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A FINITE DIMENSIONAL REALIZATION OF THE MOLLIFIER METHOD FOR COMPACT OPERATOR EQUATIONS

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ABSTRACT. We introduce and analyze a stable procedure for the approximation of $\langle f^\dagger, \varphi\rangle$ where $f^\dagger$ is the least residual norm solution of the minimal norm of the ill-posed equation $Af = g$, with compact operator $A : X \to Y$ between Hilbert spaces, and $\varphi \in X$ has some smoothness assumption. Our method is based on a finite number of singular values of A and some finite rank operators. Our results are in a more general setting than the one considered by Rieder and Schuster (2000) and Nair and Lal (2003) with special reference to the mollifier method, and it is also applicable under fewer smoothness assumptions on φ.

1. Introduction

Many inverse problems in science and engineering have their mathematical formulation as an operator equation

$$Af = g \tag{1.1}$$

where $A : X \to Y$ is a compact operator between Hilbert spaces X and Y (cf. [3, 5, 7, 6, 20]). It is well known that if A is of infinite rank, then the problem of solving the above equation is ill-posed, in the sense that the generalized solution $f^\dagger := A^\dagger g$ for $g \in D(A^\dagger) := R(A) + R(A)^\perp$ does not depend continuously on the data g (cf. [3, 4, 14]). Here $A^\dagger$ denotes the Moore-Penrose generalized inverse of A. We may recall that for $g \in D(A^\dagger)$, $f^\dagger := A^\dagger g$ is the minimum norm solution of the normalized equation $A^*Af^\dagger = A^*g$. A typical example of a compact operator with infinite rank which often occurs in practical problems is the Fredholm integral equation of the first kind,

$$\int_\Omega k(s,t)f(t)dt = g(s), \quad s \in \Omega,$$

where $k(\cdot,\cdot)$ is a non-degenerate kernel, so that the corresponding integral operator is a compact operator between suitable function spaces. So, in order to obtain stable approximate solutions for such problems, one has to apply certain regularization procedures. There are many regularization procedures available in the literature ((cf. [3, 4, 5, 7]) and the references therein).

In regularization methods, such as the Tikhonov regularization (cf. [3, 4]) or Lavrentiev regularization (cf. [13, 19]) or in any of their variants and generalizations

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(cf. [3, 4]), one finds a family of regularized solutions of the form

$$f_\eta^\delta := R_\eta g^\delta,$$

where g^δ is the available data such that $\|g - g^\delta\| \le \delta$ for some known error level $\delta > 0$, and the parameter $\eta > 0$ is chosen depending on (g^δ, δ) in such a way that $f_\eta^\delta \to f^\dagger$ as $\delta \to 0$. In the above procedures, in obtaining the regularized solutions f_η^δ, one has to solve certain well-posed problems for each data g^δ. So, it is desirable to have a procedure which gives regularized approximations for different data without having to solve different problems. Also, usually in practical problems, the space X is a space of functions defined on certain subset Ω of $\mathbb{R}^k$, and one is interested in the value of the solution $f^\dagger$ at certain points x in Ω. Having such situations in mind, Louis and Maass [8] considered a new procedure, the so-called *mollifier method*, for approximating $f^\dagger(x)$. In this procedure, one considers a family $\mathcal{E} := \{e_\gamma(\cdot,\cdot) : \gamma > 0\}$ of functions on $\Omega \times \Omega$, called *mollifiers*, such that $e_\gamma(x,\cdot)$ is well defined, $e_\gamma(x,\cdot) \in X$ for each $x \in \Omega$ and $\langle f, e_\gamma(x,\cdot)\rangle \to f(x)$ as $\gamma \to 0$. Moreover, $\mathcal{E}$ is required to have some additional property so that $\langle f, e_\gamma(x,\cdot)\rangle$ can be expressed as $\langle g, \psi_\gamma(x)\rangle$ for some $\psi_\gamma(x) \in Y$. By the first requirement $E_\gamma f^\dagger$ defined by $(E_\gamma f)(x) = \langle f, e_\gamma(x,\cdot)\rangle$, $x \in \Omega$, can be considered as an approximation of $f^\dagger$ for small enough γ, and by the latter requirement the computation of $E_\gamma f^\dagger$ is reduced in terms of known quantities.

It has been proved by the authors (cf. [15]) that if the data g and the mollifier $e_\gamma(x,\cdot)$ are in the domains of the generalized inverses of A and A^*, respectively, then

$$\langle A^\dagger g, e_\gamma(x,\cdot)\rangle = \langle g, \psi_\gamma(x)\rangle,$$

where $\psi_\gamma(x) = (A^*)^\dagger e_\gamma(x,\cdot)$. Note that the computation of $\langle g, \psi_\gamma(x)\rangle$ involves the problem of solving an ill-posed equation

$$AA^*\psi_\gamma(x) = Ae_\gamma(x,\cdot)$$

in the infinite dimensional setting. Thus, the need arises to have a finite dimensional procedure to approximate the quantity $\langle g, \psi_\gamma(x)\rangle$. As we have already pointed out, one may be interested in computing the values of $\langle g, \psi_\gamma(x)\rangle$ for a certain finite number of x's, say for $x \in \{x_1, \ldots, x_\ell\}$. Thus, the problem is to get approximations for $\langle g, \psi_\gamma(x_i)\rangle$, $i = 1, \ldots, \ell$.

Rieder and Schuster (cf. [17], [18]) suggested a new method for finding an approximation to $\psi_\gamma(x)$ by solving a finite dimensional system. In their analysis, first they consider the case of $e_\gamma(x_i,\cdot) \in R(A^*)$ and then the general case $e_\gamma(x_i,\cdot) \in X$. But the main theorem on error estimates for the general case $e_\gamma(x_i,\cdot) \in X$ involves an assumption which amounts to the assumption $e_\gamma(x_i,\cdot) \in R(A^*)$. In [15] the authors suggested a new method in the finite dimensional setting for a bounded linear operator A which is applicable for the case of $e_\gamma(x_i,\cdot) \notin R(A^*)$ as well. The purpose of this paper is to consider another procedure for the specific case when A is a compact linear operator. In this case we arrive at better estimates than the ones available in [15]. A particular case of the procedure yields an error bound of the form obtained by Rieder and Schuster (cf. [17, 18]) under certain weaker assumptions.

We shall be carrying out our analysis in a slightly more general setting, in which we are approximating $\langle f^\dagger, \varphi\rangle$ which is the same as $\langle g, (A^*)^\dagger\varphi\rangle$ (cf. [15]) whenever g and φ are in the domains of $A^\dagger$ and $(A^*)^\dagger$, respectively, and X is not necessarily a function space. In the usual mollifier method, φ could be $e_\gamma(x_i,\cdot)$, whenever X is a space of functions on Ω and $e_\gamma(\cdot,\cdot)$ is a mollifier.

2. Background materials

In our procedure for obtaining stable approximate solutions for the operator equation (1.1) where $A : X \to Y$ is a compact operator between the Hilbert spaces X and Y, we further require a subspace Z of Y which is a Hilbert space with respect to the norm $\|\cdot\|_Z$. Norms on X and Y are denoted by $\|\cdot\|$. For example, if $Y = L^2(\Omega)$, then Z may be a Sobolev space.

In the mollifier method, we may have X as a function space defined on some subset $\Omega \subseteq \mathbb{R}^d$. Our objective is to find $\langle f,\varphi\rangle$ which is the same as $\langle g,(A^*)^\dagger\varphi\rangle$. Also, the evaluation of the inner product $\langle g,(A^*)^\dagger\varphi\rangle$ may involve the integral. Hence, in order to approximate $\langle g,(A^*)^\dagger\varphi\rangle$, we are going to replace it by an inner product in $\mathbb{C}^n$ which is easily computable. For that we require n linear functionals $\psi_{n,1},\dots,\psi_{n,n}$ on Z and n elements $\phi_{n,1},\dots,\phi_{n,n}$ in Y such that the linear operator $\Pi_n : Z \to Y$ defined by

$$\Pi_n v = \sum_{k=0}^{n} \langle \psi_{n,k}, v\rangle \phi_{n,k}, \quad v \in Z, \tag{2.2}$$

has the following properties:

- $\exists$ a sequence (ρ_n) in $[0,1]$ which converges to 0 such that

$$\|\Pi_n v - v\| \le \rho_n \|v\|_Z \quad \forall\, v \in Z; \tag{2.3}$$

- $\exists$ $c > 0$ such that

$$\|\Pi_n v\| \le c\|v\|_Z \quad \forall\, v \in Z. \tag{2.4}$$

We shall also make use of the operators $\Psi_n : Z \to \mathbb{C}^n$ and $\Phi_n : \mathbb{C}^n \to Y$ defined by

$$(\Psi_n v)_k = \langle \psi_{n,k}, v\rangle := \psi_{n,k}(v),\ v \in Z,\ k = 1,\dots,n, \tag{2.5}$$

and

$$\Phi_n x = \sum_{k=1}^{n} x_k \phi_{n,k},\ x \in \mathbb{C}^n,$$

respectively. Here, x_k denotes the k-th coordinate of $x \in \mathbb{C}^n$. With the above notations, we can write Π_n as

$$\Pi_n v = \Phi_n \Psi_n v, \quad \forall\, v \in Z.$$

For instance, in the 2D-computerized tomography problem (cf. [16, 17]), one has to solve the operator equation (1.1) with A as the *Radon transform* defined by

$$(Af)(s,\theta) = \int_{-w(s)}^{w(s)} f(s\omega + t\omega^\perp)dt \tag{2.6}$$

where $\omega := (\cos\theta,\sin\theta)$, $\omega^\perp := (-\sin\theta,\cos\theta)$ with $(s,\theta) \in W := (-1,1)\times(0,\pi)$, and $w(s) := \sqrt{1-s^2}$. Here $X = L^2(\Omega)$, $Y = L^2(W)$ and $Z = H^{m+1}(W)$ for some $m > 0$ where Ω is the unit ball in $\mathbb{R}^2$ centered at the origin and $H^{m+1}(W)$ denotes the Sobolev space of order $m+1$ defined on the domain W. For $p,q \in \mathbb{N}$, define $\Pi_{p,q} : H^{m+1}(W) \longrightarrow L^2(W)$ by

$$\Pi_{p,q} v = \sum_{i=-q}^{q-1}\sum_{j=0}^{p-1} v(s_i,\theta_j) B_{q,i} \otimes B_{p,j}, \quad v \in H^{m+1}(W), \tag{2.7}$$

where $B_{p,j} = \chi_{[\theta_j,\ \theta_{j+1})}$ and $B_{q,i} = \chi_{[s_i,\ s_{i+1})}$ for $s_i = \frac{i}{q}$, and $\theta_j = j\frac{\pi}{p}, i = -q, \ldots, q-1;\ j = 0, \ldots, p-1$. Here, for $u, v \in L^2(W)$, $u \otimes v \in L^2(W)$ is defined as $(u \otimes v)(s, \theta) = u(s)v(\theta)$ for $(s, \theta) \in W$ and $\chi_{[a,b]}$ denotes the characteristic function on the interval $[a, b]$. Then for $v \in H^{m+1}(W)$ we have (cf. [17]),

$$\|\Pi_{p,q}v - v\| \leq h\|v\|_{H^{m+1}(W)}, \qquad h = \max\left\{\frac{1}{q}, \frac{\pi}{p}\right\},$$

and there exists $c_0 > 0$ such that

$$\|\Pi_{p,q}v\| \leq c_0\, \|v\|_{H^{m+1}(W)}.$$

Let G_n denotes the Gramm matrix related to the family $\{\phi_{n,1}, \ldots, \phi_{n,n}\}$. That is, G_n is the $n \times n$ matrix whose ij-th entry, $(G_n)_{ij}$, is given by

$$(G_n)_{ij} = \langle \phi_{n,j}, \phi_{n,i}\rangle,\ i, j = 1, \ldots, n. \tag{2.8}$$

Using the above definitions, it can be seen (cf. [15]) that

$$\langle \Pi_n u, \Pi_n v\rangle = \langle \Psi_n u, G_n \Psi_n v\rangle \quad \forall\, u, v \in Z. \tag{2.9}$$

We know that the compact linear operator $A : X \to Y$ between Hilbert spaces X and Y has the singular value representation as

$$Af = \sum_{k=1}^{\infty} \sigma_k \langle f, v_k\rangle u_k, \quad x \in X,$$

where $\{v_k : k \in \mathbb{N}\}$ and $\{u_k : k \in \mathbb{N}\}$ are orthonormal bases for $N(A)^\perp$ and $\overline{R(A)}$, respectively. Also, the *singular values* σ_k and *singular vectors* v_k, u_k are related by

$$Av_k = \sigma_k\, u_k, \qquad A^*u_k = \sigma_k\, v_k \tag{2.10}$$

for $k \in \mathbb{N}$. In the case of the Radon transform defined in (2.6) the singular values and singular vectors are known explicitly (see e.g., [1]; also see [2]).

For the purpose of imposing some conditions on φ, we require a family $\{X_\nu : \nu > 0\}$ of Hilbert spaces defined as follows: For $\nu > 0$, let

$$X_\nu := \{f \in N(A)^\perp : \sum_{k=1}^{\infty} \sigma_k^{-4\nu} |\langle f, v_k\rangle|^2 < \infty\}. \tag{2.11}$$

Then it is seen that X_ν is a Hilbert space with inner product and norm defined by

$$\langle f, v\rangle_\nu := \sum_{k=1}^{\infty} \sigma_k^{-4\nu} \langle f, v_k\rangle\langle v_k, v\rangle, \quad f, v \in X_\nu, \tag{2.12}$$

$$\|f\|_\nu := \sum_{k=1}^{\infty} \sigma_k^{-4\nu} |\langle f, v_k\rangle|^2, \quad f \in X_\nu, \tag{2.13}$$

respectively. We may observe that $X_\nu \subseteq X_\mu$ whenever $\mu \leq \nu$, and the inclusion map from X_ν to X_μ is continuous. Thus, the family $\{X_\nu : \nu > 0\}$ is a *Hilbert scale.*

3. Approximation of $\langle f^\dagger, \varphi\rangle$

Suppose $\varphi \in X$ and $g \in D(A^\dagger)$. Our main objective is to find an approximation for $\langle f^\dagger, \varphi\rangle$, where $f^\dagger := A^\dagger g$. As we have already mentioned, in the mollifier method, we may have X a function space defined on some subset $\Omega \subseteq \mathbb{R}^d$ and $\varphi(y) := e_\gamma(x_i, y)$ for $x_i, y \in \Omega$, $i = 1, \ldots, \ell$.

Throughout the rest of the paper we assume that $\varphi \in X_\nu$, where X_ν and its inner product and norm are defined as in (2.11), (2.12), (2.13). We observe that, since $\varphi \in X_\nu$, it can be represented as

$$\varphi = \sum_{k=1}^{\infty} \sigma_k^{2\nu} \langle u, v_k\rangle v_k$$

with $u \in X$ given by

$$u = \sum_{k=1}^{\infty} \sigma_k^{-2\nu} \langle \varphi, v_k\rangle v_k. \tag{3.14}$$

We may observe that

$$\|u\|^2 = \sum_{k=1}^{\infty} \sigma_k^{-4\nu} |\langle \varphi, v_k\rangle|^2 = \|\varphi\|_\nu^2.$$

We assume that $u_k \in Z$ for all $k \in \mathbb{N}$ and that the u_k's are orthogonal in Z. For $\alpha > 0$ and $M \in \mathbb{N}$, we define an element $v_M^{(\alpha)} \in Y$ as follows:

$$v_M^{(\alpha)} := \sum_{k=0}^{M-1} \frac{\sigma_k}{\sigma_k^2 + \alpha} \langle \varphi, v_k\rangle u_k. \tag{3.15}$$

Since $u_k \in Z$ for all $k \in \mathbb{N}$, $v_M^{(\alpha)}$ is an element in Z.

For finding an approximation to $\langle f, \varphi\rangle$, we make use of the following result available in [15]. For the sake of completion of exposition, we are including its proof as well.

Theorem 3.1. *Let Ψ_n and G_n be as in* (2.5) *and* (2.8)*, respectively, $g \in Z \cap D(A^\dagger)$ and $v \in Z$. Let $f^\dagger = A^\dagger g$. Then*

$$|\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v\rangle| \le c_1\{\rho_n \|v\|_Z + \|A^* v - \varphi\|\}, \tag{3.16}$$

where $c_1 \ge \max\{(c\|g\|_Z + \|g\|), \|f^\dagger\|\}$ and $c > 0$ is as in (2.4)*.*

Proof. Let Π_n be as in (2.2). Then by (2.9) we have

$$\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v\rangle = \langle f^\dagger, \varphi\rangle - \langle \Pi_n g, \Pi_n v\rangle.$$

Hence

$$\begin{aligned} |\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v\rangle| &= |\langle f^\dagger, \varphi\rangle - \langle \Pi_n g, \Pi_n v\rangle| \\ &\le |\langle f^\dagger, \varphi\rangle - \langle f^\dagger, A^* v\rangle| + |\langle f^\dagger, A^* v\rangle - \langle \Pi_n g, \Pi_n v\rangle| \\ &\le \|f^\dagger\|\, \|A^* v - \varphi\| + |\langle f^\dagger, A^* v\rangle - \langle \Pi_n g, \Pi_n v\rangle|. \end{aligned}$$

Since $f^\dagger = A^\dagger g$,

$$\langle f^\dagger, A^* v\rangle = \langle A f^\dagger, v\rangle = \langle Pg, v\rangle = \langle g, Pv\rangle = \langle g, v\rangle$$

where $P : Y \longrightarrow Y$ is the orthogonal projections onto $\overline{R(A)}$. Using the above relation, we get

$$|\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v\rangle| \quad \leq \quad \|f^\dagger\| \, \|A^* v - \varphi\| + |\langle g, v\rangle - \langle \Pi_n g, \Pi_n v\rangle|.$$

Using the relations (2.3) and (2.4), we have

$$\begin{aligned} |\langle g, v\rangle - \langle \Pi_n g, \Pi_n v\rangle| &\leq |\langle g, v\rangle - \langle g, \Pi_n v\rangle| + |\langle g, \Pi_n v\rangle - \langle \Pi_n g, \Pi_n v\rangle| \\ &\leq \|g\| \, \|\Pi_n v - v\| + \|\Pi_n g - g\| \, \|\Pi_n v\| \\ &\leq \rho_n \|g\| \, \|v\|_Z + c\rho_n \|g\|_Z \|v\|_Z \\ &= (c\|g\|_Z + \|g\|)\rho_n \|v\|_Z. \end{aligned}$$

Thus,

$$|\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v\rangle| \quad \leq \quad c_1\{\rho_n \|v\|_Z + \|A^* v - \varphi\|\},$$

where $c_1 \geq \max\{(c\|g\|_Z + \|g\|), \|f^\dagger\|\}$. This completes the proof. □

Now, we derive estimates for $\|v_M^{(\alpha)}\|_Z$ and $\|A^* v_M^{(\alpha)} - \varphi\|$. We do this in the next two lemmas.

Lemma 3.2. *Let* $v_M^{(\alpha)}$ *be as in* (3.15). *Suppose* $\varphi \in X_\nu$ *for some* $\nu \in (0, 1]$ *and* $u \in X$ *as in* (3.14). *Then*

$$\|A^* v_M^{(\alpha)} - \varphi\| \leq \sqrt{\alpha^{2\nu} + \sigma_M^{4\nu}} \, \|u\|. \tag{3.17}$$

Proof. Let $\varphi \in X$. From (3.15) and (2.10), we have

$$A^* v_M^{(\alpha)} = \sum_{k=0}^{M-1} \frac{\sigma_k^2}{\sigma_k^2 + \alpha} \langle \varphi, v_k\rangle v_k.$$

Since $\varphi \in X_\nu$, $0 < \nu \leq 1$, with $\varphi = \sum_{k=0}^{\infty} \sigma_k^{2\nu} \langle u, v_k\rangle v_k$ for $u \in X$ as in (3.14), we have

$$\begin{aligned} A^* v_M^{(\alpha)} - \varphi &= \sum_{k=0}^{M-1} \frac{\sigma_k^{2+2\nu}}{\sigma_k^2 + \alpha} \langle u, v_k\rangle v_k - \sum_{k=0}^{\infty} \sigma_k^{2\nu} \langle u, v_k\rangle v_k \\ &= \sum_{k=0}^{M-1} \frac{-\alpha \, \sigma_k^{2\nu}}{\sigma_k^2 + \alpha} \langle u, v_k\rangle v_k - \sum_{k=M}^{\infty} \sigma_k^{2\nu} \langle u, v_k\rangle v_k \end{aligned}$$

so that

$$\begin{aligned} \|A^* v_M^{(\alpha)} - \varphi\|^2 &= \sum_{k=0}^{M-1} \left(\frac{\alpha \, \sigma_k^{2\nu}}{\sigma_k^2 + \alpha}\right)^2 |\langle u, v_k\rangle|^2 + \sum_{k=M}^{\infty} \sigma_k^{4\nu} |\langle u, v_k\rangle|^2 \\ &\leq \sum_{k=0}^{M-1} \left(\frac{\alpha \, \sigma_k^{2\nu}}{\sigma_k^2 + \alpha}\right)^2 |\langle u, v_k\rangle|^2 + \sigma_M^{4\nu} \, \|u\|^2. \end{aligned} \tag{3.18}$$

We observe that

$$\begin{aligned} \sum_{k=0}^{M-1} \left(\frac{\alpha \, \sigma_k^{2\nu}}{\sigma_k^2 + \alpha}\right)^2 |\langle u, v_k\rangle|^2 &= \alpha^{2\nu} \sum_{k=0}^{M-1} \left(\frac{(\frac{\sigma_k^2}{\alpha})^\nu}{\frac{\sigma_k^2}{\alpha} + 1}\right)^2 |\langle u, v_k\rangle|^2 \\ &\leq \alpha^{2\nu} \|u\|^2. \end{aligned}$$

Hence, from (3.18) it follows that

$$\|A^* v_M^{(\alpha)} - \varphi\|^2 \leq \alpha^{2\nu} \|u\|^2 + \sigma_M^{4\nu} \, \|u\|^2 = (\alpha^{2\nu} + \sigma_M^{4\nu}) \|u\|^2.$$

This completes the proof. □

The above lemma shows that if M is such that $\sigma_M^2 \le \alpha$, then

$$\|A^* v_M^{(\alpha)} - \varphi\| \quad \le \quad \sqrt{2}\alpha^\nu \|u\|.$$

Lemma 3.3. *Let $v_M^{(\alpha)}$ be as in* (3.15)*. Assume that $\|u_k\|_Z \le \sigma_k^{-\beta}$ for all $k \in \mathbb{N}$ for some β with $0 \le \beta \le 1$, $\varphi \in X_\nu$ with $0 < \nu \le \frac{\beta+1}{2}$, and $u \in X$ as in* (3.14)*. Then*

$$\|v_M^{(\alpha)}\|_Z \le \alpha^{\nu - \frac{\beta+1}{2}} \|u\|. \tag{3.19}$$

Proof. Since $\varphi \in X_\nu$ with $\varphi = \sum_{k=0}^\infty \sigma_k^{2\nu} \langle u, v_k \rangle v_k$ for $u \in X$ as in (3.14), we have

$$v_M^{(\alpha)} = \sum_{k=0}^{M-1} \frac{\sigma_k}{\sigma_k^2 + \alpha} \langle \varphi, v_k \rangle u_k = \sum_{k=0}^{M-1} \frac{\sigma_k^{2\nu+1}}{\sigma_k^2 + \alpha} \langle u, v_k \rangle u_k.$$

Since the u_k's are orthogonal in Z and $\|u_k\|_Z \le \sigma_k^{-\beta}$ for all k,

$$\begin{aligned}
\|v_M^{(\alpha)}\|_Z^2 &= \sum_{k=0}^{M-1} \left(\frac{\sigma_k^{2\nu+1}}{\sigma_k^2 + \alpha} \right)^2 |\langle u, v_k \rangle|^2 \|u_k\|_Z^2 \\
&\le \sum_{k=0}^{M-1} \left(\frac{\sigma_k^{2\nu+1}}{\sigma_k^2 + \alpha} \right)^2 |\langle u, v_k \rangle|^2 \sigma_k^{-2\beta} \\
&= \sum_{k=0}^{M-1} \left(\frac{\sigma_k^{2(\nu - \frac{\beta-1}{2})}}{\sigma_k^2 + \alpha} \right)^2 |\langle u, v_k \rangle|^2 \\
&= (\alpha^{\nu - \frac{\beta+1}{2}})^2 \sum_{k=0}^{M-1} \left(\frac{(\frac{\sigma_k^2}{\alpha})^{\nu - \frac{\beta-1}{2}}}{\frac{\sigma_k^2}{\alpha} + 1} \right)^2 |\langle u, v_k \rangle|^2.
\end{aligned}$$

Since $0 \le \beta \le 1$ and $0 < \nu \le \frac{\beta+1}{2}$, we have $0 < \nu - \frac{\beta-1}{2} \le 1$ so that

$$\begin{aligned}
\|v_M^{(\alpha)}\|_Z^2 &= (\alpha^{\nu - \frac{\beta+1}{2}})^2 \sum_{k=0}^{M-1} |\langle u, v_k \rangle|^2 \\
&\le (\alpha^{\nu - \frac{\beta+1}{2}})^2 \|u\|^2.
\end{aligned}$$

Thus

$$\|v_M^{(\alpha)}\|_Z \le \alpha^{\nu - \frac{\beta+1}{2}} \|u\|.$$

□

Now we are in a position to find an approximation for $\langle f, \varphi \rangle$.

Theorem 3.4. *Let Ψ_n and G_n be as in* (2.5) *and* (2.8)*, respectively, and $g \in Z \cap D(A^\dagger)$. Let $f^\dagger = A^\dagger g$. Assume that $\|u_k\|_Z \le \sigma_k^{-\beta}$ for all $k \in \mathbb{N}$ for some β with $0 \le \beta \le 1$, and $\varphi \in X_\nu$ with $0 < \nu \le \frac{\beta+1}{2}$, and $u \in X$ as in* (3.14)*. Then*

$$|\langle f^\dagger, \varphi \rangle - \langle \Psi_n g, G_n \Psi_n v_M^{(\alpha)} \rangle| \le c_1 \{ \rho_n \alpha^{\nu - \frac{\beta+1}{2}} + \alpha^\nu + \sigma_M^{2\nu} \} \|u\| \tag{3.20}$$

where $c_1 > 0$ is as in (3.16) *and $c > 0$ is as in* (2.4)*. In particular, we have the following:*

(*a*) *If $\alpha = c_0 \rho_n^{\frac{2}{\beta+1}}$ for some $c_0 > 0$, then*

$$|\langle f^\dagger, \varphi \rangle - \langle \Psi_n g, G_n \Psi_n v_M^{(\alpha)} \rangle| \le c_1 \|u\| (\rho_n^{\frac{2\nu}{\beta+1}} + \sigma_M^{2\nu}).$$

(b) *If* $\alpha = c_0 \rho_n^{\frac{2}{\beta+1}}$ *for some* $c_0 > 0$ *and* M *is such that* $\sigma_M^2 \le \rho_n^{\frac{2}{\beta+1}}$, *then*

$$|\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v_M^{(\alpha)}\rangle| = \mathcal{O}\big(\rho_n^{\frac{2\nu}{\beta+1}}\big). \tag{3.21}$$

Proof. Let $v_M^{(\alpha)}$ be as in (3.15). Using the relations (3.17) and (3.19), we have

$$\|A^* v_M^{(\alpha)} - \varphi\| \le \sqrt{\alpha^{2\nu} + \sigma_M^{2\nu}}\ \|u\|$$

and

$$\|v_M^{(\alpha)}\|_Z \quad \le \quad \alpha^{\nu - \frac{\beta+1}{2}} \|u\|.$$

Then by Theorem 3.1 we have

$$\begin{aligned} |\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v_M^{(\alpha)}\rangle| &\le c_1 \{\rho_n \|v_M^{(\alpha)}\|_Z + \|A^* v_M^{(\alpha)} - \varphi\|\} \\ &\le c_1 \{\rho_n \alpha^{\nu - \frac{\beta+1}{2}} + \sqrt{\alpha^{2\nu} + \sigma_M^{4\nu}}\}\|u\| \\ &\le c_1 \{\rho_n \alpha^{\nu - \frac{\beta+1}{2}} + \alpha^{\nu} + \sigma_M^{2\nu}\}\|u\| \end{aligned}$$

where $c_1 \ge \max\{(c\|g\|_Z + \|g\|), \|f^\dagger\|\}$ and $c > 0$ is as in (2.4).

The particular case follow from (3.20). □

If $\varphi \in X_\nu$ for some $\nu \ge \frac{\beta+1}{2}$, then it is obvious that $\varphi \in X_{\frac{\beta+1}{2}}$. Hence the following corollary is an immediate consequence of the above theorem.

Corollary 3.5. *Let* $v_M^{(\alpha)}$ *be as in* (3.15). *Assume that* $\varphi \in X_\nu$ *with* $0 < \nu \le \frac{\beta+1}{2}$, *and* $u \in X$ *as in* (3.14). *If* $\sigma_M^2 \le \rho_n^{\frac{2}{\beta+1}}$ *and* $\alpha \le \rho_n^{\frac{2}{\beta+1}}$, *then*

$$|\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v_M^{(\alpha)}\rangle| = \mathcal{O}\left(\rho_n\right).$$

4. Concluding remarks

Remark 1. We observe that the method of this paper is based on the singular value representation of the compact linear operator A, and it is applicable only when the singular elements σ_k, v_k, u_k are known for $k = 1, \ldots, M$ for some large $M \in \mathbb{N}$. It is to be remarked that in computerized tomography problems, where A is the Radon transform in suitable spaces, it is often the case that the singular elements of A are known explicitly (see e.g., [1]; also see [2]).

Remark 2. In [15] the authors considered general bounded operators A and obtained an approximation for $\langle f^\dagger, \varphi\rangle$ in the form $\langle \Psi_n g, G_n \Psi_n v_{\alpha,n}\rangle$ with corresponding estimation

$$|\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v_{\alpha,n}\rangle| = \mathcal{O}\big(\rho_n^{\frac{\nu}{2}}\big).$$

We may observe that the error estimate (3.21) is better than the above estimate in [15] since $\frac{\nu}{2} \le \frac{2\nu}{\beta+1}$ for $0 \le \beta \le 1$.

Remark 3. An estimate of the form

$$|\langle f^\dagger, \varphi\rangle - \langle \Psi_n g, G_n \Psi_n v_M^{(\alpha)}\rangle| = \mathcal{O}(\rho_n)$$

(see Corollary 3.5) has also been obtained by Rieder and Schuster [17] with an element ω_M in place of $v_M^{(\alpha)}$, where

$$\omega_M := \sum_{k=0}^{M-1} \sigma_k^{-1} \langle \varphi, v_k\rangle u_k$$

which is the M-th truncated form of the singular value decomposition of $(A^*)^\dagger\varphi$. But, for obtaining such a result, they required ν to satisfy $\nu > \frac{\beta}{2}+\frac{1}{2}+\frac{1}{4\eta}$ for some $\eta > 0$. Clearly, this assumption of Rieder and Schuster [17] is stronger than our assumption, namely, $\nu \geq \frac{\beta+1}{2}$.

Remark 4. Suppose we make a specific choice of Z as

$$Y_\beta := \{y \in Y : \sum_{k=0}^{\infty} \sigma_k^{-2\beta} |\langle y, u_k\rangle|^2 < \infty\}$$

for $0 \leq \beta \leq 1$ with $Y_0 = N(A^*)^\perp$. We observe that $R(A) \subset Y_\beta$ for $0 \leq \beta \leq 1$. On Y_β, we define an inner product $\langle \cdot, \cdot \rangle_\beta$ as

$$\langle y, u\rangle_\beta := \sum_{k=0}^{\infty} \sigma_k^{-2\beta} \langle y, u_k\rangle\langle u_k, u\rangle, \quad y, u \in Y_\beta.$$

With this inner product, Y_β is a Hilbert space and the corresponding norm is given by

$$\|y\|_\beta = \sum_{k=0}^{\infty} \sigma_k^{-2\beta} |\langle y, u_k\rangle|^2, \quad y \in Y_\beta.$$

We note that $u_k \in Y_\beta$ for all $k \in \mathbb{N}$ and $\{\sigma_k^\beta u_k : k \in \mathbb{N}\}$ is an orthonormal basis for Y_β with respect to the norm $\|\cdot\|_\beta$. Thus we can apply Theorem 3.4. With this choice of Z as Y_β, the error estimate (3.21) is the same as that in the Nair and Lal paper ([15], Theorem 3.7). However, the conditions under which that result in [15] holds is stronger. In fact, in [15], it is required that $\beta/2 \leq \nu \leq (\beta+1)/2$ whereas the estimate (3.21) is valid for more values of ν, namely, $0 \leq \nu \leq (\beta+1)/2$.

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