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COMPUTING PERIODIC SOLUTIONS OF LINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS BY WAVEFORM RELAXATION

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ABSTRACT. We propose an algorithm, which is based on the waveform relaxation (WR) approach, to compute the periodic solutions of a linear system described by differential-algebraic equations. For this kind of two-point boundary problems, we derive an analytic expression of the spectral set for the periodic WR operator. We show that the periodic WR algorithm is convergent if the supremum value of the spectral radii for a series of matrices derived from the system is less than 1. Numerical examples, where discrete waveforms are computed with a backward-difference formula, further illustrate the correctness of the theoretical work in this paper.

1. INTRODUCTION

We often need to compute periodic solutions of a linear dynamic system under a known periodic excitation in engineering applications, such as circuit simulation and mechanical modelling. These systems are described by linear differential-algebraic equations (DAEs). They have a common form as follows:

$$\begin{cases} M\dot{x}(t) + Ax(t) + By(t) = f_1(t), \\ Cx(t) + Ny(t) = f_2(t), \quad x(0) = x(T), \quad t \in [0, T], \end{cases} \tag{1}$$

where M and N are, respectively, $n_1 \times n_1$ and $n_2 \times n_2$ nonsingular matrices, A is an $n_1 \times n_1$ matrix, B is an $n_1 \times n_2$ matrix, C is an $n_2 \times n_1$ matrix, $f_1(t) \in \mathbf{R}^{n_1}$ and $f_2(t) \in \mathbf{R}^{n_2}$ for all $t \in [0, T]$ are two known input functions with period T, and $x(t) \in \mathbf{R}^{n_1}$ and $y(t) \in \mathbf{R}^{n_2}$ for all $t \in [0, T]$ are to be computed. This is a two-point boundary problem [1]. It is also obvious that $y(0) = N^{-1}(f_2(0) - Cx(0))$ for (1). Furthermore, $y(0) = y(T)$ due to $x(0) = x(T)$ and $f_2(0) = f_2(T)$. We assume that the boundary condition on periodic solutions of (1) is consistent in what follows. The consistency of boundary conditions for a periodic solution of (1) means that the condition $x(0) = x(T)$ implies $\dot{x}(0) = \dot{x}(T)$ and $y(0) = y(T)$.

Waveform relaxation (WR) is a novel splitting technique in engineering applications. Numerical algorithms incorporated with WR are relaxation-based methods and they are suitable for scientific computations of transient responses for very large

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dynamic systems. In fact WR was first proposed to simulate MOS VLSI circuits [3, 6, 16]. It consists of a "divide-and-conquer" approach; namely, at differential equations level, it decouples a large system into a number of simplified subsystems in time-domain [2, 5, 8, 11, 13]. Therefore the latency and multirate behaviors of systems can be effectively exploited. Based on WR, we can also directly simulate lossless transmission lines with nonlinear terminations [10].

For a periodic system, after its transient response has practically decayed to zero, the steady-state response is periodic with the same period as the excitation. Since we are interested in the steady-state periodic solution, the numerical algorithm should only compute the response over one period directly without first computing the transient response that precedes it in order to save a great deal of computations. The periodic WR approach to be reported here is an effectively iterative way in function space. The resulting iterative systems with periodic constraint can be numerically solved by the sophisticated codes of DAEs or ODEs on boundary problems in the public domain (see, e.g., [1]).

Let

$$M = M_1 - M_2, A = A_1 - A_2, B = B_1 - B_2, C = C_1 - C_2, N = N_1 - N_2,$$

and $[(x^{(0)})^t(\cdot), (y^{(0)})^t(\cdot)]^t$ is a given initial guess. Now, we present a periodic WR algorithm to compute the steady-state periodic response over one period for (1). The periodic WR algorithm of (1) is

$$\begin{cases} M_1\dot{x}^{(k)}(t) + A_1x^{(k)}(t) + B_1y^{(k)}(t) \\ \qquad = M_2\dot{x}^{(k-1)}(t) + A_2x^{(k-1)}(t) + B_2y^{(k-1)}(t) + f_1(t), \\ C_1x^{(k)}(t) + N_1y^{(k)}(t) = C_2x^{(k-1)}(t) + N_2y^{(k-1)}(t) + f_2(t), \\ x^{(k)}(0) = x^{(k)}(T), \quad t \in [0,T], \quad k = 1, 2, \ldots, \end{cases} \tag{2}$$

where we suppose that M_1 and N_1 are nonsingular in this paper. In order to preserve the consistency of the boundary conditions for every periodic iteration, an initial guess $[(x^{(0)})^t\ (t),\ (y^{(0)})^t\ (t)]^t$ in (2) should satisfy $[(x^{(0)})^t\ (0),\ (y^{(0)})^t\ (0)]^t = [(x^{(0)})^t\ (T),\ (y^{(0)})^t\ (T)]^t$ and $\dot{x}^{(0)}(0) = \dot{x}^{(0)}(T)$. For any constant guess, the required boundary conditions are naturally held.

Often, for a linear system we consider its WR solutions in $C([0,T], \mathbf{C}^n)$ or $L^2([0,T], \mathbf{C}^n)$ where $n = n_1 + n_2$. This treatment can greatly simplify the theoretical analyses on WR. That is, the convergence behaviors of WR are mainly decided by the corresponding periodic WR operators in these function spaces.

The WR solutions of initial value problems of equations as in (1) were reported in [7]. The expressions of spectra and pseudospectra for their WR operators were also clearly understood [9]. However, there are few papers to theoretically analyze the spectra of the periodic WR operator for linear dynamic systems in the WR literature. The linear system of ordinary differential equations (ODEs) in [14] is a special one with an identity coefficient matrix for $\dot{x}(t)$.

In the paper we discuss the periodic WR operator derived from (2) where an analytic expression of its spectra is obtained. According to this expression we can conveniently provide the convergence condition of (2), namely if the supremum value of the spectral radii for a series of matrices derived from the splitting matrices of (1) is less than 1. We also present the spectral set of discrete periodic WR operators. A finite-difference method is then used for solving the decoupled systems in (2) for all k in our test examples. Numerical experiments are given to support the correctness of the expressions established here.

2. Spectra of periodic WR operators and convergent splittings

For a matrix $P \in \mathbf{C}^{n\times n}$ it is called noncritical w.r.t. (with respect to) T if $ip\omega \notin \sigma(P)$ for $p = 0, \pm 1, \ldots$ where $\omega = \dfrac{2\pi}{T}, i = \sqrt{-1}$, and $\sigma(P)$ is the spectral set of P. In other words, P is noncritical if for all p the matrices $ip\omega I + P$ are invertible where I is the identity matrix with dimensions $n \times n$.

The existence condition of periodic solutions for a system of ODEs with an identity coefficient matrix for its derivative term depends on the following lemma [14].

Lemma 1. *If P is noncritical w.r.t. T, then the system of ODEs*

$$\begin{cases} \dot{w}(t) + Pw(t) = h(t), \\ w(0) = w(T), \quad t \in [0, T] \end{cases} \tag{3}$$

has a unique solution for any $h \in C([0,T], \mathbf{C}^n)$ or $L^2([0,T], \mathbf{C}^n)$ with period T.

2.1. The general case: DAEs.

First, based on Lemma 1 and using elementary operations we derive the existence conditions of solutions on (1) and (2) for any given k.

Lemma 2. *The system of DAEs given in* (1) *has a unique periodic solution if $M^{-1}(A - BN^{-1}C)$ is noncritical w.r.t. T. Similarly, for any fixed k the decoupled system in* (2) *has a unique periodic solution if $M_1^{-1}(A_1 - B_1N_1^{-1}C_1)$ is noncritical w.r.t. T.*

In this paper we suppose that periodic solutions of (1) and (2) always exist. We will study under what condition the iterative solutions of (2) converge to the periodic solution of (1). We denote $D_1 = A_1 - B_1N_1^{-1}C_1$, $D_2 = A_2 - B_1N_1^{-1}C_2$, $E = B_2 - B_1N_1^{-1}N_2$, and $g(t) = f_1(t) - B_1N_1^{-1}f_2(t)$ on $[0,T]$.

By eliminating $y^{(k)}(t)$ from the first and second equations of (2), we have

$$\begin{cases} M_1\dot{x}^{(k)}(t) + D_1x^{(k)}(t) = M_2\dot{x}^{(k-1)}(t) + D_2x^{(k-1)}(t) + Ey^{(k-1)}(t) + g(t), \\ x^{(k)}(0) = x^{(k)}(T), \quad t \in [0,T]. \end{cases} \tag{4}$$

The solution of (4) is

$$\begin{aligned} x^{(k)}(t) = {} & M_1^{-1}M_2x^{(k-1)}(t) + \int_0^t e^{M_1^{-1}D_1(s-t)}M_1^{-1}(D_2 - D_1M_1^{-1}M_2)x^{(k-1)}(s)ds \\ & + e^{-M_1^{-1}D_1t}(x^{(k)}(0) - M_1^{-1}M_2x^{(k-1)}(0)) \\ & + \int_0^t e^{M_1^{-1}D_1(s-t)}M_1^{-1}(Ey^{(k-1)}(s) + g(s))ds. \end{aligned} \tag{5}$$

We use the constraint condition $x^{(k)}(0) = x^{(k)}(T)$ to solve for $x^{(k)}(0)$ in (5):

$$\begin{aligned} x^{(k)}(0) = {} & M_1^{-1}M_2x^{(k-1)}(0) \\ & + (I - e^{-M_1^{-1}D_1T})^{-1}\int_0^T e^{M_1^{-1}D_1(s-T)}M_1^{-1}(D_2 - D_1M_1^{-1}M_2) \\ & \times x^{(k-1)}(s)ds \\ & + (I - e^{-M_1^{-1}D_1T})^{-1}\int_0^T e^{M_1^{-1}D_1(s-T)}M_1^{-1}(Ey^{(k-1)}(s) + g(s))ds. \end{aligned} \tag{6}$$

After substituting (6) into (5), we arrive at a formula for $x^{(k)}(t)$ on $[0,T]$ without any boundary values. It is

$$x^{(k)} = \mathcal{K}_1 x^{(k-1)} + \mathcal{K}_2 y^{(k-1)} + \varphi_1, \tag{7}$$

where, for $u \in C([0,T],\mathbf{C}^{n_1})$ or $L^2([0,T],\mathbf{C}^{n_1})$

$$\begin{aligned}\mathcal{K}_1 u(t) = {} & M_1^{-1}M_2 u(t) + \int_0^t e^{M_1^{-1}D_1(s-t)} M_1^{-1}(D_2 - D_1 M_1^{-1} M_2)u(s)ds \\ & + e^{-M_1^{-1}D_1 t}(I - e^{-M_1^{-1}D_1 T})^{-1} \\ & \times \int_0^T e^{M_1^{-1}D_1(s-T)} M_1^{-1}(D_2 - D_1 M_1^{-1} M_2)u(s)ds,\end{aligned} \tag{8}$$

for $v \in C([0,T],\mathbf{C}^{n_2})$ or $L^2([0,T],\mathbf{C}^{n_2})$

$$\begin{aligned}\mathcal{K}_2 v(t) \quad = \quad & \int_0^t e^{M_1^{-1}D_1(s-t)} M_1^{-1} E v(s)ds \\ & + e^{-M_1^{-1}D_1 t}(I - e^{-M_1^{-1}D_1 T})^{-1} \int_0^T e^{M_1^{-1}D_1(s-T)} M_1^{-1} E v(s)ds,\end{aligned} \tag{9}$$

and on $[0,T]$

$$\begin{aligned}\varphi_1(t) \quad = \quad & \int_0^t e^{M_1^{-1}D_1(s-t)} M_1^{-1} g(s)ds \\ & + e^{-M_1^{-1}D_1 t}(I - e^{-M_1^{-1}D_1 T})^{-1} \int_0^T e^{M_1^{-1}D_1(s-T)} M_1^{-1} g(s)ds.\end{aligned} \tag{10}$$

On the other hand, we also have

$$y^{(k)} = \mathcal{K}_3 x^{(k-1)} + \mathcal{K}_4 y^{(k-1)} + \varphi_2, \tag{11}$$

where $\mathcal{K}_3 u(t) = (N_1^{-1}C_2 - N_1^{-1}C_1\mathcal{K}_1)u(t)$ for $u \in C([0,T],\mathbf{C}^{n_1})$ or $L^2([0,T],\mathbf{C}^{n_1})$, $\mathcal{K}_4 v(t) = (N_1^{-1}N_2 - N_1^{-1}C_1\mathcal{K}_2)v(t)$ for $v \in C([0,T],\mathbf{C}^{n_2})$ or $L^2([0,T],\mathbf{C}^{n_2})$, and $\varphi_2(t) = -N_1^{-1}C_1\varphi_1(t) + N_1^{-1}f_2(t)$ on $[0,T]$.

We can compactly write together (7) and (11) as

$$z^{(k)} = \mathcal{K} z^{(k-1)} + \varphi, \tag{12}$$

where $z^{(l)}(t) = [(x^{(l)})^t(t), (y^{(l)})^t(t)]^t$, $\mathcal{K} = \begin{bmatrix} \mathcal{K}_1 & \mathcal{K}_2 \\ \mathcal{K}_3 & \mathcal{K}_4 \end{bmatrix}$, and $\varphi(t) = [\varphi_1^t(t), \varphi_2^t(t)]^t$ on $[0,T]$. The linear operator $\mathcal{K}$ is a periodic WR operator of (1). It is the sum of a constant matrix multiplication operator and a linear periodic convolution operator where the periodic convolution operator is compact (see [14]). The spectral set of $\mathcal{K}$ is bounded and closed in $C([0,T],\mathbf{C}^n)$ or $L^2([0,T],\mathbf{C}^n)$ since $\mathcal{K}$ is a linear bounded operator. We can analytically write out its spectral set.

Definition 1. For a periodic integral function $w(t)$ on $[0,T]$, its Fourier coefficients are

$$\tilde{w}_p = \frac{1}{T}\int_0^T w(t)e^{-ip\omega t}dt, \quad \omega = \frac{2\pi}{T}, \quad p = 0, \pm 1, \ldots.$$

Let us also define the matrices $Q(ip\omega)$ for $p = 0, \pm 1, \pm 2, \ldots$ as

$$\begin{aligned}Q(ip\omega) = {} & \begin{bmatrix} M_1^{-1}M_2 & 0 \\ N_1^{-1}C_2 - N_1^{-1}C_1 M_1^{-1}M_2 & N_1^{-1}N_2 \end{bmatrix} \\ & + \begin{bmatrix} (ip\omega I + M_1^{-1}D_1)^{-1}M_1^{-1}(D_2 - D_1 M_1^{-1}M_2) & (ip\omega I + M_1^{-1}D_1)^{-1}M_1^{-1}E \\ -N_1^{-1}C_1(ip\omega I + M_1^{-1}D_1)^{-1}M_1^{-1}(D_2 - D_1 M_1^{-1}M_2) & -N_1^{-1}C_1(ip\omega I + M_1^{-1}D_1)^{-1}M_1^{-1}E \end{bmatrix}.\end{aligned} \tag{13}$$

Theorem 1. *The spectral set of the linear operator $\mathcal{K}$ in* (12) *is*

$$\sigma(\mathcal{K}) = \overline{\bigcup\{\sigma(Q(ip\omega)) : p = 0, \pm 1, \ldots\}}. \tag{14}$$

Proof. Because the linear operator $\mathcal{K}$ is a compact perturbation of the constant matrix multiplication operator $\tilde{Q}$ where

$$\tilde{Q} = \begin{bmatrix} M_1^{-1}M_2 & 0 \\ N_1^{-1}C_2 - N_1^{-1}C_1M_1^{-1}M_2 & N_1^{-1}N_2 \end{bmatrix}, \tag{15}$$

any element $\lambda \in \sigma(\mathcal{K})\backslash\sigma(\tilde{Q})$ is an isolated eigenvalue of $\mathcal{K}$ (see [12]). Let $z(t) = [x^t(t), y^t(t)]^t$ and assume (λ, z) is an eigenpair of the operator $\mathcal{K}$, by calculating the Fourier series coefficients of the relationship $\mathcal{K}z = \lambda z$. Then we can derive that $\sigma(\mathcal{K})\backslash\sigma(\tilde{Q}) \subseteq \bigcup\{\sigma(Q(p)) : p = 0, \pm 1, \ldots\}$. In fact, for $p = 0, \pm 1, \ldots$,

$$\widetilde{\mathcal{K}z}_p = \frac{1}{T}\int_0^T \mathcal{K}z(t)e^{-ip\omega t}dt = \begin{bmatrix} \widetilde{\mathcal{K}_1x}_p + \widetilde{\mathcal{K}_2y}_p \\ \widetilde{\mathcal{K}_3x}_p + \widetilde{\mathcal{K}_4y}_p \end{bmatrix}, \tag{16}$$

where

$$\widetilde{\mathcal{K}_1x}_p = \frac{1}{T}\int_0^T \mathcal{K}_1x(t)e^{-ip\omega t}dt, \quad \widetilde{\mathcal{K}_2y}_p = \frac{1}{T}\int_0^T \mathcal{K}_2y(t)e^{-ip\omega t}dt,$$

and

$$\widetilde{\mathcal{K}_3x}_p = \frac{1}{T}\int_0^T \mathcal{K}_3x(t)e^{-ip\omega t}dt, \quad \widetilde{\mathcal{K}_4y}_p = \frac{1}{T}\int_0^T \mathcal{K}_4y(t)e^{-ip\omega t}dt.$$

Let us calculate the term $\widetilde{\mathcal{K}_1x}_p$ in (16). We denote $\tilde{z}_p = [\tilde{x}_p^t, \tilde{y}_p^t]^t$. From (8), by exchanging the order of integrals and using of the equality $e^{ip\omega T} = 1$, we have

(17)

$$\begin{aligned}
\widetilde{\mathcal{K}_1x}_p &= \frac{1}{T}\int_0^T \mathcal{K}_1x(t)e^{-ip\omega t}dt \\
&= M_1^{-1}M_2\tilde{x}_p + \frac{1}{T}\int_0^T\Big(\int_s^T e^{-(ip\omega I + M_1^{-1}D_1)t}dt\Big)e^{M_1^{-1}D_1 s} \\
&\quad \times M_1^{-1}(D_2 - D_1M_1^{-1}M_2)x(s)ds \\
&\quad + \frac{1}{T}\int_0^T\Big(\int_0^T e^{-(ip\omega I + M_1^{-1}D_1)t}dt\Big)(I - e^{-M_1^{-1}D_1T})^{-1}e^{M_1^{-1}D_1(s-T)} \\
&\quad \times M_1^{-1}(D_2 - D_1M_1^{-1}M_2)x(s)ds \\
&= M_1^{-1}M_2\tilde{x}_p + \Big[\frac{1}{T}\int_0^T (ip\omega I + M_1^{-1}D_1)^{-1}e^{-ip\omega s}M_1^{-1}(D_2 - D_1M_1^{-1}M_2)x(s)ds \\
&\quad - \frac{1}{T}\int_0^T (ip\omega I + M_1^{-1}D_1)^{-1}e^{M_1^{-1}D_1(s-T)}M_1^{-1}(D_2 - D_1M_1^{-1}M_2)x(s)ds\Big] \\
&\quad + \frac{1}{T}\int_0^T (ip\omega I + M_1^{-1}D_1)^{-1}e^{M_1^{-1}D_1(s-T)}M_1^{-1}(D_2 - D_1M_1^{-1}M_2)x(s)ds \\
&= [M_1^{-1}M_2 + (ip\omega I + M_1^{-1}D_1)^{-1}M_1^{-1}(D_2 - D_1M_1^{-1}M_2)]\tilde{x}_p.
\end{aligned}$$

Similarly,

$$\widetilde{\mathcal{K}_2y}_p = \frac{1}{T}\int_0^T \mathcal{K}_2y(t)e^{-ip\omega t}dt = (ip\omega I + M_1^{-1}D_1)^{-1}M_1^{-1}E\tilde{y}_p. \tag{18}$$

According to the expressions of $\mathcal{K}_3x(t)$ and $\mathcal{K}_4y(t)$, by (17) and (18) we can also calculate the terms $\widetilde{\mathcal{K}_3x}_p$ and $\widetilde{\mathcal{K}_4y}_p$ for $p = 0, \pm 1, \ldots$. Thus,

$$\widetilde{\mathcal{K}z}_p = Q(ip\omega)\tilde{z}_p, \quad p = 0, \pm 1, \ldots. \tag{19}$$

Because there exists some p such that $\tilde{z}_p \neq 0$ since $z \neq 0$, the equality $\widetilde{\mathcal{K}z}_p = \lambda\widetilde{z}_p$ and (19) imply $\lambda \in \sigma(Q(ip\omega))$. It follows that

$$\sigma(\mathcal{K}) \subseteq \sigma(\tilde{Q}) \bigcup \{\sigma(Q(ip\omega)) : p = 0, \pm 1, \ldots\} = \overline{\bigcup \{\sigma(Q(ip\omega)) : p = 0, \pm 1, \ldots\}}.$$

On the other hand, for any fixed p ($p \in \{0, \pm 1, \ldots\}$) let (λ_p, Z_p) where $Z_p = [X_p^t, Y_p^t]^t$ be an eigenpair of the matrix $Q(ip\omega)$. If we define $z_p(t) = Z_p e^{ip\omega t}$ such that $z_p(t) = [x_p^t(t), y_p^t(t)]^t$ ($x_p(t) = X_p e^{ip\omega t}, y_p(t) = Y_p e^{ip\omega t}$), then $\mathcal{K}z_p = \lambda_p z_p$. Let us see the case of $\mathcal{K}_1 x_p(t)$. By (8) and the equality $e^{ip\omega T} = 1$, we have

(20)

$$\begin{aligned}
\mathcal{K}_1 x_p(t) &= M_1^{-1} M_2 X_p e^{ip\omega t} \\
&\quad + \left(\int_0^t e^{(ip\omega I + M_1^{-1} D_1)s} ds\right) e^{-M_1^{-1} D_1 t} M_1^{-1} (D_2 - D_1 M_1^{-1} M_2) X_p \\
&\quad + e^{-M_1^{-1} D_1 t} (I - e^{-M_1^{-1} D_1 T})^{-1} \left(\int_0^T e^{(ip\omega I + M_1^{-1} D_1)s} ds\right) e^{-M_1^{-1} D_1 T} M_1^{-1} \\
&\quad \times (D_2 - D_1 M_1^{-1} M_2) X_p \\
&= M_1^{-1} M_2 X_p e^{ip\omega t} + [(ip\omega I + M_1^{-1} D_1)^{-1} M_1^{-1} (D_2 - D_1 M_1^{-1} M_2) X_p e^{ip\omega t} \\
&\quad - (ip\omega I + M_1^{-1} D_1)^{-1} e^{-M_1^{-1} D_1 t} M_1^{-1} (D_2 - D_1 M_1^{-1} M_2) X_p] \\
&\quad + e^{-M_1^{-1} D_1 t} (ip\omega I + M_1^{-1} D_1)^{-1} M_1^{-1} (D_2 - D_1 M_1^{-1} M_2) X_p \\
&= [M_1^{-1} M_2 + (ip\omega I + M_1^{-1} D_1)^{-1} M_1^{-1} (D_2 - D_1 M_1^{-1} M_2)] x_p(t).
\end{aligned}$$

Similarly, we may calculate the terms $\mathcal{K}_2 y_p(t)$, $\mathcal{K}_3 x_p(t)$, and $\mathcal{K}_4 y_p(t)$. These calculations result in $\mathcal{K}z_p = \lambda_p z_p$, namely $\sigma(Q(ip\omega)) \subseteq \sigma(\mathcal{K})$. It follows that

$$\overline{\bigcup \{\sigma(Q(ip\omega)) : p = 0, \pm 1, \ldots\}} \subseteq \sigma(\mathcal{K}),$$

since the spectral set $\sigma(\mathcal{K})$ is closed. This completes the proof of Theorem 1.

Because $\rho(\mathcal{K}) = \sup\{\rho(Q(ip\omega)) : p = 0, \pm 1, \ldots\}$, the convergence condition (i.e., $\rho(\mathcal{K}) < 1$) of the periodic WR algorithm (2) can be derived from the above theorem. Now we conclude that a splitting of (1), such as in (2), is convergent if the resulting algorithm converges to a periodic solution of (1).

Corollary 1. *The periodic WR splitting in* (2) *is convergent if*

$$\sup\{\rho(Q(ip\omega)) : p = 0, \pm 1, \ldots\} < 1. \tag{21}$$

Often, a splitting of (1) in (2) is called a convergent splitting if it satisfies (21). Making use of Theorem 1, we may compare $\rho(\mathcal{K})$ and $\rho(\mathcal{K}_\infty)$, where $\mathcal{K}_\infty$ is the WR operator of the initial value problem on $[0, +\infty)$ for linear DAEs as (1) under the same splitting as before. Suppose that the eigenvalues of $M_1^{-1} D_1$ have positive real parts. In $C([0, T], \mathbf{C}^n)$ or $L^2([0, T], \mathbf{C}^n)$ we have established an expression (see [9, Theorem 1]):

$$\sigma(\mathcal{K}_\infty) = \overline{\bigcup \{\sigma(\mathbf{K}(\xi)) : \mathrm{Re}(\xi) \geq 0\}}, \tag{22}$$

where $\xi \in \mathbf{C}$ and the matrix-valued symbol $\mathbf{K}(\xi)$ satisfies

$$\mathbf{K}(\xi) = \begin{bmatrix} M_1 \xi + A_1 & B_1 \\ C_1 & N_1 \end{bmatrix}^{-1} \begin{bmatrix} M_2 \xi + A_2 & B_2 \\ C_2 & N_2 \end{bmatrix}. \tag{23}$$

On the other hand, it is elementary to show the following relationship:

$$\begin{bmatrix} ip\omega M_1 + A_1 & B_1 \\ C_1 & N_1 \end{bmatrix} Q(ip\omega) = \begin{bmatrix} ip\omega M_2 + A_2 & B_2 \\ C_2 & N_2 \end{bmatrix}. \tag{24}$$

That is,

$$Q(ip\omega) = \mathbf{K}(ip\omega). \tag{25}$$

In other words, we can rewrite (14) and (21) as

$$\sigma(\mathcal{K}) = \overline{\bigcup\{\sigma(\mathbf{K}(ip\omega)) : p = 0, \pm 1, \ldots\}} \tag{26}$$

and

$$\sup\{\rho(\mathbf{K}(ip\omega)) : p = 0, \pm 1, \ldots\} < 1. \tag{27}$$

Based on (14), (22), and (25), it implies $\sigma(\mathcal{K}) \subseteq \sigma(\mathcal{K}_\infty)$. We also pay attention to the fact that ξ in (22) belong to the right half complex plane and $ip\omega$ in (14) are some points of the imaginary axis. That is, $\sigma(\mathcal{K})$ is only a subset of $\sigma(\mathcal{K}_\infty)$.

Corollary 2. *If the $M_1^{-1}D_1$ have eigenvalues with positive real parts, then*

$$\sigma\left(\begin{bmatrix} A_1 & B_1 \\ C_1 & N_1 \end{bmatrix}^{-1} \begin{bmatrix} A_2 & B_2 \\ C_2 & N_2 \end{bmatrix}\right) \subseteq \sigma(\mathcal{K}) \subseteq \sigma(\mathcal{K}_\infty). \tag{28}$$

By the above corollary, we yield a lower bound and an upper bound of $\rho(\mathcal{K})$ as follows:

$$\rho\left(\begin{bmatrix} A_1 & B_1 \\ C_1 & N_1 \end{bmatrix}^{-1} \begin{bmatrix} A_2 & B_2 \\ C_2 & N_2 \end{bmatrix}\right) \leq \rho(\mathcal{K}) \leq \rho(\mathcal{K}_\infty). \tag{29}$$

For (2), if we want to achieve its convergence, the spectral radius of the corresponding iterative operator $\mathcal{K}$ has to be less than 1. It is often impossible to numerically compute the spectral radius for a general operator in function space without its spectral expression. Now, the expressions (14) and (13) in theory provide us a very useful tool to do such computation task. By looking at (13), we know $\lim_{|p|\to+\infty} Q(ip\omega) = \tilde{Q}$ where $\sigma(\tilde{Q}) = \sigma(M_1^{-1}M_2)\bigcup\sigma(N_1^{-1}N_2)$ in which $\tilde{Q}$ appears in (15). It says there is a positive integer $\tilde{p}$ such that the spectra of the matrices $Q(ip\omega) - \tilde{Q}$ for all p, where $|p| > \tilde{p}$, have no contribution for the computed value $\rho(\mathcal{K})$. Thus, we could save a great deal of computation since only the finite value $\sup\{\rho(Q(ip\omega)) : p = 0, \pm 1, \ldots, \pm\tilde{p}\}$ needs to be computed. We learn the convergence of (2) in advance if the computed spectral diagram of $\overline{\bigcup\{\sigma(Q(ip\omega)) : p = 0, \pm 1, \ldots, \tilde{p}\}}$ is strictly located in the unite cycle of the complex plane. If so, the corresponding WR splittings are convergent. We then adopt these convergent splittings to compute the periodic solution of (1).

Besides, for a general spitting of (1) we can also yield the convergence of (2) if its WR solutions on initial value problem are convergent on the infinite time interval $[0, +\infty)$ due to Corollary 2. Furthermore, by Corollary 2 again we realize that a necessary condition of convergence on periodic WR is

$$\rho\left(\begin{bmatrix} A_1 & B_1 \\ C_1 & N_1 \end{bmatrix}^{-1} \begin{bmatrix} A_2 & B_2 \\ C_2 & N_2 \end{bmatrix}\right) < 1.$$

In other words, in order to continue our computation process, in practice we should choose those splittings which first satisfy the above necessary condition on convergence.

For some typical matrices, which usually come from semi-discrete forms of PDEs in engineering applications, fortunately we may further avoid directly computing the unknown value $\rho(\mathcal{K})$ shown in Corollary 1. Meanwhile, we can also conveniently choose the corresponding convergent splittings from these typical matrices. The following text of this subsection is just for that purpose.

Let $\Upsilon = \begin{bmatrix} A & B \\ C & N \end{bmatrix}$ and $\Upsilon_l = \begin{bmatrix} A_l & B_l \\ C_l & N_l \end{bmatrix}$. We now state a simple and useful result on the spectral radius $\rho(\mathcal{K})$.

Theorem 2. *Assume that M and Υ are symmetric. If M_1 and Υ_1 are symmetric positive (negative) definite matrices, then*

$$\rho(\mathcal{K}) = \max\{\rho(M_1^{-1}M_2), \rho(\Upsilon_1^{-1}\Upsilon_2)\}. \tag{30}$$

Proof. We show only the case of symmetric positive definite matrices. For any fixed p ($p \in \{0, \pm 1, \ldots\}$), let $\lambda(ip\omega)$ be an eigenvalue of $Q(ip\omega)$ with the eigenvector $u(ip\omega)$. According to (24), we have

$$\left(ip\omega \begin{bmatrix} M_2 & 0 \\ 0 & 0 \end{bmatrix} + \Upsilon_2\right) u(ip\omega) = \lambda(ip\omega)\left(ip\omega \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} + \Upsilon_1\right) u(ip\omega).$$

It follows that

$$\lambda(ip\omega) = \frac{\langle \Upsilon_2 u(ip\omega), u(ip\omega)\rangle + ip\omega\langle M_2 u_1(ip\omega), u_1(ip\omega)\rangle}{\langle \Upsilon_1 u(ip\omega), u(ip\omega)\rangle + ip\omega\langle M_1 u_1(ip\omega), u_1(ip\omega)\rangle}, \tag{31}$$

where $u(ip\omega) = [u_1^t(ip\omega), u_2^t(ip\omega)]^t$. The denominator of (31) is not zero because M_1 and Υ_1 are symmetric positive definite.

If we note that the inner products in (31) are real, we can yield

$$|\lambda(ip\omega)| = \sqrt{\frac{\langle \Upsilon_2 u(ip\omega), u(ip\omega)\rangle^2 + p^2\omega^2\langle M_2 u_1(ip\omega), u_1(ip\omega)\rangle^2}{\langle \Upsilon_1 u(ip\omega), u(ip\omega)\rangle^2 + p^2\omega^2\langle M_1 u_1(ip\omega), u_1(ip\omega)\rangle^2}}. \tag{32}$$

Based on (32), it holds that

$$|\lambda(ip\omega)| \le \begin{cases} \max\left\{ \dfrac{|\langle \Upsilon_2 u(ip\omega), u(ip\omega)\rangle|}{\langle \Upsilon_1 u(ip\omega), u(ip\omega)\rangle}, \dfrac{|\langle M_2 u_1(ip\omega), u_1(ip\omega)\rangle|}{\langle M_1 u_1(ip\omega), u_1(ip\omega)\rangle} \right\}, \\ \qquad\qquad\qquad\qquad \text{if } u_1(ip\omega) \neq 0, \\ \dfrac{|\langle \Upsilon_2 u(ip\omega), u(ip\omega)\rangle|}{\langle \Upsilon_1 u(ip\omega), u(ip\omega)\rangle}, \quad \text{otherwise.} \end{cases} \tag{33}$$

It is known that for any symmetric positive definite matrix P_1 there exists a symmetric positive definite matrix $\tilde{P}_1$ such that $\tilde{P}_1^2 = P_1$. Thus, for a symmetric matrix P_2 and $x \neq 0$ we deduce

$$\frac{|\langle P_2 x, x\rangle|}{\langle P_1 x, x\rangle} = \frac{|\langle \tilde{P}_1(P_1^{-1}P_2)\tilde{P}_1^{-1} y, y\rangle|}{\langle y, y\rangle} \le \rho(P_1^{-1}P_2), \tag{34}$$

where $y = \tilde{P}_1 x$. It is a fact that the matrix $\tilde{P}_1(P_1^{-1}P_2)\tilde{P}_1^{-1}$ $(= \tilde{P}_1^{-1}P_2\tilde{P}_1^{-1})$ is also symmetric. By use of condition (34), inequality (33) becomes

$$|\lambda(ip\omega)| \le \max\{\rho(M_1^{-1}M_2), \rho(\Upsilon_1^{-1}\Upsilon_2)\}. \tag{35}$$

On the other hand, we have the inequality

(36)
$$\begin{aligned}\sup\{\rho(Q(ip\omega)) : p = 0, \pm 1, \ldots\} &\geq \max\{\rho(\tilde{Q}), \rho(\Upsilon_1^{-1}\Upsilon_2)\}\\ &= \max\{\rho(M_1^{-1}M_2), \rho(N_1^{-1}N_2), \rho(\Upsilon_1^{-1}\Upsilon_2)\}\\ &\geq \max\{\rho(M_1^{-1}M_2), \rho(\Upsilon_1^{-1}\Upsilon_2)\}\end{aligned}$$

by (13), (15), and (24). This completes the proof of Theorem 2.

Corollary 3. *Assume that M, M_1, Υ, and Υ_1 are symmetric positive (negative) definite matrices. The condition $\rho(\mathcal{K}) < 1$ holds if and only if (iff) $2M_1 - M$ and $2\Upsilon_1 - \Upsilon$ are symmetric positive (negative) definite matrices.*

Proof. We consider only the symmetric positive definite case. By (32), it is sufficient to show that for two symmetric positive definite matrices P and P_1 the relationship $|\langle P_2x, x\rangle| < \langle P_1x, x\rangle$ where $x \neq 0$ holds iff $2P_1 - P$ $(= P_1 + P_2)$ is a symmetric positive definite matrix where $P = P_1 - P_2$.

Because $|\langle P_2x, x\rangle| < \langle P_1x, x\rangle$ iff $-\langle P_1x, x\rangle < \langle P_1x, x\rangle - \langle Px, x\rangle$ since $\langle Px, x\rangle > 0$, it follows that $|\langle P_2x, x\rangle| < \langle P_1x, x\rangle$ iff $0 < \langle (2P_1 - P)x, x\rangle$. It is equivalent to $2P_1 - P$ being a symmetric positive definite matrix. This completes the proof of Corollary 3.

2.2. **The special case: ODEs.** Let us discuss a typical and important special case of (1) as follows:

$$M\dot{x}(t) + Ax(t) = f(t), \quad x(0) = x(T), \quad t \in [0, T], \tag{37}$$

where the matrix M is nonsingular. This is a system of ODEs, which also comes from circuit simulation and spatially semi-discrete approximations of parabolic partial differential equations (PPDEs) by finite element methods or preconditioned techniques in spatial variables. Its periodic WR algorithm is read as

$$\begin{cases} M_1\dot{x}^{(k)}(t) + A_1x^{(k)}(t) = M_2\dot{x}^{(k-1)}(t) + A_2x^{(k-1)}(t) + f(t),\\ x^{(k)}(0) = x^{(k)}(T), \quad t \in [0, T], \quad k = 1, 2, \ldots, \end{cases} \tag{38}$$

where $M = M_1 - M_2$ in which the matrix M_1 is nonsingular and $A = A_1 - A_2$. The following lemma can be deduced from Lemma 2.

Lemma 3. *The system of ODEs* (37) *has a unique periodic solution if $M^{-1}A$ is noncritical w.r.t. T. Similarly, for any fixed k the decoupled system in* (38) *has a unique periodic solution if $M_1^{-1}A_1$ is noncritical w.r.t. T.*

We assume that the system (37) and for any fixed k the system of (38) have periodic solutions. As shown previously in Theorem 1, we specifically have

Theorem 3. *The spectral set of the periodic WR operator $\mathcal{K}$ for* (37) *is*

(39)
$$\sigma(\mathcal{K}) = \overline{\bigcup\{\sigma(M_1^{-1}M_2 + (ip\omega I + M_1^{-1}A_1)^{-1}M_1^{-1}(A_2 - A_1M_1^{-1}M_2)) : p = 0, \pm 1, \ldots\}}.$$

This theorem is a direct generalization of Theorem 2.4 in [14], where a constraint on M_1 and M_2, i.e., M_1 is the identity matrix and M_2 is the zero matrix, had been made.

Corollary 4. *The periodic WR splitting in* (38) *is convergent if*

(40)
$$\sup\{\rho(M_1^{-1}M_2 + (ip\omega I + M_1^{-1}A_1)^{-1}M_1^{-1}(A_2 - A_1M_1^{-1}M_2)) : p = 0, \pm 1, \ldots\} < 1.$$

By a special case of the relationship (24), we can also rewrite (39) and (40) as

$$\sigma(\mathcal{K}) = \overline{\bigcup\{\sigma((ip\omega M_1 + A_1)^{-1}(ip\omega M_2 + A_2)) : p = 0, \pm 1, \ldots\}} \tag{41}$$

and

$$\sup\{\rho((ip\omega M_1 + A_1)^{-1}(ip\omega M_2 + A_2)) : p = 0, \pm 1, \ldots\} < 1. \tag{42}$$

The lower bound and upper bound of $\mathcal{K}$ are simple in this case. We need to define a matrix-valued symbol $\mathbf{K}(\xi)$ as

$$\mathbf{K}(\xi) = (M_1\xi + A_1)^{-1}(M_2\xi + A_2), \quad \xi \in \mathbf{C}. \tag{43}$$

Corollary 5. *If the $M_1^{-1}A_1$ have eigenvalues with positive real parts, then the spectral radius of the periodic WR operator $\mathcal{K}$ for* (38) *satisfies*

$$\rho(A_1^{-1}A_2) \le \rho(\mathcal{K}) \le \sup\{\rho(\mathbf{K}(\xi)) : \mathrm{Re}(\xi) \ge 0\}. \tag{44}$$

Under the condition of Corollary 5, we also have

$$\sigma(A_1^{-1}A_2) \subseteq \sigma(\mathcal{K}) \subseteq \overline{\bigcup\{\sigma(\mathbf{K}(\xi)) : \mathrm{Re}(\xi) \ge 0\}}. \tag{45}$$

Corollaries 4 and 5 imply that $\rho(A_1^{-1}A_2) < 1$ is a necessary condition of convergence for the iterative algorithm (38) for any splitting. A sufficient condition on the convergence of (38) is presented in the following corollary.

Corollary 6. *Assume that M, M_1, A, and A_1 are symmetric positive (negative) definite matrices. The periodic WR algorithm* (38) *is convergent if $2M_1 - M$ and $2A_1 - A$ are symmetric positive (negative) definite matrices.*

Now we study another special case, namely linear second-order ODEs. Its spectral expressions on periodic WR operators can be conveniently deduced from (39). The form of the equations is

$$L\ddot{x}(t) + S\dot{x}(t) + Gx(t) = f(t), \quad \dot{x}(0) = \dot{x}(T), \quad x(0) = x(T), \quad t \in [0, T], \tag{46}$$

where the matrix L is nonsingular. The above system also occurs in circuit simulation and mechanical models. The periodic WR algorithm of (46) may be written as

(47)
$$\begin{cases} L_1\ddot{x}^{(k)} + S_1\dot{x}^{(k)}(t) + G_1x^{(k)}(t) = L_2\ddot{x}^{(k-1)} + S_2\dot{x}^{(k-1)}(t) + G_2x^{(k-1)}(t) + f(t), \\ \dot{x}^{(k)}(0) = \dot{x}^{(k)}(T), \quad x^{(k)}(0) = x^{(k)}(T), \quad t \in [0, T], \quad k = 1, 2, \ldots, \end{cases}$$

where $L = L_1 - L_2$ in which the matrix L_1 is nonsingular, $S = S_1 - S_2$, and $G = G_1 - G_2$.

Let $y(t) = \dot{x}(t)$ and $y^{(l)}(t) = \dot{x}^{(l)}(t)$. Then (46) and (47), respectively, become

$$\begin{gathered} \dot{x}(t) - y(t) = 0, \quad L\dot{y} + Gx(t) + Sy(t) = f(t), \\ x(0) = x(T), \quad y(0) = y(T), \quad t \in [0, T] \end{gathered} \tag{48}$$

and

(49)
$$\begin{cases} \dot{x}^{(k)}(t) - y^{(k)}(t) = 0, \\ L_1\dot{y}^{(k)} + G_1x^{(k)}(t) + S_1y^{(k)}(t) = L_2\dot{y}^{(k-1)} + G_2x^{(k-1)}(t) + S_2y^{(k-1)}(t) + f(t), \\ x^{(k)}(0) = x^{(k)}(T), \quad y^{(k)}(0) = y^{(k)}(T), \quad t \in [0, T], \quad k = 1, 2, \ldots. \end{cases}$$

We denote $z(t) = [x^t(t), y^t(t)]^t$ and $z^{(l)}(t) = [(x^{(l)})^t(t), (y^{(l)})^t(t)]^t$. Then, we can further write (48) and (49) as in the forms of (37) and (38) by

$$\begin{bmatrix} I & 0 \\ 0 & L \end{bmatrix} \dot{z}(t) + \begin{bmatrix} 0 & -I \\ G & S \end{bmatrix} z(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \quad z(0) = z(T), \quad t \in [0, T] \tag{50}$$

and

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ 0 & L_1 \end{bmatrix} \dot{z}^{(k)}(t) + \begin{bmatrix} 0 & -I \\ G_1 & S_1 \end{bmatrix} z^{(k)}(t) \\ &\quad = \begin{bmatrix} 0 & 0 \\ 0 & L_2 \end{bmatrix} \dot{z}^{(k-1)}(t) + \begin{bmatrix} 0 & 0 \\ G_2 & S_2 \end{bmatrix} z^{(k-1)}(t) + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \\ &z^{(k)}(0) = z^{(k)}(T), \quad t \in [0, T], \quad k = 1, 2, \ldots. \end{aligned} \tag{51}$$

The following lemma states the existence of periodic solutions of (46) and the decoupled system in (47) for any given k by use of (50) and (51) according to Lemma 3. For simplicity, we omit its proof.

Lemma 4. *The system of second-order ODEs* (46) *has a unique periodic solution if* $\begin{bmatrix} 0 & -I \\ L^{-1}G & L^{-1}S \end{bmatrix}$ *is noncritical w.r.t.* T. *Similarly, for any fixed* k *the decoupled system in* (47) *has a unique periodic solution if* $\begin{bmatrix} 0 & -I \\ L_1^{-1}G_1 & L_1^{-1}S_1 \end{bmatrix}$ *is noncritical w.r.t.* T.

For $p = 0, \pm 1, \ldots$, we denote $\Theta(ip\omega) = -p^2\omega^2 L + ip\omega S + G$ and $\Theta_1(ip\omega) = -p^2\omega^2 L_1 + ip\omega S_1 + G_1$. The above lemma has a direct corollary. We now present it and give a complete proof.

Corollary 7. *The system of second-order ODEs* (46) *has a unique periodic solution if for all* p *the matrices* $\Theta(ip\omega)$ *are invertible. Similarly, for any fixed* k *the decoupled system in* (47) *has a unique periodic solution if for all* p *the matrices* $\Theta_1(ip\omega)$ *are invertible.*

Proof. We prove only the unique existence of periodic solutions on (46). By Lemma 4, it is sufficient to show that the matrix $\begin{bmatrix} 0 & -I \\ L^{-1}G & L^{-1}S \end{bmatrix}$ is noncritical w.r.t. T. First, we know

$$\begin{aligned} ip\omega \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & -I \\ L^{-1}G & L^{-1}S \end{bmatrix} &= \begin{bmatrix} ip\omega I & -I \\ L^{-1}G & ip\omega I + L^{-1}S \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} ip\omega I & -I \\ G & ip\omega L + S \end{bmatrix}. \end{aligned}$$

Next, we need to prove that for all p the matrices $\begin{bmatrix} ip\omega I & -I \\ G & ip\omega L + S \end{bmatrix}$ are invertible if the matrices $\Theta(ip\omega)$ are invertible. If the matrices $\Theta(ip\omega)$ for all p are invertible, it is easy to check

$$\begin{bmatrix} ip\omega I & -I \\ G & ip\omega L + S \end{bmatrix}^{-1} = \begin{bmatrix} \Theta^{-1}(ip\omega)(ip\omega L + S) & \Theta^{-1}(ip\omega) \\ -\Theta^{-1}(ip\omega)G & ip\omega\Theta^{-1}(ip\omega) \end{bmatrix}. \tag{52}$$

In other words, the matrix $\begin{bmatrix} 0 & -I \\ L^{-1}G & L^{-1}S \end{bmatrix}$ is noncritical w.r.t. T if for all p the matrices $\Theta(ip\omega)$ are invertible. This completes the proof of Corollary 7.

Now, we assume that (46) and for any fixed k the system of (47) have periodic solutions. Namely, the matrices $\Theta(ip\omega)$ and $\Theta_1(ip\omega)$ are invertible for all p.

Theorem 4. *The spectral set of the periodic WR operator $\mathcal{K}$ for* (47) *is*

(53)
$$\sigma(\mathcal{K}) = \overline{\bigcup\left\{\sigma\left(\begin{bmatrix} ip\omega I & -I \\ G_1 & ip\omega L_1 + S_1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ G_2 & ip\omega L_2 + S_2 \end{bmatrix}\right) : p = 0, \pm 1, \dots\right\}}.$$

Proof. Based on (51) and referring to (38), we have

$$M_1 = \begin{bmatrix} I & 0 \\ 0 & L_1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & -I \\ G_1 & S_1 \end{bmatrix},\ M_2 = \begin{bmatrix} 0 & 0 \\ 0 & L_2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ G_2 & S_2 \end{bmatrix}.$$

Furthermore, we know

$$ip\omega M_1 + A_1 = \begin{bmatrix} ip\omega I & -I \\ G_1 & ip\omega L_1 + S_1 \end{bmatrix}, \quad ip\omega M_2 + A_2 = \begin{bmatrix} 0 & 0 \\ G_2 & ip\omega L_2 + S_2 \end{bmatrix}.$$

Namely, for $p = 0, \pm 1, \dots$, it follows that

$$(ip\omega M_1 + A_1)^{-1}(ip\omega M_2 + A_2) = \begin{bmatrix} ip\omega I & -I \\ G_1 & ip\omega L_1 + S_1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ G_2 & ip\omega L_2 + S_2 \end{bmatrix}.$$

Thus, (53) is directly deduced from (41). This completes the proof of Theorem 4.

Corollary 8. *The periodic WR splitting in* (47) *is convergent if*

(54)
$$\sup\left\{\rho\left(\begin{bmatrix} ip\omega I & -I \\ G_1 & ip\omega L_1 + S_1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ G_2 & ip\omega L_2 + S_2 \end{bmatrix}\right) : p = 0, \pm 1, \dots\right\} < 1.$$

The following theorem provides another expression of $\sigma(\mathcal{K})$ in (53). For $p = 0, \pm 1, \dots$, we also denote $\Theta_2(ip\omega) = -p^2\omega^2 L_2 + ip\omega S_2 + G_2$.

Theorem 5. *For all p, if the matrices $\Theta_1(ip\omega)$ are invertible, then we have*

$$\begin{aligned} &\overline{\bigcup\left\{\sigma\left(\begin{bmatrix} ip\omega I & -I \\ G_1 & ip\omega L_1 + S_1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ G_2 & ip\omega L_2 + S_2 \end{bmatrix}\right) : p = 0, \pm 1, \dots\right\}} \\ &\qquad = \overline{\bigcup\{\sigma(\Theta_1^{-1}(ip\omega)\Theta_2(ip\omega)) : p = 0, \pm 1, \dots\}}\bigcup\{0\}. \end{aligned} \tag{55}$$

Proof. In order to prove the theorem, it is sufficient to show that the following relationship holds for any fixed p:

(56)
$$\sigma\left(\begin{bmatrix} ip\omega I & -I \\ G_1 & ip\omega L_1 + S_1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ G_2 & ip\omega L_2 + S_2 \end{bmatrix}\right) = \sigma(\Theta_1^{-1}(ip\omega)\Theta_2(ip\omega))\bigcup\{0\}.$$

By (52), similarly we know

$$\begin{bmatrix} ip\omega I & -I \\ G_1 & ip\omega L_1 + S_1 \end{bmatrix}^{-1} = \begin{bmatrix} \Theta_1^{-1}(ip\omega)(ip\omega L_1 + S_1) & \Theta_1^{-1}(ip\omega) \\ -\Theta_1^{-1}(ip\omega)G_1 & ip\omega\Theta_1^{-1}(ip\omega) \end{bmatrix}.$$

Furthermore, we have

$$\begin{aligned} &\begin{bmatrix} ip\omega I & -I \\ G_1 & ip\omega L_1 + S_1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ G_2 & ip\omega L_2 + S_2 \end{bmatrix} \\ &\qquad = \begin{bmatrix} \Theta_1^{-1}(ip\omega)G_2 & \Theta_1^{-1}(ip\omega)(ip\omega L_2 + S_2) \\ ip\omega\Theta_1^{-1}(ip\omega)G_2 & ip\omega\Theta_1^{-1}(ip\omega)(ip\omega L_2 + S_2) \end{bmatrix}. \end{aligned} \tag{57}$$

For $p = 0$, from (57) we obtain

$$\begin{bmatrix} 0 & -I \\ G_1 & S_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ G_2 & S_2 \end{bmatrix} = \begin{bmatrix} G_1^{-1}G_2 & G_1^{-1}S_2 \\ 0 & 0 \end{bmatrix}$$

since $\Theta_1(0) = G_1$. Thus, we have

$$\sigma\left(\begin{bmatrix} 0 & -I \\ G_1 & S_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ G_2 & S_2 \end{bmatrix}\right) = \sigma(G_1^{-1}G_2)\bigcup\{0\} = \sigma(\Theta_1^{-1}(0)\Theta_2(0))\bigcup\{0\}$$

since $\Theta_2(0) = G_2$. It says that (56) is valid for $p = 0$.

For $p \neq 0$, we first know $\begin{bmatrix} ip\omega I & 0 \\ -ip\omega I & I \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{ip\omega}I & 0 \\ I & I \end{bmatrix}$. For (57), we do basic calculation to yield

$$\begin{aligned} &\begin{bmatrix} ip\omega I & 0 \\ -ip\omega I & I \end{bmatrix} \begin{bmatrix} \Theta_1^{-1}(ip\omega)G_2 & \Theta_1^{-1}(ip\omega)(ip\omega L_2+S_2) \\ ip\omega\Theta_1^{-1}(ip\omega)G_2 & ip\omega\Theta_1^{-1}(ip\omega)(ip\omega L_2+S_2) \end{bmatrix} \begin{bmatrix} ip\omega I & 0 \\ -ip\omega I & I \end{bmatrix}^{-1} \\ &\quad = \begin{bmatrix} ip\omega\Theta_1^{-1}(ip\omega)G_2 & ip\omega\Theta_1^{-1}(ip\omega)(ip\omega L_2+S_2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{ip\omega}I & 0 \\ I & I \end{bmatrix} \\ &\quad = \begin{bmatrix} \Theta_1^{-1}(ip\omega)\Theta_2(ip\omega) & ip\omega\Theta_1^{-1}(ip\omega)(ip\omega L_2+S_2) \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Namely, from (57) and the above relationship we have

$$\sigma\left(\begin{bmatrix} ip\omega I & -I \\ G_1 & ip\omega L_1+S_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ G_2 & ip\omega L_2+S_2 \end{bmatrix}\right) = \sigma(\Theta_1^{-1}(ip\omega)\Theta_2(ip\omega))\bigcup\{0\}.$$

It says that (56) is also valid for $p \neq 0$. This completes the proof of Theorem 5.

By invoking (55), we can rewrite (53) and (54) as

(58)
$$\begin{aligned} \sigma(\mathcal{K}) = &\overline{\bigcup\{\sigma((-p^2\omega^2L_1+ip\omega S_1+G_1)^{-1}(-p^2\omega^2L_2+ip\omega S_2+G_2)) : p=0,\pm1,\ldots\}} \\ &\times\bigcup\{0\} \end{aligned}$$

and

(59)
$$\sup\{\rho((-p^2\omega^2L_1 + ip\omega S_1 + G_1)^{-1}(-p^2\omega^2L_2 + ip\omega S_2 + G_2)) : p = 0, \pm1, \ldots\} < 1.$$

3. Spectra of discrete periodic WR operators and finite-difference for solving periodic WR solutions

In this section we consider the discrete case of Section 2 and give a finite-difference formula for solving the periodic WR solution of (1).

3.1. **Spectra of discrete periodic WR operators.** Now we discuss the application of the linear multistep method in the periodic WR algorithm (2). For this purpose, let us fix the time increment $\tau = T/N$ and discretize (2) by a linear multistep method, where its characteristic polynomials are $a(\xi)$ and $b(\xi)$, i.e., $a(\xi) = \sum_{j=0}^{m} \alpha_j \xi^j$ and $b(\xi) = \sum_{j=0}^{m} \beta_j \xi^j$, to obtain

(60)
$$\begin{cases} \dfrac{1}{\tau}\displaystyle\sum_{j=0}^{m} \alpha_j M_1 x_{p-m+j}^{(k)} + \sum_{j=0}^{m} \beta_j A_1 x_{p-m+j}^{(k)} + \sum_{j=0}^{m} \beta_j B_1 y_{p-m+j}^{(k)} \\ \qquad = \dfrac{1}{\tau}\displaystyle\sum_{j=0}^{m} \alpha_j M_2 x_{p-m+j}^{(k-1)} + \sum_{j=0}^{m} \beta_j A_2 x_{p-m+j}^{(k-1)} \\ \qquad\quad + \sum_{j=0}^{m} \beta_j B_2 y_{p-m+j}^{(k-1)} + \sum_{j=0}^{m} \beta_j (f_1)_{p-m+j}, \\ C_1 x_p^{(k)} + N_1 x_p^{(k)} = C_2 x_p^{(k-1)} + N_2 x_p^{(k-1)} + (f_2)_p, \ p = 0, \pm 1, \ldots, \ k = 1, 2, \ldots. \end{cases}$$

In the above algorithm we assume that $a(\xi)$ and $b(\xi)$ have no common roots where $a(1) = 0$ and $\dot{a}(1) = b(1)$. In practical codes one adopts a convergent linear multistep method to solve DAEs of (2). A special case of the linear multistep method is the backward differentiation formula (BDF) where $b(\xi) = \xi^m$. The m-step constant BDF method converges to $O(\tau^m)$ for $m < 7$ [1].

Let $x_\tau^{(k)}$ and $y_\tau^{(k)}$ stand for the infinite sequences $\{x_p^{(k)}\}_{p=-\infty}^{\infty}$ and $\{y_p^{(k)}\}_{p=-\infty}^{\infty}$, and similarly for $x_\tau^{(k-1)}$, $y_\tau^{(k-1)}$, $(f_1)_\tau$ and $(f_2)_\tau$. These infinite sequences are N-periodic, for example it means that $x_{p+N}^{(k)} = x_p^{(k)}$ $(p = 0, \pm 1, \ldots)$ for the sequence $\{x_p^{(k)}\}_{p=-\infty}^{\infty}$. Now we simply rewrite (60) as

$$\begin{cases} \dfrac{1}{\tau} a M_1 x_\tau^{(k)} + b A_1 x_\tau^{(k)} + b B_1 y_\tau^{(k)} \\ \qquad = \dfrac{1}{\tau} a M_2 x_\tau^{(k-1)} + b A_2 x_\tau^{(k-1)} + b B_2 y_\tau^{(k-1)} + b (f_1)_\tau, \\ C_1 x_\tau^{(k)} + N_1 x_\tau^{(k)} = C_2 x_\tau^{(k-1)} + N_2 x_\tau^{(k-1)} + (f_2)_\tau, \quad k = 1, 2, \ldots, \end{cases} \tag{61}$$

where we denote the infinite sequences

$$\left\{ \sum_{j=0}^{m} \alpha_j M_s x_{p-m+j}^{(l)} \right\}_{p=-\infty}^{\infty}, \qquad \left\{ \sum_{j=0}^{m} \beta_j A_s x_{p-m+j}^{(l)} \right\}_{p=-\infty}^{\infty},$$

and

$$\left\{ \sum_{j=0}^{m} \beta_j B_s y_{p-m+j}^{(l)} \right\}_{p=-\infty}^{\infty}$$

by $a M_s x_\tau^{(l)}$, $b A_s x_\tau^{(l)}$, and $b B_s y_\tau^{(l)}$.

Definition 2. For an N-periodic sequence w_τ, its discrete Fourier coefficients are

$$\tilde{w}_p = \frac{1}{N} \sum_{q=0}^{N-1} w_q e^{-ipq(2\pi/N)}, \quad p = 0, \pm 1, \ldots.$$

By use of Definition 2, we know that

$$w_\tau = \sum_{q=0}^{N-1} \tilde{w}_q \epsilon_{\tau,q},$$

where $\epsilon_{\tau,q} = \{e^{ipq(2\pi/N)}\}_{p=-\infty}^{\infty}$.

Condition (S). *For the characteristic polynomials $a(\xi)$ and $b(\xi)$, we assume that the matrix*

$$\begin{bmatrix} \frac{1}{\tau}\frac{a}{b}(\xi^q)M_1 + A_1 & B_1 \\ C_1 & N_1 \end{bmatrix}^{-1}, \quad q = 0, 1, \ldots, N-1 \tag{62}$$

exists for the splitting matrices M_1, A_1, B_1, C_1, and N_1 in which $\xi = e^{i(2\pi/N)}$.

Let $z_\tau^{(l)} = [(x_\tau^{(l)})^t, (y_\tau^{(l)})^t]^t$. If Condition (S) holds, the solution of (61) for any fixed k can be written as

$$z_\tau^{(k)} = \mathcal{K}_\tau z_\tau^{(k-1)} + \varphi_\tau, \tag{63}$$

where

$$\mathcal{K}_\tau z_\tau = \sum_{q=0}^{N-1} \begin{bmatrix} \frac{1}{\tau}\frac{a}{b}(\xi^q)M_1 + A_1 & B_1 \\ C_1 & N_1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\tau}\frac{a}{b}(\xi^q)M_2 + A_2 & B_2 \\ C_2 & N_2 \end{bmatrix} \tilde{z}_q \epsilon_{\tau,q}$$

and

$$\varphi_\tau = \sum_{q=0}^{N-1} \begin{bmatrix} \frac{1}{\tau}\frac{a}{b}(\xi^q)M_1 + A_1 & B_1 \\ C_1 & N_1 \end{bmatrix}^{-1} \tilde{f}_q \epsilon_{\tau,q}$$

in which $\tilde{f}_q = [(\tilde{f}_1)_q^t, (\tilde{f}_2)_q^t]^t$. Using the same approach given in [14], we can show the following theorem (we omit the proof here).

Theorem 6. *Under Condition* (S) *the spectral set of the discrete periodic WR operator $\mathcal{K}_\tau$ in* (63) *is*

$$\sigma(\mathcal{K}_\tau) = \bigcup \left\{ \sigma\left(\begin{bmatrix} \frac{1}{\tau}\frac{a}{b}(\xi^q)M_1 + A_1 & B_1 \\ C_1 & N_1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\tau}\frac{a}{b}(\xi^q)M_2 + A_2 & B_2 \\ C_2 & N_2 \end{bmatrix} \right) : q = 0, 1, \ldots, N-1 \right\}, \tag{64}$$

where $\xi = e^{i(2\pi/N)}$.

3.2. **Finite-difference for solving periodic WR solutions.** In this subsection we compute the iterative waveforms $[(\dot{x}^{(k)})^t(t), (\dot{y}^{(k)})^t(t)]^t (k = 1, 2, \ldots)$ in (2) at $m+1$ time-points, $t_0 = 0, t_1, t_2, \ldots, t_m = T$, with a constant step-size τ. For any fixed k we take the implicit Euler method to approximate the derivatives $\dot{x}^{(k)}$ and $\dot{x}^{(k-1)}$ in (2). As a simple case of the linear multistep method presented in subsection 3.1, we now may write out the iterative matrix for discrete waveforms without using the discrete Fourier series technique. We will make use of this form to do our computations in the next section.

For this purpose, we denote $X^{(l)} = [(x^{(l)})^t(t_1), \ldots, (x^{(l)})^t(t_m)]^t \in \mathbf{R}^{mn_1}$, $Y^{(l)} = [(y^{(l)})^t(t_1), \ldots, (y^{(l)})^t(t_m)]^t \in \mathbf{R}^{mn_2}$, $F_1 = [f_1^t(t_1), \ldots, f_1^t(t_m)]^t \in \mathbf{R}^{mn_1}$, and $F_2 = [f_2^t(t_1), \ldots, f_2^t(t_m)]^t \in \mathbf{R}^{mn_2}$. It is mentioned here that the order of discrete equations is different from that of subsection 3.1 for the differential part and the algebraic part. By $x^{(l)}(t_0) = x^{(l)}(t_m)$ the discrete form of (2) is

$$\begin{cases} H_1 X^{(k)} + H_2 Y^{(k)} = J_1 X^{(k-1)} + J_2 Y^{(k-1)} + \tau F_1, \\ H_3 X^{(k)} + H_4 Y^{(k)} = J_3 X^{(k-1)} + J_4 Y^{(k-1)} + F_2, \quad k = 1, 2, \ldots, \end{cases} \tag{65}$$

where

$$H_1 = \begin{bmatrix} (M_1+\tau A_1) & & & -M_1 \\ -M_1 & (M_1+\tau A_1) & & \\ & \ddots & \ddots & \\ & & -M_1 & (M_1+\tau A_1) \end{bmatrix},$$

$$H_2 = \begin{bmatrix} \tau B_1 & & \\ & \ddots & \\ & & \tau B_1 \end{bmatrix}, H_3 = \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_1 \end{bmatrix}, H_4 = \begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_1 \end{bmatrix}$$

and

$$J_1 = \begin{bmatrix} (M_2+\tau A_2) & & & -M_2 \\ -M_2 & (M_2+\tau A_2) & & \\ & \ddots & \ddots & \\ & & -M_2 & (M_2+\tau A_2) \end{bmatrix},$$

$$J_2 = \begin{bmatrix} \tau B_2 & & \\ & \ddots & \\ & & \tau B_2 \end{bmatrix}, J_3 = \begin{bmatrix} C_2 & & \\ & \ddots & \\ & & C_2 \end{bmatrix}, J_4 = \begin{bmatrix} N_2 & & \\ & \ddots & \\ & & N_2 \end{bmatrix}.$$

For any fixed step-size τ, the convergence condition of the above iterative algorithm can be concluded in the following theorem.

Theorem 7. *The discrete algorithm* (65) *is convergent if*

$$\rho\left(\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}^{-1} \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \right) < 1. \tag{66}$$

4. Numerical experiments

Our numerical experiments are based on three examples, each of which has the form of (1), (37), and (46), respectively. We define that the iterative error is the sum of the squared difference of successive waveforms taken over all time-points.

4.1. **Example 1.** The first example is a test circuit shown in Figure 1 where n is even. This circuit is taken from [15]. It is a general form of a uniformly dissipative low-pass ladder filter circuit with a current-source input and a voltage output.

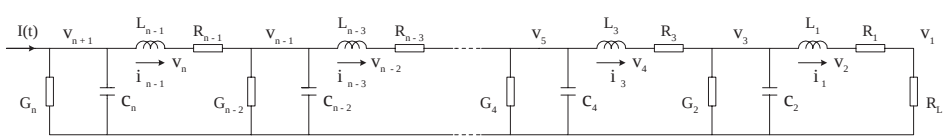

Figure 1. A linear periodic DAEs circuit with n even.

The circuit equations have a form as in (1) where $x(t) = [i_1(t),\, v_3(t),\, i_3(t),\, v_5(t),\, \ldots,\, i_{n-1}(t),\, v_{n+1}(t)]^t \in \mathbf{R}^n$ and $y(t) = [v_1(t), v_2(t),\, v_4(t),\, \ldots,\, v_{n-2}(t), v_n(t)]^t \in \mathbf{R}^{\frac{n}{2}+1}$, $f_1(t) = [0, \ldots, 0, I(t)]^t \in \mathbf{R}^n$, and $f_2(t) = [0, \ldots, 0]^t \in \mathbf{R}^{\frac{n}{2}+1}$, for any given $t \in [0, T]$.

Now, M, A, B, C, and N in (1) are some concrete matrices. For $M, A \in \mathbf{R}^{n\times n}$ we have $M = \operatorname{diag}(L_1, C_2, L_3, C_4, \ldots, L_{n-1}, C_n)$ and $A = \operatorname{diag}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{\frac{n}{2}})$ where

$$\tilde{A}_i = \begin{bmatrix} 0 & -1 \\ 1 & G_{2i} + \dfrac{1}{R_{2i+1}} \end{bmatrix} (i = 1, 2, \ldots, \frac{n}{2} - 1), \quad \tilde{A}_{\frac{n}{2}} = \begin{bmatrix} 0 & -1 \\ 1 & G_n \end{bmatrix}.$$

The matrices $B \in \mathbf{R}^{n\times(\frac{n}{2}+1)}$ and $C \in \mathbf{R}^{(\frac{n}{2}+1)\times n}$ are

$$B = \begin{bmatrix} 0 & 1 & & & \\ & & -\dfrac{1}{R_3} & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & -\dfrac{1}{R_{n-1}} \\ & & & & 1 \\ & & & & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & & & & & & \\ -1 & & & & & & \\ & -\dfrac{1}{R_3} & -1 & & & & \\ & & & \ddots & & & \\ & & & & -\dfrac{1}{R_{n-1}} & -1 & 0 \end{bmatrix}.$$

For $N \in \mathbf{R}^{(\frac{n}{2}+1)\times(\frac{n}{2}+1)}$ we have $N = \operatorname{diag}(\tilde{N}_1, \tilde{N}_2)$ where

$$\tilde{N}_2 = \operatorname{diag}\Big(\frac{1}{R_3}, \frac{1}{R_5}, \ldots, \frac{1}{R_{n-1}}\Big) \in \mathbf{R}^{(\frac{n}{2}-1)\times(\frac{n}{2}-1)}$$

and

$$\tilde{N}_1 = \begin{bmatrix} \dfrac{1}{R_1} + \dfrac{1}{R_2} & -\dfrac{1}{R_1} \\ -\dfrac{1}{R_1} & \dfrac{1}{R_1} \end{bmatrix}.$$

We seek its periodic responses by periodic WR. In our computations we use the discrete algorithm (65). For simplicity we let $n = 10$, $T = 2\pi$, and all circuit parameters are set to be 1. The boundary values satisfy $x(0) = x(2\pi)$ and $y(0) = -N^{-1}Cx(0)$ $(= y(2\pi))$.

For this example, we use the Jacobi splitting to split the matrices M and N, i.e., M_1 and N_1 are diagonal matrices of M and N if we adopt the symbols in (2). The

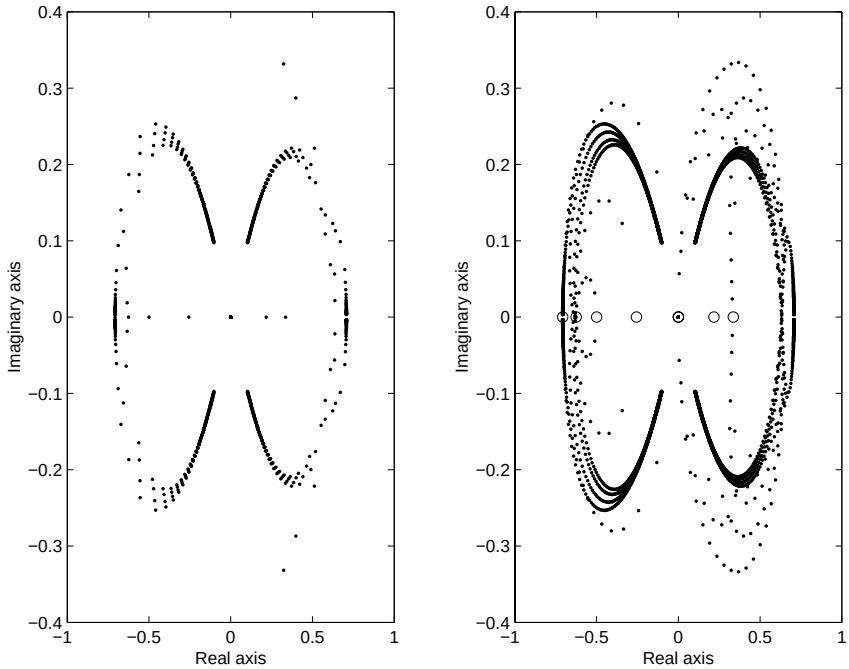

FIGURE 2. Sampled values of $\sigma(\mathcal{K})$ (left) and $\sigma(\widetilde{\mathcal{K}}_\infty)$ (right) for Case I in Example 1.

matrices B_1 and C_1 are

$$B_1 = \begin{bmatrix} 0 & 1 & & & \\ & & 0 & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \\ & & & & 1 \\ & & & & 0 \end{bmatrix} \in \mathbf{R}^{10\times 6},$$

$$C_1 = \begin{bmatrix} 0 & & & & & & \\ -1 & & & & & & \\ & 0 & -1 & & & & \\ & & & \ddots & & & \\ & & & & 0 & -1 & 0 \end{bmatrix} \in \mathbf{R}^{6\times 10}.$$

For the matrix A we have two ways to treat its splitting, (a) Case I: we simply do not split it, i.e., $A_1 = A$; (b) Case II:

$$A_1 = \begin{bmatrix} 2 & & \\ 1 & \ddots & \\ & 1 & 2 \end{bmatrix} \in \mathbf{R}^{10\times 10}.$$

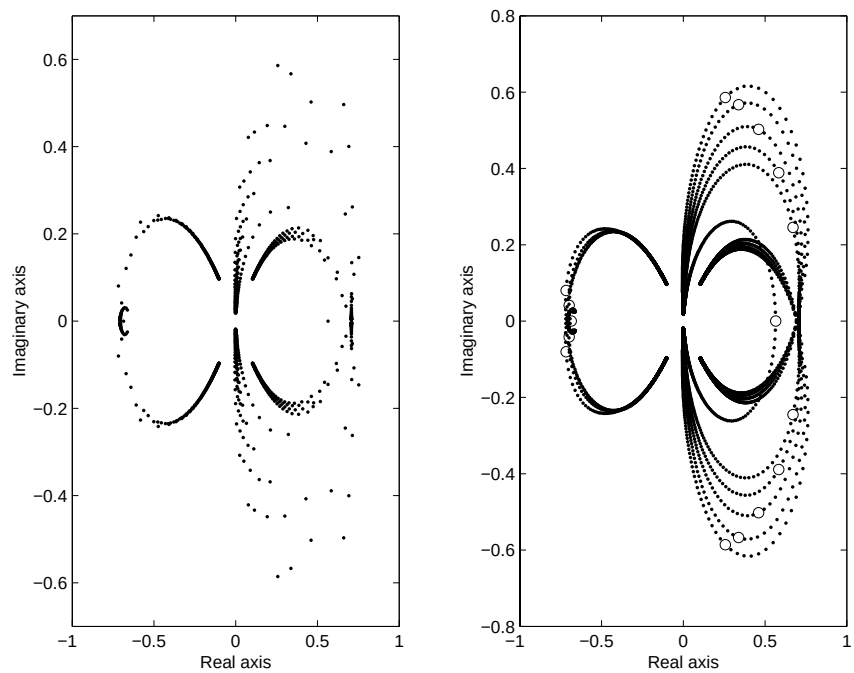

FIGURE 3. Sampled values of $\sigma(\mathcal{K})$ (left) and $\sigma(\widetilde{\mathcal{K}}_\infty)$ (right) for Case II in Example 1.

By (14) and (22), in the present situation we let $\xi = i\zeta$ ($\zeta \in \mathbf{R}$). The spectral drawings on

$$\sigma(\mathcal{K}) \quad \text{and} \quad \sigma(\widetilde{\mathcal{K}}_\infty),$$

where

$$\sigma(\widetilde{\mathcal{K}}_\infty) = \overline{\bigcup\{\sigma(\mathbf{K}(i\zeta)) : \zeta \in \mathbf{R}\}},$$

are given in Figure 2 for Case I and Figure 3 for Case II in which $p = 0, \pm 1, \ldots, \pm 50$ and $\zeta = 0, \pm 0.1, \ldots, \pm 49.9, \pm 50$. The spectral set for the case of $p = 0$ is also indicated with the symbol "o" in the right parts of Figures 2 and 3.

To compute the periodic WR solution of the system, we let the input function $I(t) = I(t + 2\pi)$ satisfy

$$I(t) = \begin{cases} t, & 0 \le t \le 0.5\pi, \\ 0.5\pi, & 0.5\pi \le t \le 1.5\pi, \\ (2\pi - t), & 1.5\pi \le t \le 2\pi. \end{cases} \tag{67}$$

The time-step is 0.02π sec and the initial guess is the zero function. The convergence results and two approximate waveforms for the voltage $v_1(t)$ are shown in Figure 4.

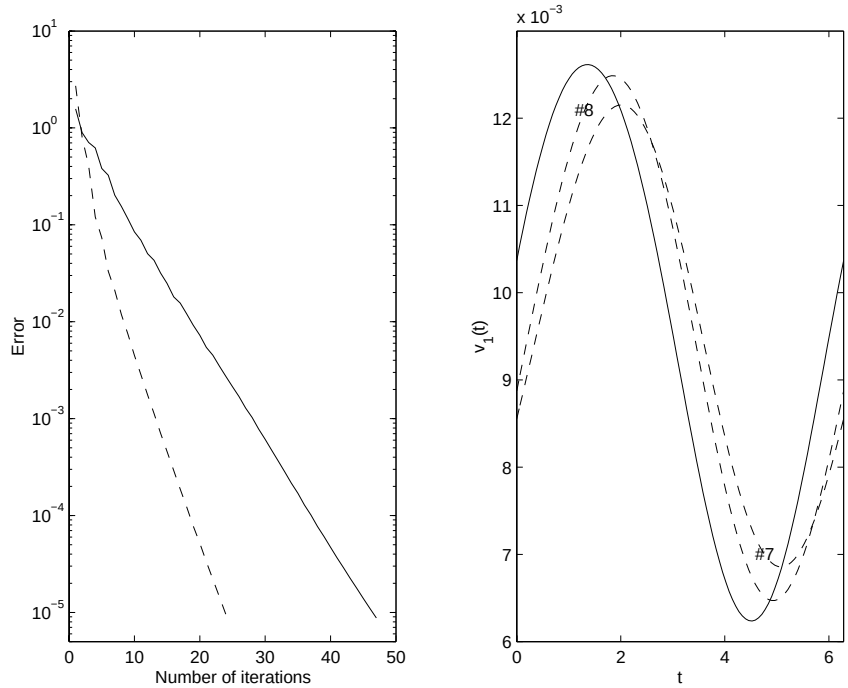

FIGURE 4. Computed results for Example 1. Left: Periodic WR iterations (dashed line for Case I and solid line for Case II). Right: Approximate waveforms ($k = 7$ and $k = 8$) of the voltage $v_1(t)$ (solid line).

4.2. **Example 2.** If we study numerical solutions of linear PPDEs by a spatial finite element method, we will meet the form of (37).

Let us consider the one-dimensional heat equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \dfrac{\partial^2 u(x,t)}{\partial x^2} = \sin(2\pi x)\sin(t), \quad (x,t) \in [0,1] \times [0,2\pi], \\ u(0,t) = u(1,t) = 0, \quad t \in [0,2\pi], \\ u(x,0) = u(x,2\pi), \quad x \in [0,1]. \end{cases} \tag{68}$$

The equation is spatially discretized on a grid $\Omega_h = \{x_j = jh | j = 0, 1, \ldots, N_s\}$ with the constant mesh size $h = 1/N_s$. The sequence of linear basis functions $\{\varphi_j\}_{j=1}^{N_s-1}$ is defined as

$$\varphi_j(x) = \begin{cases} 1 + \dfrac{x - x_j}{h}, & x_{j-1} \le x \le x_j, \\ 1 - \dfrac{x - x_j}{h}, & x_j \le x \le x_{j+1}, \\ 0, & \text{otherwise}. \end{cases} \tag{69}$$

The solution of the weak formulation for (68) is $u_h(x,t) = \sum_{j=1}^{N_s-1} \varphi_j(x)\mu_j(t)$. The function vector which is to be computed, $x(t) = [\mu_1(t), \ldots, \mu_{N_s-1}(t)]^t$, should

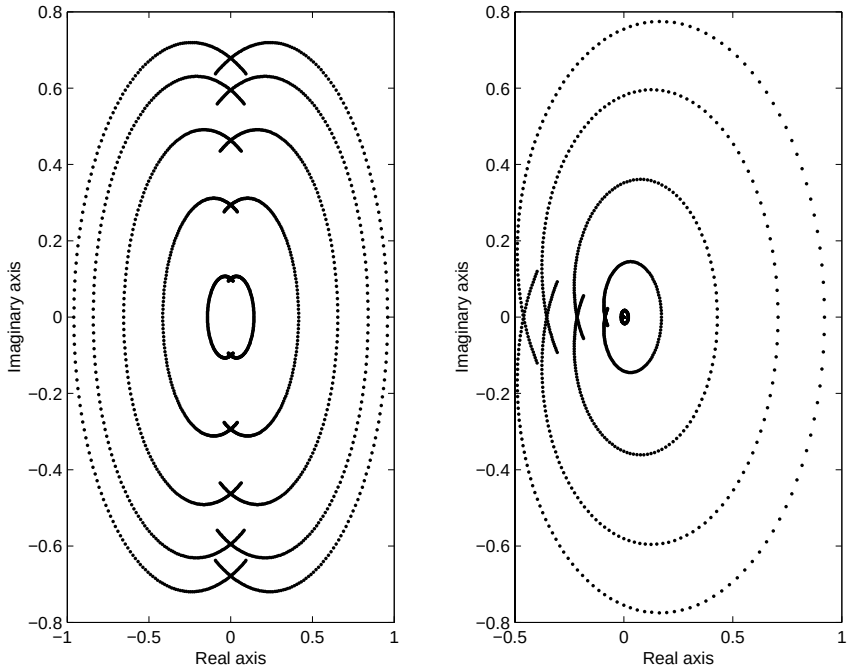

FIGURE 5. Sampled values of $\sigma(\mathcal{K})$ for Example 2 under the Jacobi splitting (left) and the Gauss-Seidel splitting (right).

satisfy (37) where

$$M = h \begin{bmatrix} 4 & 1 & \\ 1 & \ddots & 1 \\ & 1 & 4 \end{bmatrix} \in \mathbf{R}^{(N_s-1)\times(N_s-1)},$$

$$A = \frac{1}{h} \begin{bmatrix} 2 & -1 & \\ -1 & \ddots & -1 \\ & -1 & 2 \end{bmatrix} \in \mathbf{R}^{(N_s-1)\times(N_s-1)},$$

and $f(t) = -\dfrac{\sin(t)}{(2\pi)^2 h}[f_1, \ldots, f_{N_s-1}]^t$ in which $f_j = \sin(2(j-1)h\pi) - 2\sin(2jh\pi) + \sin(2(j+1)h\pi)$ $(j = 1, \ldots, N_s - 1)$. In our numerical experiments we set h to be $1/11$, namely $N_s = 11$.

We respectively adopt the Jacobi splitting and the Gauss-Seidel splitting to solve the periodic WR solution for this example. Their spectra on $\mathcal{K}$ are pictured in Figure 5 where the left part stands for Jacobi and the right part stands for Gauss-Seidel in which $p = 0, \pm 1, \ldots, \pm 100$. We can see that their spectral radii nearly approach 1.

To observe convergence behaviors on periodic WR, we let the time-step be 0.05π sec and the initial guess be the zero function. The periodic WR iteration process stops when the residual error reaches below 10^{-5}. The iteration results and two approximate waveforms for $\mu_4(t)$ are shown in Figure 6. For this example the Jacobi iteration number is less than that of the Gauss-Seidel iteration number.

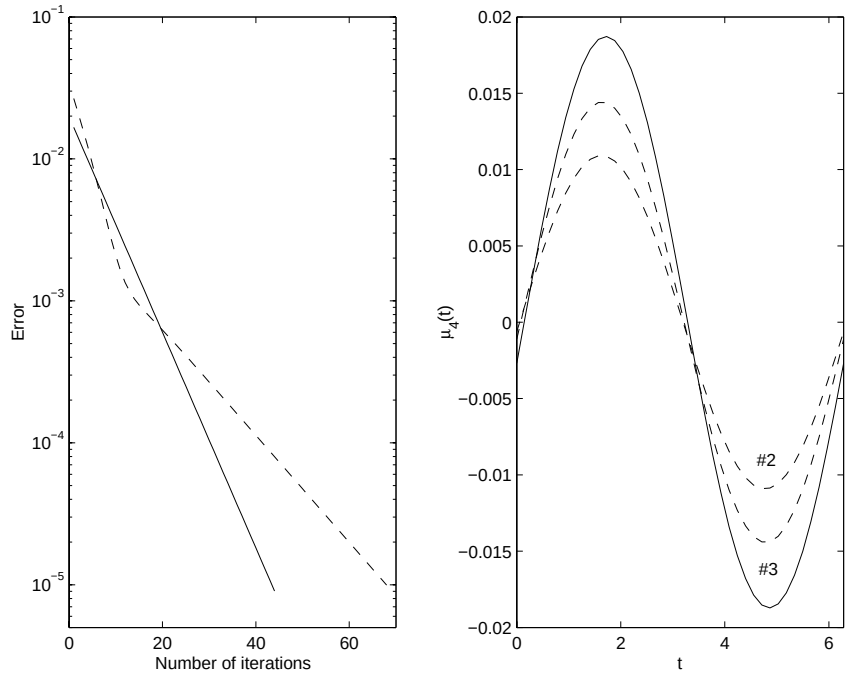

FIGURE 6. Computed results for Example 2. Left: Periodic WR iterations (dashed line for Gauss-Seidel and solid line for Jacobi). Right: Approximate waveforms ($k = 2$ and $k = 3$) of $\mu_4(t)$ (solid line).

4.3. **Example 3.** We also consider the periodic response of a linear second-order system because the second-order system often describes mechanical models [4].

By use of the symbols in (46) we now let the matrix $L = \mathrm{diag}(1, 2, \ldots, 1, 2) \in \mathbf{R}^{10\times 10}$ and the function $f(t) = [\cos(2\pi t), 0, \ldots, 0]^t \in \mathbf{R}^{10}$ for any given $t \in [0, 1]$. The matrices $S \in \mathbf{R}^{10\times 10}$ and $G \in \mathbf{R}^{10\times 10}$ are

$$S = \begin{bmatrix} 2 & -2 & 1 & 2 & 1 & & & & & \\ -2 & 2 & -2 & 1 & 2 & 1 & & & & \\ 1 & -2 & 2 & -2 & 1 & 2 & 1 & & & \\ 2 & 1 & -2 & 2 & -2 & 1 & 2 & 1 & & \\ 1 & 2 & 1 & -2 & 2 & -2 & 1 & 2 & 1 & \\ & 1 & 2 & 1 & -2 & 2 & -2 & 1 & 2 & 1 \\ & & 1 & 2 & 1 & -2 & 2 & -2 & 1 & 2 \\ & & & 1 & 2 & 1 & -2 & 2 & -2 & 1 \\ & & & & 1 & 2 & 1 & -2 & 2 & -2 \\ & & & & & 1 & 2 & 1 & -2 & 2 \end{bmatrix},$$

$$G = \begin{bmatrix} 2 & -0.5 & \\ -0.5 & \ddots & -0.5 \\ & -0.5 & 2 \end{bmatrix}.$$

We take two splittings, i.e., Jacobi and Gauss-Seidel, to solve its periodic response. Let the time-step be 0.01 sec and the initial waveform be the zero function.

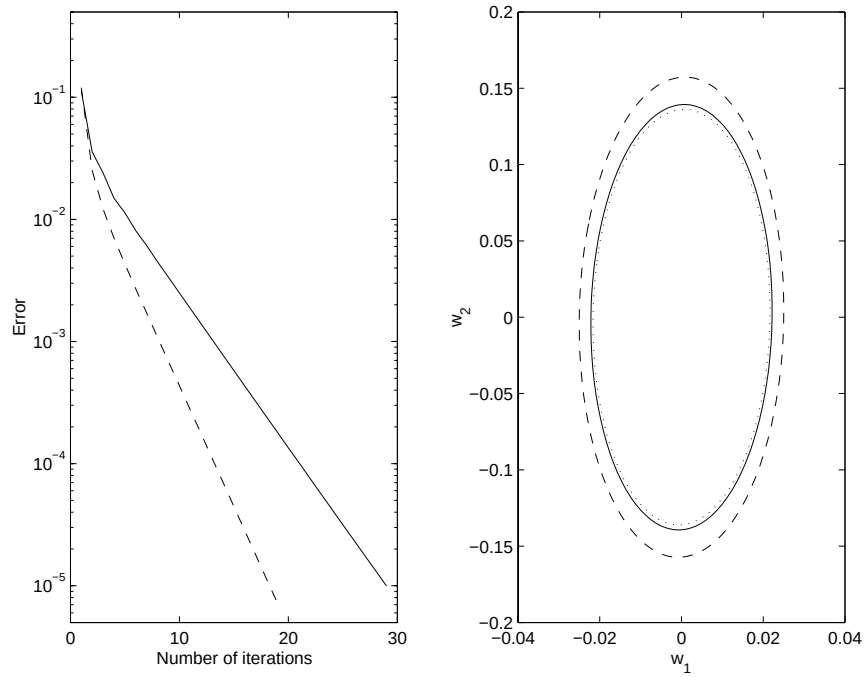

FIGURE 7. Computed results for Example 3. Left: Periodic WR iterations (dashed line for Gauss-Seidel and solid line for Jacobi). Right: Approximate (dashed line for $k = 1$ and dotted line for $k = 2$) and exact (solid line) phase drawings.

We use the same stopping criteria as in Examples 1 and 2. Numerical results on the periodic WR convergence are presented in Figure 7. The approximate and actual phase drawings of w_1 and w_2, where $w_1(t) = x_1(t)$ and $w_2(t) = \dot{x}_1(t)$ on $[0, 1]$, are also pictured in Figure 7.

5. Conclusions

We have successfully deduced an analytic expression of the spectral set on a periodic WR operator for a linear system of DAEs under a normal periodic constraint. The convergent splittings of the periodic WR algorithm on periodic solutions can be conveniently chosen from this useful expression; namely the periodic WR algorithm converges to the exact periodic response if the supremum value of spectral radii for a series of complex matrices is less than 1. The convergent condition of the paper is necessary and sufficient for this kind of relaxation-based algorithms. The practical partitions or splittings of dynamic systems can benefit from the analytic expression of the periodic WR operator; for example we may choose convergent splittings by directly viewing the spectral drawings of these operators. The simple and powerful finite-difference method is adopted to compute the discrete waveforms for these systems of DAEs. Numerical experiments on three examples further illustrate that the spectral expression is essential for WR in transient computations of periodic responses.

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