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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the American Mathematical Society classification scheme. The 2000 Mathematics Subject Classification can be found in print starting with the 1999 annual index of *Mathematical Reviews*. The classifications are also accessible from `www.ams.org/msc/`.

3[05C35, 90C27, 52B60]—*Report on global methods for combinatorial isoperimetric problems*, by L. H. Harper, Cambridge Studies in Advanced Mathematics, Vol. 90, Cambridge University Press, 2004, xiv+231 pp., hardcover, \$60.00, ISBN 0-521-83268-3

It is a very nice and useful book, written by a real expert in the field. A typical problem can be described as follows. Let G be a graph with the edge set E and the vertex set V, and let k be a given positive integer. The task is to choose a k-element set $S \subseteq V$ that minimizes the number of edges which are defined by one vertex from S and by one vertex outside of S.

The book has been based on many years of teaching this material to graduate students and this is certainly reflected in the style in which it is written. In a series of lemmas and theorems, the author leads the reader through rather simple cases to more complicated concepts. On the other hand, it offers a rich and varied selection of problems from this beautiful branch of combinatorial optimization to which a certain unifying "global" approach is developed. Informal comments at the end of each chapter provide a nice supplement to the main text and also help to gain some historical perspective on the subject. I believe that both specialists in the area and mathematicians with other backgrounds will find lots of new interesting material in this book.

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4[65L60, 65L70, 65M60, 65Nxx, 74S05, 76M10]—*Adaptive finite element methods for differential equations*, by Wolfgang Bangerth and Rolf Rannacher, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2003, viii+207 pp., softcover, EUR 22.00/SF 35.00, ISBN 3-7643-7009-2

Finite element methods are, undoubtedly, one of the most general and powerful techniques for the numerical solution of partial differential equations. Their historical roots can be traced back to the 1943 paper of Richard Courant [6] on variational methods for the approximation of problems of equilibrium and vibration. Given that V is an infinite dimensional Hilbert space, $a(\cdot,\cdot)$ is a continuous and coercive bilinear functional on $V \times V$ and $\ell(\cdot)$ is a continuous linear functional on V, the archetypal linear variational problem consists of finding u in V such that

$$(\mathrm{P}): \qquad a(u,v) = \ell(v) \quad \text{for all } v \in V.$$

Boundary value problems for scalar linear elliptic partial differential equations or elliptic systems, such as the Stokes problem modelling the flow of a viscous incompressible fluid in a bounded open set $\Omega \subset \mathbb{R}^n$, naturally fit into this abstract variational framework.

In engineering and scientific applications it is frequently the case that, instead of the field u itself, the quantity of interest is a certain *output functional* $u \mapsto J(u)$; typical examples include the weighted integral-mean-value of u, a point-value of u, the normal flux of u through (part of) the boundary $\partial\Omega$ of Ω, or, in problems that arise from fluid mechanics, the lift and the drag exerted on a body that is immersed into a viscous or inviscid fluid.

The finite element approximation of the variational problem (P) consists of selecting a finite dimensional space V_h (of dimension $N = N(h,p)$) of the space V consisting of a piecewise polynomial function of a certain degree p on a triangulation $\mathcal{T}_h$ of granularity h of the computational domain Ω, and seeking $u_h \in V_h$ such that

$$(\mathrm{P}_h): \qquad a(u_h, v_h) = \ell(v_h) \quad \text{for all } v_h \in V_h.$$

Adaptive finite element methods, driven by *a posteriori* error bounds, aim to automatically adapt the local mesh-size h or the local polynomial degree p, or both h and p, so as to accurately capture the analytical solution u, or a certain functional $u \mapsto J(u)$ of the solution.

It is this topic that forms the subject of the book by Bangerth and Rannacher under review. The book grew out of a lecture series given by the second author during the summer of 2002 at the Department of Mathematics of the ETH in Zürich. It comprises a brief Preface, followed by twelve chapters, a 24-page Appendix, a Bibliography with 138 entries, and a 5-page Index of terms; each chapter is about 15 pages long and is supplemented by computational examples as well as exercises whose model solutions are supplied in the Appendix.

As is highlighted by the authors in Chapter 1 of the book, the goal of adaptivity is the "optimal" use of computing resources according to either one of the following principles:

- Minimal work N subject to a prescribed positive tolerance `TOL`: $N \to \min$, `TOL` given; or,
- Maximal accuracy subject to prescribed work: `TOL` $\to \min$, N given.

These goals are, traditionally, approached by mesh adaptivity driven by "local refinement indicators" based on the computed solution u_h. The process of adaptivity has three main ingredients:

- a rigorous *a posteriori* bound on the error in the quantity of interest in terms of the data and the computed solution;
- a local refinement indicator extracted from the *a posteriori* error bound;
- automatic mesh adaptation (in the form of local h-refinement, or local p-refinement, or their combination referred to as hp-refinement) according to certain refinement strategies based on the local refinement indicators.

The idea of *a posteriori* error estimation stems from the early work of Babuška and Rheinboldt [2, 3]; see also the monographs of Ainsworth and Oden [1], Babuška and Strouboulis [4], and Verfürth [21] for further detail on the subject of *a posteriori* error analysis of the finite element method. The focus of this book by Bangerth and Rannacher is a general technique for goal-oriented *a posteriori* error estimation for finite element approximations of differential equations, called the dual-weighted-residual (DWR) method, and the implementation of this technique into adaptive finite element algorithms.

To give a brief sketch of the DWR method, consider the variational problem (P) and its finite element approximation (P_h) and suppose that the goal of the computation is to find an accurate approximation to the real number $J(u)$ where $J : V \to \mathbb{R}$ is (for the sake of simplicity of presentation) a linear functional and u is the solution to problem (P).

The derivation of an *a posteriori* bound by means of the DWR method on the error $J(u) - J(u_h)$ between the unknown value $J(u)$ and its known finite element approximation $J(u_h)$ rests on considering the associated *dual* problem: find $z \in V$ such that

$$\text{(D):} \qquad a(w, z) = J(w) \quad \text{for all } w \in V.$$

Clearly, setting $w = u - u_h$ in (D), we deduce that

$$\begin{aligned} J(u) - J(u_h) = J(u - u_h) &= a(u - u_h, z) \\ &= a(u - u_h, z - v_h) \end{aligned}$$

for all $v_h \in V_h$, where, in the transition to the last line, we made use of the *Galerkin orthogonality* property: $a(u - u_h, v_h) = 0$ for all $v_h \in V_h$, which is a straightforward consequence of subtracting (P_h) from (P) with $v = v_h \in V_h \subset V$. Proceeding then, using (P), we obtain

$$J(u) - J(u_h) = \ell(z - v_h) - a(u_h, z - v_h) \quad \forall v_h \in V_h.$$

Thus we have eliminated the analytical solution u, at the expense of involving the dual solution z. The last identity can be written in a more compact form on introducing the linear functional $R(u_h) : V \to \mathbb{R}$, defined by

$$R(u_h)(v) = \ell(v) - a(u_h, v) \quad \forall v \in V,$$

referred to as the *finite element residual*, or, simply, *residual*; it measures the extent to which the numerical solution u_h fails to satisfy the equation (P). Hence,

$$\begin{aligned} J(u) - J(u_h) &= R(u_h)(z - v_h) \\ &= \langle R(u_h), z - v_h \rangle \quad \forall v_h \in V_h, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the dual space V' of V and V. This *error representation formula* is at the heart of the DWR method, highlighting the fact that the error in the approximation of the value $J(u)$ depends on the interplay between the finite element residual $R(u_h)$ and the error $z - v_h$, with $v_h \in V_h$, in the approximation of the dual solution z, which acts as a weight function for the residual. Hence the terminology *dual-weighted-residual* method. In particular, the last identity implies that

$$(1) \qquad |J(u) - J(u_h)| = \inf_{v_h \in V_h} |\langle R(u_h), z - v_h \rangle|.$$

In earlier incarnations of duality-based error estimation—particularly in the pioneering research pursued by the Gothenburg school (see, for example, the articles by Johnson [16], Eriksson and Johnson [13, 14], and the illuminating survey paper by Eriksson, Estep, Hansbo, and Johnson [12])—the objective was to eliminate the explicit appearance of the dual solution z from the right-hand side of (1) through a succession of upper bounds. The first of these upper bounds involved making a particular choice of v_h such as the finite element interpolant or quasi-interpolant $P_h z$

of z; this step was followed by localizing the expression $|\langle R(u_h), z - P_h z\rangle|$ through decomposing it, as a sum of analogous terms defined locally, over the elements T in the triangulation; the next step was to apply the Cauchy–Schwarz inequality to each of these local terms in tandem with an interpolation-error bound such as $\|z - P_h z\|_{L_2(T)} \leq C_{\mathrm{int}} h_T^s \|z\|_{H^s(T)}$, where $h_T = \mathrm{diam}(T)$, with $T \in \mathcal{T}_h$, and C_{int} is an interpolation constant; and, finally, to exploit the strong stability of the dual problem to bound the Sobolev norm $\|z\|_{H^s(\Omega)}$ in terms of the data of the dual problem and the stability constant C_{stab} of the dual problem, resulting in an *a posteriori* error bound of the form

$$|J(u) - J(u_h)| \leq C_{\mathrm{int}} C_{\mathrm{stab}} \left(\sum_{T \in \mathcal{T}_h} h_T^{2s} \|R(u_h)\|^2_{L^2(T)} \right)^{1/2}$$

with no explicit dependence on the dual solution. While such an *a posteriori* error bound is *reliable* in the sense that the right-hand side of the inequality is a guaranteed upper bound on the left-hand side, numerical experiments will quickly reveal that, typically, the right-hand side will overestimate the left-hand side—sometimes by orders of magnitude—even if the sharpest available values of the constants C_{int} and C_{stab} are used. A further observation in connection with the last bound is that the original feature of (1), namely that it is the interplay between $R(u_h)$ and $z - v_h$, with $v_h \in V_h$, that governs the error $J(u) - J(u_h)$, rather than the size of $R(u_h)$ alone, is completely lost through successive applications of the Cauchy–Schwarz inequality aimed at eliminating the presence of the dual solution z. The importance of preserving the dual solution z as a locally varying weight to the residual is particularly important in instances when the dual solution exhibits complex behavior over the computational domain Ω. Whether or not this is so, of course, depends entirely on the nature of the problem (P) and the choice of the output functional J. For example, when (P) is the weak formulation of an elliptic convection-dominated diffusion equation and $J(u) = u(\mathbf{x}_0)$, $\mathbf{x}_0 \in \Omega$, the dual solution z will contain a thin internal layer which will be aligned with the subcharacteristic curve passing through $\mathbf{x}_0$. It would be unreasonable to expect that the presence of such a localized and anisotropic structure in the dual solution could be represented by, or encoded into, a single constant, C_{stab}, the stability constant of the dual problem featuring in the last *a posteriori* error bound.

These recognitions motivated, in the mid-1990s, the work of Becker and Rannacher [7] where the dual-weighted-residual method was first introduced (see also [8] and the survey articles [8] and [15]). At about the same time, other researchers have also embarked on closely related investigations (see, for example, [17], [19] and [20]).

In particular, in order to derive a sharp *a posteriori* error bound from the error representation formula (1) while retaining the presence of the dual solution in the bound as a local weight to the finite element residual, it was recognized in [7] that the number of applications of the Cauchy-Schwarz inequality in the derivation of the bound has to be kept to the minimum. An *a posteriori* error bound based on the DWR method which meets these objectives can be inferred from (1); it has the form

$$|J(u) - J(u_h)| \leq \sum_{T \in \mathcal{T}_h} |\langle R(u_h)|_T, z - P_h z\rangle_T|, \tag{2}$$

where $\langle \cdot, \cdot \rangle_T$ is a localized counterpart of the duality pairing $\langle \cdot, \cdot \rangle$, $R(u_h)|_T$ is the restriction of the (global) finite element residual $R(u_h)$ to element $T \in \mathcal{T}_h$, and $P_h z \in V_h$ is the finite element interpolant or quasi-interpolant of z.

Chapters 2–4 of the book are devoted to explaining the application of the DWR method to an ODE model problem (Chapter 2) and a PDE model problem (Chapter 3), and to discussing practical aspects of the method (Chapter 4), including the evaluation of the DWR error bound (2) and other DWR error bounds akin to (2). For, strictly speaking, inequality (2), as it stands, is *not* an *a posteriori* error bound in the classical sense of the word, given that it involves the unknown analytical solution z to the dual problem (D). Clearly, z has to be computed numerically; in particular, if a finite element method is used to compute an approximation to z, then a finite element space different from V_h must be used for this purpose; once such an approximation to z is available, it has to be projected onto V_h to obtain $z_h \in V_h$ which can be used *in lieu* of $P_h z$ in (2). The additional errors incurred through the numerical approximation of the dual solution are difficult to quantify unless one embarks on reliable *a posteriori* error estimation for the dual problem; for reasons of economy, this is rarely attempted in practice. Indeed, there is very little in the current literature in the way of rigorous analytical quantification of the impact of replacing the exact dual solution z in the DWR error bound by its numerical approximation; see, however, the recent analytical work of Carstensen [5] on the estimation of higher Sobolev norm from lower order approximation, and the application of this in the context of the DWR method. A second issue is that the necessity to compute a "reasonably" accurate approximation to the dual solution results in added computational work. The authors of the book provide a convincing computational demonstration through a wide range of model problems that, except on very coarse meshes, *a posteriori* error bounds obtained by the DWR method remain reliable and very sharp even on replacement of z by its numerical approximation. In addition, when implemented into adaptive finite element algorithms, error bounds derived by the DWR method lead to economical computational meshes.

An analysis aimed at gaining further theoretical insight into the performance of the DWR method is performed in Chapter 5 of the book. The chapter also discusses the current limits of theoretical analysis of the method focusing, in particular, on convergence under mesh refinement of the finite element residual and of the weights which incorporate the numerical approximation to the dual solution z. As is noted by the authors at the end of Section 5.3, further challenges include the convergence analysis of the method on locally refined meshes, particularly in the presence of singularities in the solutions to the primal problem (P) and/or the dual problem (D). Indeed, the convergence analysis of adaptive algorithms has been the subject of active research in recent years (see, for example, the papers of Morin, Nochetto, and Siebert [18], Cohen, Dahmen, and DeVore [11], and Binev, Dahmen, and DeVore [10] in this direction in the context of energy-norm-based *a posteriori* error estimation and adaptivity for elliptic problems).

Chapter 6 is concerned with the extension of the DWR method to nonlinear variational problems. A particularly appealing feature of the DWR method from the practical point of view is that, when applied to nonlinear PDEs, the dual problem, which is simply the adjoint of the linearization of the primal problem, is still a linear problem. Hence the computational overhead of obtaining an approximate dual solution is merely a fraction of the computational complexity of solving the primal nonlinear problem itself.

Chapters 7 to 11 discuss the application of the DWR method to, respectively, eigenvalue problems, optimization problems, time-dependent problems, linear and nonlinear problems in structural mechanics, and problems in fluid dynamics including the computation of drag and lift coefficients in a viscous incompressible flow.

The book closes, in Chapter 12, with an overview of miscellaneous and open problems, including historical remarks and a survey of current developments. Some of the open problems identified by the authors include the use of the DWR method for multidimensional time-dependent problems, its application in the context of the *hp*-version finite element method, the organization of anisotropic mesh refinement, the effective control of variational crimes, the control of the error incurred in the solution of algebraic equations which result from finite element discretizations of differential equations, the application of the DWR method to nonvariational problems, and, finally, the solution of the theoretical problems raised in Chapter 5 so as to provide complete theoretical underpinning of the DWR method. Some of these are already the subject of ongoing research.

This well-written book is highly suitable as supporting text for an advanced undergraduate or a basic graduate course on adaptive finite element methods for partial differential equations. The material is clearly structured and well organized, and the numerous computational examples and exercises induce the reader to further explore the subject. The discussions of open or incompletely understood problems are particularly stimulating and raise the understanding of the reader to the forefront of current research in the field. I warmly recommend this book to anyone with interest in the analysis of finite element methods and their application to partial differential equations.

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5[11B85, 11Z05, 37A45, 37B10, 68Q45, 68R15, 94A45]—*Automatic sequences. Theory, applications, generalizations*, by Jean-Paul Allouche and Jeffrey Shallit, Cambridge University Press, Cambridge, 2003, xvi+571 pp., \$50.00, ISBN 0-521-82332-3

> Sequences come in all flavors. Some, such as periodic sequences, are highly organized, while others are unordered and have no simple description. The subject of this book is *automatic sequences* and their generalizations. Automatic sequences form a class of sequences somewhere between simple order and chaotic disorder. This class contains such celebrated sequences as the Thue–Morse sequence and the Rudin–Shapiro sequence. . . . [from the Introduction]

The subjects of this fine book include combinatorics on words, formal languages, useful parts of number theory, formal power series, ... The chapter titles give some hint of the breadth of material appropriately touched upon: Stringology, Number Theory and Algebra, Numeration Systems, Finite Automata and Other Models of Computation, Automatic Sequences, Uniform Morphisms and Automatic Sequences, Cobham's Theorem for (k, l) Numeration Systems, Morphic Sequences, Frequency of Letters, Characteristic Words, Subwords, Cobham's Theorem, Formal Power Series, Automatic Real Numbers, Multidimensional Automatic Sequences, Automaticity, k-Regular Sequences, Physics.

Automatic Sequences is both an introduction to the study of the said sequences and related mathematics and a careful survey of known results and applications with beautifully optimized proofs, many of which can lay claim to coming from *the Book*.

Let me continue by giving an entry to the subject rather different from those emphasised in the book being reviewed (but see its Chapter 12).

Consider a formal power series $\sum_{j_1,j_2,\ldots,j_n} a_{j_1,j_2,\ldots,j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$, or $\sum_\nu a_\nu x^\nu$ in brief, representing a rational function, thus a quotient of polynomials. Fear of functions of many variables might lead one to study its "main diagonal" $\sum_j a_{j,j,\ldots,j} x^j$. That diagonal represents an algebraic function if $n = 2$; in general it represents a solution of a Fuchsian linear differential equation with rational function coefficients. If, moreover, the base field is finite, of characteristic p, then that diagonal is algebraic over the field of rational functions over that finite field. Rather more to the point, the multisequence of coefficients a_ν of an algebraic power series defined over a finite field is p-automatic, and conversely such an "automatic" power series is necessarily algebraic.

One sees this surprisingly readily. Take the base field to be $\mathbb{F}_p$, the field of p elements and set $S = \{0, 1, \ldots, p-1\}^n$. If the power series $y(x) = y(x_1, \ldots, x_n)$ is algebraic, then it satisfies an equation $\sum_{i-0}^s f_i(x) y^{p^i} = 0$. Set $f = -f_0$. Then multiply by f^{p-1}, obtaining an equation $f^p y = L(y^p, y^{p^2}, \ldots, y^{p^s})$. Now recall that $\{x^\alpha = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} : \alpha \in S\}$ is a basis for $\mathbb{F}_p[[X]]$ over $(\mathbb{F}_p[[X]])^p$. After again multiplying by f^{p-1} that yields equations $f^p y_\alpha = L_\alpha(y^p, y^{p^2}, \ldots, y^{p^s})$, $\ldots$. If one checks the multidegree in x of $L_{\alpha_1,\ldots,\alpha_k}$ as k increases, one finds it is bounded; hence there are only finitely many distinct $y_{\alpha_1,\ldots,\alpha_k}$. That says there is a positive integer e so that for every n-tuple j with the j_i less than p^e there is an $e' < e$ and an n-tuple j' with the j'_i less than $p^{e'}$ so that for all n-tuples ν,

$$a_{p^e \nu + j} = a_{p^{e'} \nu + j'}.$$

In other words, the multisequence of coefficients a_ν is p-automatic. Conversely one sees easily that each $y_{\alpha_1,\ldots,\alpha_e}$ satifies an equation of the form

$$y_{\alpha_1,\ldots,\alpha_e} = \sum x^\gamma y_{\beta_1,\ldots,\beta_e} p^{e-e'}.$$

It follows—the Jacobian determinant of such a system of equations is 1—that the $y_{\alpha_1,\ldots,\alpha_e}$ and hence y must be algebraic.

I should also mention a manuscript settling a question irritatingly open at the time the book was completed. Given an infinite string, denote by $p(n)$ the number of its distinct subwords of length n. Recently, Boris Adamczewski, Yann Bugeaud, and Florian Luca in "Sur la complexité des nombres algébriques" (*C. R. Acad. Sci. Paris, Ser. I* **336** (2004)) applied Schlickewei's p-adic generalisation of Wolfgang Schmidt's subspace theorem (which itself is a multidimensional generalisation of Roth's theorem) to proving that for the base b expansion of an irrational algebraic number $\limsup_{n\to\infty} p(n)/n = \infty$, whereas for any number generated by a finite automaton $p(n) = O(n)$.

I highly recommend *Automatic Sequences*, whether as text, reference, or all the more as an excellent read, both to rank beginners and to those already acquainted with parts of the subject.

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6[40A05, 40A10, 40A25, 40A30, 40B05, 40G05, 41A20, 41A21, 41A25, 41A55, 41A58, 41A60, 65B05, 65B10, 65B15, 65D25, 65D30, 65R10, 65R20]—*Practical extrapolation methods: theory and applications*, by Avram Sidi, Cambridge Monographs on Applied and Computational Mathematics, Vol. 10, Cambridge University Press, Cambridge, 2003, xxii+519 pp., \$95.00, ISBN 0-521-66159-5

A fundamental problem one faces in different scientific and engineering disciplines is that of finding limits of sequences $\{A_m\}$; in many cases, solutions are simply defined to be limits of sequences of approximations. Such sequences may arise in different forms. They may arise, for example,

- from the solution of linear or nonlinear systems of equations by iterative methods, in which case $\{A_m\}$ is the sequence of iterates. Here $\lim_{m\to\infty} A_m$ are the solutions to the systems being considered.
- from trapezoidal rule approximations of one- or multi-dimensional integrals, in which case $A_m = Q(h_m)$, where $Q(h_m)$ is the approximation with integration stepsize h_m, and $\{h_m\}$ is a decreasing null sequence.
- in the summation of infinite series $\sum_{k=1}^{\infty} a_k$, in which case, $A_m = \sum_{k=1}^{m} a_k$, $m = 1, 2, \ldots$. Here a_k can be numbers or they can be of the general form $a_k = c_k\phi_k(x)$, where c_k are numbers and $\phi_k(x)$ are elementary functions such as powers, trigonometric functions, orthogonal polynomials, or other special functions.
- from infinite-range integrals $\int_0^\infty f(t)\,dt$, where $A_m = \int_0^{x_m} f(t)\,dt$ for some increasing positive sequence of points $\{x_m\}$ such that $\lim_{m\to\infty} x_m = \infty$.

In many cases of practical interest, these sequences converge extremely slowly; therefore, one needs to compute a large number of the terms A_m to approximate $\lim_{m\to\infty} A_m$ with reasonable accuracy, and this makes their direct use very expensive computationally. In some cases, $\{A_m\}$ may even diverge, which makes their direct use irrelevant. (In case of divergence, one speaks about the *antilimit* of $\{A_m\}$ instead of its limit, and the antilimit is a quantity of relevance; for a power series, the antilimit may be the *Abel sum* of the series on the circle of convergence or its analytic continuation outside its circle of convergence when the radius of convergence is nonzero, or it may be its Borel sum when the radius of convergence is zero, etc.)

In all cases, whether $\{A_m\}$ converges or not, suitable *extrapolation methods* (equivalently, *convergence acceleration methods* or *sequence transformations*) are needed to obtain the limit or antilimit of $\{A_m\}$ with high accuracy by using the terms $A_1, A_2, \ldots, A_s$, where s is a small integer.

Generally speaking, an extrapolation method, when applied to a sequence $\{A_m\}$, produces another sequence $\{\hat{A}_m\}$ that converges to the limit or antilimit of $\{A_m\}$ faster than $\{A_m\}$ itself. A practical extrapolation method is one for which $\hat{A}_m$ is a linear or nonlinear combination of a *finite* (and preferably “small”) number of the A_k. It is known that methods that are linear in the A_k have very limited scope and perform poorly in general. The book under review concentrates on *nonlinear* methods. The only linear method covered in the book is the famous transformation of Euler, which seems to be the most effective linear method.

This book, written by a leading expert in the field, is an excellent up-to-date account of the most useful extrapolation methods for sequences of scalars. (As the author has written in the preface, extrapolation methods for vector sequences, because of their special theory and applications, deserve a fully dedicated book. The reviewer hopes that the author, whose work in this subject is highly regarded and well known, will undertake the writing of such a book eventually.) It presents an excellent synthesis of all aspects of the subject of convergence acceleration. It discusses a large number of recently published methods and results, as well as known ones, of both the author and other researchers. It presents their theory in a thorough and unified manner in most places, and discusses them from the point of view of the practitioner by giving important tips about their effective use. It also provides numerous examples of the types of sequences that arise in practical applications. The author has produced a book that is in complete agreement with its title: the methods it deals with are practical, and so are the problems they are applied to. Everything taken into account, this book will benefit both the theoretician and the practitioner.

Here are the unique features of the book that distinguish it from the earlier literature:

- It includes the most important aspects of extrapolation methods, namely, derivation of methods, design of efficient algorithms for their implementation, and rigorous convergence and stability analyses.
- It covers many different types of sequences that arise in practical applications along with detailed analyses of their asymptotic properties. (Most sequences treated in the common literature belong to a very limited class of sequences, namely, the class $\mathbf{b}^{(1)}$ discussed in the book.)
- It treats divergent sequences on an equal footing with convergent ones, providing at the same time an illuminating discussion of their antilimits.
- The proofs of many of the theorems are provided in the text; the reader is referred to the literature in only a few cases.
- All known extrapolation methods, either explicitly or implicitly, rely on the form of the asymptotic expansion of A_m as $m \to \infty$. In Sidi's book great emphasis is placed on the asymptotic analysis of A_m as $m \to \infty$. Theorems and simple recipes are given by which one can deduce the form of the asymptotic expansion of A_m, from which the user can decide easily on the right extrapolation method to employ.
- The issue of stability is formalized and treated in detail for the first time. The conclusions drawn from the analysis of stability are used to devise strategies the author calls *arithmetic progression sampling (APS)* and *geometric progression sampling (GPS)* that enable high accuracy in finite-precision arithmetic. (It must be noted that, without the use of these strategies, the best accuracy that can be obtained from extrapolation methods is limited and is completely destroyed eventually.)
- In addition, to make the work of the practitioner easy, the author has provided important practical information in a series of brief appendices. Of these, Appendix G contains the algorithms of the methods discussed in the book. Appendix H classifies the important types of sequences $\{A_m\}$ that arise in applications and points to the most effective convergence acceleration methods for each type. In Appendix I the author has also provided a well-documented user-friendly FORTRAN 77 code that implements the d-transformation on infinite series.

The reviewer has found the style of this book agreeable. The author has succeeded in retaining a great amount of unity throughout. Concepts are defined clearly and are illustrated with several interesting and nontrivial examples. The notation is consistent and remains the same when going from one chapter to another. Things are explained with clarity. Quite a few examples are included in every chapter, and this makes the study of the material quite pleasant. So far as the reviewer has noted, there seem to be no errors; there seem to be some misprints that the reader can correct easily by himself. The list of references contains most of the relevant literature. In order to follow the theoretical arguments with ease, some working knowledge of asymptotic expansions (what they are and what one can do with them) is required.

Here is a short description of the contents of the book:

The Introduction gives the motivation for using extrapolation methods, discusses antilimits via concrete examples, gives an overview of the major subjects to be encountered in the remaining part of the book. Among these, the issue of stability is discussed in detail (and should be studied with care). This chapter actually lays out the course of action the author takes in the study of each individual method. It is important that the reader spend some time to understand the Introduction's message.

Following the Introduction, the book is divided into four parts: Part I, which forms the bulk, gives a very detailed discussion of the Richardson extrapolation and its various generalizations. Part II presents the subject of sequence transformations and includes most of the known transformations. Part III is a single long chapter that treats quite a few applications of the methods discussed in Parts I and II that are not explicitly covered in these parts. Among the applications covered here, we mention especially those involving multi-dimensional integrals with surface, line, and corner singularities, periodic Fredholm integral equations (such as boundary integral equations on closed curves) with singularities, numerical inversion of Laplace transforms, summation of slowly convergent series with special sign patterns, summation of rearrangement series, and computation of time-periodic steady states from iterative schemes. Part IV is a collection of appendices that the reader will find very helpful in following the material of Parts I and II. Of these, Appendix A gives a summary of asymptotic expansions, Appendix D is an excellent compendium of one-dimensional Euler–Maclaurin expansions, old and new, Appendix E is about the Zeta function and related asymptotic expansions; all these are used extensively throughout the book.

In the remainder of this review, we will briefly cover the contents of Parts I and II, making some informative remarks when appropriate.

Part I. The discussion of the classical Richardson extrapolation process given in Chapters 1 and 2 is the most complete treatment that has been given in book form. Chapter 3 presents what the author calls *first generalization of the Richardson extrapolation process* and what is known in the literature by the name of one of the algorithms that implement it, namely, the E-algorithm. Chapter 4 gives a further generalization of the Richardson extrapolation process known by the name GREP and that is the author's work. From the examples given in this chapter, it becomes clear that GREP is a very comprehensive extrapolation framework.

Chapters 5 and 6 discuss two important GREPs, namely, the D-transformation for infinite-range integrals $\int_0^\infty f(t)\,dt$ and the d-transformation for infinite series

$\sum_{k=1}^{\infty} a_k$, both due to Levin and Sidi. The various numerical examples given in the original paper of Levin and Sidi and in other additional works show that these methods are capable of accelerating the convergence of a very large class of infinite-range integrals and infinite series, thus have a much larger scope than most other methods. These chapters need special attention, because they introduce important classes of functions $f(x)$ [denoted $\mathbf{B}^{(m)}$] and of sequences $\{a_k\}$ [denoted $\mathbf{b}^{(m)}$] that the reader encounters in the rest of the book and that occur frequently in scientific and engineering applications. The D-transformation is designed to compute integrals $\int_0^{\infty} f(t)\,dt$ with $f \in \mathbf{B}^{(m)}$, while the d-transformation is designed to sum infinite series $\sum_{k=1}^{\infty} a_k$ with $\{a_k\} \in \mathbf{b}^{(m)}$, $m = 1, 2, \ldots$, whether these are convergent or divergent. Concerning the issue of summation of infinite series, it must be added that most of the literature on sequence transformations has not gone beyond treatment of series $\sum_{k=1}^{\infty} a_k$ for which $\{a_k\}$ is in $\mathbf{b}^{(1)}$, the simplest of sequence classes $\mathbf{b}^{(m)}$. For commonly occurring Fourier series and orthogonal polynomial series and other similar ones, in most cases, it follows that $\{a_k\} \in \mathbf{b}^{(m)}$ with $m \geq 2$ but $\{a_k\} \notin \mathbf{b}^{(1)}$, which means that such series can be summed economically with the help of the d-transformation (and the transformation of Shanks), most other transformations being ineffective.

Even though the original paper of Levin and Sidi was published more than twenty years ago, this is the first time the D- and d-transformations have appeared in book form. As mentioned already, most researchers in sequence transformations have chosen to limit themselves to the simplest class of sequences $\mathbf{b}^{(1)}$. Recent books in this field simply ignore the d- and D-transformations. From the many papers the reviewer has refereed during the last two decades, he has gotten the impression that most researchers in the field simply do not understand these important classes of transformations. Sidi's book is the first to present the d- and D-transformations, their fascinating theory, their application, and their implementation. The reviewer hopes that it will help to put an end to their excommunication.

Chapter 7 describes the W-algorithm of the author and the W$^{(m)}$-algorithm of Ford and Sidi, two sophisticated recursive schemes, by which one can implement the D- and d-transformations, among others. Chapters 8, 9, and 10 discuss the general convergence and stability theory and efficient application of the simplest form of GREP, namely, of GREP$^{(1)}$, in different circumstances. Chapter 11 discusses the efficient use of extrapolation methods in computing oscillatory infinite-range integrals and describes the $\bar{D}$-, $\tilde{D}$-, W-, and mW-transformations that achieve very high accuracy economically. (Note that these methods are also GREPs.) Chapter 12 is about the application of the d-transformation to power series and about the resulting rational Padé-like approximations, while Chapter 13 treats the summation of Fourier series and their generalizations, via what the author calls the *complex series approach*, in an economical manner. Chapter 14 discusses some special topics in Richardson extrapolation. One such topic concerns an interesting approach of the author for the efficient computation of derivatives of limits and antilimits via extrapolation methods.

Part II. Chapter 15 discusses the Euler transformation, the only linear method considered in the book, the Aitken Δ^2-process, and the transformation of Lubkin. All three methods are of historical importance and still used whenever appropriate. Chapter 16 gives the most thorough treatment of the transformation of Shanks (implemented most efficiently via the famous ϵ-algorithm of Wynn) that has appeared in book form. Now, when the transformation of Shanks is applied to a power series, the approximations obtained are nothing but the Padé approximants. On account of this, it is natural that a discussion of this subject be included in a book dealing with acceleration of convergence. Thus, Chapter 17 is a good summary of part of the algebraic and analytic theory of Padé approximants and of related subjects, such as interpolation by exponential functions (Prony's method), continued fractions and the quotient-difference (qd) algorithm, and Gaussian quadrature. A detailed summary of the classical theorems of de Montessus and of Koenig, with the author's refinements, including the treatment of the so-called intermediate rows, and summaries of convergence of Padé approximants from moment series, from Pólya frequency series, and from entire functions, form an important part of this chapter. In Chapter 18, a brief discussion of the various generalizations of Padé approximants, such as multi-point approximants, Hermite-Padé approximants, Padé approximants from orthogonal polynomial expansions, is provided. Chapters 19–22 give detailed accounts of the $\mathcal{L}$-transformation of Levin and the more recent $\mathcal{S}$-transformation of Sidi, the ρ-algorithm of Wynn and Osada's modifications of it, the θ-algorithm of Brezinski, the higher-order G-transformation of Gray, Atchison, and McWilliams, and the transformations of Overholt and of Wimp. (The $\mathcal{S}$-transformation has been observed to be the most effective method for summing a class of wildly divergent power series that have zero radius of convergence. Such divergent series arise, for example, from perturbation analysis in theoretical physics.) Chapter 23 discusses the confluent forms of some of the sequence transformations and includes new but unpublished results of the author too. Finally, Chapter 24 provides a short description of the so-called formal theory of sequence transformations. In the reviewer's opinion, this last chapter is rather weak; it has no place in a book dealing with practical methods.

This book is quite pleasant to read. This should not be taken as a sign that it is easy reading, however; in many places, one may have to spend some time to verify the results that are put forth, whether their proofs are supplied or not. In addition to being an excellent research monograph, hence of interest to the theoretician, this book will also serve as a practical guide for those scientists and engineers who wish to apply extrapolation methods in the solution of their problems, but are not interested in the mathematical details. It can also serve as a textbook for those advanced undergraduate and graduate students who would like to undertake an in-depth study of the subject of extrapolation methods.

The reviewer recommends *Practical Extrapolation Methods* very highly to all those interested in applying and/or learning the subject. He believes that it will remain the state-of-the-art reference in extrapolation methods and their applications for many years to come.

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7[65F15, 65F50]—*The Lanczos method: evolution and application*, by Louis Komzsik, Software, Environments, and Tools, Vol. 15, SIAM, Philadelphia, PA, 2003, xii+87 pp., $42.00, ISBN 0-89871-537-7

This book of roughly 80 pages is dedicated to Cornelius Lanczos and aims to show the impact of the Lanczos method in linear algebra computations for large scale finite element problems in structural dynamics. Its goal is to illustrate the important differences between a theoretical algorithm and an industrial implementation. The author has 30 years experience in the structural dynamics software industry.

The plan of the book is as follows. Chapter 1, *The classical Lanczos method*, reviews Lanczos' famous 1950 paper on the computation of eigenvalues using Lanczos' notation. Chapter 2, *The Lanczos method in exact arithmetic*, revisits the Lanczos method using current linear algebra notation. Chapter 3, *The Lanczos method in finite precision arithmetic*, introduces breakdown and the loss of orthogonality. It also discusses partial reorthogonalization. Chapter 4, *Block real symmetric Lanczos method*, is about the solution of large symmetric definite generalized eigenvalue problems, mainly based on the work by Grimes, Lewis, and Simon (1994). Chapter 5, *Block unsymmetric Lanczos method*, applies the Lanczos method to non-Hermitian matrices. In particular, it aims to discuss ABLE by Bai, Day, and Ye (1999). It includes an adaptive block size strategy and maintains biorthogonality by a reorthogonalization scheme. Then Chapter 6, *Industrial implementation of the Lanczos method*, briefly shows some aspects of the implementation, including the spectral or shift-and-invert transformation, and in more detail the parallelization of the Lanczos method. Next, Chapter 7, *Free undamped vibrations*, and Chapter 8, *Free damped vibrations*, discuss the solution of applications and numerical results using Msc.NASTRAN on an IBM SP. Chapter 8, also discusses the quadratic eigenvalue problem. Chapter 9, *Forced vibration analysis*, is an attempt to introduce Padé via Lanczos. This method is used for algebraic model reduction of dynamical systems and matches the moments of the transfer function around a shift point. Chapter 10, *Linear systems and the Lanczos method*, discusses the conjugate gradient method. An example from a static analysis is given.

Perhaps the most interesting part is Chapter 7, which shows timings on the parallelization of the Lanczos method. The chapter discusses geometric and frequency parallelism. Usually, the eigenvalues in a given interval are of interest. Geometric parallelism is the partitioning of the matrices and the Lanczos vectors among the processes, while frequency parallelism divides the interval into parts. The former reduces the memory requests per process, while the latter reduces the communication cost. The author showed an example for which the best computation time is obtained by combining both ideas.

I was quite surprised to see so many typos and elementary errors in this book. The Padé via Lanczos method was not correctly introduced in Chapter 9.

I do not think the book is very useful for the numerical analysis community since it contains very few references, it is not mathematically correct and has a number of confusing typos. The author is clearly most familiar with the solution of the symmetric definite eigenvalue problem by the block Lanczos method.

The author does not talk about the dangers of the Lanczos method for the solution of non-Hermitian eigenvalue problems. For example, it does not even mention the Arnoldi method as a valuable (and perhaps preferable) method. Similarly, the book does not discuss the interesting and important work on improving the method for the solution of linear systems. For example, BiCGStab is not even cited. This is rather surprising for a book that wants to demonstrate the difference between a theoretical algorithm in exact arithmetic and an industrial implementation.

There might be an interest from the applications communities, in particular structural analysis and acoustics. The book may also lift the suspicion from some communities that the Lanczos method is less reliable than subspace iteration. However, the book does not allow an engineer to fully understand the Lanczos method due to a number of errors. It gives the wrong impression that the Lanczos method is a universal method that can solve all problems. The 50 citations should help the interested reader with his search for detailed information.

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8[65N30, 78A25, 78M10]—*Finite element methods for Maxwell's equations*, by Peter Monk, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003, xiv+450 pp., hardcover, \$119.50, ISBN 0-19-850888-3

During the past two decades, the use of novel finite element techniques in electromagnetic field computations has attracted a lot of attention beginning with the systematic introduction of edge elements by Nédélec in [7, 8]. Edge elements have been systematically used in computational electromagnetics as documented, for instance, in the textbooks by Bossavit [1, 2]. A modern approach, based on discrete differential forms, can be found in the recent survey article by Hiptmair [5]. The application of finite elements in the numerical solution of Maxwell's equations has also been addressed by Jin [6] and Silvester–Ferrari [10]. However, taking into account that [2], though held in high regard by the reviewer, uses its own terminology founded on a specific point of view of computational electromagnetics and that [6, 10] aim at the electroengineering community, the new book by Monk provides a comprehensive exposition of the mathematical foundations of the finite element method for Maxwell's equations and a presentation of the state-of-the-art in this area.

The book is broken into fourteen chapters. Chapters 1–12 can be roughly grouped into three parts. The first part, consisting of Chapters 1–4, is devoted to the mathematical formulation of Maxwell's equations and the appropriate functional analytic background as well as the variational formulation of the cavity problem representing the class of interior domain problems. The second part comprises Chapters 5–8 and is concerned with conforming tetrahedral and hexahedral finite elements and their application to the cavity problem. Scattering problems as the most important class of exterior and coupled interior/exterior domain problems are dealt with in the third part, comprising Chapters 9–12. Particular aspects related to the algorithmic realization of the finite element approach are addressed in Chapter 13, whereas the final Chapter 14 gives a brief introduction into inverse scattering.

Chapter 1 is devoted to mathematical models of electromagnetism with a clear emphasis on the time-harmonic Maxwell system and associated interior and exterior domain problems that will become the focus of appropriate finite element approximations in the subsequent parts of the book.

I found Chapter 2 useful in making the book self-contained. It provides the necessary tools from functional analysis and linear operator theory in a Hilbert space framework, and it states basic finite element convergence results in an abstract setting. The experienced reader can easily skip this chapter and proceed to the next one which is more subtle in so far as it introduces to the function spaces $H(\mathrm{div};\Omega)$ and $H(\mathrm{curl};\Omega)$ of vector-valued functions on bounded, simply connected

Lipschitz domains $\Omega \subset \mathbb{R}^3$ and their trace spaces which play a vital role in the finite element analysis of Maxwell's equations. Furthermore, the indispensable Helmholtz decomposition and the associated de Rham diagram are discussed in due depth.

Equipped with the arsenal provided before, Chapter 4 deals with the variational formulation of the cavity problem with varying material coefficients and impedance boundary conditions on part of the boundary. The intrinsic difficulty related to the nontrivial kernel of the curl-operator is addressed in detail and how to overcome it by means of a suitable Helmholtz decomposition. An existence and uniqueness result is proved relying on the Fredholm alternative, and the issue of cavity eigenvalues and resonances is presented as well.

Although edge elements had been known before, it was Nédélec in his seminal paper [7] who introduced these $H(\mathrm{curl})$-conforming elements in a rigorous mathematical setting and also generalized to 3D the $H(\mathrm{div})$-conforming elements due to Raviart and Thomas in the 2D case. Chapters 5 and 6 are devoted to these elements on both simplicial and hexahedral triangulations of the computational domain. Again, those readers who are familiar with this machinery may skim over the text and go directly to Chapter 7 where the cavity problem is resumed in terms of its edge element approximation. Emphasis is on the error analysis which can be accessed in several different ways. Here, in order to be consistent with the apparatus provided in Chapter 4, discrete analogues of compactness are used (compact perturbations of coercive bilinear forms and the theory of collectively compact operators). Chapter 8 discusses various extensions of edge elements such as Nédélec's edge elements of the second family [8], the *hp*-edge elements which have recently gained a lot of attention, and edge elements on curved computational domains.

It is not because of the marked preference of the author for scattering problems, but rather due to the eminent importance in applications that their mathematical formulation and discrete approximation occupy the next four chapters of the book as well as the final chapter. There are excellent textbooks on the topic, e.g., by Cessenat [3], Colton and Kress [4], and Nédélec [9], but their focus is rather on analytic results than on numerical approaches. A significant strength of Monk's book is that he illuminates the theoretical background as much as necessary to elaborate on the discrete approximations. Chapter 9 serves as an introduction and collects basic material such as the celebrated Stratton–Chu representation formula and spherical harmonics and spherical Bessel functions for the series solution of the exterior domain problem. Another topic is about the electromagnetic Calderon operators that are then used in Chapter 10 for a variational formulation of the scattering problem on a bounded domain with an artificial boundary off the scatterer. This formulation paves the way for an edge element discretization which is analyzed by means of a discrete inf-sup condition. The following two chapters treat scattering by a bounded inhomogeneity (Chapter 11) and by a buried object (Chapter 12) by means of coupled interior/exterior domain problems. In Chapter 11, a nonoverlapping domain decomposition is used with the matching achieved by a Lagrange multiplier on the artificial boundary, whereas Chapter 12 relies on an admittedly not-so-standard overlapping method based on finite element discretizations of the buried objects as well as a domain containing them and an integral representation formula of the field off the scatterers in terms of the dyadic Green's function. It is acknowledged that there are alternative approaches such as an edge element/boundary element coupling, but the exposition of such techniques would go beyond the scope of the book. With the bulk of the book being concerned with direct problems, the final Chapter 14 gives a brief outline of the mathematical aspects of inverse scattering and presents the linear sampling method as a powerful tool for numerical purposes.

With the main focus of the book on finite element analysis rather than algorithmic aspects, the intriguing Chapter 13 provides a tour d'horizon of efficient algorithmic developments including fast solvers such as multigrid and domain decomposition methods, the issues of phase error and a posteriori error estimation, special approaches to exterior domain problems (absorbing boundary conditions, perfectly matched layers), and post-processing techniques (flux recovery).

The book is very well written, thoughtfully organized, and technically sound. It contains a large bibliography which is adequately referenced with many useful comments for the reader at the beginning and sometimes the end of the individual chapters. Undoubtedly, it will become a standard reference text and should be on the bookshelves of those who are interested in the numerical solution of electromagnetic field problems.

In assessing its potential use in the classroom, this reviewer has to report on his own experience when he has used the book as an accompanying text in a graduate course on Computational Electromagnetics which was attended both by mathematicians and electrical engineers. What is indeed very appealing is that the book was very well received by all of them.

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9[65N30, 65N12, 65N15]—*Higher-order finite element methods*, by Pavel Šolín, Karel Segeth, and Ivo Doležel, Studies in Advanced Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2004, xxii+382 pp., hardcover, \$89.95, ISBN 1-58488-438-X; with 1 CD-ROM (Windows, Macintosh, UNIX, and LINUX)

In the Preface the authors state that the "modest" goal of this book is to "present the basic principles of higher order finite element methods and the technology of conforming discretizations based on hierarchic elements in spaces H^1, $\mathbf{H}$(curl), and $\mathbf{H}$(div)." After closing the book I felt that the authors achieved what they aimed at. In the book a large number of constructions on *hp* finite elements, scattered in the mathematical and engineering literature, are collected and presented in a unified and consistent way. In this respect the book is timely, very useful, and quite unique. In my opinion, this book is more about construction, implementation, and application of high order finite elements than theoretical study of the approximation, stability, and solution methods.

Chapter 1 is introductory and contains three relatively short sections. Some notation, background material on functional spaces, and the usual material on finite elements are presented in the first section. Finite elements are treated as triplets (triad) $\mathcal{K} = (K, P, \Sigma)$, where K is a subdomain in R^n, P is a space of polynomials of certain degree that are used on K, and Σ is the set of degrees of freedom. The functions of the finite element spaces are defined elementwise, and their conformity is treated in four different cases, as functions in H^1, $\mathbf{H}(\mathrm{curl})$, $\mathbf{H}(\mathrm{div})$, and L^2. A number of conforming and nonconforming elements are given as illustrations. Very important in the book is the concept of hierarchical basis of the space P, which should result in reducing the condition number of the algebraic problem and helping to build p-adaptive methods. A brief but quite exhaustive presentation of orthogonal polynomials (Gegenbauer, Chebyshev, Legendre) and their use to construct various shape functions is given. At the end of this section the concept of finite element approximation is illustrated on two point boundary value problems for ordinary differential equations of second order. This includes computing the element stiffness and mass matrices, assembly processes for forming the global matrix, its sparsity structure and connectivity. This part contains numerous figures and several algorithms. I would have put slightly more effort into describing and illustrating visually the concept of hierarchical basis.

Chapter 2 is devoted to the construction of a master finite element of arbitrary order. The goal here is to build conforming spaces of polynomials in which degrees might differ from element to element. This is a necessary and crucial property for p-adaptivity. The cases of H^1-, $\mathbf{H}(\mathrm{curl})$-, $\mathbf{H}(\mathrm{div})$-, and L^2-conforming elements are presented as related via the de Rham diagram.

The H^1-conforming elements are the first link in the diagram. The space H^1 imposes the most severe conformity requirments—global continuity. Thus, the hierarchical basis functions are the most involved to construct since they involve vertex, edge, and bubble functions in 2D and vertex, edge, face, and bubble functions in 3D. The authors treat separately, in detail, quadrilateral, triangular, hexahedral, tetrahedral, and prismatic master elements. Despite the complexity of the constructions, the presentation is quite clear with many nice illustrations.

A substantial part of this chapter is devoted to the construction of the $\mathbf{H}(\mathrm{curl})$-conforming element $\mathcal{K}^{\mathrm{curl}} = (K, \mathbf{Q}, \Sigma^{\mathrm{curl}})$ for K being rectangular, triangular, hexahedral, tetrahedral, and prismatic master elements. First, the de Rham diagram $\nabla : \ H^1 \mapsto \mathbf{H}(\mathrm{curl})$ is used to suggest a polynomial space $\mathbf{Q}$ based in the construction in H^1. Then, in 2D, the set of basis functions for $\mathbf{Q}$ is split into two groups, edge functions and bubble functions. The explicit formulas for these hierarchical basis functions use a recurrent definition of the Legendre polynomials in terms of the affine coordinates λ_j. Finally, the $\mathbf{H}(\mathrm{div})$ and L^2 conforming finite elements are constructed again using the de Rham diagram. The former is simpler than the $\mathbf{H}(\mathrm{curl})$ conforming case since the functions in $\mathbf{H}(\mathrm{div})$ need to have only continuous normal components across the interelement faces, while the latter is the simplest one since no continuity is required.

Chapter 2 contains very important material from a practical point of view. The authors have done an excellent job in explaining the principles of the constructions of the hierarchical basis functions, keeping very good accounting for the degrees of the polynomials involved in each group (edge, face, and bubbles) and writing the exact expressions for all basis functions. I think this is a difficult task. I found the notation for simplicial finite elements slightly inconvenient. It is widely accepted in the literature to order the vertexes v_i first and then use consistent ordering of the faces (or edges), e.g., face s_i is the one that does not contain vertex v_i. I suppose that such notation would have made many formulas such as (2.34) more symmetric and transparent.

Chapter 3 is the most technical part of the book. This is the part that describes the construction of a local finite element "interpolant" for various finite elements in H^1, $\mathbf{H}(\mathrm{curl})$, $\mathbf{H}(\mathrm{div})$, and L^2. Since the degrees of freedom, an important ingredient of the finite element construction, have not been discussed thoroughly in the book up to this point, this construction is not obvious at all. The construction of the so-called "projection-based" interpolant Πu is described through three main properties:

(1) locality (or for a given function u, Πu is constructed elementwise);
(2) global conformity (or it should be in one of the spaces H^1, $\mathbf{H}(\mathrm{curl})$, $\mathbf{H}(\mathrm{div})$, and L^2); and
(3) optimality, i.e., it should have some minimization properties.

Again the discussion is carried for H^1, $\mathbf{H}(\mathrm{curl})$, and $\mathbf{H}(\mathrm{div})$ separately.

The case of H^1 conforming finite elements is the most complicated. In 2D the hierarchical basis consists of vertex, edge, and bubble functions and Πu is taken in the form $\Pi u = u^v + u^e + u^b$. The vertex part, u^v, is simply the Lagrange interpolant (linear for triangles and bilinear for squares) based on the vertex values of u. The most difficult is the choice of u^e. The authors first claim that once u^v is chosen, then u^e should minimize the $H^{1/2}_{00}(e)$-norm of the residual $u - u^v$ consecutively over all edges e of K. However, since this is a complicated task, it is replaced by minimization of the scaled $H^1_0(e)$-seminorm. The explanations of this point are mathematically vague and have mostly heuristic value. The choice of u^e, in my opinion, is based on some moments of the residual $u - u^v$ along the edges. In the 3D case the situation is more complicated and the definition of Πu is even less clear. Besides, authors use various Sobolev spaces and their norms such as $H^{3/2+\epsilon}(K)$, $H^{1/2+\epsilon}(s)$, $H^{\epsilon}(e)$, etc., which have not been defined. In my opinion, the construction (or at least the one presented here) of the projection-based interpolant Πu needs more mathematical rigor and clarification. I think that an attempt to compare the interpolant discussed in the book with the existing known projections (say, Clement, Fortin, Nédélec, etc.) would have added to its better understanding.

Further curved finite elements are introduced and the corresponding transformations to the reference elements are constructed. A concept of transfinite interpolation is introduced and used. Practically, this reduces to the following: we assume that a curved finite element is given by its vertexes and by parametrized curves of its edges. The transformation then has two parts: an affine part that transforms the vertexes of the finite element into the vertexes of the reference one, and a higher order part that maps the curved edges into straight lines. Curvilinear variants of all elements presented in Chapter 2 are discussed and their mappings presented. Isoparametric transformations are just a particular case of this general approach. Further, transformation of the master element polynomial spaces for all four cases of H^1, $\mathbf{H}(\mathrm{curl})$, $\mathbf{H}(\mathrm{div})$, and L^2 conforming finite elements are presented very briefly. The derivation is based on the commutativity of the de Rham diagram applied to the spaces on reference and physical domains. All these constructions are summarized into several algorithms for assembly of the resulting discrete system. Finally, in this chapter the problem of constrained approximation is discussed. This has an important practical role in the adaptation procedure.

Chapters 4 and 5 have mostly reference value on quadratures and solvers of systems of linear equations. Exhaustive sets of Newton–Cotes, Chebyshev, Lobatto (Radau), and Gauss quadrature formulas for all types of finite elements are given (and supplied on a CD-ROM). Conjugate gradient, MINRES, GMRES, ORTHODIR and several other general iterative methods are given as algorithms. Finally, a multigrid method is presented as a general concept for solving finite element systems. Unfortunately, there are no comments or suggestions how these will work in the complex finite element approximation with higher order finite elements, presented in the book.

The final Chapter 6 is an important and necessary part of the book. First, it presents several approaches to automatic mesh optimization based on h-, p-, and hp-adaptivity based on reference solutions. Next, the basic principles of hp-adaptive computations of goal-oriented adaptivity are discussed. They are based on recovery with a guaranteed accuracy of "quantities of interest", namely some functionals of the desired solution. This part is based entirely on the works of Demkowicz, Oden, and Rahowicz and illustrates the applicability and the merits of high order finite elements.

I found a number of misprints and minor inaccuracies. For example, formula (2.52), p. 80 contains a strange symbol; in the last inequality of p. 324 $|\Omega_s|$ should be replaced by $|\Omega_s|^{1/2}$; the projector defined by (3.9) is not well explained in the text above; on pp. 132, 134, 135 there are inconsistent notations for inner product and norm in $H^{1/2}(s)$, $H^{1+r}(K_T)$. Also in several places (e.g., pp. 128, 132) the authors state "the theory requires that $u \in H^{1+\epsilon}$" (or $u \in H^{3/2+\epsilon}$) with a strange reference, while this is a simple result of the Sobolev embedding theorem. Finally, I think the book would have benefited from a better presentation and explanation of the lowest order finite elements such as the well known and popular Nédélec and Raviart–Thomas elements used in $\mathbf{H}(\mathrm{curl})$ and $\mathbf{H}(\mathrm{div})$. However, these do not compromise the overall good and clear presentation.

As noted above, the book has various merits, contains many detailed constructions, interesting concepts, and practical discussions. This is a useful and timely text in the area of high order finite element methods for partial differential equations. It could be used as a reference book and/or a supplement to a course on discretization methods for differential equations.

Rigorous mathematical study of stability, optimal error estimates, conditioning and preconditioning of the resulting systems is not a goal of the book. Nevertheless, the authors have stated various theorems and conjectures (mostly due to Demkowicz and his collaborators) that give flavor for some of the existing results and the open mathematical problems in this relatively new field of numerical analysis for PDEs.

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