consists again of two steps: the corresponding nonconforming finite element space in $\tilde{\mathcal{E}}_j$ is denoted by $\bar{M}_j \subset L^2(\Omega_{j-1})$, and inserting the piecewise affine (multi-) linear mapping $S_j \colon \Omega_{j-1} \to \Omega_j$ defined in Section 2.9, we obtain $\hat{M}_j = \{\hat{v}_j \in L^2(\Omega_j)^2 \mid \hat{v}_j \circ S_j \in \bar{M}_j\}$.

The prolongation $\bar{I}_j \colon \hat{M}_{j-1} \to \bar{M}_j$ on Ω_{j-1} (constructed in [7]) satisfies

$$(\hat{B}) \qquad \|\bar{I}_j \hat{v}_{j-1}\|_{k,\Omega_{j-1}} \lesssim \|\hat{v}_{j-1}\|_{k,\Omega_{j-1}}, \qquad \hat{v}_{j-1} \in \hat{M}_{j-1},\ k = 0, 1$$

(following from [7, formula (5.38)] and the inverse inequality). Again, we define the prolongation $\hat{I}_j \colon \hat{M}_{j-1} \to \hat{M}_j$ by $\hat{I}_j v_{j-1} = (\bar{I}_j v_{j-1}) \circ S_j^{-1}$ (note that the evaluation of $\hat{I}_j$ does not require the computation of S_j^{-1}).

Lemma 19. *We have for $v \in H^k(\Omega)^2 \cap H_0^1(\Omega)^2$*

$$\|\hat{I}_j \hat{\pi}_{j-1} \pi_{j-1} v - \hat{\pi}_j \pi_j v\|_{0,\Omega_j} \lesssim h_j^k \ \|v\|_{k,\Omega}, \qquad k = 1, 2.$$

Proof. For $v \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2$ and $\tilde{v} = \eta v \in H^2(\mathbf{R}^2)^2 \cap H_0^1(\Omega)^2$ we have $\bar{I}_j \hat{\pi}_{j-1} \psi_{j-1} v = \psi_{j-1} v = \psi_{j-1} \tilde{v} = \psi_{j-1}(\tilde{v} \circ S_j)$ and

$$\begin{aligned}(\hat{I}_j \hat{\pi}_{j-1} \pi_{j-1} v - \hat{\pi}_j \pi_j v) \circ S_j &= \bar{I}_j \hat{\pi}_{j-1} \pi_{j-1} v - (\hat{\pi}_j \pi_j v) \circ S_j \\ &= \bar{I}_j \hat{\pi}_{j-1}(\pi_{j-1} v - \psi_{j-1} v) + \psi_{j-1}(\tilde{v} \circ S_j) - \tilde{v} \circ S_j + (\tilde{v} - \hat{\pi}_j \pi_j v) \circ S_j.\end{aligned}$$

This gives the assertion for $k = 2$ by inserting $(\hat{B})$, (Q), (36), (15), and (30). Since $\hat{I}_j \hat{\pi}_{j-1} \pi_{j-1} - \hat{\pi}_j \pi_j$ is stable in $L^2(\Omega)^2$, we obtain the case $k = 1$ by interpolation. □

The application of the lemmata (combined with suitable interpolation arguments) gives (C), (D) and (P) for the interpolation operator $\hat{\pi}_j \circ \pi_j$. The stability assumptions (B), (Π) and (Φ) as well as the scaling (S) are obvious on quasi-uniform meshes. Together, this proves all requirements for the approximation property.

Acknowledgments

The authors are very grateful to the anonymous referee for the extremely careful reading of the original manuscript and the revised version, and for many detailed and helpful comments.

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