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ON THE ABSOLUTE MAHLER MEASURE OF POLYNOMIALS HAVING ALL ZEROS IN A SECTOR. II

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Abstract. Let α be an algebraic integer of degree d, not 0 or a root of unity, all of whose conjugates α_i are confined to a sector $|\arg z| \leq \theta$. In the paper *On the absolute Mahler measure of polynomials having all zeros in a sector*, G. Rhin and C. Smyth compute the greatest lower bound $c(\theta)$ of the absolute Mahler measure $(\prod_{i=1}^{d} \max(1, |\alpha_i|))^{1/d}$ of α, for θ belonging to nine subintervals of $[0, 2\pi/3]$. In this paper, we improve the result to thirteen subintervals of $[0, \pi]$ and extend some existing subintervals.

1. Introduction

Let $P(z) \neq z$ be a monic polynomial with integer coefficients, irreducible over the rationals, of degree $d \geq 1$, and having zeros $\alpha_1, \cdots, \alpha_d$. Its relative Mahler measure $M(P)$, given by

$$M(P) = \prod_{i=1}^{d} \max(1, |\alpha_i|),$$

is either 1 (if P is cyclotomic) or thought to be bounded away from 1 by an absolute constant (if P is not cyclotomic) [B1], [B2]. When the zeros of P are restricted to a closed set V which does not contain the whole unit circle, however, one can say much more. Then, from a result of Langevin [LA] there is a constant $C_V > 1$ such that the absolute Mahler measure $\Omega(P) := M(P)^{1/d}$ for such P is either 1 or else satisfies

$$\Omega(P) \geq C_V.$$

So we try to find the largest value for the constants C_V when V is the sector $\{z : |\arg z| \leq \theta\}$, where $0 \leq \theta < \pi$. We denote this best value by $c(\theta)$. It is clear that $c(\theta)$ is a nonincreasing function of θ and, using the polynomials $z^{2k+1} - 2$ as $k \to \infty$, that $c(\theta) \to 1$ as $\theta \to \pi$.

In a previous paper [RS], G. Rhin and C. Smyth succeeded in finding $c(\theta)$ exactly for θ in nine intervals. They conjectured that $c(\theta)$ is a “staircase” function of θ which is constant except for finitely many left discontinuities in any interval $[0, \Theta)$ for $\Theta < \pi$. They used auxiliary functions of the type

$$f_i(\theta) = \left\{ \max_{z \in W_\theta} \left| z^{a_i} \prod_j P_{ij}(z)^{e_{ij}} \right| \right\}^{-1/(2a_i + \sum_j e_{ij} \deg P_{ij})} \tag{1}$$

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in the sector $W_\theta = \{|z| < 1, |\arg z| \le \theta\}$. Then they find:

Theorem. *There is a continuous, monotonically decreasing function $f(\theta) > 1$ for $0 \le \theta \le 2\pi/3$ and there is a staircase function $g(\theta) > 1$ such that*

$$\min(f(\theta), g(\theta)) \le c(\theta) \le g(\theta) \qquad (0 \le \theta < \pi).$$

The function $f(\theta)$ is given by $f(\theta) := \max_{i=1}^{9} f_i(\theta)$. The function $g(\theta)$ is a decreasing staircase having left discontinuities at the angles given (in degrees) in Table 4 of [RS]. The corresponding absolute measure is the new smaller value of $g(\theta)$ which is the smallest value of $\Omega(P)$ that could be found, for P having all its zeros in $|\arg z| \le \theta$.

In the proof of the Theorem, Rhin and Smyth referred to Langevin's proof [LA], which has three basic ingredients:

(i) the observation that the set $V_1 = V \cap \{z \in \mathbb{C} : |z| \le 1\}$ has transfinite diameter less than 1,

(ii) a result of Kakeya to the effect that for any set W of transfinite diameter less than 1 and symmetric about the real axis there is a nonzero polynomial A with integer coefficients such that $\sup_{z \in W} |A(z)| < 1$,

(iii) deduction of $\Omega(P) \ge C_V$ from (i) and (ii) using $W := \{z : z \in V \text{and } \bar{z} \in V\}$.

For the computation of $f(\theta) = \max_{i=1}^{9} f_i(\theta)$, they use, for each f_i, an auxiliary polynomial A as in (ii), and they choose such A of the form $z^a R(z)$, where a is a positive integer and R is a reciprocal polynomial of degree r with integer coefficients, i.e.,

$$A(z) = z^a \prod_j P_j(z)^{e_j}.$$

The function

$$m(\theta) = \sup_{z \in W_\theta} |A(z)|^{\frac{1}{2a+r}} \tag{2}$$

is then associated with A. Then Langevin's argument of (iii) above gives

$$\Omega(P) \ge \frac{1}{m(\theta)} \qquad \text{if } \gcd(P, A) = 1$$

for P irreducible, of degree d, with integer coefficients. For, if $\alpha_1, \cdots, \alpha_d$ are the zeros of P, then, since $R(z) = z^r R(z^{-1})$, one has

$$\begin{aligned} 1 &\le \left| \prod_{i=1}^{d} \alpha_i^a R(\alpha_i) \right| = \prod_{|\alpha_i| \le 1} |\alpha_i^a R(\alpha_i)| \times \prod_{|\alpha_i| > 1} |\alpha_i^{a+r} R(\alpha_i^{-1})| \\ &= \prod_{|\alpha_i| \le 1} |\alpha_i^a R(\alpha_i)| \times \prod_{|\alpha_i| > 1} |(\alpha_i^{-1})^a R(\alpha_i^{-1})| \times \prod_{|\alpha_i| > 1} \alpha_i^{2a+r} \\ &\le m(\theta)^{(2a+r)d} M(P)^{2a+r} \end{aligned}$$

whence $\Omega(P) \ge 1/m(\theta)$.

Then each $f_i(\theta)$ was defined, as in equation (1), to be the function $1/m(\theta)$ corresponding to a polynomial A chosen so that $f(\theta_i) > g(\theta_i)$ and so that the length of the interval $[\theta_i, \theta_i']$ over which $f(\theta) > g(\theta)$ was as long as possible. Thus, if $g(\theta_i) = \Omega(P_*)$ (Table 4 in [RS]), then $\Omega(P_*) < f_i(\theta_i)$. From (2) it follows that P is a factor of A and that, among polynomials with all conjugates in $|\arg z| \le \theta_i$, only factors of A can have absolute measure less than $f_i(\theta_i)$. Now P_* does indeed divide

A, and in fact it has the smallest absolute measure among factors A of measure > 1. It follows that $\Omega(P_*)$ is the smallest value of the absolute measure for polynomials having all zeros in $|\arg z| \le \theta$ for $\theta \in [\theta_i, \theta_i']$. Hence, $c(\theta) = \Omega(P_*)$ for these θ.

One of the main problems in the previous paper was to find for each interval suitable polynomials to use to obtain a good auxiliary function. In fact they only used a heuristic process and produced a table of good P_j which were for almost all polynomials of one of the following six types:

$$\begin{array}{lll} z^n Q(z + z^{-1} - k) & (k = 3, 2, 1, 0) & \text{(types } 1, 2, 3, 4), \\ z^n S(z + z^{-1} - 2) & \text{where } S(x) = Q(1) x^n Q(1 + 1/x) & \text{(type 5)}, \\ z^n (Q(z) + Q(1/z)) & & \text{(type 6)}. \end{array}$$

Here Q is a degree n monic polynomial with small coefficients, also with $Q(1) = \pm 1$ for the fifth type. As pointed out in [RS, p. 301] "The reason for polynomials of these types giving good polynomials appears mysterious, however."

The second author gave in [WU] an algorithm improving the ones given by P. Borwein and T. Erdelyi [BE] and L. Habsieger and B. Salvy [HS] to find polynomials which have to be involved in such auxiliary functions $f_i(\theta)$. This method gives better lower bounds for $c(\theta)$ for four new intervals of θ between 0 and $7\pi/9$.

Table 1 shows the 13 intervals $[\theta_i,\ \theta_i']$ where $f(\theta) > g(\theta)$, so that $c(\theta) = g(\theta) = g(\theta_i)$ for θ in those intervals; i.e., $c(\theta)$ is known exactly. Here $c(\theta) = c(\theta_i) = \Omega(p)$ for $\theta \in [\theta_i, \theta_i']$. The fifth column presents the results from [RS]. The polynomial P is read off from Table 3. The function $f(\theta)$ is given by $f(\theta) := \max_{i=1}^{13} f_i(\theta)$ where the $f_i(\theta)$ are defined as in (1) and the a_i, P_{ij} and the e_{ij} are given by Table 2, using the polynomials of Table 3. The function $g(\theta)$ employs the polynomials listed in Table 4 of [RS], where we add the polynomials P_{25} and P_{31} of our Table 3.

Table 2 gives the auxiliary functions

$$A_i(z) = z^{a_i} \prod_j P_{ij}(z)^{e_{ij}}$$

used to compute $f_i(\theta)$ for $i = 1, \cdots, 13$.

Table 3 shows the reciprocal polynomials used in Tables 1 and 2, where $d = \deg P$ and $\varphi(P) = \max\{|\arg z| : P(z) = 0\}$.

TABLE 1.

i	$c(\theta)$	θ_i	θ_i'	θ_i' in [RS]	P
1	1.618034	0.000000	17.40	17.39	P_2
2	1.539222	26.408740	26.65	26.65	P_7
3	1.493633	30.440145	30.74	30.59	P_8
4	1.303055	47.941432	49.46	49.46	P_9
5	1.300734	50.830684	50.96		P_{12}
6	1.259269	60.890196	63.87	63.87	P_{15}
7	1.210608	73.631615	74.04	73.99	P_{19}
8	1.154618	80.241034	82.43	81.40	P_{21}
9	1.129338	86.708519	91.40	91.40	P_{23}
10	1.096504	101.353607	101.99		P_{25}
11	1.055423	112.647119	115.32	115.32	P_{29}
12	1.033097	127.355699	129.47		P_{31}
13	1.020306	137.102805	137.15		P_{37}

Table 2.

i	θ'_i	Polynomials P_{ij}	Exponents e_{ij}	a_i
1	17.40	P_1 P_2 P_3 P_4 P_5	21021 05610 00054 00140 00258	20829
2	26.65	P_1 P_6 P_7	26358 00726 00255	19499
3	30.74	P_1 P_8	29817 00605	18366
4	49.46	P_1 P_9 P_{12} P_{14}	19000 00964 00642 13732	11807
5	50.96	P_1 P_9 P_{10} P_{11} P_{12} P_{13} P_{14}	15859 01071 00267 00231 00287 00223 14466	11684
6	63.87	P_1 P_{14} P_{15} P_{16} P_{18}	10218 18924 00572 00369 00988	13958
7	74.04	P_1 P_{14} P_{17} P_{19} P_{20} P_{23}	06927 25721 00009 00460 00577 00257	12853
8	82.43	P_1 P_{14} P_{21} P_{22} P_{23} P_{24}	06227 18812 00865 01032 00584 05931	11749
9	91.40	P_1 P_{14} P_{21} P_{23} P_{24} P_{28}	06647 12953 00039 02344 09209 00918	12165
10	101.99	P_1 P_{14} P_{24} P_{25}	05043 06353 10453 00268	10268
		P_{26} P_{27} P_{28} P_{29}	00747 00563 04769 00344	
11	115.32	P_1 P_{14} P_{24} P_{28} P_{29} P_{30} P_{32}	03973 05717 05892 06225 01039 04497 00688	11251
12	129.47	P_1 P_{14} P_{24} P_{28} P_{30} P_{31}	01916 03282 02376 02271 03763 01257	10725
		P_{32} P_{33} P_{34} P_{35} P_{38}	03267 00301 00159 00756 00641	
13	137.15	P_1 P_{14} P_{24} P_{28} P_{30} P_{32}	01491 03082 01982 01696 02448 01997	15026
		P_{35} P_{36} P_{37} P_{38} P_{39}	01777 00576 00770 01624 00323	

Table 3.

P	$\Omega(P)$	$\varphi(P)$	d	Highest half of coefficients of P								
P_1	1.000000	0.000000	2	1	-2							
P_2	1.618034	0.000000	2	1	-3							
P_3	1.634404	17.665834	16	1	-25	281	-1873	8238	-25211	55246	-88031	102749
P_4	1.610559	18.863408	8	1	-12	58	-143	193				
P_5	1.611995	20.717188	12	1	-18	141	-628	1756	-3219	3935		
P_6	1.547928	26.301669	10	1	-14	85	-287	585	-739			
P_7	1.539222	26.408740	4	1	-5	9						
P_8	1.493633	30.440145	6	1	-8	26	-37					
P_9	1.303055	47.941432	6	1	-5	13	-17					
P_{10}	1.322672	49.112713	12	1	-11	62	-212	487	-788	923		
P_{11}	1.312282	49.353680	4	1	-3	5						
P_{12}	1.300734	50.830684	8	1	-7	26	-53	67				
P_{13}	1.308589	52.798885	14	1	-12	76	-302	832	-1669	2510	-2871	
P_{14}	1.000000	60.000000	2	1	-1							
P_{15}	1.259269	60.890196	6	1	-4	10	-13					
P_{16}	1.245865	68.365783	12	1	-7	30	-85	175	-268	309		
P_{17}	1.241661	72.761003	8	1	-5	16	-29	35				
P_{18}	1.238359	73.295530	8	1	-4	13	-23	28				
P_{19}	1.210608	73.631615	6	1	-3	7	-9					
P_{20}	1.208398	74.983796	8	1	-4	12	-21	25				
P_{21}	1.154618	80.241034	8	1	-3	8	-13	15				
P_{22}	1.189207	81.578941	4	2	-4	5						
P_{23}	1.129338	86.708519	6	1	-2	4	-5					
P_{24}	1.000000	90.000000	2	1	0							
P_{25}	1.096504	101.353606	10	1	-2	5	-9	12	-13			
P_{26}	1.106899	101.562999	6	1	-1	2	-3					
P_{27}	1.101001	106.852539	12	1	-2	6	-12	20	-26	29		
P_{28}	1.000000	108.000000	4	1	-1	1						
P_{29}	1.055423	112.647119	8	1	-1	2	-3	3				
P_{30}	1.000000	120.000000	2	1	1							
P_{31}	1.033097	127.355699	12	1	-1	2	-3	4	-5	5		
P_{32}	1.000000	128.571429	6	1	-1	1	-1					
P_{33}	1.040011	131.102998	10	1	-1	2	-3	3	-3			
P_{34}	1.039015	131.327187	14	1	0	1	-2	2	-3	3	-3	
P_{35}	1.000000	135.000000	4	1	0	0						
P_{36}	1.034105	136.742591	10	1	-1	2	-2	2	-3			
P_{37}	1.020306	137.102805	12	1	0	1	-1	1	-2	1		
P_{38}	1.000000	140.000000	6	1	0	0	-1					
P_{39}	1.000000	144.000000	4	1	1	1						

2. Search for good polynomials for the auxiliary functions

Let $\theta < \pi$ be a fixed angle. For a nonzero polynomial $A \in \mathbb{Z}[z]$ we define $a = a(A)$ to be the multiplicity of the root 0 of A and $\|A\| = \sup_{z \in W_\theta} |A(z)|$.

As we have seen in the introduction, the search for a good auxiliary function f for W_θ is equivalent to seeking a polynomial A (such that $z^{-a}A$ is reciprocal) in $\mathbb{Z}[z]$ and such that $\|A\|^{1/(a(A)+\deg A)}$ is as small as possible.

Let A_n be the polynomial of degree n such that

$$\|A_n\|^{\frac{1}{a(A_n)+n}} = \min_{\substack{A \in \mathbb{Z}[z] \\ \deg A = n}} \|A\|^{\frac{1}{a(A)+n}}.$$

We can define

$$\tau_\theta = \lim_{n\to\infty} \|A_n\|^{\frac{1}{a(A_n)+n}}$$

as a generalization of $t_{\mathbb{Z}}(W_\theta)$ which is the *integer transfinite diameter* of W_θ (in this case the exponent of $\|A_n\|$ is $1/n$).

Then the factors of the polynomials A_n lead to good auxiliary functions as follows. It is difficult to compute the polynomials A_n for n large, so we will compute some polynomials $A_n^{'}$ of sufficiently large degree (say 40) where the norm $\|A_n^{'}\|$ is sufficiently small and use their factors Q_j inside the function f. For this we use the following algorithm, which was already described in [WU].

Step 1. We use the LLL algorithm to find a polynomial $Q(x)$ of degree m (say 30) in $\mathbb{Z}[x]$ which has a small sup norm in the interval $[2\cos\theta, 2]$. Then we choose the integer a such that the polynomial $A = z^{a+m}Q(z+1/z)$ has a norm $\|A\|^{1/(2a+2m)}$ as small as possible.

It is well known that the LLL algorithm gives better results when used in low dimension. So, in Step 2 we will show that $A_n^{'}$ has an explicit factor of large degree.

Step 2. We use the previous bound and a generalization of the orthogonal Müntz-Legendre polynomials to find polynomials that must divide $A_n^{'} = A$ (where $n = a + 2m$) in $\mathbb{Z}[z]$.

Step 3. We use now the LLL algorithm to find new polynomial factors of $A_n^{'}$.

By this algorithm, we find the polynomials P_{25} and P_{31}, which not only improve the function $g(\theta)$, but also give better bounds for $c(\theta)$ in the intervals $[101.35, 101.99]$ and $[127.35, 129.47]$. We also find polynomials (for example P_{13}) that do not improve the function $g(\theta)$ but give us a new interval in which $c(\theta)$ is known exactly. Furthermore, we find some other polynomials for the auxiliary function which enable us to extend existing intervals.

3. Computation of the auxiliary functions

We use (for a fixed θ) the auxiliary function

$$f(z) = |z|^a \prod_{j=1}^{J} |Q_j(z)|^{e_j}$$

where the polynomials Q_j are those which have been computed in Section 2 and such that the positive rationals a and e_j satisfy the linear condition

$$2a + \sum_{j=1}^{J} e_j \deg Q_j = 1.$$

The optimal function f is obtained by semi-infinite linear programming [WU], [RS]. This gives four new intervals for $c(\theta)$. Moreover, technical improvements allow us to enlarge some intervals found earlier.

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