MATHEMATICS OF COMPUTATION
Volume 74, Number 252, Pages 1923–1935
S 0025-5718(05)01736-9
Article electronically published on March 15, 2005

EVEN MOMENTS OF GENERALIZED RUDIN–SHAPIRO POLYNOMIALS

CHRISTOPHE DOCHE

Abstract. We know from Littlewood (1968) that the moments of order 4 of the classical Rudin–Shapiro polynomials $P_n(z)$ satisfy a linear recurrence of degree 2. In a previous article, we developed a new approach, which enables us to compute exactly all the moments $\mathcal{M}_q(P_n)$ of even order q for $q \leqslant 32$. We were also able to check a conjecture on the asymptotic behavior of $\mathcal{M}_q(P_n)$, namely $\mathcal{M}_q(P_n) \sim C_q 2^{nq/2}$, where $C_q = 2^{q/2}/(q/2+1)$, for q even and $q \leqslant 52$. Now for every integer $\ell \geqslant 2$ there exists a sequence of generalized Rudin–Shapiro polynomials, denoted by $P^{(\ell)}_{0,n}(z)$. In this paper, we extend our earlier method to these polynomials. In particular, the moments $\mathcal{M}_q(P^{(\ell)}_{0,n})$ have been completely determined for $\ell = 3$ and $q = 4, 6, 8, 10$, for $\ell = 4$ and $q = 4, 6$ and for $\ell = 5, 6, 7, 8$ and $q = 4$. For higher values of ℓ and q, we formulate a natural conjecture, which implies that $\mathcal{M}_q(P^{(\ell)}_{0,n}) \sim C_{\ell,q}\ell^{nq/2}$, where $C_{\ell,q}$ is an explicit constant.

1. Introduction

Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ be the complex torus and let f be an $L^q(\mathbb{T})$–function. The moment of order $q \in \mathbb{N}$ of f satisfies

$$\mathcal{M}_q(f) = \int_0^1 \left| f\left(e^{2i\pi t}\right)\right|^q \, dt. \tag{1}$$

The Rudin–Shapiro polynomials [8] are defined by the recurrence relations

$$P_{n+1}(z) = P_n(z) + z^{2^n} Q_n(z), \qquad Q_{n+1}(z) = P_n(z) - z^{2^n} Q_n(z) \tag{2}$$

and the first values $P_0(z) = Q_0(z) = 1$. Obviously $\mathcal{M}_2(P_n) = 2^n$. In 1968, Littlewood [5] evaluated $\mathcal{M}_4(P_n)$ and established that $\mathcal{M}_4(P_n) \sim 4^{n+1}/3$. In 1980, Saffari [7] conjectured that

$$\mathcal{M}_q(P_n) \sim \frac{2^{(n+1)q/2}}{q/2+1}.$$

In [2] we were able to prove this result for q even less than or equal to 52.

Received by the editor March 6, 2004 and, in revised form, May 31, 2004.
2000 *Mathematics Subject Classification.* Primary 11B83, 11B37, 42C05.
Key words and phrases. Rudin–Shapiro polynomials, signal theory, Krawtchouk polynomials.

In this article, we investigate the moments of generalized Rudin–Shapiro polynomials as defined in [4]. More precisely let $\ell \geqslant 2$, put $\theta = e^{2i\pi/\ell}$ and

$$M(z) = \begin{bmatrix} 1 & z & z^2 & \cdots & z^{(\ell-1)} \\ 1 & \theta z & \theta^2 z^2 & \cdots & \theta^{\ell-1} z^{(\ell-1)} \\ 1 & \theta^2 z & \theta^4 z^2 & \cdots & \theta^{2(\ell-1)} z^{(\ell-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \theta^{\ell-1} z & \theta^{(\ell-1)2} z^2 & \cdots & \theta^{(\ell-1)(\ell-1)} z^{(\ell-1)} \end{bmatrix}. \tag{3}$$

Then the generalized Rudin–Shapiro polynomials are defined by

$$P^{(\ell)}_{0,0}(z) = \cdots = P^{(\ell)}_{\ell-1,0}(z) = 1,$$
$$\begin{bmatrix} P^{(\ell)}_{0,n+1}(z) \\ \vdots \\ P^{(\ell)}_{\ell-1,n+1}(z) \end{bmatrix} = M\big(z^{\ell^n}\big) \times \begin{bmatrix} P^{(\ell)}_{0,n}(z) \\ \vdots \\ P^{(\ell)}_{\ell-1,n}(z) \end{bmatrix} \text{ for } n \geqslant 0. \tag{4}$$

The classical Rudin–Shapiro polynomials correspond to the choice $\ell = 2$.

Since all the sequences $\big(P^{(\ell)}_{i,n}(z)\big)_{n\in\mathbb{N}}$ play a quite symmetric role, we shall focus our attention on $\big(P^{(\ell)}_{0,n}(z)\big)_{n\in\mathbb{N}}$ in the remainder.

In this paper we extend the methods used in [2] for $\ell = 2$ to larger ℓ. The results show that the moments $\mathcal{M}_q\big(P^{(\ell)}_{0,n}\big)$ satisfy a linear recurrence and support the following conjecture.

Conjecture 1. Let ℓ be an integer greater than 1, let q be an even integer and let $\big(P^{(\ell)}_{0,n}(z)\big)_{n\in\mathbb{N}}$ be defined as in (4). We then have

$$\mathcal{M}_q\big(P^{(\ell)}_{0,n}\big) \sim \frac{\ell^{(n+1)q/2}}{\dbinom{q/2+\ell-1}{\ell-1}}.$$

In the next part, we describe an efficient algorithm to compute the moments.

In the third part, we show that these moments must satisfy a linear recurrence and we describe how to obtain it.

In the fourth part, we check Conjecture 1 for some small values of ℓ and q; see Theorem 1.

In the last part, we introduce a generalization of Krawtchouk polynomials in several variables to prove Conjecture 1 under a quite natural assumption; see Conjecture 2 and Theorem 2.

Rudin–Shapiro polynomials are widely used in signal theory, and the corresponding moments are useful to approximate their maximum modulus on $\mathbb{T}$. Furthermore, they are related to deep problems in harmonic analysis.

2. Moments computation algorithm

Since the number of terms of $P^{(\ell)}_{0,n}(z)$ grows exponentially with n, it becomes quickly impossible to compute the moments in a naïve way using (1). Instead, let us introduce

$$\begin{aligned} S_n(\underline{a},\underline{a}',z) &= \left[\left(\sum_{i=0}^{\ell-1} a_i P_{i,n}^{(\ell)}(z)\right)\left(\sum_{i=0}^{\ell-1} a_i'\overline{P_{i,n}^{(\ell)}}(1/z)\right)\right]^{q/2} \\ &= \sum_k c_{k,n}(\underline{a},\underline{a}')z^k, \end{aligned}$$

where $\overline{P_{i,n}^{(\ell)}}$ is the complex conjugate of $P_{i,n}^{(\ell)}$ and where $\underline{a}$ and $\underline{a}'$ are two vectors of formal parameters, namely $\underline{a} = (a_0, \ldots, a_{\ell-1})$ and $\underline{a}' = (a_0', \ldots, a_{\ell-1}')$. The quantity

$$\mathcal{M}_q\big(P_{0,n}^{(\ell)}\big) = \int_0^1 \big|P_{0,n}^{(\ell)}(e^{2i\pi t})\big|^q \, dt$$

is then obviously equal to the constant term of the Laurent polynomial $S_n(\underline{\alpha}_0, \underline{\alpha}_0, z)$ where $\underline{\alpha}_0 = (1, 0, \ldots, 0)$. More generally, the moment of $P_{i,n}^{(\ell)}(z)$ is simply equal to $c_{0,n}\big(\underline{\alpha}_i, \underline{\alpha}_i\big)$ where $\underline{\alpha}_i$ is the vector whose coefficients are 0 except the i–th one, which is equal to 1. Now for $n \geqslant 0$, let

$$T_n(\underline{a},\underline{a}',z) = \frac{1}{\ell^n}\sum_{y^{\ell^n}=z} S_n(\underline{a},\underline{a}',y) = \sum_k c_{k\ell^n,n}(\underline{a},\underline{a}')z^k.$$

The constant terms of $S_n\big(\underline{a},\underline{a}',z\big)$ and of $T_n\big(\underline{a},\underline{a}',z\big)$ are equal for all n but, unlike S_n, the number of terms of T_n is bounded above independently of n. In addition the T_n's can be obtained in a very simple way.

Lemma 1. *The polynomial $T_{n+1}(\underline{a},\underline{a}',z)$ can be deduced from $T_n(\underline{a},\underline{a}',z)$ by the following operations:*

Let $\underline{A}(z) = {}^tM(z)\,\underline{a}$ and $\underline{A}'(z) = \overline{{}^tM}(z)\,\underline{a}'$, where $\overline{{}^tM}$ is the complex conjugate of ${}^tM$. Set

$$\sum_k d_k(\underline{a},\underline{a}')z^k = T_n\left(\underline{A}(z), \underline{A}'(1/z), z\right).$$

Then we have

$$T_{n+1}(\underline{a},\underline{a}',z) = \sum_k d_{k\ell}(\underline{a},\underline{a}')z^k.$$

Proof. By definition

$$\begin{aligned} T_{n+1}(\underline{a},\underline{a}',z) &= \frac{1}{\ell^{n+1}}\sum_{y^{\ell^{n+1}}=z} S_{n+1}(\underline{a},\underline{a}',y) \\ &= \frac{1}{\ell}\sum_{y^\ell=z}\frac{1}{\ell^n}\sum_{w^{\ell^n}=y} S_{n+1}(\underline{a},\underline{a}',w). \end{aligned}$$

From (4), we have also

$$\sum_{i=0}^{\ell-1} a_i P_{i,n+1}^{(\ell)}(z) = \sum_{j=0}^{\ell-1}\left(\sum_{i=0}^{\ell-1}\theta^{ij} z^{j\ell^n} a_i\right) P_{j,n}^{(\ell)}(z)$$

and

$$\sum_{i=0}^{\ell-1} a_i'\overline{P_{i,n+1}^{(\ell)}}(1/z) = \sum_{j=0}^{\ell-1}\left(\sum_{i=0}^{\ell-1}\theta^{-ij}/z^{j\ell^n} a_i'\right)\overline{P_{j,n}^{(\ell)}}(1/z).$$

So $S_{n+1}(\underline{a},\underline{a}',z) = S_n\left(\underline{A}\big(z^{\ell^n}\big), \underline{A}'\big(1/z^{\ell^n}\big), z\right)$.

We deduce that

$$\begin{aligned} T_{n+1}(\underline{a},\underline{a}',z) &= \frac{1}{\ell}\sum_{y^\ell=z}\frac{1}{\ell^n}\sum_{w^{\ell^n}=y} S_n\big(\underline{A}(w^{\ell^n}),\underline{A}'(1/w^{\ell^n}),w\big)\\ &= \frac{1}{\ell}\sum_{y^\ell=z} T_n\big(\underline{A}(y),\underline{A}'(1/y),y\big).\end{aligned}$$

As $T_n\big(\underline{A}(y),\underline{A}'(1/y),y\big)=\sum_k d_k(\underline{a},\underline{a}')y^k$ this shows that

$$T_{n+1}(\underline{a},\underline{a}',z)=\sum_k d_{k\ell}(\underline{a},\underline{a}')z^k,$$

as claimed. □

We immediately derive from this lemma an efficient algorithm to compute the moments of order q of the polynomials $P_{0,n}^{(\ell)}(z)$ for $n \geqslant 0$.

Algorithm 1 (Computation of $\mathcal{M}_q\big(P_{0,n}^{(\ell)}\big)$)**.** Input: An integer ℓ, the order q of the moments and a bound $N \geqslant 0$. Output: The moments of order q of $P_{0,n}^{(\ell)}(z)$ for $n \leqslant N$.

Step 1. $T(\underline{a},\underline{a}',z) \quad\leftarrow\quad \big[(a_0+\cdots+a_{\ell-1})(a_0'+\cdots+a_{\ell-1}')\big]^{q/2}$
$\underline{A}(z) \leftarrow {}^tM(z)\,\underline{a}$
$\underline{A}'(z) \leftarrow \overline{{}^tM}(z)\,\underline{a}'$
$\mathcal{M}_q\big(P_{0,0}^{(\ell)}\big) \leftarrow 1$
$n \leftarrow 1$
Step 2. while $n \leqslant N$ **do**
  $T(\underline{a},\underline{a}',z) \leftarrow \dfrac{1}{\ell}\sum_{y^\ell=z} T\big(\underline{A}(y),\underline{A}'(1/y),y\big)$
  $\mathcal{M}_q\big(P_{0,n}^{(\ell)}\big) \leftarrow$ the constant term of $T(\underline{\alpha}_0,\underline{\alpha}_0,z)$
  $n \leftarrow n+1$
endwhile
Step 3. **return** $\mathcal{M}_q\big(P_{0,0}^{(\ell)}\big),\ldots,\mathcal{M}_q\big(P_{0,N}^{(\ell)}\big)$

Example. Suppose that $\ell = 3$. To obtain the moments of the polynomials $P_{0,n}^{(3)}(z)$ of order $q = 4$, first set

$$T(a_0,a_1,a_2,a_0',a_1',a_2',z) \quad\leftarrow\quad [(a_0+a_1+a_2)(a_0'+a_1'+a_2')]^2.$$

Then in $T(a_0,a_1,a_2,a_0',a_1',a_2',z)$ substitute

- a_0 by $a_0+a_1+a_2$
- a_1 by $(a_0+a_1e^{2i\pi/3}+a_2e^{4i\pi/3})z$
- a_2 by $(a_0+a_1e^{4i\pi/3}+a_2e^{2i\pi/3})z^2$
- a_0' by $a_0'+a_1'+a_2'$
- a_1' by $(a_0'+a_1'e^{4i\pi/3}+a_2'e^{2i\pi/3})/z$
- a_2' by $(a_0'+a_1'e^{2i\pi/3}+a_2'e^{4i\pi/3})/z^2$.

After that, the polynomial $T(a_0,a_1,a_2,a_0',a_1',a_2',z)$ has 324 terms. The transformation

$$T(a_0,a_1,a_2,a_0',a_1',a_2',z) \leftarrow \frac{1}{3}\sum_{y^3=z} T(a_0,a_1,a_2,a_0',a_1',a_2',y) \tag{5}$$

discards the coefficients of degree k in z such that $k \not\equiv 0 \mod 3$ and divides the degree of the other terms by 3. So $T(a_0, a_1, a_2, a'_0, a'_1, a'_2, z)$ has now only 108 terms and $T(1,0,0,1,0,0,z) = 19 + 4(z + 1/z)$. So the moment of order 4 of $P^{(3)}_{0,1}(z)$, that is the constant term of $T(1,0,0,1,0,0,z)$ is equal to 19. Another iteration gives a polynomial with 382 terms. This number reduces to 100 after applying (5), and $T(1,0,0,1,0,0,z) = 93 - 6(z+1/z)$. Thus the moment of $P^{(3)}_{0,2}(z)$ of order 4 is equal to 93. If we keep iterating these transformations, we successively get the moments of order $q = 4$ of all the $P^{(3)}_{0,n}(z)$'s.

Remark. It is possible to extend Lemma 1 and Algorithm 1 to a wider class of polynomial sequences with very little change. In particular, we can choose any polynomials in $\mathbb{C}[z]$ for $P^{(\ell)}_{0,0}(z), \ldots, P^{(\ell)}_{\ell-1,0}(z)$ and any matrix with coefficients in $\mathbb{C}[z]$ for $M(z)$. In this case, replace the assignment of $T(\underline{a}, \underline{a}', z)$ and $\mathcal{M}_q\big(P^{(\ell)}_{0,0}\big)$ in the first step of Algorithm 1 by

$$T(\underline{a}, \underline{a}', z) \quad \leftarrow \quad \left[\left(\sum_{i=0}^{\ell-1} a_i P^{(\ell)}_{i,0}(z)\right)\left(\sum_{i=0}^{\ell-1} a'_i \overline{P^{(\ell)}_{i,0}}(1/z)\right)\right]^{q/2}$$

and

$$\mathcal{M}_q\big(P^{(\ell)}_{0,0}\big) \leftarrow \text{ the constant term of } \left[P^{(\ell)}_{0,0}(z)\overline{P^{(\ell)}_{0,0}}(1/z)\right]^{q/2}.$$

3. Applications

Let $\mathcal{E}_{\ell,q}$ be the vector space over $\mathbb{C}$ generated by the basis $\mathcal{B}_{\ell,q}$ whose elements are

$$z^m a_0^{n_0} \cdots a_{\ell-1}^{n_{\ell-1}} a_0'^{n'_0} \cdots a_{\ell-1}'^{n'_{\ell-1}},$$

where $n_0, \ldots, n_{\ell-1}, n'_0, \ldots, n'_{\ell-1}$ are nonnegative integers such that

$$n_0 + \cdots + n_{\ell-1} = n'_0 + \cdots + n'_{\ell-1} = q/2$$

and m belongs to $[-q/2+1, q/2-1]$. Obviously $T_n(\underline{a}, \underline{a}', z) \in \mathcal{E}_{\ell,q}$ for every n.

Now let $\underline{n} = (n_0, \ldots, n_{\ell-1})$, $\underline{n}' = (n'_0, \ldots, n'_{\ell-1})$ and

$$V(\underline{a}, \underline{a}', z) = \sum_{m,\underline{n},\underline{n}'} c_{m,\underline{n},\underline{n}'} z^m a_0^{n_0} \cdots a_{\ell-1}^{n_{\ell-1}} a_0'^{n'_0} \cdots a_{\ell-1}'^{n'_{\ell-1}}$$

be an element of $\mathcal{E}_{\ell,q}$ and define the map $T_{\ell,q}$ by

$$T_{\ell,q}\big(V(\underline{a}, \underline{a}', z)\big) = \frac{1}{\ell} \sum_{y^\ell = z} V\big(\underline{A}(y), \underline{A}'(1/y), y\big).$$

Clearly $T_{\ell,q}$ is linear and the image of $T_n(\underline{a}, \underline{a}', z)$ by $T_{\ell,q}$ is $T_{n+1}(\underline{a}, \underline{a}', z)$. This implies that the polynomials $T_n(\underline{a}, \underline{a}', z)$, hence the moments of order q of the $P^{(\ell)}_{0,n}(z)$'s, satisfy a linear recurrence $R_{\ell,q}(x)$ which must divide the minimal polynomial of the linear map $T_{\ell,q}$.

In [2] we were able to determine these recurrences and therefore to compute exactly all the even moments $q \leqslant 32$ of the classical Rudin–Shapiro polynomials.

In the following, we establish some recurrences satisfied by the moments of the generalized Rudin–Shapiro polynomials for $\ell = 3,4,5,6,7,8$ and some values of q from 4 to 10. Computations have been performed with `GP-PARI` [1] and `Maple`.

In theory, it is sufficient to know the minimal polynomial of the matrix of $T_{\ell,q}$ in the basis $\mathcal{B}_{\ell,q}$ to compute the linear recurrence satisfied by the moments. When the dimension of the matrix is small, one computes directly its characteristic polynomial to deduce its minimal polynomial. But quickly the size of the basis and of the coefficients make such a computation impossible. In particular,

$$\dim \mathcal{E}_{\ell,q} = \binom{q/2+\ell-1}{\ell-1}^2 (q-1).$$

However, it is possible to get rid of a great number of vectors since many elements of $\mathcal{B}_{\ell,q}$ belong to the kernel of $T_{\ell,q}$. More precisely, it is easy to see that

$$T_{\ell,q}(z^m a_0^{n_0} \cdots a_{\ell-1}^{n_{\ell-1}} a_0'^{n_0'} \cdots a_{\ell-1}'^{n_{\ell-1}'}) = 0 \quad \text{if and only if} \quad \sum_{i=0}^{\ell-1} i(n_i' - n_i) \not\equiv m \mod \ell.$$

Even if, in general, this reduction is not sufficient to find a recurrence satisfied by the moments, the number of terms in the new basis cleared from unnecessary vectors gives a smaller upper bound $D_{\ell,q}$ on the degree of the recurrence.

Now given a sequence $u_0, u_1, \ldots, u_N, \ldots$ generated by an unknown linear recurrence, we are left with the problem of finding the minimal polynomial R of the recurrence. For this, consider the matrix of maximal rank r:

$$\begin{bmatrix} u_0 & u_1 & \cdots & u_{r-1} \\ u_1 & u_2 & \cdots & u_r \\ \vdots & \vdots & \ddots & \vdots \\ u_{r-1} & u_r & \cdots & u_{2r-2} \end{bmatrix}.$$

Inverting it, we immediately deduce a conjectural minimal polynomial for the recurrence.

When r is large, say $r > 100$, the computation of the inverse is difficult, especially since the coefficients grow exponentially. So it is better to perform the computations modulo small primes, then to apply the Chinese remainder theorem. In addition, for $\ell = 5, 7$ and 8 the moments do not belong to $\mathbb{Z}$. In this case, we determine the minimal equation over $\mathbb{Z}[e^{2i\pi/\ell}]$ at the cost of longer computations.

Once we have the conjectural recurrence polynomial we check that this relation is the right one for all n and not only for the very first values. To do this we check that all the terms of order up to $D_{\ell,q}$ satisfy the equation. This is sufficient since the degree of the recurrence is known to be less than or equal to $D_{\ell,q}$.

4. Experimental results

Given ℓ, the moments of order 2 of the generalized Rudin–Shapiro polynomials $P_{0,n}^{(\ell)}(z)$ are trivially equal to ℓ^n, so that they satisfy the recurrence $u_{n+1} = \ell u_n$ with $u_0 = 1$. The corresponding polynomial is therefore $R_{\ell,2}(x) = x - \ell$.

Now it is not hard to ensure that $R_{\ell,q}(x)$ splits into $R_{\ell,q-2}(x/\ell)$ (up to a suitable power of ℓ to make it monic) times a new factor $F_{\ell,q}(x)$. This relies on the fact that

$$E_\ell = a_0 a_0' + \cdots + a_{\ell-1} a_{\ell-1}'$$

is an eigenvector of $T_{\ell,2}$ with eigenvalue ℓ. Indeed,

$$\begin{aligned} T_{\ell,2}(E_\ell) &= \frac{1}{\ell}\sum_{y^\ell=z}\sum_{i=0}^{\ell-1}\left(\sum_{j=0}^{\ell-1} a_j\theta^{ij}\right)\left(\sum_{k=0}^{\ell-1} a'_k\theta^{-ik}\right) \\ &= \sum_{i=0}^{\ell-1}\sum_{j,k=0}^{\ell-1}\theta^{i(j-k)}a_j a'_k \\ &= \ell\sum_{i=0}^{\ell-1} a_i a'_i + \sum_{\substack{j,k=0\\ j\neq k}}^{\ell-1}\sum_{i=0}^{\ell-1}\theta^{i(j-k)}a_j a'_k \\ &= \ell E_\ell. \end{aligned}$$

So if V_λ is an eigenvector of $T_{\ell,q}$ with eigenvalue λ, then $E_\ell V_\lambda$ will be an eigenvector of $T_{\ell,q+2}$ with eigenvalue $\ell\lambda$. As a consequence, $E_\ell^{q/2}$ is an eigenvector of $T_{\ell,q}$ with eigenvalue $\ell^{q/2}$.

For $\ell = 2$, we found the recurrences $R_{2,q}(x)$ for q even and less than or equal to 32 [2]. We noticed that $F_{2,q} = 1$ for $q \equiv 2 \mod 4$; that is why only the values corresponding to $q \equiv 0 \pmod 4$ are reported for $\ell = 2$. Further information on $R_{2,q}(x)$ can be found in Table 1 where $\rho(F_{2,q})$ is the maximal modulus of the roots of $F_{2,q}(x)$.

For larger ℓ, namely for

- $\ell = 3$, $q = 4, 6, 8$ and 10,
- $\ell = 4$, $q = 4$ and 6,
- $\ell = 5, 6, 7$ and 8, $q = 4$,

we compute $R_{\ell,q}(x)$ and verify that $\rho(F_{\ell,q})$ is less than $\ell^{q/2}$. So $\ell^{q/2}$ is simple and is the root of $R_{\ell,q}(x)$ of largest modulus. This implies that the asymptotic behavior of $\mathcal{M}_q\big(P_{j,n}^{(\ell)}\big)$ is $C_{\ell,q}\ell^{nq/2}$, where $C_{\ell,q}$ can be deduced from the minimal recurrence polynomial. Indeed,

$$C_{\ell,q} = \left(\sum_{n=0}^{d_{\ell,q}-1} r_{n,\ell,q}\mathcal{M}_q\big(P_{j,n}^{(\ell)}\big)\right) \Bigg/ \left(\sum_{n=0}^{d_{\ell,q}-1} r_{n,\ell,q}\ell^{nq/2}\right),$$

TABLE 1. Properties of the polynomials $R_{2,q}(x)$

q	$\deg R_{2,q}$	$\deg F_{2,q}$	$\rho(F_{2,q})/2^{q/2}$	$C_{2,q}$
4	2	1	0.50	$\frac{4}{3}$
8	12	10	0.69	$\frac{16}{5}$
12	36	24	0.74	$\frac{64}{7}$
16	78	42	0.76	$\frac{256}{9}$
20	144	66	0.75	$\frac{1024}{11}$
24	240	96	0.72	$\frac{4096}{13}$
28	369	129	0.73	$\frac{16384}{15}$
32	536	167	0.75	$\frac{65536}{17}$

where

$$\sum_{n=0}^{d_{\ell,q}-1} r_{n,\ell,q} x^n = R_{\ell,q}(x)/\big(x-\ell^{q/2}\big).$$

Example. Take $\ell = 3$. The recurrence $R_{3,4}(x)$ is equal to $(x-9)F_{3,4}(x)$ with

$$\begin{aligned}F_{3,4}(x) = {} & x^{19}+3x^{18}-48x^{17}-54x^{16}+270x^{15}-2970x^{14}+21303x^{13}\\ & +66096x^{12}-75087x^{11}+2991816x^{10}-14545737x^9-65800269x^8\\ & +215587899x^7-1021961043x^6-3231692721x^5+30032262351x^4\\ & -43907655420x^3-505583738145x^2+2353579470675\end{aligned}$$

and

$$\mathcal{M}_4\big(P_{0,n}^{(3)}\big) \sim \left(\frac{1441593785878509060}{961062523919006040}\right) 9^n = \frac{3}{2} 9^n.$$

All the results are gathered in the following theorem.

Theorem 1. *For $\ell = 3$, $q = 4, 6, 8, 10$, $\ell = 4$, $q = 4, 6$ and $\ell = 5, 6, 7, 8$, $q = 4$ the asymptotic behavior of the moments of the generalized Rudin–Shapiro polynomials $P_{0,n}^{(\ell)}(z)$ defined as in (4) satisfy*

$$\mathcal{M}_q\big(P_{0,n}^{(\ell)}(z)\big) \sim C_{\ell,q}\ell^{nq/2} \text{ with } C_{\ell,q} = \frac{\ell^{q/2}}{\binom{q/2+\ell-1}{\ell-1}}. \tag{6}$$

See Tables 2, 3 and 4 for additional information.

In the next section we formulate a natural conjecture which is true for all the cases we have investigated and implies (6) for all ℓ and all q.

TABLE 2. Properties of the polynomials $R_{3,q}(x)$

q	$\deg R_{3,q}$	$\deg F_{3,q}$	$\rho(F_{3,q})/3^{q/2}$	$D_{3,q}$	$C_{3,q}$
4	20	19	0.74	36	$\frac{3}{2}$
6	83	63	0.73	166	$\frac{27}{10}$
8	240	157	0.64	525	$\frac{27}{5}$
10	574	334	0.65	1323	$\frac{81}{7}$

TABLE 3. Properties of the polynomials $R_{4,q}(x)$

q	$\deg R_{4,q}$	$\deg F_{4,q}$	$\rho(F_{4,q})/4^{q/2}$	$D_{4,q}$	$C_{4,q}$
4	35	34	0.76	74	$\frac{8}{5}$
6	208	173	0.62	500	$\frac{16}{5}$

TABLE 4. Comparison of the properties of $R_{\ell,4}(x)$ for $\ell = 2, \dots, 8$

ℓ	$\deg R_{\ell,4}$	$\deg F_{\ell,4}$	$\rho(F_{\ell,4})/\ell^2$	$D_{\ell,4}$	$C_{\ell,4}$
2	2	1	0.50	6	$\frac{4}{3}$
3	20	19	0.74	36	$\frac{3}{2}$
4	35	34	0.76	74	$\frac{8}{5}$
5	57	56	0.80	135	$\frac{5}{3}$
6	80	79	0.84	219	$\frac{12}{7}$
7	113	112	0.86	336	$\frac{7}{4}$
8	145	144	0.88	484	$\frac{16}{9}$

5. Determination of the constant $C_{\ell,q}$

This work was first carried out with Laurent Habsieger in the case $\ell = 2$. In particular, he used classical properties of Krawtchouk polynomials to prove that $C_{2,q} = \frac{2^{q/2}}{q/2+1}$. In the following we introduce a generalization of Krawtchouk polynomials for larger ℓ and we get an explicit expression for $C_{\ell,q}$ mimicking his proof when $\ell = 2$.

Let $n_0, \dots, n_{\ell-1}$ and N be nonnegative integers. We shall use the following notation for the multinomial coefficient:

$$\binom{N}{n_0 \cdots n_{\ell-1}} = \begin{cases} \dfrac{N!}{n_0! \cdots n_{\ell-1}!} & \text{if } n_0 + \dots + n_{\ell-1} = N, \\ 0 & \text{if } n_0 + \dots + n_{\ell-1} \neq N. \end{cases}$$

Let $\mathcal{B}'_{\ell,q}$ be the basis of $\mathcal{E}_{\ell,q}$ built from $\mathcal{B}_{\ell,q}$, where we replace all the vectors of the form $(a_0a'_0)^{n_0} \cdots (a_{\ell-1}a'_{\ell-1})^{n_{\ell-1}}$ by

$$(a_0a'_0)^{n_0} \cdots (a_{\ell-1}a'_{\ell-1})^{n_{\ell-1}} - \frac{(a_0a'_0 + \dots + a_{\ell-1}a'_{\ell-1})^{q/2}}{\dbinom{q/2}{n_0 \cdots n_{\ell-1}}\dbinom{q/2+\ell-1}{\ell-1}} \tag{7}$$

except $(a_{\ell-1}a'_{\ell-1})^{q/2}$, which is replaced by $(a_0a'_0 + \dots + a_{\ell-1}a'_{\ell-1})^{q/2}$.

Let $\mathcal{F}_{\ell,q}$ be the vector space generated by $(a_0a'_0 + \dots + a_{\ell-1}a'_{\ell-1})^{q/2}$ and $\mathcal{G}_{\ell,q}$ the vector space generated by $\mathcal{B}'_{\ell,q} \setminus (a_0a'_0 + \dots + a_{\ell-1}a'_{\ell-1})^{q/2}$.

Lemma 2. *In the new basis $\mathcal{B}'_{\ell,q}$ the projection of $(a_0a'_0)^{n_0} \cdots (a_{\ell-1}a'_{\ell-1})^{n_{\ell-1}}$ on $\mathcal{F}_{\ell,q}$ has coordinate*

$$\frac{1}{\dbinom{q/2}{n_0 \cdots n_{\ell-1}}\dbinom{q/2+\ell-1}{\ell-1}}.$$

Proof. These equalities are trivial except for $(a_{\ell-1}a'_{\ell-1})^{q/2}$. In this case we see that the coordinate of $(a_{\ell-1}a'_{\ell-1})^{q/2}$ along $(a_0a'_0+\cdots+a_{\ell-1}a'_{\ell-1})^{q/2}$ is equal to

$$1-\sum_{\substack{\underline{n}\\ n_{\ell-1}\neq q/2}}\frac{\binom{q/2}{n_0\cdots n_{\ell-1}}}{\binom{q/2}{n_0\cdots n_{\ell-1}}\binom{q/2+\ell-1}{\ell-1}} = 1-\frac{\binom{q/2+\ell-1}{\ell-1}-1}{\binom{q/2+\ell-1}{\ell-1}} = \frac{1}{\binom{q/2+\ell-1}{\ell-1}},$$

which is equal to

$$\frac{1}{\binom{q/2}{0\cdots 0\,q/2}\binom{q/2+\ell-1}{\ell-1}},$$

as expected. □

Lemma 3. *We have $\mathcal{E}_{\ell,q}=\mathcal{F}_{\ell,q}\oplus\mathcal{G}_{\ell,q}$ with $T_{\ell,q}(\mathcal{F}_{\ell,q})=\mathcal{F}_{\ell,q}$ and $T_{\ell,q}(\mathcal{G}_{\ell,q})\subset\mathcal{G}_{\ell,q}$.*

The first two statements are evident but to prove the last result we need to introduce additional material.

Recall that N is a positive integer, and take k_0 and k_1 such that $k_0+k_1=N$. Then the classical Krawtchouk polynomials $K_n(k_0,k_1,N)$, see for example [3], are defined by

$$\sum_{n=0}^{\infty}K_n(k_0,k_1,N)z^n=(1-z)^{k_0}(1+z)^{k_1}.$$

At present, let us introduce the quantities $K_{\underline{n}}(\underline{k},N)$, where as usual $\underline{n}$ stands for $(n_0,\ldots,n_{\ell-1})$ and where $k_0+\cdots+k_{\ell-1}=N$. The $K_{\underline{n}}(\underline{k},N)$'s can be viewed as a generalization of Krawtchouk polynomials in several variables. Indeed, they are defined by the generating sequence

$$\sum_{\underline{n}}K_{\underline{n}}(\underline{k},N)a_0^{n_0}a_1^{n_1}\cdots a_{\ell-1}^{n_{\ell-1}}=\prod_{j=0}^{\ell-1}\left(a_0+\theta^j a_1+\cdots+\theta^{j(\ell-1)}a_{\ell-1}\right)^{k_j}. \tag{8}$$

Note that they are explicitly defined by the formula

$$K_{\underline{n}}(\underline{k},N)=\sum_{\substack{0\leqslant i,j\leqslant \ell-1\\ n_{i,j}\geqslant 0,\ \sum_j n_{i,j}=n_i}}\theta^{\sum ijn_{i,j}}\prod_{j=0}^{\ell-1}\binom{k_j}{n_{0,j}\cdots n_{\ell-1,j}}.$$

The symmetry and orthogonality properties of the classical Krawtchouk polynomials, which are the core of the proof of Lemma 3 for $\ell=2$, can also be generalized to $K_{\underline{n}}(\underline{k},N)$ for larger ℓ.

Lemma 4. *Let $\underline{k}$, $\underline{n}$ and $\underline{n}'$ be vectors of integers. Then we have*

$$\binom{N}{k_0\cdots k_{\ell-1}}K_{\underline{n}}(\underline{k},N)=\binom{N}{n_0\cdots n_{\ell-1}}K_{\underline{k}}(\underline{n},N) \tag{9}$$

and

$$\sum_{\underline{k}}\binom{N}{k_0\cdots k_{\ell-1}}K_{\underline{n}}(\underline{k},N)\overline{K_{\underline{n}'}(\underline{k},N)}=\begin{cases}0 & \text{if } \underline{n}\neq\underline{n}',\\ \ell^N\binom{N}{n_0\cdots n_{\ell-1}} & \text{otherwise.}\end{cases} \tag{10}$$

Proof. The symmetry property (9) is immediate using (8). To show the orthogonality property (10), consider

$$\sum_{\underline{n},\underline{n}'}\sum_{\underline{k}}\binom{N}{k_0\cdots k_{\ell-1}}K_{\underline{n}}(\underline{k},N)\overline{K_{\underline{n}'}(\underline{k},N)}a_0^{n_0}a_0'^{n_0'}\cdots a_{\ell-1}^{n_{\ell-1}}a_{\ell-1}'^{n_{\ell-1}'}. \tag{11}$$

Using (8) and swapping the sums, we see that (11) can be written as

$$\sum_{\underline{k}}\binom{N}{k_0\cdots k_{\ell-1}}\prod_{j=0}^{\ell-1}\Big[\big(a_0+\theta^j a_1+\cdots+\theta^{j(\ell-1)}a_{\ell-1}\big)\big(a_0'+\theta^{-j}a_1'+\cdots+\theta^{-j(\ell-1)}a_{\ell-1}'\big)\Big]^{k_j}.$$

This sum is equal to

$$\left[\sum_{j=0}^{\ell-1}\big(a_0+\theta^j a_1+\cdots+\theta^{j(\ell-1)}a_{\ell-1}\big)\big(a_0'+\theta^{-j}a_1'+\cdots+\theta^{-j(\ell-1)}a_{\ell-1}'\big)\right]^N,$$

i.e.,

$$\ell^N(a_0a_0'+a_1a_1'+\cdots+a_{\ell-1}a_{\ell-1}')^N. \tag{12}$$

Expanding (12), one remarks that for each monomial involved in this expression the degrees of a_i and a_i' are equal. In particular, if $\underline{n}\neq\underline{n}'$, then

$$a_0^{n_0}a_0'^{n_0'}\cdots a_{\ell-1}^{n_{\ell-1}}a_{\ell-1}'^{n_{\ell-1}'}$$

does not appear in (12); hence the coefficient before this term in (11) is zero. If $\underline{n}=\underline{n}'$, it is clear that the coefficient of

$$(a_0a_0')^{n_0}\cdots(a_{\ell-1}a_{\ell-1}')^{n_{\ell-1}}$$

in (11) is

$$\ell^N\binom{N}{n_0\cdots n_{\ell-1}}.$$

This proves the result. □

Proof of Lemma 3. Given m, $\underline{n}$ and $\underline{n}'$, let us compute the image of the monomial $z^m a_0^{n_0}a_0'^{n_0'}\cdots a_{\ell-1}^{n_{\ell-1}}a_{\ell-1}'^{n_{\ell-1}'}$ by $T_{\ell,q}$. The result is a sum of monomials having all the same degree in z, which is

$$m+\sum_{i=0}^{\ell-1}i(n_i-n_i'). \tag{13}$$

Therefore, if (13) is nonzero, $T_{\ell,q}(z^m a_0^{n_0}a_0'^{n_0'}\cdots a_{\ell-1}^{n_{\ell-1}}a_{\ell-1}'^{n_{\ell-1}'})$ belongs to $\mathcal{G}_{\ell,q}$. This also easily implies that the monomial we are considering is an element of $\mathcal{G}_{\ell,q}$.

If (13) is zero, then

$$T_{\ell,q}(z^m a_0^{n_0}a_0'^{n_0'}\cdots a_{\ell-1}^{n_{\ell-1}}a_{\ell-1}'^{n_{\ell-1}'})=\prod_{j=0}^{\ell-1}\left(\sum_{i=0}^{\ell-1}a_i\theta^{ij}\right)^{n_j}\left(\sum_{i=0}^{\ell-1}a_i'\theta^{-ij}\right)^{n_j'}.$$

Using (8) and Lemma 2 it is easy to ensure that the coefficient of the projection of $T_{\ell,q}(z^m a_0^{n_0}a_0'^{n_0'}\cdots a_{\ell-1}^{n_{\ell-1}}a_{\ell-1}'^{n_{\ell-1}'})$ on $\mathcal{F}_{\ell,q}$ is equal to

$$C_{\underline{n},\underline{n}'}=\sum_{\underline{k}}\frac{K_{\underline{k}}(\underline{n},q/2)\overline{K_{\underline{k}}(\underline{n}',q/2)}}{\binom{q/2}{k_0\cdots k_{\ell-1}}\binom{q/2+\ell-1}{\ell-1}}.$$

With the help of (9), we get

$$C_{\underline{n},\underline{n}'} = \sum_{\underline{k}} \binom{q/2}{k_0 \cdots k_{\ell-1}} \frac{K_{\underline{n}}(\underline{k}, q/2)\overline{K_{\underline{n}'}(\underline{k}, q/2)}}{\binom{q/2}{n_0 \cdots n_{\ell-1}}\binom{q/2}{n'_0 \cdots n'_{\ell-1}}\binom{q/2+\ell-1}{\ell-1}}.$$

From (10) we know that $C_{\underline{n},\underline{n}'}$ is zero if $\underline{n} \neq \underline{n}'$.

If $\underline{n} = \underline{n}'$ we have

$$\begin{aligned} C_{\underline{n}} &= \sum_{\underline{k}} \binom{q/2}{k_0 \cdots k_{\ell-1}} \frac{K_{\underline{n}}(\underline{k}, q/2)\overline{K_{\underline{n}}(\underline{k}, q/2)}}{\binom{q/2}{n_0 \cdots n_{\ell-1}}^2\binom{q/2+\ell-1}{\ell-1}} \\ &= \frac{\ell^{q/2}}{\binom{q/2}{n_0 \cdots n_{\ell-1}}\binom{q/2+\ell-1}{\ell-1}}. \end{aligned}$$

But if $\underline{n} = \underline{n}'$ with $n_{\ell-1} \neq q/2$, the monomial $z^m a_0^{n_0} a_0'^{n'_0} \cdots a_{\ell-1}^{n_{\ell-1}} a_{\ell-1}'^{n'_{\ell-1}}$ has been replaced by the vector of the form (7), which can be written

$$z^m a_0^{n_0} a_0'^{n'_0} \cdots a_{\ell-1}^{n_{\ell-1}} a_{\ell-1}'^{n'_{\ell-1}} - C_{\underline{n}} \frac{(a_0 a'_0 + \cdots + a_{\ell-1} a'_{\ell-1})^{q/2}}{\ell^{q/2}}. \tag{14}$$

Since $T_{\ell,q}$ is a linear map such that $(a_0 a'_0 + \cdots + a_{\ell-1} a'_{\ell-1})^{q/2}$ is one of its eigenvectors for the eigenvalue $\ell^{q/2}$, this shows that the image of (14) by $T_{\ell,q}$ has a nil coordinate on $\mathcal{F}_{\ell,q}$.

So the image by $T_{\ell,q}$ of any vector basis of $\mathcal{G}_{\ell,q}$ has a zero component on $\mathcal{F}_{\ell,q}$, and this ensures that $T_{\ell,q}(\mathcal{G}_{\ell,q}) \subset \mathcal{G}_{\ell,q}$. □

The computations we have done for small values of ℓ and q suggest that all the eigenvalues of $T_{\ell,q}$ are less than or equal to $\ell^{q/2}$ in modulus. It seems also that $(a_0 a'_0 + \cdots + a_{\ell-1} a'_{\ell-1})^{q/2}$ is the only eigenvector of $\mathcal{E}_{\ell,q}$ associated to $\ell^{q/2}$. We formulate this result in the following conjecture.

Conjecture 2. The eigenvalues of the restriction of $T_{\ell,q}$ to $\mathcal{G}_{\ell,q}$ have a modulus smaller than $\ell^{q/2}$.

Proving this result would imply Conjecture 1. More precisely,

Theorem 2. *Assume Conjecture 2 is true. We then have*

$$\mathcal{M}_q\big(P_{0,n}^{(\ell)}\big) \sim C_{\ell,q} \ell^{qn/2} \quad \textit{with} \quad C_{\ell,q} = \frac{\ell^{q/2}}{\binom{q/2+\ell-1}{\ell-1}}.$$

Proof. We know that $\ell^{q/2}$ is the largest eigenvalue of $T_{\ell,q}$ and that its eigenspace is $\mathcal{F}_{\ell,q}$ of dimension 1. So

$$\mathcal{M}_q\big(P_{0,n}^{(\ell)}\big) \sim C_{\ell,q} \ell^{nq/2},$$

where $C_{\ell,q}$ is the coordinate of the projection of the initial vector

$$\Big[\big(a_0 + \cdots + a_{\ell-1}\big)\big(a'_0 + \cdots + a'_{\ell-1}\big)\Big]^{q/2}$$

on $\mathcal{F}_{\ell,q}$. Namely,

$$\begin{aligned} C_{\ell,q} &= \sum_{\underline{n}} \binom{q/2}{n_0 \cdots n_{\ell-1}}^2 \frac{1}{\binom{q/2}{n_0 \cdots n_{\ell-1}}\binom{q/2+\ell-1}{\ell-1}} \\ &= \frac{1}{\binom{q/2+\ell-1}{\ell-1}} \sum_{\underline{n}} \binom{q/2}{n_0 \cdots n_{\ell-1}} \\ &= \frac{\ell^{q/2}}{\binom{q/2+\ell-1}{\ell-1}}. \end{aligned}$$

□

A collection of `GP-PARI` [1] functions to compute the moments of the generalized Rudin–Shapiro polynomials for various values of ℓ and q is available at [6]. For instance, the procedure `findmoment` gives the exact moment of order q of $P_{0,n}^{(\ell)}(z)$, for any n and for the values of ℓ and q discussed in this article. The files include all the factors $F_{\ell,q}(x)$ and the requested initial moments.

Acknowledgments

I would like to thank Bahman Saffari, who suggested to me to investigate the moments of generalized Rudin–Shapiro polynomials and Laurent Habsieger, with whom I had many discussions on the subject. I am also grateful for the many helpful comments and suggestions of the referee.

References

1. C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier, *User's guide to PARI / GP, ver. 2.1.3*, 2002. `http://www.parigp-home.de/`
2. C. Doche and L. Habsieger, *Moments of the Rudin–Shapiro polynomials*. Journal of Fourier Analysis and Applications **10** (2004).
3. I. Krasikov and S. Litsyn, *On integral zeros of Krawtchouk polynomials*. Journal of Combinatorial Theory, Ser. A **150** (1996), 441–447. `http://www.eng.tau.ac.il/~litsyn/publications.html` MR1383506 (97i:33005)
4. Z. X. Lei, *Some properties of generalized Rudin–Shapiro polynomials*, Chinese Ann. Math. Ser. A **12** N°2 (1991), 145–153. MR1112426 (92k:11027)
5. J. E. Littlewood, *Some problems in real and complex analysis*, D. C. Heath and Co. Raytheon Education Co., Lexington, Mass., 1968. MR0244463 (39:5777)
6. *A GP–PARI program to compute the moments of some generalized Rudin–Shapiro polynomials.* `http://www.math.u-bordeaux.fr/~cdoche/moments.gp`
7. B. Saffari, *personal communication*, (2001).
8. H. S. Shapiro, *Extremal problems for polynomials and power series*, Ph.D. thesis, MIT, 1951.

Division of ICS, Building E6A, Macquarie University, New South Wales 2109 Australia

E-mail address: `doche@ics.mq.edu.au`