ON THE ABSOLUTE MAHLER MEASURE OF POLYNOMIALS HAVING ALL ZEROS IN A SECTOR. II

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Abstract. Let \( \alpha \) be an algebraic integer of degree \( d \), not 0 or a root of unity, all of whose conjugates \( \alpha_i \) are confined to a sector \( |\arg z| \leq \theta \). In the paper *On the absolute Mahler measure of polynomials having all zeros in a sector*, G. Rhin and C. Smyth compute the greatest lower bound \( c(\theta) \) of the absolute Mahler measure \( \prod_{i=1}^{d} \max(1, |\alpha_i|)^{1/d} \) of \( \alpha \), for \( \theta \) belonging to nine subintervals of \([0, 2\pi/3]\). In this paper, we improve the result to thirteen subintervals of \([0, \pi]\) and extend some existing subintervals.

1. Introduction

Let \( P(z) \neq z \) be a monic polynomial with integer coefficients, irreducible over the rationals, of degree \( d \geq 1 \), and having zeros \( \alpha_1, \ldots, \alpha_d \). Its relative Mahler measure \( M(P) \), given by

\[
M(P) = \prod_{i=1}^{d} \max(1, |\alpha_i|),
\]

is either 1 (if \( P \) is cyclotomic) or thought to be bounded away from 1 by an absolute constant (if \( P \) is not cyclotomic) \([B1], [B2]\). When the zeros of \( P \) are restricted to a closed set \( V \) which does not contain the whole unit circle, however, one can say much more. Then, from a result of Langevin \([LA]\) there is a constant \( C_V > 1 \) such that the absolute Mahler measure \( \Omega(P) := M(P)^{1/d} \) for such \( P \) is either 1 or else satisfies

\[
\Omega(P) \geq C_V.
\]

So we try to find the largest value for the constants \( C_V \) when \( V \) is the sector \( \{z : |\arg z| \leq \theta\} \), where \( 0 \leq \theta < \pi \). We denote this best value by \( c(\theta) \). It is clear that \( c(\theta) \) is a nonincreasing function of \( \theta \) and, using the polynomials \( z^{2k+1} - 2 \) as \( k \to \infty \), that \( c(\theta) \to 1 \) as \( \theta \to \pi \).

In a previous paper \([RS]\), G. Rhin and C. Smyth succeeded in finding \( c(\theta) \) exactly for \( \theta \) in nine intervals. They conjectured that \( c(\theta) \) is a “staircase” function of \( \theta \) which is constant except for finitely many left discontinuities in any interval \([0, \Theta)\) for \( \Theta < \pi \). They used auxiliary functions of the type

\[
f_i(\theta) = \max_{z \in W_\theta} \left| z^{a_i} \prod_{j} P_{ij}(z)^{e_{ij}} \right|^{-1/(2a_i + \sum_j e_{ij} \deg P_{ij})}
\]

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in the sector $W_\theta = \{ |z| < 1, \arg z | \leq \theta \}$. Then they find:

**Theorem.** There is a continuous, monotonically decreasing function $f(\theta) > 1$ for $0 \leq \theta \leq 2\pi/3$ and there is a staircase function $g(\theta) > 1$ such that

$$\min(f(\theta), g(\theta)) \leq c(\theta) \leq g(\theta) \quad (0 \leq \theta < \pi).$$

The function $f(\theta)$ is given by $f(\theta) : = \max_{i=1}^9 f_i(\theta)$. The function $g(\theta)$ is a decreasing staircase having left discontinuities at the angles given (in degrees) in Table 4 of [RS]. The corresponding absolute measure is the new smaller value of $g(\theta)$ which is the smallest value of $\Omega(P)$ that could be found, for $P$ having all its zeros in $|\arg z| \leq \theta$.

In the proof of the Theorem, Rhin and Smyth referred to Langevin’s proof [LA], which has three basic ingredients:

(i) the observation that the set $V_1 = V \cap \{ z \in \mathbb{C} : |z| \leq 1 \}$ has transfinite diameter less than $1$,

(ii) a result of Kakeya to the effect that for any set $W$ of transfinite diameter less than $1$ and symmetric about the real axis there is a nonzero polynomial $A$ with integer coefficients such that $\sup_{z \in W} |A(z)| < 1$,

(iii) deduction of $\Omega(P) \geq C_V$ from (i) and (ii) using $W : = \{ z : z \in V \text{ and } \bar{z} \in V \}$.

For the computation of $f(\theta) = \max_{i=1}^9 f_i(\theta)$, they use, for each $f_i$, an auxiliary polynomial $A$ as in (ii), and they choose such $A$ of the form $z^a R(z)$, where $a$ is a positive integer and $R$ is a reciprocal polynomial of degree $r$ with integer coefficients, i.e.,

$$A(z) = z^a \prod_j P_j(z)^{\epsilon_j}.$$ 

The function

$$m(\theta) = \sup_{z \in W_\theta} |A(z)|^{\frac{1}{2\pi r}}$$

is then associated with $A$. Then Langevin’s argument of (iii) above gives

$$\Omega(P) \geq \frac{1}{m(\theta)} \quad \text{if } \gcd(P, A) = 1$$

for $P$ irreducible, of degree $d$, with integer coefficients. For, if $\alpha_1, \cdots, \alpha_d$ are the zeros of $P$, then, since $R(z) = z^r R(z^{-1})$, one has

$$1 \leq \prod_{i=1}^d |a_i^z R(\alpha_i)| = \prod_{|\alpha_i| \leq 1} |a_i^z R(\alpha_i)| \times \prod_{|\alpha_i| > 1} |a_i^{a+r} R(\alpha_i^{-1})|$$

$$= \prod_{|\alpha_i| \leq 1} |a_i^z R(\alpha_i)| \times \prod_{|\alpha_i| > 1} |(\alpha_i^{-1})^a R(\alpha_i^{-1})| \times \prod_{|\alpha_i| > 1} \alpha_i^{2a+r}$$

$$\leq m(\theta)^{(2a+r)d} M(P)^{2a+r}$$

whence $\Omega(P) \geq 1/m(\theta)$.

Then each $f_i(\theta)$ was defined, as in equation (1), to be the function $1/m(\theta)$ corresponding to a polynomial $A$ chosen so that $f(\theta_i) > g(\theta_i)$ and so that the length of the interval $[\theta_i, \theta'_i]$ over which $f(\theta) > g(\theta)$ was as long as possible. Thus, if $g(\theta_i) = \Omega(P_\star)$ (Table 4 in [RS]), then $\Omega(P_\star) < f_i(\theta_i)$. From (2) it follows that $P_\star$ is a factor of $A$ and that, among polynomials with all conjugates in $|\arg z| \leq \theta_i$, only factors of $A$ can have absolute measure less than $f_i(\theta_i)$. Now $P_\star$ does indeed divide
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A, and in fact it has the smallest absolute measure among factors A of measure > 1. It follows that \( \Omega(P) \) is the smallest value of the absolute measure for polynomials having all zeros in \( |\arg z| \leq \theta \) for \( \theta \in [\theta_i, \theta'_i] \). Hence, \( c(\theta) = \Omega(P) \) for these \( \theta \).

One of the main problems in the previous paper was to find for each interval suitable polynomials to use to obtain a good auxiliary function. In fact they only used a heuristic process and produced a table of good \( P_j \) which were for almost all polynomials of one of the following six types:

\[
\begin{align*}
  z^nQ(z+z^{-1}-k) & \quad (k=3,2,1,0) \quad \text{(types 1, 2, 3, 4)}, \\
  z^nS(z+z^{-1}-2) & \quad \text{where } S(x) = Q(1)x^nQ(1+1/x) \quad \text{(type 5)}, \\
  z^n(Q(z)+Q(1/z)) & \quad \text{(type 6)}. 
\end{align*}
\]

Here \( Q \) is a degree \( n \) monic polynomial with small coefficients, also with \( Q(1) = \pm 1 \) for the fifth type. As pointed out in [RS] p. 301 “The reason for polynomials of these types giving good polynomials appears mysterious, however.”

The second author gave in [WU] an algorithm improving the ones given by P. Borwein and T. Erdelyi [BE] and L. Habsieger and B. Salvy [RS] to find polynomials which have to be involved in such auxiliary functions \( f_i(\theta) \). This method gives better lower bounds for \( c(\theta) \) for four new intervals of \( \theta \) between 0 and \( 7\pi/9 \).

Table 1 shows the 13 intervals \([\theta_i, \theta'_i]\) where \( f(\theta) > g(\theta) \), so that \( c(\theta) = g(\theta_i) = g(\theta_i) \) for \( \theta \) in those intervals; i.e., \( c(\theta) \) is known exactly. Here \( c(\theta) = c(\theta_i) = \Omega(p) \) for \( \theta \in [\theta_i, \theta'_i] \). The fifth column presents the results from [RS]. The polynomial \( P \) is read off from Table 3. The function \( f(\theta) \) is given by \( f(\theta) := \max_{i=1}^{13} f_i(\theta) \) where the \( f_i(\theta) \) are defined as in (1) and the \( a_i, P_j \) and the \( e_{ij} \) are given by Table 4 using the polynomials of Table 3. The function \( g(\theta) \) employs the polynomials listed in Table 4 of [RS], where we add the polynomials \( P_{25} \) and \( P_{31} \) of our Table 5.

Table 2 gives the auxiliary functions

\[
A_i(z) = z^{a_i} \prod_j P_{ij}(z)^{e_{ij}}
\]

used to compute \( f_i(\theta) \) for \( i = 1, \cdots, 13 \).

Table 3 shows the reciprocal polynomials used in Tables 1 and 2 where \( d = \deg P \) and \( \omega(P) = \max\{|\arg z|: P(z) = 0\} \).

Table 1.

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<tr>
<th>( i )</th>
<th>( \theta(\theta) )</th>
<th>( \theta_i )</th>
<th>( \theta'_i ) in [RS]</th>
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</table>
2. Search for good polynomials for the auxiliary functions

Let $\theta < \pi$ be a fixed angle. For a nonzero polynomial $A \in \mathbb{Z}[z]$ we define $a = a(A)$ to be the multiplicity of the root 0 of $A$ and $\|A\| = \sup_{z \in W_0} |A(z)|$.

As we have seen in the introduction, the search for a good auxiliary function $f$ for $W_0$ is equivalent to seeking a polynomial $A$ (such that $z^{-a}A$ is reciprocal) in $\mathbb{Z}[z]$ and such that $\|A\|^{1/(a(A)+\deg A)}$ is as small as possible.

Let $A_n$ be the polynomial of degree $n$ such that

$$\|A_n\|^{1/(n^{a_n}+n)} = \min_{A \in \mathbb{Z}[z]} \|A\|^{1/(a(A)+n)}.$$
We can define
\[ \tau_{\theta} = \lim_{n \to \infty} \| A_n \|_{\pi + \theta + \infty} \]
as a generalization of \( t_2(W_\theta) \) which is the integer transfinite diameter of \( W_\theta \) (in this case the exponent of \( \| A_n \| \) is \( 1/n \)).

Then the factors of the polynomials \( A_n \) lead to good auxiliary functions as follows. It is difficult to compute the polynomials \( A_n \) for \( n \) large, so we will compute some polynomials \( A'_n \) of sufficiently large degree (say 40) where the norm \( \| A'_n \| \) is sufficiently small and use their factors \( Q_j \) inside the function \( f \). For this we use the following algorithm, which was already described in [WU].

**Step 1.** We use the LLL algorithm to find a polynomial \( Q(x) \) of degree \( m \) (say 30) in \( \mathbb{Z}[x] \) which has a small sup norm in the interval \( [2 \cos \theta, 2] \). Then we choose the integer \( a \) such that the polynomial \( A = z^{a+m}Q(z + 1/z) \) has a norm \( \| A \|^{1/(2a+2m)} \) as small as possible.

It is well known that the LLL algorithm gives better results when used in low dimension. So, in Step 2 we will show that \( A'_n \) has an explicit factor of large degree.

**Step 2.** We use the previous bound and a generalization of the orthogonal Muntz-Legendre polynomials to find polynomials that must divide \( A'_n = A \) (where \( n = a + 2m \)) in \( \mathbb{Z}[z] \).

**Step 3.** We use now the LLL algorithm to find new polynomial factors of \( A'_n \).

By this algorithm, we find the polynomials \( P_{25} \) and \( P_{31} \), which not only improve the function \( g(\theta) \), but also give better bounds for \( c(\theta) \) in the intervals \([101.35, 101.99]\) and \([127.35, 129.47]\). We also find polynomials (for example \( P_{13} \)) that do not improve the function \( g(\theta) \) but give us a new interval in which \( c(\theta) \) is known exactly. Furthermore, we find some other polynomials for the auxiliary function which enable us to extend existing intervals.

### 3. Computation of the auxiliary functions

We use (for a fixed \( \theta \)) the auxiliary function
\[ f(z) = |z|^a \prod_{j=1}^{J} |Q_j(z)|^{e_j} \]
where the polynomials \( Q_j \) are those which have been computed in Section 2 and such that the positive rationals \( a \) and \( e_j \) satisfy the linear condition
\[ 2a + \sum_{j=1}^{J} e_j \deg Q_j = 1. \]

The optimal function \( f \) is obtained by semi-infinite linear programming [WU], [RS]. This gives four new intervals for \( c(\theta) \). Moreover, technical improvements allow us to enlarge some intervals found earlier.

### References


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