MATHEMATICS OF COMPUTATION
Volume 74, Number 249, Pages 263–278
S 0025-5718(04)01636-9
Article electronically published on January 27, 2004

IMPROVED METHODS AND STARTING VALUES TO SOLVE THE MATRIX EQUATIONS $X \pm A^*X^{-1}A = I$ ITERATIVELY

IVAN G. IVANOV, VEJDI I. HASANOV, AND FRANK UHLIG

ABSTRACT. The two matrix iterations $X_{k+1} = I \mp A^*X_k^{-1}A$ are known to converge linearly to a positive definite solution of the matrix equations $X \pm A^*X^{-1}A = I$, respectively, for known choices of X_0 and under certain restrictions on A. The convergence for previously suggested starting matrices X_0 is generally very slow. This paper explores different initial choices of X_0 in both iterations that depend on the extreme singular values of A and lead to much more rapid convergence. Further, the paper offers a new algorithm for solving the minus sign equation and explores mixed algorithms that use Newton's method in part.

1. Introduction

Solving the matrix equations $X + A^*X^{-1}A = I$ and $X - A^*X^{-1}A = I$ is a problem of practical importance. These two equations were studied in [1], [5], [6], [7]. They arise in control theory [1], [7], dynamic programming, and statistics, for example (see the references given in [15]). The second equation (with the minus sign) arises specifically in the analysis of stationary Gaussian reciprocal processes over a finite interval [7]. Finally, following [11], trying to solve special linear systems leads to solving nonlinear matrix equations of the above types as follows: For a linear system $Mx = f$ with $M = \begin{pmatrix} I & A \\ A^* & I \end{pmatrix}$ positive definite we can rewrite $M = \tilde{M} + \text{diag}[I - X, 0]$ for $\tilde{M} = \begin{pmatrix} X & A \\ A^* & I \end{pmatrix}$. Clearly

$$\tilde{M} = \begin{pmatrix} X & A \\ A^* & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ A^*X^{-1} & I \end{pmatrix} \begin{pmatrix} X & A \\ 0 & X \end{pmatrix}.$$

Such a decomposition of $\tilde{M}$ exists if and only if X is a positive definite solution of the matrix equation $X + A^*X^{-1}A = I$. Solving the linear system $\tilde{M}y = f$ is equivalent to solving two linear systems with a lower and upper block triangular system matrix. To compute the solution of $Mx = f$ from y, the Woodbury formula can be applied.

We write $B > 0$ ($B \geq 0$) if the matrix B is Hermitian positive definite (semidefinite). If $B - C$ is Hermitian positive definite (semidefinite), then we write

Received by the editor May 29, 2001 and, in revised form, May 7, 2003.
2000 *Mathematics Subject Classification.* Primary 65F10.
Key words and phrases. Matrix equation, positive definite solution, iterative method, Newton's method.
This work is partially supported by Shumen University under Grant #3/04.06.2001.

$B > C$ ($B \geq C$). This defines a partial order on the set of Hermitian matrices. Moreover, if $B \geq C$ in the sense that $B - C$ is positive semidefinite, then $X^*BX \geq X^*CX$ for any nonsingular matrix X since $X^*(B-C)X \geq 0$ by Sylvester's law of inertia.

In this paper we generalize the iterative methods [6], [7], [3], [15]. We propose new rules to choose the iteration start X_0 for computing a positive definite solution of the matrix equations

$$X + A^*X^{-1}A = I \tag{1}$$

and

$$X - A^*X^{-1}A = I, \tag{2}$$

where I is $n \times n$ identity matrix and A is a given square matrix. Theoretical properties of the solutions to equations (1) and (2) have been studied in several papers [1], [5], [6], [7], [3], [15]. Engwerda, Ran, and Rijkeboer [5] have proved that if equation (1) has a solution, then it has a maximal positive definite solution X_L and a minimal positive definite solution X_s, such that for any solution X, $X_L \geq X \geq X_s$. Note that there may be no other solutions but X_L and X_s to (1). We show that if $\|A\| \leq \frac{1}{2}$ for the ℓ_2 induced operator matrix norm $\|.\|$, then the solution X_L is the unique positive definite solution of (1) with $\frac{1}{2}I \leq X_L \leq I$, and that X_s is the unique positive definite solution of (1) with $O \leq X_s \leq \frac{1}{2}I$. In [3], [15] the convergence rate of numerical algorithms for solving these two equations has been analyzed.

In this paper we describe starting values for iterations that ensure quick convergence to a positive definite solution of (1) and (2), respectively, when A is nonsingular. The rates of convergence for the proposed starting matrices X_0 depend on one parameter α or β that is derived from the singular values of A. A new method for solving (2) will be proposed, as well as mixed methods that rely in part on Newton's method. Numerical examples will be discussed and the results of experiments given. We use $\|A\|$ to denote the ℓ_2 induced operator norm of the matrix A, i.e., $\|A\| = \sigma_1(A)$, where $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0$ are the singular values of A in nonincreasing order and $\rho(A)$ is the spectral radius of A. Throughout the paper we use the fact that $\sigma_n^2(A)I \leq AA^*$ and $A^*A \leq \sigma_1^2(A)I$ in our partial order.

2. The matrix equation $X + A^*X^{-1}A = I$

Here we discuss equation (1) and consider new starting values for the iteration proposed by [6].

Specifically we consider

$$X_{k+1} = I - A^*X_k^{-1}A, \quad k = 0, 1, 2, \ldots,\ X_0 = \alpha I, \quad \frac{1}{2} \leq \alpha \leq 1. \tag{3}$$

Our theorems give sufficient condition for the existence of a solution of (1).

Theorem 2.1. *Assume that the equation* (1) *has a positive definite solution. Then the iteration* (3) *with* $X_0 = \alpha I$, $\frac{1}{2} < \alpha \leq 1$, *and*

$$\alpha(1-\alpha) \ \leq\ \sigma_n^2(A) \tag{4}$$

defines a monotonically decreasing matrix sequence which converges to the maximal positive definite solution X_L *of equation* (1).

Proof. To prove convergence of the sequence $\{X_k\}$ defined by (3) and (4), we first show that $X_0 \geq X$ for any positive definite solution X of (1). The formulas (3) and (4) imply that $A^*A \geq \alpha(1-\alpha)I$ and

$$X_0 + A^* X_0^{-1} A = \alpha I + \frac{1}{\alpha} A^* A \geq \alpha I + (1-\alpha) I = I = X + A^* X^{-1} A.$$

Hence

$$(X_0 - X) - A^* X^{-1}(X_0 - X) X_0^{-1} A \geq 0$$

and

$$(X_0 - X) - \left[A^* X_0^{-1}(X_0 - X) X_0^{-1} A + A^* X_0^{-1}(X_0 - X) X^{-1}(X_0 - X) X_0^{-1} A\right] \geq 0.$$

Since $A^* X_0^{-1}(X_0 - X) X^{-1}(X_0 - X) X_0^{-1} A$ is congruent to the positive definite inverse X^{-1} of X, we have $A^* X_0^{-1}(X_0 - X) X^{-1}(X_0 - X) X_0^{-1} A \geq 0$. Looking at the previously displayed inequality, we conclude then that

$$(X_0 - X) - A^* X_0^{-1}(X_0 - X) X_0^{-1} A = C \tag{5}$$

and $C \geq 0$ as well. In (5) put $Y := X_0 - X$ for brevity. The equation (5) $Y - A^* X_0^{-1} Y X_0^{-1} A = C$ is a Stein equation [13], which has a unique solution if and only if $\lambda_r \bar{\lambda}_s \neq 1$ for any two eigenvalues λ_r and λ_s of $X_0^{-1}A$. If $\rho(X_0^{-1}A) < 1$ for the spectral radius ρ, then Y is the unique solution. In addition if $C \geq 0$, then $Y \geq 0$ [13].

Since $\frac{1}{2} < \alpha \leq 1$, we have $\rho(X_0^{-1}A) = \rho(\frac{1}{\alpha}A) < 2\rho(A)$. It is known [6] that if the equation (1) has a positive definite solution, then $\rho(A) \leq \frac{1}{2}$. Hence $\rho(X_0^{-1}A) < 1$ by our assumption and consequently $X_0 \geq X$.

Secondly, if $X_k \geq X > 0$, then $X_{k+1} - X = I - A^* X_k^{-1} A - I + A^* X^{-1} A = A^*(X^{-1} - X_k^{-1})A \geq 0$ and thus $X_k \geq X_L$, where X_L denotes the largest possible positive definite solution; see [5].

Thirdly, we prove that the sequence of iteration matrices $\{X_k\}$, defined by (3) and (4), is decreasing. Using (4), we know that $\alpha(1-\alpha)I \leq A^*A$. Thus $X_1 = I - \frac{A^*A}{\alpha} \leq \alpha I = X_0$. For X_2 we obtain $X_2 = I - A^* X_1^{-1} A \leq I - A^* X_0^{-1} A = X_1$. If we assume that $X_{k+1} \leq X_k$, then $X_{k+1}^{-1} \geq X_k^{-1}$ and thus

$$X_{k+1} = I - A^* X_k^{-1} A \leq I - A^* X_{k-1}^{-1} A = X_k.$$

Hence the sequence $\{X_k\}$ is monotonically decreasing. It is bounded from below by the matrix X_L, hence it is convergent to a matrix $\widehat{X}$ that satisfies $\widehat{X} = I - A^* \widehat{X}^{-1} A$, i.e., $\widehat{X}$ solves (1) and $\widehat{X} \geq X_L$. But X_L is the maximal solution. Thus $\widehat{X} = X_L$. □

Theorem 2.2. *Let A be a nonsingular matrix. Then the iteration* (3) *with start $X_0 = \alpha I$, $\frac{1}{2} < \alpha < 1$, and α defined by* (4) *converges more rapidly than the iterative method $Y_{k+1} = I - A^* Y_k^{-1} A$ with $Y_0 = I$. Specifically we have*

$$\|X_k - X_L\| < \|Y_k - X_L\| \quad \textit{for all} \quad k.$$

Proof. It is known that X_k converges from above to X_L (Theorem 2.1) and that Y_k converges to X_L [6]. If A is nonsingular, then $\sigma_n(A) > 0$ and there exists a number $\alpha \in (\frac{1}{2}, 1)$ such that $\alpha(1-\alpha) \leq \sigma_n^2(A)$. Thus $X_0 = \alpha I < I = Y_0$. And therefore

$$X_1 = I - A^* X_0^{-1} A < I - A^* Y_0^{-1} A = Y_1.$$

Assuming that $X_k < Y_k$, we have

$$X_{k+1} = I - A^* X_k^{-1} A < I - A^* Y_k^{-1} A = Y_{k+1}.$$

Consequently,

$$X_L \le X_k < Y_k \quad \text{for all} \quad k$$

and

$$\|X_k - X_L\| < \|Y_k - X_L\|. \qquad \square$$

Definition ([12, Section 9])**.** Let $\{x^k\}_{k=1}^{\infty}$ be a sequence of vectors $x^k \in \mathbb{R}^n$ that converges to x^*. Then

$$R_1 = \lim_{k\to\infty} \sup \sqrt[k]{\|x^k - x^*\|}$$

is called an *R-multiplier* of this sequence. The convergence rate of $\{x^k\}$ is called *R-linear* if $0 < R_1 < 1$, and *R-sublinear* (sublinear) if $R_1 = 1$.

The following theorem has been proved in [9].

Theorem 2.3. *For all $k \ge 0$,*

$$\|X_{k+1} - X_L\| \le \|X_L^{-1}A\|^2 \|X_k - X_L\|.$$

Moreover,

$$\lim_{k\to\infty} \sup \sqrt[k]{\|X_k - X_L\|} \le (\rho(X_L^{-1}A))^2.$$

If $\rho(X_L^{-1}A) < 1$, then the convergence of (3) is R-linear and if $\rho(X_L^{-1}A) = 1$, then its convergence rate is sublinear.

Theorem 2.4. *Let A have the singular values $\sigma_1, \sigma_2, \dots, \sigma_n$ with $\frac{1}{2} \ge \sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0$, and assume the numbers $\tilde{\alpha}, \tilde{\beta} \in [\frac{1}{2}, 1]$ are such that $\tilde{\alpha}\ (1 - \tilde{\alpha})\ = \sigma_n^2$ and $\tilde{\beta}\ (1 - \tilde{\beta})\ = \sigma_1^2$. Then*

- (i) *For $\gamma \in [\tilde{\alpha}, 1]$ and $X_0 = \gamma I$, the sequence $\{X_k\}$ in* (3) *is monotonically decreasing and converges to the maximal positive definite solution X_L of* (1)*. If $\gamma = \tilde{\alpha}$, then the iteration* (3) *converges faster than the same iteration with any other $\gamma \in (\tilde{\alpha}, 1]$.*
- (ii) *For $\gamma \in [\frac{1}{2}, \tilde{\beta}]$ and $X_0 = \gamma I$, the sequence $\{X_k\}$ in* (3) *is monotonically increasing and converges to a positive definite solution X_β of* (1)*.*
- (iii) *If $\gamma \in (\tilde{\beta}, \tilde{\alpha})$ and $\sigma_1 < \frac{1}{2}$, then the sequence $\{X_k\}$ with $X_0 = \gamma I$ converges to a positive definite solution X_γ of* (1)*.*
- (iv) *X_L is the unique positive definite solution of* (1) *with $\frac{1}{2} I \le X_L \le I$.*

In Theorem 2.4 we show that with $X_0 = \tilde{\beta} I$ in part (ii) we achieve faster convergence in (3) than with any other $X_0 = \gamma I$ for $\gamma \in [\frac{1}{2}, \tilde{\beta})$.

Proof. If $\frac{1}{2} \ge \sigma$, note that the equation $x(1-x) = \sigma^2$ always has the positive solution $\frac{1}{2} + \sqrt{\frac{1}{4} - \sigma^2}$. Since $f(x) = x(1-x)$ is monotonically decreasing on $[\frac{1}{2}, 1]$, we conclude that for all real α, β with $\frac{1}{2} \le \beta \le \tilde{\beta} \le \tilde{\alpha} \le \alpha \le 1$ the inequalities

$$\alpha(1-\alpha)\ I \le \sigma_n^2(A) I \le AA^* \le \sigma_1^2(A) I \le \beta(1-\beta)\ I$$

are satisfied.

(i) For $\gamma \in [\tilde{\alpha}, 1]$ we consider the matrix sequence (3). In particular

$$\begin{aligned} X_0 &:= \gamma I \geq \tilde{\alpha} I \geq \tilde{\beta} I, \\ X_1 &= I - \frac{1}{\gamma} A^* A \leq I - \frac{1}{\gamma} \tilde{\alpha}(1-\tilde{\alpha}) I \leq I - \frac{1}{\gamma} \gamma(1-\gamma) I = \gamma I, \\ X_1 &= I - \frac{1}{\gamma} A^* A \geq I - \frac{1}{\tilde{\beta}} A^* A \geq I - \frac{1}{\tilde{\beta}} \tilde{\beta}(1-\tilde{\beta}) I = \tilde{\beta} I. \end{aligned}$$

Hence $X_0 \geq X_1 \geq \tilde{\beta} I$.

Assuming that $\tilde{\beta} I \leq X_k \leq X_{k-1}$, we obtain for X_{k+1}

$$X_{k+1} = I - A^* X_k^{-1} A \leq I - A^* X_{k-1}^{-1} A = X_k$$

and

$$X_{k+1} = I - A^* X_k^{-1} A \geq I - \frac{1}{\tilde{\beta}} A^* A \geq I - \frac{1}{\tilde{\beta}} \tilde{\beta}(1-\tilde{\beta}) I = \tilde{\beta} I.$$

Hence the sequence $\{X_k\}$ is monotonically decreasing and bounded from below by the matrix $\tilde{\beta} I$. Consequently the sequence $\{X_k\}$ converges to a positive definite solution of (1).

Next we prove that X_k converges to X_L. First, assume $\tilde{\alpha} > \frac{1}{2}$. In Theorem 2.1 we saw that for any positive definite solution X of (1) we must have $X \leq \tilde{\alpha} I \leq \gamma I$ for any $\gamma \in [\tilde{\alpha}, 1]$.

Hence $X_L \leq \gamma I = X_0$.

Assume that $X_L \leq X_k$. For X_{k+1} compute

$$X_{k+1} - X = A^*(X^{-1} - X_k^{-1}) A \geq 0.$$

Thus $\{X_k\}$ converges to X_L for $\gamma \in [\tilde{\alpha}, 1]$.

Now we prove that the iteration (3) with $X_0 = \tilde{\alpha} I$ is converging faster than the same iteration with $\gamma \in (\tilde{\alpha}, 1]$. We denote by $\{X_k^{'}\}$ the matrix sequence (3) with initial matrix $X_0 = \tilde{\alpha} I$ and by $\{X_k^{''}\}$ the matrix sequence with $X_0 = \gamma I$ for $\gamma \in (\tilde{\alpha}, 1]$. We shall prove that

$$X_k^{'} < X_k^{''} \quad \text{and} \quad \|X_k^{'} - X_L\| < \|X_k^{''} - X_L\|$$

for $k = 0, 1, \ldots$.

At the start we have $X_0^{'} = \tilde{\alpha} I < \gamma I = X_0^{''}$.

Assuming that $X_k^{'} < X_k^{''}$, we obtain

$$X_{k+1}^{'} = I - A^*(X_k^{'})^{-1} A < I - A^*(X_k^{'})^{-1} A = X_{k+1}^{'}.$$

Hence $X_L \leq X_k^{'} < X_k^{''}$ for $k = 0, 1, \ldots$.

Secondly, if $\tilde{\alpha} = \frac{1}{2}$, then $\tilde{\beta} = \frac{1}{2}$. This implies $\sigma_1 = \cdots = \sigma_n = \frac{1}{2}$, i.e., $A = \frac{1}{2} W$ for a unitary matrix W with $W^* W = I$. Clearly $A = \frac{1}{2} W$ is normal. If A is normal and $\sigma_1 \leq \frac{1}{2}$, then $X_1 = \frac{1}{2}\left[I + (I - 4A^* A)^{\frac{1}{2}}\right]$ and $X_2 = \frac{1}{2}\left[I - (I - 4A^* A)^{\frac{1}{2}}\right]$ always satisfy the equation (1) and $X_1 = X_L$, while $X_2 = X_s$ [6]. Consequently in our case we have $X_1 = X_L = X_2 = X_s = \frac{1}{2} I$.

(ii) For $\gamma \in \left[\frac{1}{2}, \tilde{\beta}\right]$ we have

$$\begin{aligned} X_0 &:= \gamma I \leq \tilde{\beta} I \leq \tilde{\alpha} I, \\ X_1 &= I - \frac{1}{\gamma} A^* A \geq I - \frac{1}{\gamma} \tilde{\beta}(1-\tilde{\beta}) I \geq I - \frac{1}{\gamma} \gamma(1-\gamma) I = \gamma I, \\ X_1 &= I - \frac{1}{\gamma} A^* A \leq I - \frac{1}{\tilde{\alpha}} A^* A \leq I - \frac{1}{\tilde{\alpha}} \tilde{\alpha}(1-\tilde{\alpha}) I = \tilde{\alpha} I. \end{aligned}$$

Hence $X_0 \leq X_1 \leq \tilde{\alpha} I$.

Assuming that $X_{k-1} \leq X_k \leq \tilde{\alpha} I$, we compute

$$X_{k+1} = I - A^* X_k^{-1} A \geq I - A^* X_{k-1}^{-1} A = X_k$$

and

$$X_{k+1} = I - A^* X_k^{-1} A \leq I - \frac{1}{\tilde{\alpha}} A^* A \leq I - \frac{1}{\tilde{\alpha}} \tilde{\alpha}(1-\tilde{\alpha}) I = \tilde{\alpha} I.$$

Hence the sequence $\{X_k\}$ is monotonically increasing and bounded above by the matrix $\tilde{\alpha} I$. Consequently the sequence $\{X_k\}$ converges to a positive definite solution X_β of (1).

(iii) For $\gamma \in (\tilde{\beta}, \tilde{\alpha})$, we have

$$\tilde{\beta} I < X_0 = \gamma I < \tilde{\alpha} I.$$

Assuming that $\tilde{\beta} I < X_k < \tilde{\alpha} I$ we see that

$$X_{k+1} = I - A^* X_k^{-1} A < I - \frac{1}{\tilde{\alpha}} A^* A \leq I - \frac{1}{\tilde{\alpha}} \tilde{\alpha}(1-\tilde{\alpha}) I = \tilde{\alpha} I,$$

and

$$X_{k+1} = I - A^* X_k^{-1} A > I - \frac{1}{\tilde{\beta}} A^* A \geq I - \frac{1}{\tilde{\beta}} \tilde{\beta}(1-\tilde{\beta}) I = \tilde{\beta} I.$$

Consequently $\tilde{\beta} I < X_k < \tilde{\alpha} I$, for $k = 0, 1, \ldots$.

Now consider

$$\begin{aligned} \|X_{k+1} - X_k\| &= \|A^*(X_{k-1}^{-1} - X_k^{-1}) A\| = \|A^* X_{k-1}^{-1} (X_k - X_{k-1}) X_k^{-1} A\| \\ &\leq \|A\|^2 \|X_{k-1}^{-1}\| \|X_k^{-1}\| \|X_k - X_{k-1}\| < \left(\frac{\|A\|}{\tilde{\beta}}\right)^2 \|X_k - X_{k-1}\|. \end{aligned}$$

Since $\|A\| < \frac{1}{2}$ and $\tilde{\beta} > \frac{1}{2}$ are assumed here, it follows that $\{X_k\}$ is a Cauchy sequence of positive definite matrices in the Banach space $\mathcal{C}^{n \times n}$. Hence this sequence has a positive definite limit X_γ with $\tilde{\beta} I \leq X_\gamma \leq \tilde{\alpha} I$.

(iv) Let $\tilde{X}$ be any positive definite solution of (1) such that $\frac{1}{2} I \leq \tilde{X} \leq I$. We have to prove that $\tilde{X} = X_L$.

It was proved in [5] that $X = X_L$ is the unique solution for which $X + \lambda A$ is invertible for all $|\lambda| < 1$. Thus we want to prove that $\det(\tilde{X} + \lambda A) \neq 0$ for $|\lambda| < 1$.

Note that if $\lambda = 0$, then $\det(\tilde{X} + \lambda A) = \det(\tilde{X}) \neq 0$.

If $\lambda \neq 0$, then

$$\begin{aligned} \det(\tilde{X} + \lambda A) &= \det(\tilde{X}) \det(I - (-\lambda) \tilde{X}^{-1} A) \\ &= (-\lambda)^n \det(\tilde{X}) \det\left(\frac{1}{-\lambda} I - \tilde{X}^{-1} A\right). \end{aligned}$$

Hence $\det(\tilde{X} + \lambda A) = 0$ if and only if $\frac{1}{-\lambda}$ is an eigenvalue of the matrix $\tilde{X}^{-1}A$. For the two matrices $\tilde{X}$ and A we have $\frac{1}{2}I \leq \tilde{X}$, $\tilde{X}^{-1} \leq 2I$, and $\|A\|^2 = \sigma_1^2(A) = \tilde{\beta}(1-\tilde{\beta}) \leq \frac{1}{4}$. Thus

$$|\lambda^{-1}| = \left|\frac{1}{-\lambda}\right| \leq \rho(\tilde{X}^{-1}A) \leq \|\tilde{X}^{-1}A\| \leq \|\tilde{X}^{-1}\|\|A\| \leq 1.$$

And $\det(\tilde{X} + \lambda A) = 0$ can only hold for $|\lambda| \geq 1$. Hence $\tilde{X} + \lambda A$ is nonsingular for $|\lambda| < 1$. But X_L is the unique solution with this property. Hence $\tilde{X} = X_L$. □

Remark. One can quickly see how important it is to choose a good starting matrix X_0 for the iteration (3) by looking at the "left edge" of the inequality $\alpha(1-\alpha)\, I \leq \sigma_n^2(A)I \leq AA^* \leq \sigma_1^2(A)I \leq \beta(1-\beta)\, I$, for example. Assume that $A = \frac{1}{2}W$ for a unitary matrix W with $W^*W = I$. To solve (1) in this case, Theorem 2.4 suggests that we use $X_0 = \frac{1}{2}I$ as a start for (3). Clearly X_0 is the unique solution of (1); i.e., $X_0 = X_L = X_s$. If we modify the coefficient $\alpha = \frac{1}{2}$ of I used in X_0 only very slightly, we obtain convergence in the sense that $\|X_{k+1} - X_k\| \leq tol = 10^{-10}$ for a randomly generated orthogonal 3-by-3 matrix W after the following number of iterations.

α	number of iterations	α	number of iterations	α	number of iterations
0.500000	1	0.500027	3843	0.500047	11723
...	1	0.500030	5695	0.600000	70709
0.500022	1	0.500035	8076	0.700000	70713
0.500023	623	0.500041	10167	0.800000	70711
0.500024	1528	0.500046	11492	0.900000	70713

This shows how slow the convergence of (3) is in general and how crucial it is to find a good starting matrix X_0 for the iteration (3).

From the theorem above we obtain

$$\lim_{k\to\infty} \sup \sqrt[k]{\|X_k - X_L\|} \leq \lim_{k\to\infty} \sup \sqrt[k]{q^{2k}\|X_0 - X_L\|} = q^2, \tag{6}$$

where $q = \frac{\|A\|}{\tilde{\beta}}$. It is well known [12] that if $q < 1$, then the convergence rate of the iteration procedure (3) is R-linear and if $q = 1$, then the convergence rate of the iteration procedure (3) is sublinear.

Theorem 2.5. *The iterative method* (3) *with* $X_0 = \tilde{\beta}I$ *(with* $\tilde{\beta}$ *defined as in* (ii) *of Theorem 2.4) converges more rapidly than the same method with any* $X_0 = \gamma I$ *for* $\gamma \in \left[\frac{1}{2}, \tilde{\beta}\right)$.

Proof. We denote the matrix sequence defined by (3) and $X_0 = \tilde{\beta}I$ by $\{X_k^{'}\}$ and the matrix sequence defined by (3) and $X_0 = \gamma I$ for $\gamma \in \left[\frac{1}{2}, \tilde{\beta}\right)$ by $\{X_k^{''}\}$. We show that

$$\|X_k^{'} - X_L\| < \|X_k^{''} - X_L\|$$

for $k = 0, 1, \ldots$.

According to conditions (i) and (ii) of Theorem 2.4, the two sequences $\{X_k^{'}\}$ and $\{X_k^{''}\}$ are monotonically increasing and bounded above by X_L. It is sufficient to prove that $X_k^{'} > X_k^{''}$ for $k = 0, 1, \ldots$.

Obviously $X_0^{'} = \tilde{\beta} I > \gamma I = X_0^{''}$.

Assuming $X_k^{'} > X_k^{''}$, we have

$$X_{k+1}{}^{'} = I - A^*(X_k^{-1})^{'} A > I - A^*(X_k^{-1})^{''} A = (X_{k+1})^{''}.$$

Thus $X_L \leq X_k^{'} > X_k^{''}$ for all $k = 0, 1, \ldots$. □

Theorem 2.6. *If A is nonsingular and $\|A\| = \sigma_1(A) \leq \frac{1}{2}$, then the minimal positive definite solution X_s of equation* (1) *is the unique solution with $0 < X_s \leq \frac{1}{2} I$.*

Proof. It is known that $\tilde{X}$ is a solution of (1) if and only if $\tilde{Y} = I - \tilde{X}$ is a solution of the equation $Y + AY^{-1}A^* = I$ [5]. Moreover, if X_s is the minimal solution of (1), then $Y_L = I - X_s$ is the maximal solution of $Y + AY^{-1}A^* = I$.

Consider the equation $Y + AY^{-1}A^* = I$. Since $\|A\| \leq \frac{1}{2}$ there exist $\tilde{\alpha}, \tilde{\beta} \in [\frac{1}{2}, 1]$ such that $\tilde{\alpha}\,(1 - \tilde{\alpha}) = \sigma_n^2(A)$ and $\tilde{\beta}\,(1 - \tilde{\beta}) = \sigma_1^2(A)$, respectively. According to Theorem 2.4 (iv), the maximal positive definite solution Y_L is the unique solution of $Y + AY^{-1}A^* = I$ with $\frac{1}{2} I \leq Y_L < I$.

Assume $\tilde{X} \neq X_s$ is any solution of (1) with $0 < \tilde{X} \leq \frac{1}{2} I$. Then $\tilde{Y} = I - \tilde{X} \neq Y_L$ is a solution of $Y + AY^{-1}A^* = I$ with $\frac{1}{2} I \leq \tilde{Y} < I$. Hence the equation $Y + AY^{-1}A^* = I$ has two different solutions Y_L and $\tilde{Y}$. Since Y_L is the unique solution with $\frac{1}{2} I \leq Y_L < I$, we must have $Y_L = \tilde{Y}$, or $\tilde{X} = X_s$.

Therefore X_s is the unique positive definite solution of (1) with $0 < X_s \leq \frac{1}{2} I$. □

3. The matrix equation $X - A^*X^{-1}A = I$

The more general equation

$$X - A^*X^{-1}A = Q \tag{7}$$

has been considered in [3], [7], [9]. El-Sayed [3] has considered this equation with $Q = I$ and has proposed the iterative method $X_{k+1} = I + A^*X_k^{-1}A$ with $X_0 = I$ for computing a positive definite solution. El-Sayed [3] has proved the following theorem.

Theorem 3.1. *If $\|A\| = \sigma_1(A) < 1$, then the equation $X - A^*X^{-1}A = I$ has a positive definite solution X.*

Levy and Ferrante [7] have described an iterative algorithm for computing the positive definite solution of this equation and they have related this matrix equation to an algebraic Riccati equation of the type arising in Kalman filtering. They have proved that the equation above has a unique positive definite solution and that for every matrix A the iteration $X_{k+1} = Q + A^*X_k^{-1}A$ with $X_0 = Q$ converges to the unique positive definite solution X_+ of (7).

Guo and Lancaster [9] have studied convergence results for the iterative method considered by Levy. The following theorem has been proved [9].

Theorem 3.2. *The iteration*

$$X_{k+1} = Q + A^*X_k^{-1}A, \quad k = 0, 1, 2, \ldots \quad \textit{with} \;\; X_0 = Q$$

satisfies

$$\|X_{2k} - X_+\| \leq \|X_+^{-1}A\|^2 \|X_{2k-1} - X_+\|$$

for all $k \geq 1$. Moreover

$$\lim_{s \to \infty} \; \sup \sqrt[s]{\|X_s - X_+\|} \leq (\rho(X_+^{-1}A))^2 < 1.$$

Here X_+ denotes the unique positive definite solution of equation (2).

Since $\rho(X_+^{-1}A) < 1$, the convergence of the above iteration is *R-linear*.

We present two iterative methods for computing the positive definite solution of equation (2). The first method uses the same iteration formula as El-Sayed's method, but it uses a different initial matrix X_0 with improved convergence. The second method is new and apparently even faster.

As our first method we consider the iteration

$$X_{k+1} = I + A^* X_k^{-1} A, \quad k = 0, 1, 2, \dots \quad \text{with} \quad X_0 = \alpha I. \tag{8}$$

First we prove a theorem.

Theorem 3.3. *Let $\alpha \geq 1$ be the real number with*

$$\alpha(\alpha - 1) = \sigma_n^2(A). \tag{9}$$

Then the iteration sequence $\{X_k\}$ defined by (8) with $X_0 = \alpha I$ converges to the positive definite solution X_+ of equation (2).

Proof. Note that the equation $x(x-1) = \sigma^2$ always has one real solution

$$\frac{1}{2} + \sqrt{\frac{1}{4} + \sigma^2} \geq 1.$$

Consider the iteration

$$Y_{k+1} = I + A^* Y_k^{-1} A, \quad k = 0, 1, 2, \dots, \quad Y_0 = I. \tag{10}$$

Then $Y_1 = I + A^*A$ and $X_1 = I + \frac{1}{\alpha} A^*A$. It is well known [7] that the inequalities

$$Y_0 = I \leq Y_2 \leq \cdots \leq Y_{2k} \leq \cdots \leq Y_{2s+1} \leq \cdots \leq Y_3 \leq Y_1 = I + A^*A$$

hold for the sequence $\{Y_k\}$.

We have two cases: A is singular or A is not. If A is singular, then $\sigma_n(A) = 0$ and subsequently $\alpha = 1$ in (9). In this case the two matrix sequences $\{Y_k\}$ and $\{X_k\}$ are identical, i.e., $Y_k = X_k,\ k = 1, 2, \dots$. If A is nonsingular, then $\alpha > 1$ since $A^*A > 0$. Hence $I = Y_0 < X_0 = \alpha I$ and

$$I + A^*A = Y_1 > I + \frac{1}{\alpha} A^*A = X_1.$$

For α we have $\alpha(\alpha - 1)I \leq A^*A$ and therefore $X_0 \leq X_1$. Hence

$$Y_0 < X_0 \leq X_1 < Y_1.$$

Moreover, $X_2 > Y_2$ since $X_1 < Y_1$, and similarly $X_3 < Y_3$ and so on. Thus we have

$$Y_{2p} < X_{2p} \leq X_{2p+1} < Y_{2p+1} \quad \text{for all} \quad p \tag{11}$$

and

$$X_{2p+2} \leq X_{2p+1} \quad \text{for all} \quad p. \tag{12}$$

According to (11) and (12), we obtain

$$Y_{2p+2} < X_{2p+2} \leq X_{2p+1} < Y_{2p+1} \quad \text{for all} \quad p. \tag{13}$$

The two subsequences $\{Y_{2k}\}$ and $\{Y_{2k+1}\}$ are monotone increasing and decreasing, respectively. They converge to the positive definite solution of the equation $X - A^* X^{-1} A = I$. Consequently, the subsequences $\{X_{2k}\}$ and $\{X_{2k+1}\}$ converge to the same limit. $\square$

Theorem 3.4. *Let A be nonsingular. Then the iteration (8) with $X_0 = \alpha I$ for $\alpha(1-\alpha) = \sigma_n^2(A)$ converges to the positive solution of (2) and there exists an integer s, $s \le k$, such that*

$$\|X_s - A^* X_s^{-1} A - I\| < \|Y_k - A^* Y_k^{-1} A - I\|$$

for all k.

Proof. Combining (11) and (13), we write

$$\|X_{k+1} - X_k\| < \|Y_{k+1} - Y_k\|$$

for $k = 0, 1, \ldots$. Since

$$\begin{aligned} \|X_{k+1} - X_k\| &= \|I + A^* X_k^{-1} A - X_k\| \\ &= \|X_k - A^* X_k^{-1} A - I\| \end{aligned}$$

and

$$\begin{aligned} \|Y_{k+1} - Y_k\| &= \|I + A^* Y_k^{-1} A - Y_k\| \\ &= \|Y_k - A^* Y_k^{-1} A - I\|, \end{aligned}$$

we have

$$\|X_k - A^* X_k^{-1} A - I\| < \|Y_k - A^* Y_k^{-1} A - I\|.$$

Consequently there exists an integer $s \le k$ for which

$$\|X_s - A^* X_s^{-1} A - I\| < \|Y_k - A^* Y_k^{-1} A - I\|. \qquad \square$$

Theorem 3.5. *Assume that equation (2) has a positive definite solution X and that*

$$\sigma_n^2(A)\left(\sigma_n^2(A) + 1\right) \ge \sigma_1^2(A) \tag{14}$$

and

$$\beta(\beta - 1) = \sigma_1^2(A). \tag{15}$$

Then the iteration

$$X_{k+1} = I + A^* X_k^{-1} A, \quad k = 0, 1, 2, \ldots, \quad X_0 = \beta I, \tag{16}$$

converges to the positive definite solution X_+ of (2) and there exists an integer s, $s \le k$, with

$$\|X_s - A^* X_s^{-1} A - I\| < \|Y_k - A^* Y_k^{-1} A - I\|$$

for all k.

The proof is similar to the proofs of Theorems 3.3 and 3.4: Consider the sequences $\{X_k\}$ defined by (16) and $\{Y_k\}$ defined by (10). From (14) it follows that $X_0 \le Y_1$, and, according to (15), we obtain $X_1 \le X_0$. Hence

$$Y_0 \le X_1 \le X_0 \le Y_1.$$

Therefore

$$Y_{2p} \le X_{2p-1} \le X_{2p} \le Y_{2p+1} \quad \text{for all} \quad p$$

and

$$Y_{2p+2} \le X_{2p+1} \le X_{2p} \le Y_{2p+1} \quad \text{for all} \quad p.$$

Thus the sequence $\{X_k\}$ converges to the positive definite solution X_+.

For our second method we consider the iteration

$$X_{k+1} = \frac{1}{2}(X_k + I + A^* X_k^{-1} A), \quad k = 0, 1, 2, \ldots, \quad X_0 = I + A^* A. \tag{17}$$

Theorem 3.6. *For every X_k in the iteration sequence defined by* (17)*, we have $I \le X_k \le I + A^* A$.*

Proof. We know that $I \le X_0 \le I + A^* A$. Suppose that $I \le X_k \le I + A^* A$. Then

$$\begin{aligned} A^* X_k^{-1} A &\le A^* A, \\ I + A^* X_k^{-1} A &\le A^* A + I. \end{aligned}$$

Since $X_k \le I + A^* A$, we have

$$\begin{aligned} X_k + I + A^* X_k^{-1} A &\le 2(I + A^* A), \\ X_{k+1} &\le I + A^* A. \end{aligned}$$

Moreover

$$\begin{gathered} I + X_k \ge 2I, \\ \frac{1}{2}(X_k + I + A^* X_k^{-1} A) \ge \frac{1}{2}(X_k + I) \ge I, \\ X_{k+1} \ge I. \quad \square \end{gathered}$$

Theorem 3.7. *Let X_+ be the positive definite solution of* (2)*. Consider the matrix sequence $\{X_k\}$ that is defined by* (17)*. Assume that $X_k \ge X_+$ holds for one integer k. Then $X_k \ge X_{k+1}$.*

Proof. Levy [7] has shown that equation (2) has a unique positive definite solution X_+ with $I \le X_+ \le I + AA^*$. Thus $X_0 \ge X_+$.

For X_1 we compute

$$X_1 = \frac{1}{2}(X_0 + I + A^* X_0^{-1} A) \le \frac{1}{2}(X_0 + I + A^* X_+^{-1} A) = \frac{1}{2}(X_0 + X_+) \le X_0.$$

Assuming that $X_k \ge X_+$, we have

$$X_{k+1} = \frac{1}{2}(X_k + I + A^* X_k^{-1} A) \le \frac{1}{2}(X_k + I + A^* X_+^{-1} A) = \frac{1}{2}(X_k + X_+) \le X_k.$$

$\square$

Theorem 3.8. *Let X_+ be the positive definite solution of* (2) *and assume that $\|X_+^{-1} A\| \|X_k^{-1} A\| < 1$ where the matrices X_k are defined by* (17)*. Then the iterative method* (17) *converges at least linearly to X_+.*

Proof. For $X_{k+1} - X_+$ we have

$$\begin{aligned} X_{k+1} - X_+ &= \frac{1}{2}(X_k - X_+ + A^*(X_k^{-1} - X_+^{-1}) A) \\ &= \frac{1}{2}(X_k - X_+ + A^* X_+^{-1}(X_+ - X_k) X_k^{-1} A). \end{aligned}$$

Therefore

$$\|X_{k+1} - X_+\| \le \frac{1}{2} \|X_k - X_+\| \left(1 + \|X_+^{-1} A\| \, \|X_k^{-1} A\|\right). \tag{18}$$

Since $\|X_+^{-1} A\| \, \|X_k^{-1} A\| < 1$, we compute

$$q = \frac{1}{2}\left(1 + \|X_+^{-1} A\| \, \|X_k^{-1} A\|\right) < 1$$

as a bound for the convergence rate of (17). Thus the matrix sequence $\{X_k\}$ converges to X_+. □

Corollary 3.9. *Let X_+ be the positive definite solution of* (2) *and assume that* $\|X_+^{-1}A\|\;\|A\| < 1$. *Then the iteration* (17) *converges to* X_+.

4. Numerical experiments

We have carried out numerical experiments for computing a positive definite solution of equations (1) and (2) in MATLAB on a PENTIUM computer and on a SUN workstation. We have used the methods described in Sections 2 and 3. Guo and Lancaster [9] have considered using the Newton method for finding positive definite solutions of the above equations. Newton's method for our problems involves a large amount of computations per iteration. Lancaster and Guo [9] estimate that the computational work per iteration for Newton's method and this problem is roughly 10 to 15 times that of the iterative method (3) with $\alpha = 1$. We compare our iterative methods for various starting matrices with Newton's method and also derive experimental data for mixed iteration schemes for solving (1) and (2).

As a practical stopping criterion for the iterations we use

$$\varepsilon_1(Z) = \|Z + A^T Z^{-1} A - I\|_\infty \le tol$$

and

$$\varepsilon_2(Z) = \|Z - A^T Z^{-1} A - I\|_\infty \le tol,$$

respectively, for various values of *tol*; i.e., the same criterion that we have used in our earlier Remark on the sensitivity of the optimal value of α in (3).

4.1. Numerical experiments on solving $X + A^T X^{-1} A = I$.

We have tested iteration (3) with different initial matrices X_0 for solving equation (1) $X + A^T X^{-1} A = I$ and a number of $n \times n$ matrices A.

Example 1. Consider the matrix

$$A = \begin{pmatrix} 0.471 & 0.002 & 0.04 \\ 0.002 & 0.472 & -0.002 \\ -0.04 & -0.001 & 0.471 \end{pmatrix}.$$

Here $\|X_L^{-1}A\| = 0.72$ and $\rho(X_L^{-1}A) = 0.71$. With $tol := 10^{-10}$, we have computed X_L using (3) for different starting values of α: Method (3) with $\alpha = 1$ needs 32 iterations; for $\alpha = 0.672$, 28 iterations. If $\alpha = \frac{1}{2}$, then 33 iterations are required, and setting $\alpha = 0.657$ takes 27 iterations.

According to Theorem 2.3 and formula (6), method (3) has *R-linear* convergence. There are values of α for which the iteration converges slightly faster than for $\alpha = 1$ or $\alpha = \frac{1}{2}$.

Example 2 ([9, Example 7.2])**.** Consider the normal matrix

$$A = \begin{pmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.15 & 0.15 \\ 0.1 & 0.15 & 0.25 \end{pmatrix}.$$

We have $\|A\| = \frac{1}{2}$ and $\rho(X_L^{-1}A) = 1$. For $tol := 10^{-8}$ we compute X_L by using (3) with different starting values α: When $\alpha = 1$ or $\alpha = 0.99$, then 7071 or 7070 iterations are required. Guo and Lancaster [9] carry out experiments with the same matrix A. The Newton method in [9] needs twelve iterations to find X_L with the same tolerance *tol*. Method (3) with $\alpha = \frac{1}{2}$ has *sublinear* convergence and it needs only five iterations, which use a small fraction of the effort with Newton's method.

Example 3. Consider the matrix

$$A = \frac{\tilde{A}}{2\|\tilde{A}\|},$$

where

$$\tilde{A} = \begin{pmatrix} 0.1 & -0.15 & -0.2598076 \\ 0.15 & 0.2125 & -0.0649519 \\ 0.2598076 & -0.0649519 & 0.1375 \end{pmatrix}.$$

The matrix $\tilde{A}$ was considered by Zhan [15, Example 1]. We have $\|A\| = \frac{1}{2}$, $\|X_L^{-1}A\| = 1$ and $\rho(X_L^{-1}A) = 1$. We use $tol := 10^{-7}$ here. The method (3) with $\alpha = 1$ needs 2398 iterations. $\alpha = 0.807$ takes 2397 iterations. And method (3) with $\alpha = \frac{1}{2}$ needs only eleven iterations. The Newton method [9] needs nine iterations for computing X_L for the same stopping accuracy. If *tol* is chosen smaller than 10^{-6}, however, then Newton's method does not converge in finite time.

Example 4. Consider the equation $Z + Z^{-1} = Q$, where Q is the circulant matrix $Q = \text{circ}(4, -1, 0, \ldots, 0, -1)$. This equation is equivalent to (1) with $X = Q^{-\frac{1}{2}}ZQ^{-\frac{1}{2}}$ and $A = Q^{-\frac{1}{2}}$.

Thus $\|A\| = \frac{1}{2}$, $\|X_L^{-1}A\| = 1$, and $\rho(X_L^{-1}A) = 1$. We use $tol := 10^{-8}$. The method (3) with $\alpha = 1$ needs 7070 iterations and for $\alpha = \frac{1}{2}$ it needs ten iterations. The Newton method needs twelve iterations.

Examples 2, 3, and 4 show that method (3) with $\alpha = \frac{1}{2}$ takes slightly fewer iterations than the Newton method when $\rho(X_L^{-1}A)$ is equal to or very close to 1. On top of this, the method (3) requires much less computational work per iteration than the Newton method.

4.2. **Numerical experiments on solving $X - A^TX^{-1}A = I$.** Here we try to solve $X - A^*X^{-1}A = I$ by using iterations (8), (16), and (17) and Newton's method [9]. In [9] Guo and Lancaster state that Newton's method has local quadratic convergence if it can be used with an initial matrix X_0 that is sufficiently close to the unique positive definite solution X_+ of (2). Specifically, we experiment with mixed algorithms that use our iterations first to approximate X_+ roughly and then accelerate convergence by using Newton's method.

Example 5. Consider the matrix

$$A = \begin{pmatrix} -3.47 & 3.47 \\ -2.89 & -3.47 \end{pmatrix}.$$

We have $\|X_+^{-1}A\| = 0.95$ and $\rho(X_+^{-1}A) = 0.9$. Here we set $tol := 10^{-8}$. Method (8) with $\alpha = 1$ needs 100 iterations, while for $\alpha = 4.944$ only 87 iterations are required to find X_+. Method (17) with $X_0 = I + A^TA$ needs thirteen iterations. Here we have $\|X_+^{-1}A\|\|X_k^{-1}A\| < 1$ for all $k = 0, 1, \ldots, 13$ and method (17) is much faster than method (8).

As an experiment of a mixed algorithm, we have carried out six iterations with each of the two methods (8) and (17), followed by Newton's method with X_6 as the start. The results are given in Table 1. The first column indicates the method and

TABLE 1.

6 iterations using:	Newton iterations	$\varepsilon_2(\tilde{X})$
(8) with $X_0 = I$	4	6.07×10^{-11}
(8) with $X_0 = 4.944I$	3	2.94×10^{-13}
(17)	3	2.99×10^{-15}

its start. The second column contains the number of further iterations in Newton's method until obtaining $\tilde{X}$ with error $\varepsilon_2(\tilde{X}) \leq 10^{-10}$.

The combination of (17) with Newton's method is more effective than the other combinations because it takes the fewest Newton iterations and achieves the best accuracy.

Example 6. Consider the matrix

$$A = \frac{1.41}{\|\tilde{A}\|}\, \tilde{A},$$

where $\tilde{A}$ is the matrix from Example 3.

We compute X_+ using (8): If $\alpha = 1$, then 28 iterations are required; if $\alpha = 1.721$, i.e., if α is chosen according to Theorem 3.3, then 26 iterations are required. Computing the solution with (16) and $\beta = 1.996$ requires 21 iterations. Here $\|X_+^{-1}A\| = 0.706 < 1$ and $\|X_+^{-1}A\|\,\|A\| = 0.996 < 1$. Hence the matrix sequence defined by (17) converges. It needs only 15 iterations to compute X_+.
Again we experiment with mixed iterations: First we make k iterations to achieve the accuracy *tol* with one of the methods (8), (16), or (17), and then we continue with Newton's method with start X_k until the accuracy of $tol := 10^{-10}$ is obtained. The results are given in Tables 2 and 3. Column (a) contains the number of iterations needed by our linearly converging methods to get close to X_+ and obtain $\varepsilon_2(X_k) \leq tol_1$. Column (b) contains the number of iterations from there using the quadratically converging Newton's method to obtain $\tilde{X}$ with $\varepsilon_2(\tilde{X}) \leq tol_2$ for the chosen value of $tol_2 = tol$.

Again the methods (16) and (17) are more effective than (8).

Example 7 ([9, Example 7.4])**.** Consider the equation $Y - B^*Y^{-1}B = Q$, where

$$B = \begin{pmatrix} 50 & 20 \\ 10 & 60 \end{pmatrix}, \qquad Q = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}.$$

This equation is equivalent to (2) with $X = Q^{-\frac{1}{2}}YQ^{-\frac{1}{2}}$ and $A = Q^{-\frac{1}{2}}BQ^{-\frac{1}{2}}$.

We have $\|Y_+^{-1}B\| = 1.00007$, $\|X_+^{-1}A\| = 1.0098$, and $\rho(Y_+^{-1}B) = \rho(X_+^{-1}A) = 0.972$. For $tol := 10^{-8}$, we compute the solution using (8): $\alpha = 1$ requires 405 iterations, while starting with $\alpha = 13.299$ needs 380 iterations. For (17) we observe that $\|X_+^{-1}A\|\|X_k^{-1}A\| < 1$ for $k = 0, 1, \ldots, 6$. Hence the condition (18) is true. From (18) it follows $\|X_7 - X_+\| \leq \|X_6 - X_+\|$. We can continue with Newton's method and X_7 as its start. After two Newton iterations we arrive at $\tilde{X}$ with $\varepsilon_2(\tilde{X}) = 6.32 \times 10^{-11}$.

We have carried out further experiments on (8): Making 100 iterations with (8) for $\alpha = 1$, followed by two iterations with Newton's method we obtain $\tilde{X}$ with $\varepsilon_2(\tilde{X}) = 9.38 \times 10^{-9}$. Alternately we can make 100 iterations with (8) for $\alpha = 13.299$ and follow with two iterations with Newton's method to obtain the solution $\tilde{X}$ with

Table 2.

method	(a)	(b)	$\varepsilon_2(\tilde{X})$
	$tol_1 = 0.1$	$tol_2 = 10^{-10}$	
(8) with $X_0 = I$	3	5	1.07×10^{-15}
(8) with $X_0 = 1.721I$	3	3	6.21×10^{-16}
(16) with $X_0 = 1.996I$	2	3	6.62×10^{-16}
(17)	3	3	1.28×10^{-15}

Table 3.

method	(a)	(b)	$\varepsilon_2(\tilde{X})$
	$tol_1 = 0.001$	$tol_2 = 10^{-10}$	
(8) with $X_0 = I$	12	2	5.19×10^{-16}
(8) with $X_0 = 1.721I$	10	2	8.68×10^{-16}
(16) with $X_0 = 1.996I$	7	2	1.45×10^{-15}
(17)	6	2	1.09×10^{-15}

$\varepsilon_2(\tilde{X}) = 2.02 \times 10^{-10}$. When iterating 85 times with (8) for $\alpha = 13.299$, followed by two iterations with Newton, the solution $\tilde{X}$ satisfies $\varepsilon_2(\tilde{X}) = 2.78 \times 10^{-9}$.

Hence the convergence rate of (8) depends on our choice of α. There are values of α for which the combined method takes fewer iterations and achieves the same accuracy.

Note that we can compute a solution $\tilde{X}$ for this problem by using (17) alone in only fourteen iterations. The iteration (17) converges for this A, but unfortunately we cannot prove convergence for (17) and this matrix theoretically.

5. Conclusions

For solving equation (1) $A + A^* X^{-1} A = I_n$ iteratively, iteration (3) benefits most from choosing the starting matrix $X_0 = \alpha I$ or $X_0 = \beta I$, where α and β are defined as in (4) or in Theorem 2.4, respectively, from the extremal singular values of A.

If A is real, then each iteration step (3) $X_{k+1} = I - A^* X_k^{-1} A$ costs $\frac{7}{3} n^3$ flops per iteration [15]. To find all singular values of A via the Golub–Reinsch algorithm in order to set up X_0 requires about $2\frac{2}{3} n^3$ operations; see, e.g., [8, ch. 5.4.5]. Thus "preconditioning" the iterations with one SVD of A adds the equivalent of about one extra iteration step to the whole procedure.

Looking at the examples in Section 4 above, we note no speed-up in Example 1; all iterations converge in 27 to 33 iterations and not much can be achieved with an "SVD preconditioned" start X_0. However, in Examples 2–4 and in the example preceding Theorem 2.5, we achieve convergence in 5, 9, 10, and 1 respective iterations from the starting matrix X_0 that is suggested from the singular values of A. In contrast, from suboptimal starting matrices $X_0 = cI_n$, we need 7,000, 2,400, 7,000, and 70,000 iterations of (3), respectively, for convergence. Thus for the price of one extra iteration, we can achieve speed-up factors of around 1,160, 240, 640, and 35,000, respectively.

Hence we recommend always starting with the optimal starting matrix of the form cI_n as suggested from A's SVD to solve (1) $A + A^*X^{-1}A = I_n$ iteratively via the standard iteration (3).

When trying to solve the "–" equation (2) $A - A^*X^{-1}A = I_n$ iteratively, our experiments show only a slight improvement in the rate of convergence between the standard iteration (8), even when starting with our optimal α and β values from (9) or (15) from $X_0 = \alpha I_n$ or $X_0 = \beta I_n$, and the special iteration (17). This speed-up is limited to a factor of at most 12 in Examples 5–7 of Section 4.

Thus for solving (2), an "SVD preconditioning" is seemingly of less value than in the previous "+" equation case (1), though we still recommend it. Moreover we also suggest experimenting here with mixing standard iterations with subsequent Newton iterations steps.

References

1. W. N. Anderson, T. D. Morley and G. E. Trapp, *Positive Solution to* $X = A - BX^{-1}B^*$, Linear Algebra Appl., **134** (1990), 53–62. MR **91c:**47031
2. B. L. Buzbee, G. H. Golub and C. W. Nielson, *On Direct Methods for Solving Poisson's Equations*, SIAM J. Numer. Anal., **7** (1970), 627–656. MR **44:**4920
3. S. M. El-Sayed, *Theorems for the Existence and Computing of Positive Definite Solutions for Two Nonlinear Matrix Equation*, Proc. of 25^{th} Spring Conference of the Union of Bulgarian Mathematicians, Kazanlak, 1996, pp.155-161, (in Bulgarian).
4. S. M. El-Sayed and A. C. M. Ran, *On an Iteration Method for Solving a Class of Nonlinear Matrix Equations*, SIAM J. Matrix Anal. Appl., **23** (2001), 632–645. MR **2002m:**15023
5. J. C. Engwerda, A. C. M. Ran and A. L. Rijkeboer, *Necessary and Sufficient Conditions for the Existence of a Positive Definite Solution of the Matrix Equation* $X + A^*X^{-1}A = Q$, Linear Algebra Appl., **186** (1993), 255–275. MR **94j:**15012
6. J. C. Engwerda, *On the Existence of a Positive Definite Solution of the Matrix Equation* $X + A^TX^{-1}A = I$, Linear Algebra Appl., **194** (1993), 91–108. MR **94j:**15013
7. A. Ferrante and B. Levy, *Hermitian Solutions of the Equation* $X = Q + NX^{-1}N^*$, Linear Algebra Appl., **247** (1996) 359–373. MR **97m:**93071
8. C. H. Golub and C. F. Van Loan, *Matrix Computations*, Second Edition, John Hopkins University Press, Baltimore, 1989. MR **90d:**65055
9. C.-H. Guo and P. Lancaster, *Iterative Solution of Two Matrix Equations*, Math. Comp., **68** (1999), 1589–1603. MR **99m:**65061
10. A. S. Householder, *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964. MR **30:**5475
11. I. G. Ivanov and S. M. El-Sayed, *Properties of Positive Definite Solutions of the Equation* $X + A^*X^{-2}A = I$, Linear Algebra Appl., **279** (1998), 303–316. MR **99c:**15019
12. J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970 (in Russian 1975). MR **42:**8686
13. O. Taussky, *Matrices C with* $C^n \to 0$, J. Algebra, **1** (1965), 5–10. MR **28:**5069
14. J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford University Press, London, 1965. MR **32:**1894
15. X. Zhan, *Computing the Extremal Positive Definite Solutions of a Matrix Equation*, SIAM J. Sci. Computing, **17** (1996), 1167–1174. MR **97g:**65074

Faculty of Economics and Business Administration, 125 Tzarigradsko chaussee, bl.3, Sofia University, Sofia 1113, Bulgaria
E-mail address: i_ivanov@feb.uni-sofia.bg

Laboratory of Mathematical Modelling, Shumen University, Shumen 9712, Bulgaria
E-mail address: v.hasanov@fmi.shu-bg.net

Department of Mathematics, Auburn University, Auburn, Alabama 36849–5310
E-mail address: uhligfd@auburn.edu