MATHEMATICS OF COMPUTATION
Volume 73, Number 248, Pages 2019–2025
S 0025-5718(04)01661-8
Article electronically published on May 7, 2004

CANONICAL VECTOR HEIGHTS ON K3 SURFACES WITH PICARD NUMBER THREE— AN ARGUMENT FOR NONEXISTENCE

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Abstract. In this paper, we investigate a K3 surface with Picard number three and present evidence that strongly suggests a canonical vector height cannot exist on this surface.

1. Introduction and background

Let V be a K3 surface over a number field K. A vector height on V/K is a function

$$\mathbf{h} : V/K \to \mathrm{Pic}(V/K) \otimes \mathbb{R}$$

with the following two properties: (1) For any $\sigma \in \mathrm{Aut}(V/K)$,

$$\mathbf{h}(\sigma P) = \sigma_* \mathbf{h}(P) + \mathbf{O}(1),$$

where $\sigma_* = (\sigma^{-1})^*$ is the pushforward of σ and $\mathbf{O}(1)$ is a vector function with bounded components; and (2) for any Weil height $h_D(P)$ associated to a divisor D, we have

$$h_D(P) = \mathbf{h}(P) \cdot D + O(1).$$

Vector heights exist and are unique up to bounded vector functions [Ba1].

Given a basis $\mathcal{D} = \{D_1, \ldots, D_n\}$ for $\mathrm{Pic}(V/K) \otimes \mathbb{R}$, let $\mathcal{D}^* = \{D_1^*, \ldots, D_n^*\}$ be the *dual basis* defined by the property that $D_i \cdot D_j^* = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol (that is, the matrix $[\delta_{ij}]$ is the identity matrix). Let $J = [D_i \cdot D_j]$ be the intersection matrix with respect to the basis $\mathcal{D}$. Then J^{-1} is the intersection matrix with respect to the basis $\mathcal{D}^*$, and J is the change of basis matrix from the basis $\mathcal{D}^*$ to the basis $\mathcal{D}$.

Given Weil heights h_{D_i}, the function

$$\mathbf{h}(P) = \sum_{i=1}^{n} h_{D_i}(P) D_i^*$$

is a vector height.

We call a vector height $\widehat{\mathbf{h}}$ a *canonical vector height* if $\widehat{\mathbf{h}}$ is a vector height and for every $\sigma \in \mathrm{Aut}(V/K)$ and $P \in V/K$,

$$\widehat{\mathbf{h}}(\sigma P) = \sigma_* \widehat{\mathbf{h}}(P).$$

Received by the editor February 14, 2003.
2000 *Mathematics Subject Classification.* Primary 14G40, 11G50, 14J28.
Key words and phrases. K3 surfaces, canonical vector heights.

The existence of a canonical vector height makes it possible or easier to answer certain arithmetic questions. If V/K is an algebraic K3 surface over a number field K that has Picard number two and an infinite group of automorphisms $\mathrm{Aut}(V/K)$, then there exists a canonical vector height on V/K [Ba1]. The goal of this paper is to numerically verify (though not rigorously prove) that a canonical vector height does not exist on a certain K3 surface with Picard number three and thereby to give convincing evidence that they do not exist in general.

Suppose $\sigma \in \mathrm{Aut}(V/K)$ and that σ^* has a maximal real eigenvalue $\omega > 1$ with associated eigenvector $E \in \mathrm{Pic}(V) \otimes \mathbb{R}$. Silverman [S] defined the height

$$\widehat{h}_E(P) = \lim_{n\to\infty} \omega^{-n} h_E(\sigma^n P), \tag{1}$$

where h_E is a Weil height with respect to E. This height is canonical with respect to σ, since $\widehat{h}_E(\sigma P) = \omega \widehat{h}_E(P)$. Of particular use to us is the property that $\widehat{h}_E(P)$ is independent of the choice one makes for Weil height h_E.

Suppose now that there exists a canonical vector height $\widehat{\mathbf{h}}$ on V/K. Then the function $\widehat{\mathbf{h}}(P) \cdot E$ is a Weil height with respect to the divisor E, so

$$\begin{aligned}
\widehat{h}_E(P) &= \lim_{n\to\infty} \omega^{-n} \widehat{\mathbf{h}}(\sigma^n P) \cdot E \\
&= \lim_{n\to\infty} \omega^{-n} \sigma_*^n \widehat{\mathbf{h}}(P) \cdot E \\
&= \lim_{n\to\infty} \omega^{-n} \widehat{\mathbf{h}}(P) \cdot (\sigma^*)^n E \\
&= \lim_{n\to\infty} \omega^{-n} \widehat{\mathbf{h}}(P) \cdot \omega^n E \\
&= \widehat{\mathbf{h}}(P) \cdot E.
\end{aligned}$$

Thus, if we can calculate $\widehat{h}_E(P)$, then this will give us a linear equation for $\widehat{\mathbf{h}}(P)$.

Our idea for demonstrating that no such $\widehat{\mathbf{h}}$ can exist is to calculate $\widehat{h}_E(P)$ for enough σ so that we arrive at an inconsistent system of linear equations.

2. The example

We will look at a surface V defined by a $(2,2,2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{Q}$. Such surfaces have been studied by Wheler [Wh], Wang [Wa], Billard [Bi], and the author [Ba2]. The proofs of some of the following statements can be found in these sources. A $(2,2,2)$ form can be written in the form

$$F(X,Y,Z) = F_{00}(Y,Z)X_0^2 + F_{01}(Y,Z)X_0X_1 + F_{11}(Y,Z)X_1^2,$$

where $X = (X_0, X_1)$, etc., and the polynomials F_{ij} are $(2,2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1$. If the variety is nonsingular, then the surface V defined by $F(X,Y,Z) = 0$ is a K3 surface. Let

$$\begin{aligned}
p_1 : \quad & V \to \mathbb{P}^1 \times \mathbb{P}^1 \\
(X,Y,Z) &\mapsto (Y,Z)
\end{aligned}$$

be the projection onto the second two coordinates. Define p_2 and p_3 in a similar fashion. Generically, the projection p_1 defines a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ by the surface V. The exception is when there exists a point $(Y', Z') \in \mathbb{P}^1 \times \mathbb{P}^1$ such that we simultaneously have

$$F_{00}(Y',Z') = F_{01}(Y',Z') = F_{11}(Y',Z') = 0.$$

Then V includes the line (X, Y', Z'), which is a -2 curve on V. In such a case, the Picard number for V is at least 4. If each of p_1, p_2, and p_3 define a double cover everywhere, then V has Picard number three. Let $\pi_1(X, Y, Z) = X$ be the projection onto the first component, and define π_2 and π_3 similarly. Let H be a point in $\mathbb{P}^1$ and let D_i be the divisor class defined by $\pi_i^*(H)$. Then $\mathcal{D} = \{D_1, D_2, D_3\}$ is a basis for $\mathrm{Pic}(V)$ and the intersection matrix with respect to this basis is

$$J = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

If (X, Y, Z) is a point on V, then $p_1^{-1}(Y, Z) = \{(X, Y, Z), (X', Y, Z)\}$ and the map

$$\begin{aligned} \sigma_1 : V &\to V \\ (X, Y, Z) &\mapsto (X', Y, Z) \end{aligned}$$

is an automorphism of V. Explicitly,

$$X' = \begin{cases} [F_{01}(Y, Z)X_1 + F_{00}(Y, Z)X_0, -F_{00}(Y, Z)X_1] & \text{if this is in } \mathbb{P}^1 \\ [F_{01}(Y, Z)X_0 + F_{11}(Y, Z)X_1, -F_{11}(Y, Z)X_0] & \text{otherwise.} \end{cases}$$

The maps σ_2 and σ_3 can be defined similarly. Let $T_i = [\sigma_i^*]_{\mathcal{D}^*}$ be the the pullback of σ_i in the basis $\mathcal{D}^*$. Then

$$T_1 = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix}.$$

One can easily verify that $T_i^2 = 1$ and $T_i^t J^{-1} T_i = J^{-1}$.

We investigate the surface $V/\mathbb{Q}$ that is defined by $F(X, Y, Z) = 0$, where (in affine coordinates)

$$F([x, 1], [y, 1], [z, 1]) = (y^2 z^2 + y + 4)x^2 + (y^2 + z^2)x + (yz^2 + z + 2y^2).$$

It is fairly easy to check that $F(X, Y, Z) = 0$ is nonsingular over $\mathbb{Z}/2\mathbb{Z}$, so it is nonsingular over $\mathbb{Q}$, and only a little tedious to check that the projections p_i are everywhere double covers. Thus, the Picard number for V is 3, as desired.

The surface V includes the point $P_0 = [[0, 1], [-1, 1], [-1, 1]]$. With the exception of these three properties (smooth, Picard number three, and with a rational point), there is nothing special about our choice for V, and we presume that it is a random representative of the class.

Let h be the usual logarithmic height on $\mathbb{P}^1(\mathbb{Q})$. That is, if $X = [X_0, X_1] \in \mathbb{P}^1(\mathbb{Q})$, then we can choose $X_0, X_1 \in \mathbb{Z}$ with $\gcd(X_0, X_1) = 1$. We define $h(X) = \log(\max\{|X_0|, |X_1|\})$. This induces several Weil heights on V:

$$h_{D_i}(P) = h(\pi_i(P)),$$

and from this, we can define the vector height

$$\mathbf{h}(P) = \sum_{i=1}^{3} h_{D_i}(P) D_i^*.$$

The matrices T_1 and $T_1 T_2$ do not have any eigenvalues larger than one. (Interpreted as isomorphisms of the hyperbolic surface $\mathbf{x}^t J^{-1} \mathbf{x} = 1$, these are, respectively, a reflection and a parabolic translation.) Thus, we must look at a combination of three of these generating matrices before we will find one with

TABLE 1. Values of $\omega^{-n}h_E(\sigma^n P_0)$ for various $\sigma = \sigma_{ijk}$ and their associated eigenvectors $E = E_{kji}$, and various n.

n	$\omega^{-n}h_{E_{123}}(\sigma_{321}^n P_0)$	$\omega^{-n}h_{E_{231}}(\sigma_{132}^n P_0)$	$\omega^{-n}h_{E_{312}}(\sigma_{213}^n P_0)$	$\omega^{-n}h_{E_{213}}(\sigma_{312}^n P_0)$
1	.672800	.316293	.711937	1.007649
2	.664771	.325946	.772532	.975334
3	.665283	.325731	.770970	.972127
4	.665303	.325732	.771149	.972045
5	.665311	.325733	.771154	.972055

a positive eigenvalue. The matrix $T_1T_2T_3$ has an eigenvalue $\omega = \alpha^3$ and associated eigenvector $[E_{123}]_{\mathcal{D}^*} = [\alpha^2, \alpha, 1]$, where $\alpha = \frac{3+\sqrt{5}}{2}$. As an element of $\mathrm{Pic}(V) \otimes \mathbb{R}$, $E_{123} = \alpha^2 D_1^* + \alpha D_2^* + D_3^*$. Since in the following all expressions will be in the basis $\mathcal{D}^*$, let us drop the explicit reference to $\mathcal{D}^*$, so by $[E_{123}]$ we mean $[E_{123}]_{\mathcal{D}^*}$. Note that $T_1T_2T_3 = [\sigma_1^*\sigma_2^*\sigma_3^*]_{\mathcal{D}^*} = [(\sigma_3\sigma_2\sigma_1)^*]_{\mathcal{D}^*}$. We therefore choose $\sigma = \sigma_{321} = \sigma_3\sigma_2\sigma_1$. We set

$$h_{E_{123}}(P) = \mathbf{h}(P) \cdot E_{123}$$

and calculate

$$\omega^{-n}h_{E_{123}}(\sigma_{321}^n(P_0)).$$

These calculations are shown in Table 1. The table is rather short, since calculating this value for $n = 6$ involves integer arithmetic with million digit integers. Similar calculations are made for $\sigma_{132} = \sigma_1\sigma_3\sigma_2$, $\sigma_{213} = \sigma_2\sigma_1\sigma_3$, and $\sigma_{312} = \sigma_3\sigma_1\sigma_2$. For each of these automorphisms, the largest eigenvalue is again $\omega = \alpha^3$, and the associated eigenvectors are, respectively, $[E_{231}] = [1, \alpha^2, \alpha]$, $[E_{312}] = [\alpha, 1, \alpha^2]$, and $[E_{213}] = [\alpha, \alpha^2, 1]$.

Let A be the matrix whose rows are $[E_{123}]$, $[E_{231}]$, and $[E_{312}]$. Let $B = [B_1, B_2, B_3] = [\widehat{h}_{E_{123}}(P_0), \widehat{h}_{E_{231}}(P_0), \widehat{h}_{E_{312}}(P_0)]$. Suppose now that a canonical vector height $\widehat{\mathbf{h}}$ exists for V. Then,

$$\widehat{h}_E(P_0) = \widehat{\mathbf{h}}(P_0) \cdot E = E \cdot \widehat{\mathbf{h}}(P_0) = [E]^t J^{-1}[\widehat{\mathbf{h}}(P_0)],$$

so

$$\begin{aligned} AJ^{-1}[\widehat{\mathbf{h}}(P_0)] &= B, \\ [\widehat{\mathbf{h}}(P_0)] &= JA^{-1}B. \end{aligned}$$

The exact value of JA^{-1} is known; B is approximately $[.665311, .325733, .771154]$. This gives us the approximation $[\widehat{\mathbf{h}}(P_0)] \approx [.169405, .326915, .176779]$. We therefore have

$$\widehat{h}_{E_{213}}(P_0) = \widehat{\mathbf{h}}(P_0) \cdot E_{213} \approx .331699,$$

which is not very close to the value .972055 shown in Table 1.

From Table 1, it would appear that our estimate for the entries of B are accurate to $\pm .00002$. Since

$$\begin{aligned} \widehat{h}_{E_{213}}(P_0) &= E_{213} \cdot \widehat{\mathbf{h}}(P_0) \\ &= [E_{213}]^t J^{-1}(JA^{-1}B) \\ &= [E_{213}]^t A^{-1}B \\ &= .345492B_1 + .904508B_2 - .25B_3, \end{aligned}$$

we expect an error in our calculation of no more than $\pm.00003$. Since our two calculations differ significantly, we conclude that no such canonical vector height could exist on this K3 surface and therefore that it is unlikely that a canonical vector height should exist on any K3 surface with Picard number greater than 2, except for perhaps in some very special cases.

Remark. The referee noted that the eigenvalue ω is a quadratic unit, and wondered whether this is always the case. For $n = 2$ and 3, it is. More generally, ω is either a quadratic unit or a Salem number. Suppose J is the intersection matrix for some surface V. Then J has one positive eigenvalue and $n-1$ negative eigenvalues (by the Hodge index theorem). Thus the surface $\mathbf{x}^t J\mathbf{x} = \mathbf{x}\cdot\mathbf{x} = 0$ is a cone, and in particular, does not contain any planes. Suppose $T = \sigma^*$ for some $\sigma \in \mathrm{Aut}(V)$. Then $T^t JT = J$ and T has integer entries. If $\mathbf{v}$ is an eigenvector for T and $T\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{v}\cdot\mathbf{v} = T\mathbf{v}\cdot T\mathbf{v} = \lambda^2\mathbf{v}\cdot\mathbf{v}$. Hence, either $\lambda^2 = 1$ or $\mathbf{v}\cdot\mathbf{v} = 0$. Suppose T has three (possibly equal) real eigenvalues α, β, and γ, none equal to either ± 1. Let their associated eigenvectors be $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, respectively (where these vectors are linearly independent). Then $\mathbf{u}\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{v} = \mathbf{w}\cdot\mathbf{w} = 0$. Note that $\mathbf{u}\cdot\mathbf{v} \neq 0$, since if it did, then the surface $\mathbf{x}\cdot\mathbf{x} = 0$ would contain the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, and as noted earlier, this surface contains no planes. Let $\mathbf{y} = \mathbf{u}+\mathbf{v}$. Then $\mathbf{y}\cdot\mathbf{y} = 2\mathbf{u}\cdot\mathbf{v}$. But $\mathbf{y}\cdot\mathbf{y} = T\mathbf{y}\cdot T\mathbf{y} = 2\alpha\beta\mathbf{u}\cdot\mathbf{v}$. Hence $\alpha\beta = 1$. Similarly, $\alpha\gamma = 1$ and $\beta\gamma = 1$, which implies $\alpha = \beta = \gamma = 1$, a contradiction. Thus, there can be at most two real eigenvalues not equal to ± 1. Suppose now that T has an eigenvalue λ that is not real. Let its associated eigenvector be $\mathbf{v} = \mathbf{v}_R + i\mathbf{v}_I$ where $\mathbf{v}_R$ and $\mathbf{v}_I$ are real vectors. Then $\mathbf{v}_R$ and $\mathbf{v}_I$ are linearly independent. Since $\mathbf{v}\cdot\mathbf{v} = \lambda^2\mathbf{v}\cdot\mathbf{v}$, and $\lambda^2 \neq 1$, we get $\mathbf{v}\cdot\mathbf{v} = 0$, which implies $\mathbf{v}_R\cdot\mathbf{v}_R + \mathbf{v}_I\cdot\mathbf{v}_I = 0$ and $\mathbf{v}_R\cdot\mathbf{v}_I = 0$. Since $\bar{\mathbf{v}}\cdot\mathbf{v} = \bar{\lambda}\lambda\bar{\mathbf{v}}\cdot\mathbf{v}$, we get $|\lambda| = 1$ or $\mathbf{v}_R\cdot\mathbf{v}_R - \mathbf{v}_I\cdot\mathbf{v}_I = 0$. If we have the latter, then the plane spanned by $\mathbf{v}_R$ and $\mathbf{v}_I$ is in the cone $\mathbf{x}\cdot\mathbf{x} = 0$, which is a contradiction. Thus $|\lambda| = 1$. Finally, since T has integer entries, the minimal polynomial for ω divides the characteristic polynomial for T. Hence, ω is an algebraic integer, it has only one other real conjugate $\pm\omega^{-1}$ (since $\det T = \pm 1$), and all its other conjugates are complex with magnitude one. Such a number is either a quadratic unit or a Salem number. When $n = 2$ or 3, there can be no complex eigenvalues, so ω must be a quadratic unit. Geometrically (for $n = 3$), T is a translation or glide reflection that translates along the line with endpoints the associated eigenvectors.

3. Further analysis of the error in B

In the previous section, we stated that the estimates for the entries of B look to be accurate to $\pm.00002$. This was based on the observation that the sequence that converges to the canonical height $\widehat{h}_E$ converges geometrically and that the difference of the fourth and fifth iteration for our various calculations is no more than $\pm.00002$. Let us now present a more sophisticated argument.

Let σ be an automorphism of V whose pull back σ^* has a maximal real eigenvalue $\omega > 1$ with associated eigenvector E. Let h_E be a Weil height with respect to the divisor E. Then

$$h_E(\sigma P) = h_{\sigma^* E}(P) + O(1) = h_{\omega E}(P) + O(1) = \omega h_E(P) + O(1). \tag{2}$$

The function implied by the $O(1)$ is bounded independent of P. To make our argument completely rigorous, we would have to find explicit bounds for this error term. This can possibly be done, following the ideas presented by Call and Silverman in

TABLE 2. The quantity described in Eq. 5 for $\sigma = \sigma_{321}$ and various values of n and m.

$n\backslash m$	0	1	2	3	4
1	12.1				
2	11.3	-2.6			
3	11.3	-2.3	3.0		
4	11.3	-2.3	2.9	2.0	
5	11.3	-2.3	2.9	2.7	15.5

[C-S], but the extra effort is probably not worth it. Instead, we will use our data to get an idea of the range of the error term. Note that

$$\begin{aligned} h_E(\sigma^n(P)) &= \omega h_E(\sigma^{n-1}P) + O(1) \\ &= \omega^2 h_E(\sigma^{n-2}P) + \omega O(1) + O(1) \\ &= \omega^n h_E(P) + \left(\frac{\omega^n - 1}{\omega - 1}\right) O(1), \end{aligned} \tag{3}$$

where the bound on the function implied by $O(1)$ in (3) is the same as the bound for the function implied by the $O(1)$ in (2). Thus, for $n > m$, we get

$$\begin{aligned} &\omega^{-n} h_E(\sigma^n P) - \omega^{-m} h_E(\sigma^m P) \\ &\qquad = \omega^{-n}\left(\omega^{n-m} h_E(\sigma^m P) + \left(\frac{\omega^{n-m} - 1}{\omega - 1}\right) O(1)\right) - \omega^{-m} h_E(\sigma^m P) \\ &\qquad = \left(\frac{\omega^{-m} - \omega^{-n}}{\omega - 1}\right) O(1), \end{aligned} \tag{4}$$

where the bound on the function implied by the $O(1)$ is again the same as in (2).

Using the values in Table 1 and turning (4) around, we get data on the function implied by the $O(1)$ in (2). More precisely, if we set

$$x_n = x_n(\sigma) = \omega^{-n} h_E(\sigma^n P),$$

then for $n > m$,

$$\left(\frac{\omega - 1}{\omega^{-m} - \omega^{-n}}\right)(x_n - x_m) = O(1), \tag{5}$$

so the left-hand side in the above equation gives us some information about the function. We tabulate these values of $\sigma = \sigma_{321}$ in Table 2. Note that $x_0 = 0$, since $\mathbf{h}(P_0) = [0, 0, 0]$.

Similar calculations were made for σ_{132}, σ_{213}, and σ_{312}. The absolute values ranged from .09 to 19.5. Unfortunately, this gives us only a lower bound on the desired bound. Still, assuming an upper bound of 100, we get an error of

$$B_1 - x_5(\sigma_{321}) = \lim_{n\to\infty} x_n(\sigma_{321}) - x_5(\sigma_{321}) = \pm\left(\frac{\omega^{-5}}{\omega - 1}\right) 100 \approx \pm .000003,$$

which is a factor of ten better than the error we arrived at in the previous section.

4. Further evidence

The point P_0 has an interesting unexpected feature:

$$\sigma_2\sigma_3\sigma_2(P_0) = P_0,$$
$$\sigma_3\sigma_2\sigma_3(P_0) = P_0.$$

Thus, if $\widehat{\mathbf{h}}$ exists, then $\widehat{\mathbf{h}}(P_0)$ must be in the eigenspace associated to the eigenvalue 1 for $\sigma_2^*\sigma_3^*\sigma_2^*$ and in the eigenspace associated to the eigenvalue 1 for $\sigma_3^*\sigma_2^*\sigma_3^*$. The intersection is one dimensional, and we find

$$\widehat{\mathbf{h}}(P_0) = kD_2^* + kD_3^*$$

for some k. Since $\widehat{\mathbf{h}}(P_0) \cdot E = \widehat{h}_E(P_0)$, we find

$$k = .194135 \pm .000001$$

when we set $E = E_{123}$ and

$$k = .651466 \pm .000006$$

when we set $E = E_{321}$, again leading to a contradiction.

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