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ON THE NONEXISTENCE OF 2-CYCLES FOR THE $3x + 1$ PROBLEM

JOHN L. SIMONS

ABSTRACT. This article generalizes a proof of Steiner for the nonexistence of 1-cycles for the $3x + 1$ problem to a proof for the nonexistence of 2-cycles. A lower bound for the cycle length is derived by approximating the ratio between numbers in a cycle. An upper bound is found by applying a result of Laurent, Mignotte, and Nesterenko on linear forms in logarithms. Finally numerical calculation of convergents of $\log_2 3$ shows that 2-cycles cannot exist.

1. INTRODUCTION

The $3x+1$ problem is a notorious problem of elementary number theory. Let x_n be a natural number and consider a sequence, generated conditionally by $x_{n+1} = \frac{1}{2}x_n$ if x_n is even and by $x_{n+1} = \frac{1}{2}(3x_n + 1)$ if x_n is odd. Numerical verification indicates that for "all" natural numbers x_n the cycle $(1, 2)$ finally appears. A formal proof is lacking so far in spite of various approaches to the problem; see [10].

We call a cyclic solution an m-cycle if the numbers x_n appear in m sequences, each consisting of a subsequence of odd numbers followed by a subsequence of even numbers. Steiner [7] assumes the existence of a 1-cycle with k odd numbers and ℓ even numbers and proves four partial results:

(1) an inequality for the ratio $(k + \ell)/k$;
(2) a numerical lower bound for k, from which it follows that $(k + \ell)/k$ must be a convergent in the continued fraction expression of $\log_2 3$;
(3) an upper bound for k by applying a theorem of Baker [1, p. 45] on linear forms in two logarithms;
(4) a (very effective) lower bound for the partial quotient of the convergent of a possible solution.

Numerical calculation of partial quotients shows that the only 1-cycle that satisfies these conditions is $(1, 2)$.

As has been remarked by Lagarias [4], the result of that proof is rather weak considering the underlying number theory. We modify and generalize Steiner's approach to prove the nonexistence of 2-cycles (consisting of k_1 odd numbers, ℓ_1 even numbers, k_2 odd numbers and ℓ_2 even numbers).

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Let $K=\sum_{i=1}^{2}k_i$, $L=\sum_{i=1}^{2}\ell_i$. We then derive

(1) a generalized inequality for the ratio $(K+L)/K$;
(2) a numerical lower bound for K, from which it follows that $(K+L)/K$ must be a convergent in the continued fraction expression of $\log_2 3$;
(3) an upper bound for K by applying a theorem of Laurent, Mignotte and Nesterenko [5] on linear forms in two logarithms;
(4) a lower bound for the partial quotient of the convergent of a possible solution.

Steiner's numerical calculation finally shows that no other 2-cycle satisfies these conditions. We show that the approach fails to prove the nonexistence of m-cycles for $m>2$.

2. The nonexistence of 2-cycles

We call the twofold 1-cycle $(1,2,1,2)$ a trivial 2-cycle and any other 2-cycle nontrivial. We will computationally exclude small values for x_n and K. The nonexistence of 2-cycles is proved by a series of lemmas along the line of Steiner's original proof, with a crucial lemma to satisfy the conditions for the continued fraction approximation part of the proof.

Lemma 1. *A necessary and sufficient condition for the existence of a 2-cycle is the existence of a solution (a_i,k_i,ℓ_i) of the diophantine system of equations*

$$\text{(1)}\qquad \left\{\begin{array}{rcrcl} -3^{k_1}a_1 & + & 2^{k_2+\ell_1}a_2 & = & 2^{\ell_1}-1,\\ 2^{k_1+\ell_2}a_1 & - & 3^{k_2}a_2 & = & 2^{\ell_2}-1.\end{array}\right.$$

Proof. Assume that such a solution exists. Then $a_i\not\equiv 0 \pmod 2$. By taking

$$x_0=a_12^{k_1}-1$$

which is an odd number, it is easily verified that

$$x_{k_1}=a_13^{k_1}-1$$

is the first even number after k_1 odd numbers.

The first row equation of (1) then generates ℓ_1-1 additional even numbers and shows that

$$x_{k_1+\ell_1}=a_22^{k_2}-1$$

is the first appearing odd number. By induction a 2-cycle exists, which proves the necessity of the condition in the lemma.

Now assume that a 2-cycle exists. The first odd number in the subsequence of k_1 odd numbers can be written in the form

$$a_12^{k_1}-1$$

with $a_1,k_1>0$, $a_1\not\equiv 0 \pmod 2$. Hence

$$x_{k_1}=a_13^{k_1}-1$$

is an even number and the beginning of a subsequence of ℓ_1 even numbers. The first odd number is then

$$x_{k_1+\ell_1}=\left(a_13^{k_1}-1\right)/2^{\ell_1}$$

which can be written in the form

$$a_22^{k_2}-1$$

with $a_2, k_2 > 0$, $a_2 \not\equiv 0 \pmod 2$. By induction a solution of the diophantine system of equations (1) exists, which proves the sufficiency of the condition in the lemma. □

Note that $a_i = k_i = l_i = 1$ is a solution of the system (1) corresponding with the trivial 2-cycle $(1, 2, 1, 2)$.

Lemma 2. *If a solution of the diophantine system* (1) *of Lemma* 1 *exists, then* a_i, k_i *and* ℓ_i *satisfy the relation*

$$1 < 2^{K+L}/3^K = \prod_{i=1}^{2} \frac{a_i - 3^{-k_i}}{a_i - 2^{-k_i}}. \tag{2}$$

Proof. The first row equation of the system (1) can be rewritten in the form

$$2^{\ell_1} = \left(a_1 3^{k_1} - 1\right) / \left(a_2 2^{k_2} - 1\right).$$

Hence

$$2^{k_2+\ell_1}/3^{k_1} = \frac{a_1 - 3^{-k_1}}{a_2 - 2^{-k_2}},$$

and similarly from the second row equation

$$2^{k_1+\ell_2}/3^{k_2} = \frac{a_2 - 3^{-k_2}}{a_1 - 2^{-k_1}}.$$

Multiplication leads to the equal sign part of the lemma. Since $3^{-k_i} < 2^{-k_i}$, the lemma is proved. □

Lemma 3. *If* a_i, k_i *and* ℓ_i *satisfy the relation* (2) *of Lemma* 2*, then* a_i, k_i *and* ℓ_i *also satisfy the inequality*

$$0 < (K+L)\log 2 - K \log 3 < \sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1}. \tag{3}$$

Proof. Since

$$1 < \frac{a_i - 3^{-k_i}}{a_i - 2^{-k_i}} < \frac{a_i}{a_i - 2^{-k_i}} = \frac{a_i 2^{k_i}}{a_i 2^{k_i} - 1},$$

it follows from relation (2) that

$$1 < 2^{K+L}/3^K < \prod_{i=1}^{2} \frac{a_i 2^{k_i}}{a_i 2^{k_i} - 1}.$$

Taking logs and using $\log(1+x) < x$ if $x < 1$ leads to

$$0 < (K+L)\log 2 - K \log 3 < \sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1},$$

which proves this lemma. □

Note that this is a generalization of the result $0 < (k+\ell)\log 2 - k \log 3 < 1/(2^k - 1)$ in Steiner's proof. From there on Steiner derives a lower bound $k_{\min}$ with the property that if $k > k_{\min}$, then $(k+\ell)/k$ must be a convergent of the continued fraction expression of $\log_2 3$. A generalization is not straightforward, since if $K = \sum_{i=1}^{2} k_i$ is large, a single k_i can still take a small value. However for the expression

$$\sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1}$$

an effective upper bound can be derived by exploiting the average values of k_i and ℓ_i.

Lemma 4. *If a nontrivial* 2*-cycle exists, then*

$$\sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1} < 1.19 \cdot 2^{(L-K)/2}.$$

Proof. Let $k = K/2$. Let $\bar{a} > 0$ be defined by

$$\bar{a}^2 = \prod_{i=1}^{2} \frac{a_i 2^{k_i} - 1}{2^{k_i}}.$$

Let ρ_i be defined by

$$\rho_i \bar{a} 2^k = a_i 2^{k_i} - 1.$$

Hence

$$\frac{\rho_1}{\rho_2} = \frac{a_1 2^{k_1} - 1}{a_2 2^{k_2} - 1} < \left(\frac{2}{3}\right)^{k_1} \frac{a_1 3^{k_1} - 1}{a_2 2^{k_2} - 1} = \left(\frac{2}{3}\right)^{k_1} 2^{\ell_1}.$$

Since $\rho_1 \rho_2 = 1$, we have

$$\rho_1^2 = \frac{\rho_1}{\rho_2} < 2^{k_1 + \ell_1 - k_1 \log_2 3} < 2^{\ell_1 - \frac{1}{2} k_1}.$$

Let $\ell = L/2$. Then we have for ρ_1 (since $\frac{1}{4} k_1 + \frac{1}{2} \ell_2 \geq \frac{3}{4}$)

$$\rho_1 < 2^{\frac{1}{2} \ell_1 - \frac{1}{4} k_1} \leq 2^{\ell - \frac{3}{4}}.$$

In a similar way we can prove this inequality holds for ρ_2 and consequently we have

$$\sum_{i=1}^{2} \frac{1}{\rho_i} = \sum_{i=1}^{2} \rho_i < 2^{\ell + \frac{1}{4}}.$$

For a nontrivial 2-cycle with $a_1 a_2 \geq 3$ we have

$$\bar{a}^2 = \prod_{i=1}^{2} \frac{a_i 2^{k_i} - 1}{2^{k_i}} > \prod_{i=1}^{2} \frac{a_i (2^{k_i} - 1)}{2^{k_i}} > \frac{a_1 \frac{1}{2} 2^{k_i}}{2^{k_1}} \frac{a_2 \frac{2}{3} 2^{k_2}}{2^{k_2}} \geq 1.$$

It follows that

$$\sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1} = \sum_{i=1}^{2} \frac{1}{\rho_i \bar{a} 2^k} < \frac{1}{\bar{a}} 2^{\ell - k + \frac{1}{4}} < 1.19 \cdot 2^{(L-K)/2},$$

which proves this lemma. □

Lemma 5. *If a nontrivial* 2*-cycle exists, then* $(K + L)/K$ *must be a convergent in the continued fraction expansion of* $\log_2 3$.

Proof. From Lemma 3 we have $0 < (K+L)\log 2 - K\log 3$; hence

$$K + L > K\frac{\log 3}{\log 2} > 1.58K.$$

Suppose $(K+L) > 1.6K$. Then we have

$$(K+L)\log 2 - K\log 3 > (1.6\log 2 - \log 3)K > 0.009K.$$

We computationally checked that for all starting values $x_0 \le 100$ the trivial cycle $(1,2)$ appears and that for all values k_1 and k_2 with $k_1 + k_2 \le 100$ no integer solutions of the system (1) of Lemma 1 exist other than $a_i = k_i = \ell_i = 1$. So we will now explicitly assume that $K > 100$ and $x_i = a_i 2^{k_i} - 1 > 100$. From Lemma 3 we have

$$(K+L)\log 2 - K\log 3 < \sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1},$$

and thus $(K+L)\log 2 - K\log 3 < 0.02$, which contradicts the lower bound $0.009K > 0.9$, and hence $K + L < 1.6K$. Consequently

$$1.19 \cdot 2^{(L-K)/2} < 1.19 \cdot 2^{-0.2K} < \frac{\frac{1}{2}\log 2}{K} \text{ if } K > 100.$$

Substitution of this result in Lemma 3 and Lemma 4 leads to

$$0 < (K+L)\log 2 - K\log 3 < \frac{\frac{1}{2}\log 2}{K}$$

or equivalently

$$0 < \frac{K+L}{K} - \frac{\log 3}{\log 2} < \frac{1}{2K^2},$$

which proves this lemma. □

Lemma 6. *If a nontrivial* 2*-cycle exists, then* $K < 86\,000$.

Proof. Let $\Lambda = (K+L)\log 2 - K\log 3$. Then $\Lambda > 0$ from Lemma 3. According to a theorem on linear forms in two logarithms of Laurent, Mignotte and Nesterenko [5], if $\Lambda > 0$, then

$$\log\Lambda \ge -24.34 D^4 \log A_1 \log A_2 \max\left\{\log\left(\frac{K+L}{\log A_2} + \frac{K}{\log A_1}\right) + 0.14, \frac{21}{D}, \frac{1}{2}\right\}^2.$$

Here $D = 1$ is the degree of the extension field of $\mathbb{Q}$, $A_1 = 3$ and $A_2 = e$.

We now distinguish two cases for $T = \log\left(\frac{K+L}{\log 3} + K\right)$.

(a) If $T \le 20.86$, since $K + L > 1.58K$ from Lemma 5, we have $K < 4.8 \cdot 10^8$. Also if $T \le 20.86$, then $-\log\Lambda \le 24.34(\log 3)21^2 < 11\,800$. From Lemma 5 we have $-\log\Lambda \ge 0.2K\log 2 - \log 1.19$. Thus $0.2K\log 2 - \log 1.19 < 11\,800$; hence $K < 86\,000$.

(b) If $T > 20.86$, since $K + L < 1.6K$ from Lemma 5, we have $K > 4.6 \cdot 10^8$. Also if $T > 20.86$, then $-\log\Lambda \le 24.34(\log 3)(T+0.14)^2 < 26.75(\log K + 1.04)^2$. From Lemma 5 we have $-\log\Lambda \ge 0.2K\log 2 - \log 1.19$. Thus, $0.2K\log 2 - \log 1.19 < 26.75(\log K + 1.04)^2$; hence $K < 24\,000$, which contradicts the lower bound $K > 4.6 \cdot 10^8$.

So $T \le 20.86$ and $K < 86\,000$, which proves this lemma. □

Lemma 7. *If a nontrivial* 2*-cycle exists, then the partial quotient* a_{n+1} *in the continued fraction expansion of* $\log_2 3$*, corresponding with the solution* $(K+L)/K$*, is greater than* 3 500*.*

Proof. According to a theorem of Legendre [3, p. 153] we have for the partial quotients a_n of a possible solution $(K+L)/K$ of Lemma 5 the inequality

$$\left|\log_2 3 - \frac{K+L}{K}\right| > \frac{1}{(a_{n+1}+2)K^2}.$$

From Lemmas 3 and 4 we have $0 < (K+L)\log 2 - K\log 3 < 2^{(L-K)/2+\frac{1}{4}}$ or equivalently

$$\left|\log_2 3 - \frac{K+L}{K}\right| < \frac{2^{(L-K)/2+\frac{1}{4}}}{(\log 2)K}$$

From Lemma 5 we have $L - K < -0.4K$. Thus we have for K the inequality

$$\frac{1}{(a_{n+1}+2)K^2} < \frac{2^{-0.2K+\frac{1}{4}}}{(\log 2)K}$$

or equivalently

$$a_{n+1} > \frac{(\log 2)2^{0.2K-\frac{1}{4}}}{K} - 2 > 3\,500$$

if $K > 100$, which proves this lemma. □

Main Theorem 1. *There are no nontrivial* 2*-cycles for the* $3x+1$ *problem.*

Proof. Suppose such a 2-cycle exists. Then according to Lemma 5 the ratio $(K+L)/K$ must be a convergent in the continued fraction expansion of $\log_2 3$. According to Lemmas 4 and 6 we only need Steiner's calculations for the range $100 < K < 86\,000$. The only values of K and $K+L$ in this range for which $\Lambda > 0$ are $(306, 485)$ and $(15601, 24727)$. The corresponding partial quotients in the continued fraction expansion of $\log_2 3$ (taken from Steiner) satisfy $a_{n+1} < 25$. This upper bound contradicts the lower bound 3 500 of Lemma 7, which proves the theorem. □

3. Remarks

Remark 1. It is known from exterior calculations [2, pp. 215–218], [10, p. 23] that the cycle length $K+L$ of a possible cycle satisfies $K+L > 357\,638\,239$. Together with the upper bound $K < 86\,000$ of Lemma 6 this proves the nonexistence of 2-cycles. If for $m > 2$ a generalization of Lemma 4 can be found, the upper bound $K < 86\,000$ of Lemma 6 will increase, so this ad hoc line of proof should vanish for some $m > 2$.

Remark 2. The nonexistence of nontrivial 2-cycles can alternatively be proved by applying a result of de Weger [9, p. 108]. He uses a result of Waldschmidt [8] to derive upper bounds for linear forms of the type $a\log 2 - b\log 3$. In particular he (implicitly) proves that the equation $1 < 2^{k+\ell}/3^k < 1 + 3^{-0.1k}$ has for $k \geq 32$ no solutions. This can shorten the proof for the nonexistence of 2-cycles (and also Steiner's proof). The result of de Weger can be reformulated as $0 < (k+\ell)\log 2 - k\log 3 < 2^{-0.158k}$ has no solutions for $k \geq 32$. From Lemmas 4 and 5 it follows that $0 < (K+L)\log 2 - K\log 3 < 2^{-0.2K+\frac{1}{4}} (< 2^{-0.158K})$, so nontrivial 2-cycles cannot exist. De Weger's method can be applied for any coefficient $0 < \alpha < 1$ in

$1+3^{-\alpha k}$. If for $m>2$ a generalization of Lemma 4 can be found, the coefficient 0.2 in the exponent $-0.2K+\frac{1}{4}$ will decrease, so this line of proof can in principle be generalized for $m>2$.

Remark 3. There is no straightforward generalization to prove the nonexistence of m-cycles ($m>2$) for the $3x+1$ problem. We will sketch a trial proof for $m=3$ to demonstrate this. It is easily verified that Lemmas 1, 2 and 3 can be generalized to

$$0<(K+L)\log 2-K\log 3<\sum_{i=1}^{3}\frac{1}{a_i 2^{k_i}-1}.$$

We now have to find an upper bound for the right-hand part of this inequality as is done in Lemma 4 for 2-cycles. Let $k=K/3$ and let $\bar{a}$ and ρ_i, respectively, be defined by

$$\bar{a}^3=\prod_{i=1}^{3}\frac{a_i 2^{k_i}-1}{2^{k_i}},$$
$$\rho_i\bar{a}2^k=a_i 2^{k_i}-1.$$

Then $\bar{a}^3>0.375$. Substitution into the generalized inequality leads to

$$0<(K+L)\log 2-K\log 3<\sum_{i=1}^{3}\frac{1}{a_i 2^{k_i}-1}<\sum_{i=1}^{3}\frac{1}{\rho_i\bar{a}2^k}.$$

In a similar way as is done in Lemma 4 we can derive upper bounds for ρ_i^{-3}:

$$\rho_1^{-3}<2^{-\frac{1}{2}k_2-k_3+\ell_2+2\ell_3},$$
$$\rho_2^{-3}<2^{-\frac{1}{2}k_3-k_1+\ell_3+2\ell_1},$$
$$\rho_3^{-3}<2^{-\frac{1}{2}k_1-k_2+\ell_1+2\ell_2}.$$

If we assume that k_i and l_i satisfy

$$-\tfrac{1}{2}k_2-k_3+l_2+2l_3\le\tfrac{3}{2}(l_1+l_2+l_3),$$
$$-\tfrac{1}{2}k_3-k_1+l_3+2l_1\le\tfrac{3}{2}(l_1+l_2+l_3),$$
$$-\tfrac{1}{2}k_1-k_2+l_1+2l_2\le\tfrac{3}{2}(l_1+l_2+l_3).$$

then we have $\rho_i^{-1}<2^{\frac{3}{2}L}$ and consequently

$$0<(K+L)\log 2-K\log 3<1.39\cdot 2^{\frac{3}{2}L-K}.$$

This is a result similar to the result of Lemma 4 for 2-cycles and the proof could continue. The exception class of k_i and ℓ_i values which do not satisfy all these relations is, however, large. Let $k_1=7M+N_1$ and $\ell_3=7M+N_2$ ($N_2>N_1$) and all other $k_i=\ell_i=M$. Then only the first inequality is not satisfied. These arbitrarily chosen values are not necessarily a solution of the original system of diophantine equations, but all possible solutions must be checked separately. An alternative approach to generalize the results for m-cycles ($m>2$) is discussed in a forthcoming paper of Simons and de Weger [6].

Remark 4. This line of reasoning can also be used to (dis)prove the nonexistence of 2-cycles (and 1- cycles) for the $px+(p-2)r$ problem with $p\ge 3$. A similar reasoning as used in Lemmas 1, 2 and 3 leads to the generalized inequality

$$0<(K+L)\log 2-K\log 3<\sum_{i=1}^{2}\frac{r}{a_i 2^{k_i}-r}.$$

It can be proved that only "small" m-cycles can exist; however, the class of small cycles for the $px + (p-2)r$ problem does contain several m-cycles. The $3x + 5$ problem has for instance the 1-cycles $(1, 4, 2)$ and $(19, 31, 49, 76, 38)$, the 2-cycle $(23, 37, 58, 29, 46)$ and the 6-cycle $(187, \ldots, 427, \ldots, 1091, \ldots, 1847, \ldots, 781, \ldots, 883, \ldots, 374)$ with period 27. From such calculations a lower bound can be derived for the continued fraction approximation, and Laurent's theorem gives an upper bound for the period length of any possible m-cycle, m fixed.

Remark 5. The remark of Lagarias about the weakness of the result remains valid.

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University of Groningen, PO Box 800, 9700 AV Groningen, The Netherlands
E-mail address: j.l.simons@bdk.rug.nl