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COMPARISON THEOREMS OF KOLMOGOROV TYPE AND EXACT VALUES OF n-WIDTHS ON HARDY–SOBOLEV CLASSES

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ABSTRACT. Let $S_\beta := \{z \in \mathbb{C} : |\mathrm{Im} z| < \beta\}$ be a strip in complex plane. $\widetilde{H}^r_{\infty,\beta}$ denotes those 2π-periodic, real-valued functions on $\mathbb{R}$ which are analytic in the strip S_β and satisfy the condition $|f^{(r)}(z)| \le 1$, $z \in S_\beta$. Osipenko and Wilderotter obtained the exact values of the Kolmogorov, linear, Gel′fand, and information n-widths of $\widetilde{H}^r_{\infty,\beta}$ in $L_\infty[0,2\pi]$, $r = 0,1,2,\dots$, and $2n$-widths of $\widetilde{H}^r_{\infty,\beta}$ in $L_q[0,2\pi]$, $r = 0$, $1 \le q < \infty$.

In this paper we continue their work. Firstly, we establish a comparison theorem of Kolmogorov type on $\widetilde{H}^r_{\infty,\beta}$, from which we get an inequality of Landau–Kolmogorov type. Secondly, we apply these results to determine the exact values of the Gel′fand n-width of $\widetilde{H}^r_{\infty,\beta}$ in $L_q[0,2\pi]$, $r = 0,1,2\dots$, $1 \le q < \infty$. Finally, we calculate the exact values of Kolmogorov $2n$-width, linear $2n$-width, and information $2n$-width of $\widetilde{H}^r_{\infty,\beta}$ in $L_q[0,2\pi]$, $r \in \mathbb{N}$, $1 \le q < \infty$.

1. Introduction

Let X be a normed linear space and X_n be an n-dimensional subspace of X. For each $x \in X$, $E(x;X_n)$ denotes the distance of the n-dimensional subspace X_n from x, defined by

$$E(x;X_n) := \inf_{y\in X_n} \|x-y\|, \tag{1}$$

and the quantity

$$E(A,X_n) := \sup_{x\in A} \inf_{y\in X_n} \|x-y\| \tag{2}$$

is said to be the deviation of A from X_n. Thus $E(A,X_n)$ measures how well the "worst element" of A can be approximated from X_n.

Given a subset A of X, one might ask how well one can approximate A by n-dimensional subspaces of X. Thus, we consider the possibility of allowing the n-dimensional subspaces X_n to vary within X. This idea, introduced by Kolmogorov in 1936, is now referred to as the Kolmogorov n-width of A in X. It is defined by

$$d_n(A,X) := \inf_{X_n} \sup_{x\in A} \inf_{y\in X_n} \|x-y\|, \tag{3}$$

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where X_n runs over all n-dimensional subspaces of X.

The Kolmogorov n-width $d_n(A, X)$ describes the minimum error of A approximated by any n-dimensional subspace X_n in X. In addition to the Kolmogorov n-width, there are three other related concepts that will be studied in this paper. The linear n-width of A in X is defined by

$$\lambda_n(A, X) := \inf_{P_n} \sup_{x \in A} \|x - P_n x\|, \tag{4}$$

where the infimum is taken over all bounded linear operators mapping X into itself whose range has dimension at most n.

The Gel′fand n-width of A in X is given by

$$d^n(A, X) := \inf_{X^n} \sup_{x \in A \cap X^n} \|x\|, \tag{5}$$

where X^n runs over all subspaces of X of codimension n (here we assume that $0 \in A$), and the information n-width is the quantity

$$i_n(A, X) := \inf_{l_1, \ldots, l_n} \inf_{m: Z^n \to X} \sup_{x \in A} \|x - m(l_1 x, \ldots, l_n x)\|, \tag{6}$$

where $l_1, \ldots, l_n$ run over all continuous linear functionals on A and m is taken over all maps of Z^n into X ($Z = \mathbb{R}$ or $\mathbb{C}$ depending on whether A is a set of real-valued or complex-valued functions). More detailed information about the n-widths is contained in the books of Pinkus [16], and Lorentz, Golischek and Makovoz [7].

Now we introduce the classes of functions to be studied. Let $S_\beta := \{z \in \mathbb{C} : |\mathrm{Im} z| < \beta\}$ be a strip in a complex plane. For an integer $r \geq 0$, the class $\widetilde{H}^r_{\infty,\beta}$ ($\widetilde{h}^r_{\infty,\beta}$) consists of those 2π-periodic, real-valued functions on $\mathbb{R}$ which are analytic in the strip S_β and satisfy the condition $|f^{(r)}(z)| \leq 1$ ($|\mathrm{Re} f^{(r)}(z)| \leq 1$), $z \in S_\beta$. For $r = 0$, we will omit the upper index in the notation of these classes.

Many extremal problems, especially for calculating the exact values of n-widths, were investigated for the classical Sobolev class of 2π-periodic functions $\widetilde{W}^r_p$ (see [6]) in L_q space, where $L_q := L_q[0, 2\pi]$ is the classical Lebesgue integral space of 2π-periodic real-valued functions with the usual norm $\|\cdot\|_q$, $1 \leq q \leq \infty$. By the efforts of many mathematicians some similar results were also built for the classes in which each function can be represented as a convolution with a kernel having a cyclic variation diminishing property or satisfying Property B (see, e.g., [15]–[17]). For example, the Kolmogorov n-widths of the class $\widetilde{h}^r_{\infty,\beta}$ in the space L_∞ were obtained by Tikhomirov [19], Forst [4] for $r = 0$, and by Osipenko [11] for all $r \in \mathbb{N}$. After this, the exact values of the even n-widths of the class of $\widetilde{h}^r_{\infty,\beta}$ in L_q, $1 \leq q < \infty$ were calculated by Pinkus [16] for $r = 0$ and by Osipenko [10] for all $r \in \mathbb{N}$.

On the other hand, recently, the exact estimates of the even n-widths of the class $\widetilde{H}^r_{\infty,\beta}$ in the space L_q were determined by Osipenko [9] for $r = 0$, $1 \leq q \leq \infty$, and by Osipenko [11]–[12], and Osipenko and Wilderotter [13] for all $r \in \mathbb{N}$, $q = \infty$.

Motivated by Osipenko [11], in this paper we determine the exact values of the Gel′fand n-width, Kolmogorov $2n$-width, linear $2n$-width, and information $2n$-width of $\widetilde{H}^r_{\infty,\beta}$ in the space L_q, $1 \leq q < \infty$, for all $r \in \mathbb{N}$.

The fundamental difference between $\widetilde{H}^r_{\infty,\beta}$ and $\widetilde{h}^r_{\infty,\beta}$ lies in the fact that functions in $\widetilde{h}^r_{\infty,\beta}$ may be characterized as convolutions with a Property B kernel (see [16]), while such a representation is not available for functions in $\widetilde{H}^r_{\infty,\beta}$. Therefore, in some cases other techniques must be applied in order to deal with $\widetilde{H}^r_{\infty,\beta}$. Our approach will consist in establishing a comparison theorem of Kolmogorov type on $\widetilde{H}^r_{\infty,\beta}$ by using Rouche's theorem. And then, we derive an inequality of Taikov type for trigonometric polynomials [6], [18], from which we get the upper estimates of Gel′fand widths. In processing the lower estimates of n-widths, we follow the method of Zensykbaev [20], Micchelli and Pinkus [8], Pinkus [15], and Osipenko [11] to consider a minimum norm question on analogue of classical polynomial perfect splines.

We now outline the rest of this paper. In Section 2, we establish a comparison theorem of Kolmogorov type on the Hardy–Sobolev class $\widetilde{H}^r_{\infty,\beta}$ and as a consequence, we derive an inequality of Landau–Kolmogorov type on this class of functions. In Section 3, using the results of Section 2, we demonstrate an inequality on nonincreasing rearrangement, from which we get a Taikov type inequality that will be used as the upper estimates of the Gel′fand n-width of the class $\widetilde{H}^r_{\infty,\beta}$. In Section 4, we solve a minimum norm question on an analogue of the classical polynomial perfect splines, and then we use it to prove the lower estimates of n-widths, which together with some results of Osipenko (see [14]) and Section 3 determine the exact values of the Gel′fand n-width, Kolmogorov $2n$-width, linear $2n$-width, and information $2n$-width of $\widetilde{H}^r_{\infty,\beta}$ in L_q, $r \in \mathbb{N}$, $1 \le q < \infty$.

2. Comparison theorem of Kolmogorov type on $\widetilde{H}^r_{\infty,\beta}$

The Kolmogorov comparison theorem (see [6]), which concerns the comparison of derivatives of differentiable functions defined on the real line, plays an important role in establishing some sharp inequalities in approximation theory. In this section we will prove a comparison theorem of Kolmogorov type on the class of analytic functions $\widetilde{H}^r_{\infty,\beta}$ and then derive an inequality of Landau–Kolmogorov type on this class of functions. Besides their own independent roles, these results will also be used in the following section to establish an inequality of Taikov type on the class $\widetilde{H}^r_{\infty,\beta}$.

Before we advance our discussion further, we introduce some notions of sign changes of vectors and functions that are very important in our research.

Definition 1 (see [16], pp. 45, 59)**.** Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \backslash \{0\}$ be a real nontrivial vectors.

(i) $S^-(x)$ indicates the number of sign changes in the sequence $x_1, \ldots, x_n$ with zero terms discarded. The number $S_c^-(x)$ of cyclic variations of the sign of x is given by

$$S_c^-(x) := \max_i S^-(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_i) = S^-(x_k, \ldots, x_n, x_1, \ldots, x_k),$$

where k is any integer for which $x_k \neq 0$. Obviously $S_c^-(x)$ is invariant under cyclic permutations, and $S_c^-(x)$ is always an even number.

(ii) $S^+(x)$ counts the maximum number of sign changes in the sequence $x_1, \ldots, x_n$, where zero terms are arbitrarily assigned values $+1$ or -1. The number $S_c^+(x)$ of maximum cyclic variations of sign of x is defined by

$$S_c^+(x) := \max_i S^+(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_i).$$

Let $f(x)$ be a piecewise continuous 2π-periodic function. We assume that $f(x) = [f(x+) + f(x-)]/2$ for all x and

$$S_c(f) := \sup S_c^-\left((f(x_1), \dots, f(x_m))\right), \tag{7}$$

where the supremum is taken over all $x_1 < \cdots < x_m < x_1 + 2\pi$ and all $m \in \mathbb{N}$.

Moreover, we need another notation about the count of zeros of a function. Suppose that $f(x)$ is a continuous periodic function of periodic 2π. We define

$$\widetilde{Z}_c(f) := \sup S_c^+\left((f(x_1), \dots, f(x_m))\right), \tag{8}$$

where the supremum runs over all $x_1 < \cdots < x_m < x_1 + 2\pi$ and all $m \in \mathbb{N}$.

Clearly,

$$S_c(f) \le \widetilde{Z}_c(f).$$

$S_c(f)$ denotes the number of sign changes of f on a period. $\widetilde{Z}_c(f)$ denotes the number of zeros of f on a period, where zeros that are sign changes are counted once, and zeros that are not sign changes are counted twice.

Now we introduce the standard function $\Phi_{n,r}^{\beta}$ of the comparison theorem on the class $\widetilde{H}_{\infty,\beta}^r$ and recall some of its properties.

Set

$$\Lambda_{2n} := \{\xi : \xi = (\xi_1, \dots, \xi_{2n}),\ 0 \le \xi_1 < \cdots < \xi_{2n} < 2\pi\}, \quad n \in \mathbb{N}.$$

For each $\xi \in \Lambda_{2n}$ we define

$$h_\xi(t) := (-1)^j, \quad t \in [\xi_{j-1}, \xi_j), \quad j = 1, \dots, 2n+1,$$

where $\xi_0 := 0$, $\xi_{2n+1} := 2\pi$. There is an especially important function of the above form, which is the function h_ξ for $\xi_j = (j-1)\pi/n, j = 1, \dots, 2n$. For convenience, we write it as h_n.

Let $\varphi_0(z) := \tan(\pi/4\, z)$ and

$$(f * g)(x) := \frac{1}{2\pi} \int_0^{2\pi} f(x-t) g(t)\, dt,$$

$$\Phi_{n,0}^{\beta} := \varphi_0(K_\beta * h_n), \qquad \Phi_{n,r}^{\beta} := D_r * \varphi_0(K_\beta * h_n), \quad r = 1, 2, \dots,$$

where

$$D_r(t) := 2 \sum_{k=1}^{\infty} \frac{\cos(kt - (\pi r/2))}{k^r}, \quad r = 1, 2, \dots \tag{9}$$

is the Bernoulli kernel, and

$$K_\beta(z) := 1 + 2 \sum_{k=1}^{\infty} \frac{\cos(kz)}{\cosh(k\beta)}. \tag{10}$$

Then we have (see [11])

$$\Phi_{n,r}^{\beta}(t) = \frac{\pi}{\sqrt{\lambda}\Lambda n^r} \sum_{k=0}^{\infty} \frac{\sin((2k+1)nt - \pi r/2)}{(2k+1)^r \sinh((2k+1)2n\beta)}, \quad r = 0, 1, \dots, \tag{11}$$

where

$$\lambda = 4e^{-2\beta n} \left(\frac{\sum_{k=0}^{\infty} e^{-4\beta n k(k+1)}}{1 + 2\sum_{k=1}^{\infty} e^{-4\beta n k^2}} \right)^2, \tag{12}$$

and

$$\Lambda = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}} \tag{13}$$

is the complete elliptic integral of the first kind with modulus λ. Using the representation (see [1], p. 226)

$$\mathrm{sn}(\frac{2n\Lambda}{\pi}t, \lambda) = \frac{\pi}{\lambda\Lambda}\sum_{k=0}^{\infty}\frac{\sin((2k+1)nt)}{\sinh((2k+1)2n\beta)},$$

[10] proved that

$$\Phi_{n,0}^{\beta}(t) = \sqrt{\lambda}\,\mathrm{sn}(\frac{2n\Lambda}{\pi}t, \lambda).$$

For $m \in \{0\} \cup \mathbb{N}$, put

$$\begin{aligned} t_{n,r}^{(j)} &:= \begin{cases} \frac{\pi(j-1)}{n}, & r = 2m, \\ \frac{\pi(j-1)}{n} + \frac{\pi}{2n}, & r = 2m+1, \end{cases} \\ x_{n,r}^{(j)} &:= \begin{cases} \frac{\pi(j-1)}{n} + \frac{\pi}{2n}, & r = 2m, \\ \frac{\pi(j-1)}{n}, & r = 2m+1, \end{cases} \end{aligned} \qquad j = 1, 2, \ldots, 2n. \tag{14}$$

It is easily seen that $t_{n,r}^{(j)}$, $j = 1, \ldots, 2n$, are zeros of $\Phi_{n,r}^{\beta}$ and $x_{n,r}^{(j)}$, $j = 1, \ldots, 2n$, are extremal points of $\Phi_{n,r}^{\beta}$ in $[0, 2\pi)$. Put

$$\Delta_{n,r}^{(j)} := \begin{cases} [\frac{\pi(j-1)}{n} + \frac{\pi}{2n}, \frac{\pi j}{n} + \frac{\pi}{2n}), & r = 2m, \\ [\frac{\pi(j-1)}{n}, \frac{\pi j}{n}), & r = 2m+1, \end{cases} \qquad j = 1, \ldots, 2n. \tag{15}$$

Then $\Phi_{n,r}^{\beta}$ is strictly monotonic in $\Delta_{n,r}^{(j)}$, $j = 1, \ldots, 2n$. By similar arguments as the Euler perfect splines (see [6], p.102), Osipenko (see [11]) has shown that

$$\|\Phi_{n,r}^{\beta}\|_{\infty} = \frac{\pi}{\sqrt{\lambda}\Lambda n^r}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^r \sinh((2k+1)2n\beta)}\,.$$

Now we are in position to prove the comparison theorem on the class $\in \widetilde{H}_{\infty,\beta}^{r}$.

Theorem 1 (Comparison theorem of Kolmogorov type)**.** *Let $r = 0, 1, 2, \ldots$ and $f \in \widetilde{H}_{\infty,\beta}^{r}$ be such that $\|f\|_{\infty} \le \|\Phi_{n,r}^{\beta}\|_{\infty}$ for some positive integer n and $f(a) = \Phi_{n,r}^{\beta}(\alpha)$ for some a, $\alpha \in \mathbb{R}$. Then*

$$|f'(a)| \le |{\Phi_{n,r}^{\beta}}'(\alpha)|. \tag{16}$$

Proof. Without loss of generality we may assume $a = \alpha$ and $r \in \mathbb{N}$ (the case where $r = 0$ is simpler). Assume that for some $f \in \widetilde{H}_{\infty,\beta}^{r}$, positive integer n and $\alpha \in \mathbb{R}$, we have $\|f\|_{\infty} \le \|\Phi_{n,r}^{\beta}\|_{\infty}$, $f(\alpha) = \Phi_{n,r}^{\beta}(\alpha)$, and $|f'(\alpha)| > |{\Phi_{n,r}^{\beta}}'(\alpha)|$. It follows from the definition that f' and ${\Phi_{n,r}^{\beta}}'$ are continuous. If $f(\alpha) \ne 0$, their continuity ensures the existence of $0 < \rho < 1$, $\alpha_0 \in \mathbb{R}$ such that

$$\rho f(\alpha_0) = \Phi_{n,r}^{\beta}(\alpha_0), \qquad \rho|f'(\alpha_0)| > |{\Phi_{n,r}^{\beta}}'(\alpha_0)|. \tag{17}$$

If $f(\alpha) = \Phi_{n,r}^{\beta}(\alpha) = 0$, we can choose some ρ satisfying $1 > \rho > \frac{|\Phi_{n,r}^{\beta}{}'(\alpha)|}{|f'(\alpha)|}$ and let $\alpha_0 = \alpha$. Then (17) also holds. Put $\overline{f} := \rho f$, then $\overline{f} \in \widetilde{H}_{\infty,\beta}^{r}$ and $\|\overline{f}\|_\infty < \|\Phi_{n,r}^{\beta}\|_\infty$. It is enough to consider one possible case $\overline{f}(\alpha_0) = \Phi_{n,r}^{\beta}(\alpha_0) \geq 0$, $\overline{f}'(\alpha_0) > \Phi_{n,r}^{\beta}{}'(\alpha_0) > 0$ (the other cases can be treated by the same way). Let $\Delta_{n,r}^{(k)}$ be the interval that contains α_0. By geometrical consideration we see that on $\Delta_{n,r}^{(k)}$ the graphs of $\overline{f}$ and $\Phi_{n,r}^{\beta}$ intersect at least three times, while on each of the other intervals $\Delta_{n,r}^{(j)}(j \in \{1,\dots,2n\}\backslash\{k\})$ these graphs intersect at least once. Hence for $F(x) := \Phi_{n,r}^{\beta}(x) - \rho f(x)$, we have $S_c(F) \geq 2n+2$. By Rolle's theorem, $S_c(F^{(r)}) \geq 2n+2$, where

$$F^{(r)}(x) = \sqrt{\lambda}\mathrm{sn}(\frac{2n\Lambda}{\pi}x, \lambda) - \rho f^{(r)}(x).$$

Denote by $H_\infty(\Delta_\beta)$ the set of functions that are analytic on the annulus

$$\Delta_\beta := \{z \in \mathbb{C} : e^{-\beta} < |z| < e^{\beta}\}$$

and satisfy the condition $|g(z)| \leq 1,\ z \in \Delta_\beta$. If $g(z) \in \widetilde{H}_{\infty,\beta}$, then $g((1/i)\ln z) \in H_\infty(\Delta_\beta)$. We know that

$$G(z) := \sqrt{\lambda}\,\mathrm{sn}(\frac{2n\Lambda}{\pi i}\ln z, \lambda) \tag{18}$$

has exactly $2n$ zeros: $\cos\frac{\pi(j-1)}{n} + i\sin\frac{\pi(j-1)}{n}$, $j = 1,\dots,2n$ on Δ_β. Since on $\partial\Delta_\beta$

$$|G(z) - F^{(r)}(\frac{1}{i}\ln z)| = |\rho f^{(r)}(\frac{1}{i}\ln z)| \leq \rho < 1 \equiv |G(z)|,$$

Rouche's theorem implies that $F^{(r)}$ has at most $2n$ zeros on a period 2π. So $S_c(F^{(r)}) \leq 2n$. This is a contradiction. Theorem 1 is proved. □

Remark 1. Among others, Sun [17] established a comparison theorem of Kolmogorov type for the class $\widetilde{h}_{\infty,\beta}$. Moreover, Fisher [3] demonstrated an inequality of Landau–Kolmogorov type for the class of nonperiodic analytic functions on the open unit disk whose rth derivatives are bounded by 1 and which are real-valued on the interval $(-1,1)$.

Our next aim is to establish a Landau–Kolmogorov type inequality and some other corollaries of the comparison theorem which will be used for establishing an inequality of Taikov type on the class $\widetilde{H}_{\infty,\beta}^{r}$ in the following section.

Corollary 1 (Landau–Kolmogorov type inequality)**.** *Let* $r = 0,1,2,\dots$ *and* $f \in \widetilde{H}_{\infty,\beta}^{r}$ *be such that* $\|f\|_\infty \leq \|\Phi_{n,r}^{\beta}\|_\infty$ *for some positive integer* n*. Then*

$$\|f'\|_\infty \leq \|\Phi_{n,r}^{\beta}{}'\|_\infty. \tag{19}$$

Proof. Assume that there exists a $\xi \in \mathbb{R}$ for which $|f'(\xi)| > \|\Phi_{n,r}^{\beta}{}'\|_\infty$. Since $\Phi_{n,r}^{\beta}(t)$ is a continuous function of t and

$$\|f\|_\infty \leq \|\Phi_{n,r}^{\beta}\|_\infty, \qquad -\|\Phi_{n,r}^{\beta}\|_\infty \leq \Phi_{n,r}^{\beta}(t) \leq \|\Phi_{n,r}^{\beta}\|_\infty, \quad t \in \mathbb{R},$$

there exists an $\eta \in \mathbb{R}$ such that $f(\xi) = \Phi_{n,r}^{\beta}(\eta)$ and $|f'(\xi)| > |\Phi_{n,r}^{\beta}{}'(\eta)|$, which contradicts Theorem 1. Corollary 1 is proved.

Let $C(\mathbb{R})$ be the set of all continuous functions on the real line $\mathbb{R}$. A function $\psi \in C(\mathbb{R})$ is said to be regular (see [6], p. 107) if it has a period $2l$ and if on some interval $(a, a+2l)$ (a is a point of an absolute extremum of ψ) there is a point c

such that ψ is strictly monotone on (a, c) and $(c, a+2l)$. In order to emphasize the length of the period $2l$, sometimes we shall speak about ψ as being $2l$-regular.

We say that a function $f \in C(\mathbb{R})$ possesses a μ-property with respect to a regular function ψ if for every $\alpha \in \mathbb{R}$ and on every interval of monotonicity of ψ the difference $\psi(t) - f(t+\alpha)$ either does not change sign or changes sign exactly once: from $+$ to $-$ if ψ decreases or from $-$ to $+$ if ψ increases. It is clear that f will possess the μ-property with respect to $\psi(t+\beta)$, $\beta \in \mathbb{R}$, if it possesses the μ-property with respect to ψ. □

Corollary 2. *Let $r = 0, 1, 2, \ldots,$ and let $f \in \widetilde{H}^r_{\infty,\beta}$ be such that $\|f\|_\infty \le \|\Phi^\beta_{n,r}\|_\infty$ for some positive integer n. Then the function f possesses a μ-property with respect to $\Phi^\beta_{n,r}$.*

Proof. If f does not possess a μ-property with respect to $\Phi^\beta_{n,r}$, then for some $\alpha \in \mathbb{R}$ the difference $\Phi^\beta_{n,r}(t) - f(t+\alpha)$ changes sign on a monotonic interval $\Delta^{(k)}_{n,r}$, $k \in \{1, \ldots, 2n\}$ of the standard function $\Phi^\beta_{n,r}$ from $+$ to $-$ if $\Phi^\beta_{n,r}$ increases and from $-$ to $+$ if $\Phi^\beta_{n,r}$ decreases. By continuity arguments this fact will also hold for the difference

$$F(t) := \Phi^\beta_{n,r}(t) - (1-\epsilon) f(t+\alpha)$$

for $0 < \epsilon < 1$ sufficiently small. Since $(1-\epsilon)\|f(\cdot+\alpha)\|_\infty < \|\Phi^\beta_{n,r}\|_\infty$, F changes sign at least three times on the interval $\Delta^{(k)}_{n,r}$ and at least once on each of the other intervals $\Delta^{(j)}_{n,r}$ $(j \in \{1, \ldots, 2n\}\backslash\{k\})$. Hence $S_c(F) \ge 2n+2$. By Rolle's theorem, $S_c(F^{(r)}) \ge 2n+2$, where

$$F^{(r)}(t) = \sqrt{\lambda}\,\mathrm{sn}(\frac{2n\Lambda}{\pi} t, \lambda) - (1-\epsilon) f^{(r)}(t+\alpha).$$

On the other hand we can prove that $S_c(F^{(r)}) \le 2n$ by the same method as that in the proof of Theorem 1. This contradiction proves the corollary. □

By virtue of Corollary 2 and [6, Lemma 3.2.1], we have

Corollary 3. *Let $r = 0, 1, 2, \ldots,$ and let $f \in \widetilde{H}^r_{\infty,\beta}$ be such that $\|f\|_\infty \le \|\Phi^\beta_{n,r}\|_\infty$ for some positive integer n and $f(\xi_0) = \Phi^\beta_{n,r}(\eta_0)$ for some $\xi_0 \in \mathbb{R}$, $\eta_0 \in \Delta^{(k)}_{n,r}$.*

(i) *If the standard function $\Phi^\beta_{n,r}$ increases on the interval $\Delta^{(k)}_{n,r}$, then*

$$\begin{aligned} &f(\xi_0+u) \le \Phi^\beta_{n,r}(\eta_0+u), && \text{for all } u \ge 0, \text{ and } \eta_0+u \in \Delta^{(k)}_{n,r} \textstyle\bigcap [\eta_0, \infty), \\ &f(\xi_0-\mu) \ge \Phi^\beta_{n,r}(\eta_0-\mu), && \text{for all } u \ge 0, \text{ and } \eta_0-u \in \Delta^{(k)}_{n,r} \textstyle\bigcap (-\infty, \eta_0]. \end{aligned}$$

(ii) *If the standard function $\Phi^\beta_{n,r}$ decreases on the interval $\Delta^{(k)}_{n,r}$, then the inequalities in* (i) *are true with opposite sign.*

It follows from Corollary 1, Corollary 2, [6, Proposition 3.2.3], and the symmetry of the graph of $\Phi^\beta_{n,r}$ that

Corollary 4. *Let $r = 0, 1, 2, \ldots,$ and let $f \in \widetilde{H}^r_{\infty,\beta}$ be such that $\|f\|_\infty \le \|\Phi^\beta_{n,r}\|_\infty$ for some positive integer n. If $f(\xi_0) = \Phi^\beta_{n,r}(\eta_0), f(\xi_1) = \Phi^\beta_{n,r}(\eta_1)$, ξ_0, η_0, ξ_1, $\eta_1 \in \mathbb{R}$, and η_0, η_1 are contained in the same monotonic interval $\Delta^{(k)}_{n,r}$, then*

$$|\xi_0 - \xi_1| \ge |\eta_0 - \eta_1|. \tag{20}$$

A function F is said to be a periodic integral of a real 2π-periodic continuous function f on $\mathbb{R}$ if $F'(x) = f(x)$ and $F(x+2\pi) = F(x)$ for all $x \in \mathbb{R}$. From Corollary 1, we have immediately the following theorem which will be used in the next section.

Theorem 2. *Suppose that $r = 0, 1, 2, \ldots$, $f \in \widetilde{H}^r_{\infty,\beta}$, and F is a periodic integral of f such that $\|F\|_\infty \le \|\Phi^\beta_{n,r+1}\|_\infty$ for some positive integer n. Then*

$$\|f\|_\infty \le \|\Phi^\beta_{n,r}\|_\infty. \tag{21}$$

3. Taikov type inequalities on $\widetilde{H}^r_{\infty,\beta}$

In this section we establish an inequality of Taikov type, which leads to the upper estimates of the Gel′fand n-widths. To do this, we need some auxiliary lemmas.

Lemma 1. *Let $n \in \mathbb{N}$ and $r = 0, 1, \ldots$. Then*

$$\int_0^{2\pi} |\Phi^\beta_{n,r}(t)|\,dt = \bigvee_0^{2\pi} \Phi^\beta_{n,r+1} = 4n\|\Phi^\beta_{n,r+1}\|_\infty. \tag{22}$$

Proof. By (11), $\Phi^\beta_{n,r}$ is $2\pi/n$-periodic, is strictly monotonic in $\Delta^{(j)}_{n,r}$, $j = 1, \ldots, 2n$, and $\Phi^\beta_{n,r}(t + \frac{\pi}{n}) = -\Phi^\beta_{n,r}(t)$ for all $t \in [0, 2\pi)$. First we assume that $r+1$ is odd. Then it follows from (14) that $\left|\Phi^\beta_{n,r+1}(\pi/n) - \Phi^\beta_{n,r+1}(0)\right| = 2\|\Phi^\beta_{n,r+1}\|_\infty$. Hence,

$$\begin{aligned}\int_0^{\frac{2\pi}{n}} |\Phi^\beta_{n,r}(t)|\,dt &= 2\int_0^{\frac{\pi}{n}} |\Phi^\beta_{n,r}(t)|\,dt = 2\left|\int_0^{\frac{\pi}{n}} \Phi^\beta_{n,r}(t)\,dt\right| \\ &= 2\left|\Phi^\beta_{n,r+1}(\frac{\pi}{n}) - \Phi^\beta_{n,r+1}(0)\right| = 4\|\Phi^\beta_{n,r+1}\|_\infty \\ &= 2\bigvee_0^{\frac{\pi}{n}} \Phi^\beta_{n,r+1} = \bigvee_0^{\frac{2\pi}{n}} \Phi^\beta_{n,r+1}.\end{aligned}$$

Consequently,

$$\int_0^{2\pi} |\Phi^\beta_{n,r}(t)|\,dt = \bigvee_0^{2\pi} \Phi^\beta_{n,r+1} = 4n\|\Phi^\beta_{n,r+1}\|_\infty.$$

By the same method as above, we can prove that (22) also holds if $r+1$ is even. The details are omitted. The proof of Lemma 1 is complete. □

Now let f be a 2π-periodic integrable function, and denote by $r(f,t)$ the non-increasing rearrangement of $|f|$ (see [6], p. 110). With this notation we have

Lemma 2 (see [6], p.112). *Let f, $g \in L_q$ $(1 \le q < \infty)$, and*

$$\int_0^x r(f,t)\,dt \le \int_0^x r(g,t)\,dt, \quad 0 \le x \le 2\pi.$$

Then

$$\|f\|_q \le \|g\|_q.$$

Lemma 3 (see [6], p. 114). *Let $f \in C(\mathbb{R})$ possess the μ-property with respect to the $2\pi/n$-regular $(n = 1, 2, \ldots)$ function ψ and*

$$\int_0^{2\pi/n} \psi(t)\,dt = 0.$$

If

$$\min_u \psi(u) \le f(t) \le \max_u \psi(u), \quad \forall\, t \in \mathbb{R},$$
$$\max_{a,b} \left| \int_a^b f(t)\, dt \right| \le \frac{1}{2} \int_0^{2\pi/n} |\psi(t)|\, dt,$$

then

$$\int_0^x r(f,t)\, dt \le \int_0^x r(\psi,t)\, dt, \quad 0 \le x \le 2\pi.$$

Theorem 3. *Let $r = 0, 1, 2, \dots,$ $f \in \widetilde{H}^r_{\infty,\beta}$, and F be a periodic integral of f such that $\|F\|_\infty \le \|\Phi^\beta_{n,r+1}\|_\infty$ for some positive integer n. Then*

$$\int_0^x r(f,t)\, dt \le \int_0^x r(\Phi^\beta_{n,r}, t)\, dt, \quad 0 \le x \le 2\pi, \tag{23}$$

$$\|f\|_q \le \|\Phi^\beta_{n,r}\|_q, \quad 1 \le q < \infty. \tag{24}$$

Proof. By virtue of Lemma 2, the inequality (24) follows from (23). So we only need to prove the inequality (23). Without loss of generality we assume that $r \in \mathbb{N}$. Since $f \in \widetilde{H}^r_{\infty,\beta}$ and F is a periodic integral of f such that $\|F\|_\infty \le \|\Phi^\beta_{n,r+1}\|_\infty$ for some positive integer n, we conclude from Theorem 2 that $\|f\|_\infty \le \|\Phi^\beta_{n,r}\|_\infty$, and it follows from Corollary 2 that f possesses a μ-property with respect to $\Phi^\beta_{n,r}$. By noticing that $\int_0^{2\pi/n} \Phi^\beta_{n,r}(t)\, dt = 0$ and the following result derived from the proof of Lemma 1,

$$\max_{a,b} |\int_a^b f(t)dt| \le 2\|F\|_\infty \le 2\|\Phi^\beta_{n,r+1}\|_\infty$$
$$= \int_0^{\pi/n} |\Phi^\beta_{n,r}(t)|\, dt = \frac{1}{2} \int_0^{2\pi/n} |\Phi^\beta_{n,r}(t)|\, dt,$$

we see that Lemma 3 is applicable for the functions f and $\Phi^\beta_{n,r}$. Hence, the inequality (23) is true. Theorem 3 is proved. □

Consider a subset of $\widetilde{H}^r_{\infty,\beta}$ defined by

$$\widetilde{H}^r_{\infty,\beta} \bigcap \mathcal{T}_n^\perp := \{ f \in \widetilde{H}^r_{\infty,\beta} : f \perp \mathcal{T}_n \}, \tag{25}$$

where the condition $f \perp \mathcal{T}_n$ means that

$$\left.\begin{aligned} &\int_0^{2\pi} f(t) \sin(kt)\, dt = 0, \\ &\int_0^{2\pi} f(t) \cos(kt)\, dt = 0, \end{aligned}\right. \qquad k = 0, 1, \dots, n-1.$$

Let F be the periodic integral of $f \in \widetilde{H}^r_{\infty,\beta} \bigcap \mathcal{T}_n^\perp$ and satisfy the condition $\int_0^{2\pi} F(t)dt = 0$. Then we have

$$\int_0^{2\pi} F(t) \sin(kt) dt = 0, \qquad \int_0^{2\pi} F(t) \cos(kt) dt = 0, \quad k = 0, 1, \dots, n-1.$$

It was proved by Osipenko and Wilderotter in [13] that

$$\|F\|_\infty \le \|\Phi^\beta_{n,r+1}\|_\infty, \tag{26}$$

which together with Theorem 3 gives an inequality of Taikov type as follows.

Theorem 4 (Taikov type inequality)**.** *Let* $n\in\mathbb{N}$, $r=0,1,2,\dots,$ *and* $f\in\widetilde{H}^r_{\infty,\beta}\bigcap\mathcal{T}_n^{\perp}$. *Then*

$$\|f\|_q \le \|\Phi^{\beta}_{n,r}\|_q, \quad 1\le q<\infty.$$

Since $\Phi^{\beta}_{n,r}$ is $2\pi/n$-periodic and $\Phi^{\beta}_{n,r}\perp 1$, $\Phi^{\beta}_{n,r}\in\widetilde{H}^r_{\infty,\beta}\bigcap\mathcal{T}_n^{\perp}$. From Theorem 4 and (26), we obtain

Corollary 5. *Let* $n\in\mathbb{N}$ *and* $r=0,1,2,\dots.$ *Then*

$$\sup_{f\in\widetilde{H}^r_{\infty,\beta}\bigcap\mathcal{T}_n^{\perp}} \|f\|_q = \|\Phi^{\beta}_{n,r}\|_q, \quad 1\le q<\infty.$$

Remark 2. Among others, Taikov [18] proved that for all $n\in\mathbb{N}$ and $r=1,2,\dots,$

$$\sup_{f\in\widetilde{W}^r_{\infty}\bigcap\mathcal{T}_n^{\perp}} \|f\|_q \le \|\phi_{n,r}\|_q, \quad 1\le q<\infty, \tag{27}$$

where $\widetilde{W}^r_{\infty}$ is the classical Sobolev class, and

$$\phi_{n,r}(t)=\frac{4}{\pi n^r}\sum_{k=0}^{\infty}\frac{\sin((2k+1)nt-\pi r/2)}{(2k+1)^{r+1}}, \quad r=0,1,\dots,$$

is the $2\pi/n$-periodic perfect Euler spline of degree r. The inequality (27) is now referred to as Taikov's inequality. For the case $q=1$, (27) was also proved by Turovets in another way. For more detail, see [6, page 172 and page 207].

4. Exact values of n-widths of $\widetilde{H}^r_{\infty,\beta}$

In this section, we will solve a minimum norm question on an analogue of the classical polynomial perfect splines, and then using this result, we will estimate the lower bounds of n-widths, which together with some results of Osipenko (see [14]) and Section 3 will determine the exact values of the Gel′fand n-width, Kolmogorov $2n$-width, linear $2n$-width, and information $2n$-width of $\widetilde{H}^r_{\infty,\beta}$ in L_q, $r\in\mathbb{N}$, $1\le q<\infty$. First we recall some definitions.

Definition 2. [15] The kernel k is called strictly sign consistent of order $2\ell+1$ ($SSC_{2\ell+1}$) if

$$\sigma\det\left(k(x_j-y_m)\right)^{2\ell+1}_{j,m=1}>0$$

whenever $0\le x_1<\cdots<x_{2\ell+1}$, $0\le y_1<\cdots<y_{2\ell+1}$, and $\sigma=1$ or -1.

With this definition, we have the following two results which will be used in the proof of the next theorem.

Lemma 4 (see [5], p. 250–270 and [15])**.** *If k is $SSC_{2\ell+1}$, h is a real 2π-periodic piecewise continuous function and $S_c(h)\le 2n$, then*

$$\widetilde{Z}_c(k*h)\le 2n,$$

where $\widetilde{Z}_c(k*h)$ is defined by (8)*.*

Corollary 6. *Let $n\in\mathbb{N}$, h is a real 2π-periodic piecewise continuous function and $S_c(h)\le 2n$. Then*

$$S_c(K_\beta*h)\le\widetilde{Z}_c(K_\beta*h)\le 2n. \tag{28}$$

Proof. It was proved by Forst [4] that K_β is $SSC_{2\ell+1}$ for all $\ell=0,1,\dots.$ By Lemma 4, we get (28). Corollary 6 is proved. □

Lemma 5 ([15])**.** *Let* $\xi = (\xi_1, \dots, \xi_{2n})$ *and* $\eta = (\eta_1, \dots, \eta_{2n}) \in \Lambda_{2n}$*. Then*

$$S_c(h_\xi \pm h_\eta) \le 2n.$$

Moreover, if $\xi_k = \eta_{k+2\ell}$ *for some* k *and* ℓ*, then*

$$S_c(h_\xi - h_\eta) \le 2(n-1).$$

Armed with these preparations, we are now ready to investigate a problem about the minimum norm on analogue of the polynomial perfect splines which is the key for getting the lower estimates of the Gel′fand n-width, Kolmogorov $2n$-width, linear $2n$-width, and information $2n$-width of $\widetilde{H}^r_{\infty,\beta}$ in L_q, $r \in \mathbb{N}$, $1 \le q < \infty$.

Theorem 5. *Let* $n \in \mathbb{N}$*,* $r = 0, 1, 2, \dots$*, and* φ *be a differentiable, odd, and strictly increasing function for which* φ' *is continuous on* $[-1,1]$*. Put*

$$\overline{\Lambda}_{2n} := \{\xi : \xi \in \Lambda_{2m},\, m \in \mathbb{N},\, m \le n\},$$
$$\overline{\Lambda}^{\varphi}_{2n} := \{\xi : \xi \in \overline{\Lambda}_{2n},\, \varphi(K_\beta * h_\xi) \perp 1\}.$$

Then

$$\inf_{a\in\mathbb{R},\ \xi\in\overline{\Lambda}^{\varphi}_{2n}} \|a + D_r * \varphi(K_\beta * h_\xi)\|_q = \min_{m\in\mathbb{N},\, m\le n} \|D_r * \varphi(K_\beta * h_m)\|_q, \quad 1 \le q \le \infty, \tag{29}$$

where when $r = 0$ (29) *means that*

$$\inf_{a\in\mathbb{R},\ \xi\in\overline{\Lambda}^{\varphi}_{2n}} \|a + \varphi(K_\beta * h_\xi)\|_q = \min_{m\in\mathbb{N},\, m\le n} \|\varphi(K_\beta * h_m)\|_q, \quad 1 \le q \le \infty.$$

Proof. The case $q = \infty$ was proved by Osipenko in [11] (in this case it is enough to assume φ is a continuous, odd, and strictly increasing function defined on $[-1,1]$).

Now we assume that $1 \le q < \infty$, $r \in \mathbb{N}$ (the case $r = 0$ is simpler), and follow the approach of Zensykbaev [20], Micchelli and Pinkus [8], and Pinkus [15].

A compactness argument shows that the minimum in (29) is attained; i.e., there exists an $a^\star \in \mathbb{R}$ and a $\xi^\star \in \overline{\Lambda}^{\varphi}_{2n}$, $\xi^\star = (\xi^\star_1, \dots, \xi^\star_{2m})$, $m \le n$, such that

$$\inf_{a\in\mathbb{R},\ \xi\in\overline{\Lambda}^{\varphi}_{2n}} \|a + D_r * \varphi(K_\beta * h_\xi)\|_q^q = \|a^\star + D_r * \varphi(K_\beta * h_{\xi^\star})\|_q^q.$$

Using the method of the Lagrange multiplier, we find that the optimal $a^\star$ and $\xi^\star = (\xi^\star_1, \dots, \xi^\star_{2m})$ must satisfy the system of nonlinear equations:

$$\int_0^{2\pi} f(x)\, dx = 0, \tag{30}$$

$$\begin{aligned} &\tfrac{(-1)^j}{2\pi^2} \int_0^{2\pi} f(x) \left[\int_0^{2\pi} D_r(t) \varphi'((K_\beta * h_{\xi^\star})(x-t)) K_\beta(x - t - \xi^\star_j)\, dt \right] dx \\ &+ \tfrac{\theta(-1)^j}{\pi} \int_0^{2\pi} \varphi'((K_\beta * h_{\xi^\star})(t)) K_\beta(t - \xi^\star_j)\, dt = 0, \quad j = 1, \dots, 2m, \end{aligned} \tag{31}$$

$$\int_0^{2\pi} \varphi((K_\beta * h_{\xi^\star})(t))\, dt = 0,$$

where θ is the Lagrange multiplier, and

$$f(x) := q|a^\star + D_r * \varphi((K_\beta * h_{\xi^\star})(x))|^{q-1} \mathrm{sgn}[a^\star + D_r * \varphi((K_\beta * h_{\xi^\star})(x))]. \tag{32}$$

Since $\varphi' \geq 0$ and φ' is continuous on $[-1,1]$, by Rolle's theorem and Corollary 6, we get

$$\begin{aligned} S_c(f) &= S_c\left(a^\star + D_r * \varphi(K_\beta * h_{\xi^\star})\right) \leq S_c(\varphi(K_\beta * h_{\xi^\star})) \\ &\leq S_c(K_\beta * h_{\xi^\star}) \leq S_c(h_{\xi^\star}) \leq 2m. \end{aligned} \tag{33}$$

First we claim that the knots of the vector $\xi^\star$ are equidistant which means that $\xi^\star_{j+1} - \xi^\star_j = \pi/m$, $j = 1, \ldots, 2m$, i.e., $h_{\xi^\star} = h_m$. By translation we may assume that $\xi^\star = (\xi^\star_1, \ldots, \xi^\star_{2m})$ satisfies $0 = \xi^\star_1 < \xi^\star_2 < \cdots < \xi^\star_{2m} < 2\pi$ and

$$\delta = \xi^\star_2 = \xi^\star_2 - \xi^\star_1 = \min\{\xi^\star_{i+1} - \xi^\star_i : i = 1, \ldots, 2m\},$$

where $\xi^\star_{2m+1} = 2\pi$. Assume that $h_{\xi^\star} \neq h_m$. It follows from Lemma 5 that

$$S_c(h_{\xi^\star}(\cdot) + h_{\xi^\star}(\cdot + \delta)) \leq 2(m-1).$$

Since φ is a continuous, odd, and strictly increasing function, by Corollary 6, we have

$$\begin{aligned} &S_c(\varphi((K_\beta * h_{\xi^\star})(\cdot)) + \varphi((K_\beta * h_{\xi^\star})(\cdot + \delta))) \\ &\quad = S_c((K_\beta * h_{\xi^\star})(\cdot) + (K_\beta * h_{\xi^\star})(\cdot + \delta)) \\ &\quad \leq S_c(h_{\xi^\star}(\cdot) + h_{\xi^\star}(\cdot + \delta)) \leq 2(m-1). \end{aligned} \tag{34}$$

Set

$$\begin{aligned} p(x) &= a^\star + D_r * \varphi((K_\beta * h_{\xi^\star})(x)), \\ r(x) &= p(x+\delta) = a^\star + D_r * \varphi((K_\beta * h_{\xi^\star})(x+\delta)). \end{aligned}$$

Thus

$$p(x) + r(x) = 2a^\star + \frac{1}{2\pi}\int_0^{2\pi} D_r(x-y)[\varphi((K_\beta * h_{\xi^\star})(y)) + \varphi((K_\beta * h_{\xi^\star})(y+\delta))]\,dy.$$

From Rolle's theorem and (34), it follows that $S_c(p+r) \leq 2(m-1)$. Since

$$\operatorname{sgn}(a+b) = \operatorname{sgn}\left(|a|^{q-1}\operatorname{sgn} a + |b|^{q-1}\operatorname{sgn} b\right),$$

for every $a,b \in \mathbb{R}$ and $q \in [1,\infty)$, it follows that

$$\operatorname{sgn}(|p(\cdot)|^{q-1}\operatorname{sgn}(p(\cdot)) + |r(\cdot)|^{q-1}\operatorname{sgn}(r(\cdot))) \leq 2(m-1)$$

for each $q \in [1,\infty)$. From (32),

$$\begin{aligned} f(x) &= q|a^\star + D_r * \varphi((K_\beta * h_{\xi^\star})(x))|^{q-1}\operatorname{sgn}[a^\star + D_r * \varphi((K_\beta * h_{\xi^\star})(x))] \\ &= q|p(x)|^{q-1}\operatorname{sgn}(p(x)). \end{aligned}$$

Therefore, we have

$$S_c\left(f(\cdot) + f(\cdot+\delta)\right) = S_c(p+r) \leq 2(m-1).$$

Set

$$P(y) = \frac{1}{2\pi}\int_0^{2\pi} f(x)G(x;y)\,dx + \theta\int_0^{2\pi} \varphi'((K_\beta * h_{\xi^\star})(t))K_\beta(t-y)\,dt$$

and

$$R(y) = \frac{1}{2\pi}\int_0^{2\pi} f(x+\delta)G(x;y)\,dx + \theta\int_0^{2\pi} \varphi'((K_\beta * h_{\xi^\star})(t+\delta))K_\beta(t-y)\,dt,$$

where

$$G(x;y) = \int_0^{2\pi} D_r(t)\varphi'((K_\beta * h_{\xi^\star})(x-t))K_\beta(x-t-y)\,dt.$$

By change of scale and Fubini's theorem we have

$$\begin{aligned}
P(y) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\left[\int_0^{2\pi} D_r(x-t)\varphi'((K_\beta * h_{\xi^\star})(t))K_\beta(t-y)\,dt\right]dx \\
&\quad +\ \theta\int_0^{2\pi}\varphi'((K_\beta * h_{\xi^\star})(t))K_\beta(t-y)\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\varphi'((K_\beta * h_{\xi^\star})(t))K_\beta(t-y)\left[\int_0^{2\pi} D_r(x-t)f(x)\,dx\right]dt \\
&\quad +\theta\int_0^{2\pi}\varphi'((K_\beta * h_{\xi^\star})(t))K_\beta(t-y)\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\varphi'((K_\beta * h_{\xi^\star})(t))f_r(t)K_\beta(t-y)\,dt \\
&\quad +\theta\int_0^{2\pi}\varphi'((K_\beta * h_{\xi^\star})(t))K_\beta(t-y)\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\varphi'((K_\beta * h_{\xi^\star})(t))K_\beta(t-y)\left(f_r(t)+2\pi\theta\right)\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\varphi'((K_\beta * h_{\xi^\star})(t+y))K_\beta(t)\left(f_r(t+y)+2\pi\theta\right)\,dt,
\end{aligned}$$

where

$$f_r(t) = \int_0^{2\pi} D_r(x-t)f(x)\,dx.$$

Since $\varphi' \geq 0$ and φ' is continuous on $[-1,1]$, by Rolle's theorem, Corollary 6, and (33), we conclude that

(35)

$$\begin{aligned}
\widetilde{Z}_c(P(\cdot)+R(\cdot)) &\leq S_c\left(\varphi'((K_\beta * h_{\xi^\star})(\cdot))(f_r(\cdot)+f_r(\cdot+\delta)+4\pi\theta)\right) \\
&\leq S_c(f_r(\cdot)+f_r(\cdot+\delta)+4\pi\theta) \leq S_c(f(\cdot)+f(\cdot+\delta)) \leq 2(m-1).
\end{aligned}$$

A simple change of variable argument shows that $R(y) = P(y+\delta)$, from which we obtain

$$\widetilde{Z}_c(P(\cdot)+P(\cdot+\delta)) \leq 2(m-1).$$

From (31) we have

$$P(\xi_i^\star) = 0, \quad i = 1,\ldots,2m.$$

By our choice of δ, $\xi_i^\star < \xi_i^\star + \delta \leq \xi_{i+1}^\star$, $i = 1,\ldots,2m$, and therefore

$$S_c^+\left(P(\xi_1^\star+\delta),\ldots,P(\xi_{2n}^\star+\delta)\right) = 2m.$$

Thus

$$S_c^+\left(P(\xi_1^\star)+P(\xi_1^\star+\delta),\ldots,P(\xi_{2n}^\star)+P(\xi_{2n}^\star+\delta)\right) = 2m,$$

which implies that

$$\widetilde{Z}_c\left(P(\cdot)+P(\cdot+\delta)\right) \geq 2m.$$

This is a contradiction, and therefore $h_{\xi^\star}(\cdot) = -h_{\xi^\star}(\cdot+\delta)$, i.e., $h_{\xi^\star} = h_m$.

Now we proceed to show that $a^\star = 0$. Let

$$f(x;a) := q|a + D_r * \varphi((K_\beta * h_m)(x))|^{q-1}\mathrm{sgn}[a + D_r * \varphi((K_\beta * h_m)(x))].$$

Since the constant term is a free variable, $f(\cdot;a^\star) \perp 1$. Because

$$D_r * \varphi((K_\beta * h_m)(x+\frac{\pi}{m})) = -D_r * \varphi((K_\beta * h_m)(x)),$$

it follows that $f(\cdot;0) \perp 1$. Thus $a^\star = 0$. Since $h_m \in \{h_\xi : \xi \in \overline{\Lambda}_{2n}^{\varphi}\}$ for all $m \in \mathbb{N}$ satisfying $m \le n$, we get (29). $\square$

Corollary 7. *Let $n \in \mathbb{N}$ and $r = 0, 1, 2, \ldots$. Then*

$$\inf_{a\in\mathbb{R},\ \xi\in\overline{\Lambda}_{2n}^{\varphi_0}} \|a + D_r * \varphi_0(K_\beta * h_\xi)\|_q = \|D_r * \varphi_0(K_\beta * h_n)\|_q = \|\Phi_{n,r}^\beta\|_q, \quad 1 \le q < \infty. \tag{36}$$

Proof. Noting that $\varphi_0(z) := \tan(\pi/4\, z)$ satisfies the conditions of φ in Theorem 5, by (29) we only need to prove that

$$\min_{m\in\mathbb{N},\, m\le n} \|D_r * \varphi_0(K_\beta * h_m)\|_q = \|D_r * \varphi_0(K_\beta * h_n)\|_q, \quad 1 \le q < \infty.$$

In fact, since $D_r * \varphi_0(K_\beta * h_n) \in \widetilde{H}^r_{\infty,\beta} \bigcap \mathcal{T}_m^\perp$ for all $m \in \mathbb{N}$ satisfying $m \le n$, by Theorem 4 we have

$$\|D_r * \varphi_0(K_\beta * h_n)\|_q \le \|D_r * \varphi_0(K_\beta * h_m)\|_q, \quad 1 \le q < \infty.$$

Corollary 7 is proved. $\square$

Our next aim is to study the exact estimates of the Gel′fand n-widths of the class of functions $\widetilde{H}^r_{\infty,\beta}$.

Theorem 6. *For $n \in \mathbb{N}$, $r = 0, 1, 2, \ldots,$ and $1 \le q \le \infty$,*

$$d^{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) = d^{2n-1}(\widetilde{H}^r_{\infty,\beta}, L_q) = \|\Phi^\beta_{n,r}\|_q. \tag{37}$$

Proof. The case of $q = \infty$, where r is a nonnegative integer and $n \in \mathbb{N}$, and the case of $1 \le q < \infty$, where $r = 0$ and n is even, were proved in [9] and [13]. We consider the case $r = 1, 2, \ldots$ (the case $r = 0$ is simpler), and $1 \le q < \infty$. We begin with the lower estimate. Set

$$S^{2n} := \left\{ x = (x_1, \ldots, x_{2n+1}) \in \mathbb{R}^{2n+1} : \sum_{k=1}^{2n+1} |x_k| = 2\pi \right\}, \tag{38}$$

$$\tau_0(x) := 0, \qquad \tau_j(x) := \sum_{k=1}^{j} |x_k|, \quad j = 1, \ldots, 2n+1.$$

For each $x \in S^{2n}$, put

$$g_x(t) := \operatorname{sgn} x_j, \quad \tau_{j-1}(x) \le t < \tau_j(x), \quad j = 1, \ldots, 2n+1, \tag{39}$$

$$f_x := D_r * \varphi_0(K_\beta * g_x). \tag{40}$$

Suppose that

$$X^{2n} := \{ f \in L_q : \langle l_j,\, f\rangle = 0,\, l_j \in {L_q}^*,\, j = 1, \ldots, 2n\}.$$

If $\langle l_j,\, 1\rangle = 0, j = 1, \ldots, 2n$, then

$$\sup_{f\in\widetilde{H}^r_{\infty,\beta}\cap X^{2n}} \|f\|_q = \infty.$$

Therefore, we only need to consider the subspace X^{2n} such that there exists a $j_0 \in \{1, \ldots, 2n\}$ for which $\langle l_{j_0}, 1\rangle \neq 0$. Then without loss of generality we may assume that $\langle l_1, 1\rangle \neq 0$. Set

$$L_j := l_j - \frac{\langle l_j, 1\rangle}{\langle l_1, 1\rangle} l_1, \quad j = 2, \ldots, 2n.$$

For each $x \in S^{2n}$ denote by A_1 the mapping

$$A_1(x) := (b(x), \langle L_2, f_x\rangle, \ldots, \langle L_{2n}, f_x\rangle),$$

where

$$b(x) := \int_0^{2\pi} \varphi_0((K_\beta * g_x)(t))dt.$$

Since A_1 is an odd and continuous map of S^{2n} into $\mathbb{R}^{2n}$, by Borsuk's theorem (see [6, p. 91]), there exists an $x^\star \in S^{2n}$ for which $A_1(x^\star) = 0$. Then $g_{x^\star} \in \{h_\xi : \xi \in \overline{\Lambda}_{2n}^{\varphi_0}\}$, where $\overline{\Lambda}_{2n}^{\varphi_0} := \{\xi \in \overline{\Lambda}_{2n} : \varphi_0(K_\beta * h_\xi) \perp 1\}$, and $\langle L_i, f_{x^\star}\rangle = 0$, $i = 2, \ldots, 2n$.

As the function $\varphi_0(z) = \tan(\pi/4\, z)$ maps the strip $|\mathrm{Re} z| < 1$ conformally onto the open unit disk, $f_{x^\star} \in \widetilde{H}^r_{\infty,\beta}$. Thus

$$f^\star := f_{x^\star} - \frac{\langle l_1, f_{x^\star}\rangle}{\langle l_1, 1\rangle} \in X^{2n} \cap \widetilde{H}^r_{\infty,\beta}.$$

Consequently, by virtue of Corollary 7,

$$\sup_{f \in \widetilde{H}^r_{\infty,\beta} \cap X^{2n}} \|f\|_q \geq \|f^\star\|_q \geq \inf_{a \in \mathbb{R},\ \xi \in \overline{\Lambda}_{2n}^{\varphi_0}} \|a + D_r * \varphi_0(K_\beta * h_\xi)\|_q = \|\Phi^\beta_{n,r}\|_q.$$

Therefore,

$$d^{2n-1}(\widetilde{H}^r_{\infty,\beta}, L_q) \geq d^{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) \geq \|\Phi^\beta_{n,r}\|_q, \tag{41}$$

which completes the lower estimate of (37). Now we turn to the upper estimate. It follows from Theorem 4 that

$$d^{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) \leq d^{2n-1}(\widetilde{H}^r_{\infty,\beta}, L_q) \leq \|\Phi^\beta_{n,r}\|_q, \tag{42}$$

which, in combination with (41), gives the proof of Theorem 6. □

Remark 3. Among others, Sun [17] determined the exact values of the Gel′fand n-widths

$$d^{2n}(\widetilde{h}_{\infty,\beta}, L_q) = d^{2n-1}(\widetilde{h}_{\infty,\beta}, L_q) = \|\phi^\beta_n\|_q, \quad 1 \leq q \leq \infty,\ n = 1, 2, \ldots,$$

where

$$\phi^\beta_n(x) = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{\cos(2\nu+1)nx}{(2\nu+1)\cosh(2\nu+1)n\beta}\,. \tag{43}$$

Let

$$k := 4e^{-\beta} \left(\frac{\sum_{m=0}^{\infty} e^{-2\beta m(m+1)}}{1 + 2\sum_{m=1}^{\infty} e^{-2\beta m^2}} \right)^2.$$

The complete elliptic integral of the first kind with modulus k is defined by formula

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Set

$$\mathrm{ctn}(z) := \frac{\mathrm{cn}(z)\cdot \mathrm{dn}(z)}{\mathrm{sn}(z)},$$

where $\mathrm{sn}(z)$, $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$ denote the Jacobi elliptic functions with modulus k (see [1] and [2]).

Lemma 6 ([14]). *There exists a subspace of continuous functions with $2n$ dimension*

$$W_{2n} = \left\{ c_0 + \sum_{j=1}^{2n} c_j (D_r * g_j) : \sum_{j=1}^{2n} c_j = 0 \right\},$$

such that for any $f \in \widetilde{H}^r_{\infty,\beta}$ there is a unique function $L_{2n}(f) \in W_{2n}$ interpolating f at $2n$ equidistant points

$$L_{2n}(f, t^{(j)}_{n,r}) = f(t^{(j)}_{n,r}), \quad j = 1, \ldots, 2n, \tag{44}$$

and satisfying

$$|f(x) - L_{2n}(f,x)| \le |\Phi^{\beta}_{n,r}(x)| \quad \forall x \in [0, 2\pi), \tag{45}$$

where $t^{(j)}_{n,r}$, $j = 1, \ldots, 2n$ are defined by (14)*, and*

$$g_j(t) = \Phi^{\beta}_{n,0}(t)\mathrm{ctn}\left(\frac{K}{\pi}(t - t^{(j)}_{n,r})\right), \quad j = 1, \ldots, 2n.$$

We are now in position to prove the exact estimates of the Kolmogorov $2n$-width, linear $2n$-width, and information $2n$-width of the class of functions $\widetilde{H}^r_{\infty,\beta}$.

Theorem 7. *For integer $r \ge 0$, $n \in \mathbb{N}$, and $1 \le q \le \infty$,*

$$d_{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) = \lambda_{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) = d^{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) = i_{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) = \|\Phi^{\beta}_{n,r}\|_q. \tag{46}$$

Proof. The case of $r = 0$, $1 \le q \le \infty$, and the case of $r \in \mathbb{N}$, $q = \infty$ were obtained in [9] and [11]–[13]. Let $r \in \mathbb{N}$. First, we will prove the lower bound for the Kolmogorov widths. Assume that $1 < q < \infty$, and S^{2n}, g_x, and f_x are the same as those in the proof of Theorem 6 (see (38)–(40)). Since the class of functions $\widetilde{H}^r_{\infty,\beta}$ $(r = 1, 2, \ldots,)$ contains all constants, in order to establish the lower bound of $d_{2n}(\widetilde{H}^r_{\infty,\beta}, L_q)$ we only have to consider the subspace $X_{2n} \subset L_q$, which also contains the constants. Let X_{2n} be any $2n$-dimensional subspace of L_q, $1 < q < \infty$, such that $1 \in X_{2n}$. Suppose that $X_{2n} = \mathrm{span}\{f_1, \ldots, f_{2n}\}$ and $f_1(t) \equiv 1$. Let $a_1(x), \ldots, a_{2n}(x)$ be the coefficients of $f_1, \ldots, f_{2n}$, respectively, in the unique best approximation to f_x from X_{2n}. The mapping

$$A(x) := (b(x), a_2(x), \ldots, a_{2n}(x)),$$

where

$$b(x) := \int_0^{2\pi} \varphi_0((K_\beta * g_x)(t))\, dt,$$

is an odd and continuous map of S^{2n} into $\mathbb{R}^{2n}$. By Borsuk's theorem there exists an $x^\star \in S^{2n}$ for which $A(x^\star) = 0$. Then $g_{x^\star} \in \{h_\xi : \xi \in \overline{\Lambda}^{\varphi_0}_{2n}\}$ and $a_i(x^\star) = 0$, $i = 2, \ldots, 2n$. As the function $\varphi_0(z) := \tan(\pi/4\, z)$ maps the strip $|\mathrm{Re} z| < 1$

conformally onto the open unit disk, $f_{x^\star} \in \widetilde{H}^r_{\infty,\beta}$. Therefore, if $1 < q < \infty$, by Corollary 7 we have

(47)
$$\begin{aligned} d_{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) &\geq \sup_{f\in\widetilde{H}^r_{\infty,\beta}} \inf_{g\in X_{2n}} \|f-g\|_q \geq \|f_{x^\star} - a_1(x^\star)\|_q \\ &\geq \inf_{a\in\mathbb{R}}\ \inf_{\xi\in\overline{\Lambda}^{\varphi_0}_{2n}} \|a + D_r * \varphi_0(K_\beta * h_\xi)\|_q = \|\Phi^\beta_{n,r}\|_q, \quad 1 < q < \infty. \end{aligned}$$

By passing to the limit $q \to 1$, $q \to \infty$, respectively, we obtain the lower estimate of the Kolmogorov widths $d_{2n}(\widetilde{H}^r_{\infty,\beta}, L_q)$, $1 \leq q \leq \infty$.

Now we prove the upper estimate of the linear widths. It follows from Lemma 6 that

$$\sup_{f\in\widetilde{H}^r_{\infty,\beta}} \|f - L_{2n}(f)\|_q \leq \|\Phi^\beta_{n,r}\|_q, \quad 1 \leq q \leq \infty,$$

where $L_{2n}(f) \in W_{2n}$. From the definition of λ_n we have

$$\lambda_{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) \leq \|\Phi^\beta_{n,r}\|_q, \quad 1 \leq q \leq \infty. \tag{48}$$

Since $d_{2n} \leq \lambda_{2n}$ and $d^{2n} \leq i_{2n} \leq \lambda_{2n}$, by virtue of (47), (48) and Theorem 6, we get Theorem 7. □

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