MATHEMATICS OF COMPUTATION
Volume 75, Number 254, Pages 997–1013
S 0025-5718(05)01811-9
Article electronically published on November 29, 2005

CM-FIELDS WITH RELATIVE CLASS NUMBER ONE

GEON-NO LEE AND SOUN-HI KWON

Abstract. We will show that the normal CM-fields with relative class number one are of degrees ≤ 216. Moreover, if we assume the Generalized Riemann Hypothesis, then the normal CM-fields with relative class number one are of degrees ≤ 96, and the CM-fields with class number one are of degrees ≤ 104. By many authors all normal CM-fields of degrees ≤ 96 with class number one are known except for the possible fields of degree 64 or 96. Consequently the class number one problem for normal CM-fields is solved under the Generalized Riemann Hypothesis except for these two cases.

1. Introduction

In [O1], Odlyzko proved that there are only finitely many normal CM-fields of a given class number (see also [S]). In [H], Hoffstein showed unconditionally that normal CM-fields with relative class number one are of degrees ≤ 434 and determined upper bounds for their root discriminants. Recently, Bessassi ([B]) improved Hoffstein's bounds: he showed that normal CM-fields with relative class number one are of degrees ≤ 266 and that if the Generalized Riemann Hypothesis is true, then they are of degrees ≤ 164. Usually, to solve the class number one problem for normal CM-fields of a given degree, one determines their possible Galois groups and tries to solve this problem for a given Galois group. However, there are too many groups of 2-power orders, e.g., 267 groups of order 64 and 2328 groups of order 128 (see [M], [HS], [JNO], [Wo], [TW], and [Ob]). It seems reasonable to try to get analytically sharper bounds for the degrees of normal CM-fields with class number one than the previously known ones, even with the assumption of the Generalized Riemann Hypothesis. Let K be a CM-field of degree $2n$ with maximal totally real subfield k, D_K the absolute value of its discriminant, $\rho_K = D_K^{1/2n}$ its root discriminant, and let $h_K^- = h_K/h_k$ be its relative class number, where h_K and h_k are the class number of K and that of k, respectively. The purpose of this paper is to show the following.

Theorem 1. *Let K be a CM-field of degree $2n$.*

(1) *We assume the Generalized Riemann Hypothesis and assume that K is normal over $\mathbb{Q}$. If $n \geq 50$, then*

$$h_K^- \geq \frac{2}{6.69671} \cdot \frac{(1.06730)^n}{\log(2n \cdot 3.77012)} \geq 1.30699 > 1. \tag{1.1}$$

Received by the editor January 19, 2005 and, in revised form, February 27, 2005.
2000 *Mathematics Subject Classification.* Primary 11R29, 11R42.
Key words and phrases. CM-fields, class numbers, relative class numbers, Dedekind zeta functions.

If $h_K^- = 1$*, then* $n \leq 48$ *and* $\rho_K \leq \alpha(n)$*, where the* $\alpha(n)$*'s are given in Table* 6 *below. If* $50 \leq n \leq 83$*, then* $h_K^- \geq h(n)$*, where the* $h(n)$*'s are given in Table* 5 *below.*

(2) *We do not assume the Generalized Riemann Hypothesis, but assume that* K *is normal over* $\mathbb{Q}$*. If* $n \geq 109$*, then*

$$h_K^- \geq \frac{(1.06136)^n}{n \cdot 5.98607} \geq 1.01035 > 1. \tag{1.2}$$

If $109 \leq n \leq 134$*, then* $h_K^- \geq h(n)$*, where the* $h(n)$*'s are given in Table* 7 *at the end of this paper.*

(3) *We assume the Generalized Riemann Hypothesis, but do not assume that* K *is normal over* $\mathbb{Q}$*. If* $n \geq 53$*, then*

$$h_K^- \geq \frac{(1.08068)^n}{\sqrt{n}(6.17730)(1.43729)^{1/\sqrt{n}}} \geq 1.29231 > 1. \tag{1.3}$$

If $53 \leq n \leq 88$*, then* $h_K^- \geq h(n)$*, where the* $h(n)$*'s are given in Table* 8 *at the end of this paper.*

All imaginary abelian number fields with class number one are known by Yamamura (see [Y] and [CK1]). By many authors all normal CM-fields of degrees < 48 with class number one are known. The class number one problem for normal CM-fields of degree 48 is partially solved. For full details see [LO1], [LOO], [LO2], [Lou1], [Lef], [LLO], [PYK], [CK2], [P], [LPCK], and [CK3]. Recently, it has been proved that there are no normal CM-fields of degree ≤ 96 with class number one except for the possible fields of degree 64 or 96 (see [PK]). We remark that in [PK] the authors used the upper bound for root discriminant $\alpha(24)$ above to solve the class number one problem for the normal CM-fields of degree 48. Note that there are 223 ($= 230 - 7$) nonabelian groups of order 96.

For a number field M we let O_M, κ_M, and ω_M be its ring of algebraic integers, the residue of $\zeta_M(s)$ (the Dedekind zeta function of M) at $s = 1$, and the number of roots of unity in M, respectively. For a CM-field K we denote by $Q_K \in \{1, 2\}$ its Hasse unit index. From the analytic class number formula we have

$$h_K^- = \frac{Q_K \omega_K}{(2\pi)^n} \sqrt{\frac{D_K}{D_k}} \frac{\kappa_K}{\kappa_k} \geq \frac{2D_K^{1/4}}{(2\pi)^n} \frac{\kappa_K}{\kappa_k}, \tag{1.4}$$

where $[K : \mathbb{Q}] = 2n$ ([W]).

To prove Theorem 1 we proceed as follows. Using Weil's explicit formula, we get explicit lower bounds for D_K in Section 2 and upper bounds for κ_k in Section 3. Using this Weil's formula Bessassi improved significantly the previously known bounds for κ_k. We take care of prime ideals of small norms when we deal with this Weil's formula, which allows us to improve upon Bessassi's upper bounds for κ_k. In Section 4 we give lower bounds for κ_K. When we do not assume the Generalized Riemann Hypothesis, we use Louboutin's bounds in [Lou3]. When we assume the Generalized Riemann Hypothesis, we use Bessassi's result in [B], and what is more we take care of prime ideals of small norms. In Section 5, using those bounds above we get explicit lower bounds for h_K^-. Ultimately Section 6 is devoted to the proof of Theorem 1 itself.

2. Lower bounds for D_K

To begin with, we recall the following result deduced from Weil's explicit formula.

Proposition 2. *Let F be a real-valued even function with $F(0) = 1$ for which the following conditions hold.*

(i) *The sum $\int_0^\infty F(x)\cosh(x/2)dx$ exists.*
(ii) *The function F is of bounded variation, the value in each point being the average of the limit to the right and the limit to the left.*
(iii) *The function $\big(1 - F(x)\big)/x$ is also of bounded variation.*
(iv) *Assuming the Generalized Riemann Hypothesis, the Fourier transform of F is nonnegative. Without this hypothesis, the Fourier transform of $f(x) = F(x)\cosh(x/2)$ is nonnegative.*

Let k be a totally real number field of degree n over $\mathbb{Q}$. Set

$$I_n(F) = \frac{4}{n}\int_0^\infty F(x)\cosh(x/2)dx + \int_0^\infty \frac{(1-F(x))e^{x/2}}{\sinh(x)}dx$$

and $B_n(F) = \log(8\pi e^\gamma) + (\pi/2) - I_n(F)$. For a positive integer $i \geq 2$ we let $g_i \geq 0$ be the number of prime ideals $\mathfrak{p}$ in k with $N_{k/\mathbb{Q}}(\mathfrak{p}) = i$, and

$$l_i(F) = \sum_{m\geq 1} \frac{\log i}{i^{m/2}} F(m\log i). \tag{2.1}$$

Then we have

$$\log D_k \geq nB_n(F) + 2\sum_{i\geq 2} g_i l_i(F). \tag{2.2}$$

Proof. See [Poi1, Propositions 4 and 5] and Section 3 in [B]. □

To get lower bounds for discriminants assuming the Generalized Riemann Hypothesis we choose

$$0 \leq F(x) = F_{O,b_O}(x) = \begin{cases} (1-|\frac{x}{b_O}|)\cos(\pi x/b_O) + \frac{1}{\pi}\sin(\pi|x/b_O|) & \text{for } |x| \leq b_O, \\ 0 & \text{for } |x| > b_O \end{cases}$$

as chosen by Odlyzko ([O2] and [Poi2]). Without assuming the Generalized Riemann Hypothesis it is known that Tartar's choice is the best one ([Poi1], [Poi2], and [O2]):

$$0 \leq F(x) = F_{T,b_T}(x) = \begin{cases} 9\Big(\frac{\sin(x/b_T)-(x/b_T)\cos(x/b_T)}{(x/b_T)^3}\Big)^2/\cosh(x/2) & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

Here $b_O > 0$ and $b_T > 0$ will be chosen for each degree n to give the best possible bound. For these b_O and b_T we let $B(n) = B_n(F_{O,b_O})$ and $l_i = l_i(F_{O,b_O})$ if we assume the Generalized Riemann Hypothesis, $B(n) = B_n(F_{T,b_T})$ and $l_i = l_i(F_{T,b_T})$ otherwise. We can easily estimate $B(n)$, e.g., $B_5(F_{O,3.13308}) = 1.89381$, $B_6(F_{O,3.51124}) = 2.09779$, $B_5(F_{T,1.00000}) = 1.79442$, and $B_6(F_{T,1.00000}) = 2.04584$. We will need the fact that $B(5) > 1$ and $B(6) > 2$ in the proof of Lemma 7 below. Note that $n \mapsto B(n)$ is increasing, and that both $l_i(F_{O,b_O})$ and $l_i(F_{T,b_T})$ are always nonnegative.

3. Upper bounds for κ_k

Usually an upper bound for κ_k is obtained by first estimating $\zeta_k(\sigma)$ with $\sigma > 1$. To estimate $\zeta_k(\sigma)$ from above we use Weil's formula with $F(x) = F_b(x) = \operatorname{sech}(x/2)/(1+(x/b)^2)$ ([B]). Since the Fourier transform of $f_b(x) = F_b(x)\cosh(x/2) = 1/(1+(x/b)^2)$ is $\hat{f}_b(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_b(x)e^{-itx}dx = \sqrt{\frac{\pi}{2}}\, be^{-b|t|} \geq 0$, we can use Proposition 2.

For real $x > 1$ and $t \geq 3$ a positive integer, we set

$$c_0(\sigma, b, x) := -\log(1-x^{-\sigma})/\big(2\sum_{m\geq 1}\frac{\log x}{x^{m/2}}F_b(m\log x)\big), \tag{3.1}$$

$$c_1(\sigma, b, t) := \sup_{x\geq t} c_0(\sigma, b, x), \tag{3.2}$$

$$L_i(\sigma, b, t) := \big(c_0(\sigma, b, i) - c_1(\sigma, b, t)\big)l_i(F_b), \tag{3.3}$$

and

$$C_2(b, n) := B_n(F_b). \tag{3.4}$$

Remark 1. Since $t \geq 3$, our $c_1(\sigma, b, t)$ is smaller than Bessassi's $c_1(\sigma, b) = c_1(\sigma, b, 2)$ in [B, Theorem 4]. This allows us to improve upon Bessassi's upper bounds for κ_k in Proposition 5 below. (See Remark 2 below.)

Lemma 3. *Let $\sigma > 1$ be given. We have*

$$\zeta_k(\sigma) \leq \left(\frac{D_k}{\exp(nC_2(b,n))}\right)^{c_1(\sigma,b,t)} \prod_{i=2}^{t-1}[\exp(2L_i(\sigma,b,t))]^{g_i}$$

for every positive integer $t \geq 3$.

Proof. We have

$$\begin{aligned}
\log\zeta_k(\sigma) &= -\textstyle\sum_{i\geq 2} g_i\log(1-i^{-\sigma}) \\
&= 2\textstyle\sum_{i\geq 2} g_i c_0(\sigma,b,i)l_i(F_b) && \text{(by (3.1))} \\
&\leq 2\textstyle\sum_{i=2}^{t-1} g_i c_0(\sigma,b,i)l_i(F_b) + 2c_1(\sigma,b,t)\sum_{i\geq t} g_i l_i(F_b) && \text{(by (3.2))} \\
&= 2\textstyle\sum_{i=2}^{t-1} g_i L_i(\sigma,b,t) + 2c_1(\sigma,b,t)\sum_{i\geq 2} g_i l_i(F_b) && \text{(by (3.3))} \\
&\leq 2\textstyle\sum_{i=2}^{t-1} g_i L_i(\sigma,b,t) + c_1(\sigma,b,t)(\log D_k - nB_n(F_b)) && \text{(by (2.2))} \\
&= 2\textstyle\sum_{i=2}^{t-1} g_i L_i(\sigma,b,t) + c_1(\sigma,b,t)(\log D_k - nC_2(b,n)) && \text{(by (3.4))}.
\end{aligned}$$

The result follows. □

Lemma 4. *Let k be a totally real number field of degree $n \geq 1$. If $\zeta_k(s)$ has no real zero in the range $1/2 < \beta < 1$, we set $E_\sigma = 1$; if β is any real zero of $\zeta_k(s)$ in this range, we set $E_\sigma = \frac{1-\beta}{\sigma-\beta}$. Set*

$$h(\sigma) = \pi^{-\sigma/2}\Gamma(\sigma/2) \qquad (\sigma > 1)$$

and $\psi(\sigma) = (\Gamma'/\Gamma)(\sigma)$. For $\tilde{\sigma}$ such that $\tilde{\sigma} \geq 1 + \sigma/\sqrt{7+4\sqrt{2}}$ and $\tilde{\sigma} \geq (5+\sqrt{12\sigma^2-5})/6$ if we do not assume the Generalized Riemann Hypothesis, and

$\tilde{\sigma} \geq 1 + (\sigma - 1)/\sqrt{3}$ if we do assume the Generalized Riemann Hypothesis, set

$$c_3(\tilde{\sigma}, n) = \frac{n}{4}\psi'\Big(\frac{\tilde{\sigma}}{2}\Big) - \frac{1}{\tilde{\sigma}^2} - \frac{1}{(\tilde{\sigma}-1)^2}.$$

Then, for $\sigma > 1$ we have

$$\kappa_k < E_\sigma \frac{\sigma(\sigma-1)\zeta_k(\sigma) {D_k}^{(\sigma-1)/2} h^n(\sigma)}{\exp\big(\sigma(\sigma-1)c_3(\tilde{\sigma}, n)/2\big)}.$$

Proof. See [O1], [Poi2], [W, Lemma 11.21], and [B, Lemmas 5 and 6]. □

Putting together Lemmas 3 and 4 we get an upper bound for κ_k:

Proposition 5. *Let $c_1 = c_1(\sigma, b, t)$, $C_2 = C_2(b, n)$, $h(\sigma)$, $c_3 = c_3(\tilde{\sigma}, n)$, and E_σ be as above. Let*

$$c_4 = c_4(\sigma, b, t) = c_1(\sigma, b, t) + \frac{\sigma - 1}{2}$$

and let

$$C_5(n, b, t, \sigma, \tilde{\sigma}) = \exp\left(\frac{1}{c_4}\Big(c_1 C_2 + \frac{\sigma(\sigma-1)}{2}\frac{c_3}{n} - \frac{1}{n}\log\big(\sigma(\sigma-1)\big) - \log h(\sigma)\Big)\right).$$

(1) *Let k be a totally real number field of degree n. We have then*

$$\kappa_k \leq E_\sigma \Big(\frac{D_k}{C_5(n, b, t, \sigma, \tilde{\sigma})^n}\Big)^{c_4(\sigma, b, t)} \prod_{i=2}^{t-1} [\exp(2L_i(\sigma, b, t))]^{g_i}$$

for every positive integer $t \geq 3$.

(2) *Let $m \geq 1$ and $t \geq 3$ be given integers and let $b > 0$ be given. Assume that for given m, b, and t we have chosen σ and $\tilde{\sigma}$. Set $c_4(m, b, t) = c_4(\sigma, b, t)$, $C_5(m, b, t) = C_5(m, b, t, \sigma, \tilde{\sigma})$, and $L_i(m, b, t) = L_i(\sigma, b, t)$. Then*

$$\kappa_k \leq E_\sigma \Big(\frac{D_k}{C_5(m, b, t)^n}\Big)^{c_4(m, b, t)} \prod_{i=2}^{t-1} [\exp(2L_i(m, b, t))]^{g_i}$$

for any totally real number field k of degree $n \geq m$.

Possible values for $c_4(\sigma, b, t)$ and $C_5(m, b, t, \sigma, \tilde{\sigma})$ for small degrees are given in Table 1 below. These values for $m \geq 108$ are unconditionally obtained and those for $48 \leq m \leq 83$ are obtained assuming the Generalized Riemann Hypothesis.

Proof. (1) To get possible values for $C_5(n, b, t, \sigma, \tilde{\sigma})$ and $c_4(\sigma, b, t)$ we proceed as follows. First, we fix n. Second, we fix b and t. (In the proof of Theorem 1 below we will explain how to choose favorable values for b and t for a given n.) Third, we find $\sigma \geq 1.01$ having three properties at once: (i) σ is as small as possible; (ii) $c_4(\sigma, b, t)$ is as small as possible; (iii) $2\pi b c_1(\sigma, b, t) + \log\big(\sigma(\sigma-1)\big) \geq 0$. (For the reason why we want $\sigma \geq 1.01$, see the proof of Proposition 8 point (2) below.) Finally we choose $\tilde{\sigma}$ so that $c_3(\tilde{\sigma}, n)$ is as large as possible. For given n, b, t, and σ, the values $L_i(\sigma, b, t)$ can be easily computed.

(2) The function $n \mapsto C_5(n, b, t, \sigma, \tilde{\sigma})$ is increasing for given b, t, σ, and $\tilde{\sigma}$. Once we have chosen b, t, σ, and $\tilde{\sigma}$ for a given m, we have then $C_5(n, b, t, \sigma, \tilde{\sigma}) \geq C_5(m, b, t)$. So,

$$\kappa_k \leq E_\sigma \Big(\frac{D_k}{C_5(m, b, t)^n}\Big)^{c_4(m, b, t)} \prod_{i=2}^{t-1} [\exp(2L_i(m, b, t))]^{g_i}$$

for any totally real number field k of degree $n \geq m$. □

TABLE 1.

m	b	t	σ	$\tilde{\sigma}$	$c_4(\sigma,b,t)$	$C_5(m,b,t,\sigma,\tilde{\sigma})$
134	8.866	47	1.0162	1.2865	0.08181	54.799
130	8.678	45	1.0167	1.2868	0.08310	54.604
120	8.480	43	1.0172	1.2872	0.08452	54.361
110	8.163	40	1.0181	1.2879	0.08689	53.967
109	8.163	40	1.0181	1.2879	0.08689	53.961
108	8.163	40	1.0181	1.2879	0.08689	53.954
83	12.103	101	1.0107	1.2882	0.06485	56.587
80	12.103	101	1.0107	1.2928	0.06485	56.561
72	12.103	101	1.0107	1.3064	0.06485	56.483
70	12.103	101	1.0107	1.3102	0.06485	56.462
64	11.890	96	1.0110	1.3226	0.06573	56.297
60	10.740	73	1.0126	1.3319	0.07096	55.627
50	8.678	45	1.0167	1.3601	0.08310	53.851
49	8.678	45	1.0167	1.3634	0.08310	53.832
48	8.480	43	1.0172	1.3668	0.08452	53.608

Remark 2. When $n = 83$, Bessassi obtained $C_5(83, b, 2, \sigma, \tilde{\sigma}) = 38.9255$ and $c_4(\sigma, b, 2) = 0.36705$ with $b = 4.78685$, $\sigma = 1.01114$, and $\tilde{\sigma} = 1.28823$ in [B, Table 4]. However we obtained $C_5(83, b, t, \sigma, \tilde{\sigma}) = 56.58718\cdots$ and $c_4(\sigma, b, t) = 0.0648497\cdots$ with $b = 12.103$, $\sigma = 1.0107405\cdots$, $\tilde{\sigma} = 1.288211\cdots$, and $t = 101$. This improvement on upper bounds for κ_k and Lemma 7 yield that $h_K^- \geq 48986$ if $[K : \mathbb{Q}] = 2 \cdot 83$ in the proof of Theorem 1 point (1) below.

4. LOWER BOUNDS FOR κ_K

Proposition 6. (1) *Let $m \geq 1$ and $\rho > e$ be given. Assume the Generalized Riemann Hypothesis. For any normal number field K of degree $n \geq m$ and of root discriminant $\rho_K \geq \rho$ we have*

$$\kappa_K \geq \frac{1}{c_6(m,\rho)\log\log D_K} \prod_{i=2}^{t-1} \Big(\frac{1}{1-i^{-2}}\Big)^{g_i},$$

where $t \geq 3$ is a positive integer and the $c_6(m,\rho)$'s can be taken as in Table 2.

(2) *For any totally imaginary number field K of degree ≥ 10 and root discriminant $\rho_K \geq 2\pi^2$, we have unconditionally*

$$\kappa_K \geq \begin{cases} \frac{1}{ce^{1/(2c)}\log D_K} & \text{if } \zeta_K(1-1/(c\log D_K)) \leq 0, \\ \frac{1-\beta}{2e^{1/(2c)}} & \text{if } \zeta_K(\beta) \leq 0 \text{ with } 1-1/(c\log D_K) \leq \beta < 1, \end{cases}$$

where $c = (2+\sqrt{3})/4$.

TABLE 2.

m	98	96	80	64	60	48	46
ρ	50.29	50.71	54.98	61.90	64.37	75.08	77.61
$c_6(m,\rho)$	6.6705	6.6778	6.7432	6.8245	6.8482	6.9303	6.9460
m	40	36	32	30	20	16	10
ρ	87.36	96.55	109.3	117.7	211.0	324.7	1163
$c_6(m,\rho)$	6.9972	7.0355	7.0779	7.1008	7.2384	7.3076	7.4320

(3) [B, Theorem 14] *For any number field K different from $\mathbb{Q}$ for which the Generalized Riemann Hypothesis for $\zeta_K(s)$ holds true, we have*

$$\kappa_K \geq \frac{e^{-3/2}}{\sqrt{\log D_K}} \exp\Big(\frac{-1}{\sqrt{\log D_K}}\Big).$$

Proof. (1) We have

$$\kappa_K = \lim_{s\to 1}(s-1)\zeta_K(s) = \lim_{s\to 1}\frac{\zeta_K(s)}{\zeta(s)} = \prod_p E(p) = M(Q)R(Q)T(Q),$$

where

$$E(p) = (1-p^{-1})\prod_{\mathcal{P}_K|p}\big(1-N_{K/\mathbb{Q}}(\mathcal{P}_K)^{-1}\big)^{-1} \geq (1-p^{-1})\prod_{\mathcal{P}_k|p}\big(1-N_{k/\mathbb{Q}}(\mathcal{P}_k)^{-2}\big)^{-1},$$

$M(Q) = \prod_{p\leq Q} E(p)$, $R(Q) = \prod_{\substack{p>Q \\ p \text{ ramified}}} E(p)$, and $T(Q) = \prod_{\substack{p>Q \\ p \text{ nonramified}}} E(p)$. For an estimate of $T(Q)$ and a lower bound for $R(Q)$ we use [B, Lemma 11 and (26)]. For $M(Q)$, we choose t and Q with $t \leq Q$ so that

$$\begin{aligned} M(Q) &= \prod_{p\leq Q} E(p) \\ &\geq \prod_{p\leq Q}(1-p^{-1})\prod_{i=2}^{t-1}(1-i^{-2})^{-g_i} \\ &\geq \tfrac{1}{e^{\gamma}\log Q}\big(1-\tfrac{1}{2(\log Q)^2}\big)\prod_{i=2}^{t-1}(1-i^{-2})^{-g_i}. \end{aligned}$$

The remainder of the proof is the same as that of [B, Corollary 13].

(2) See [Lou2, Theorem 1] and [B, Theorem 16]. □

5. Lower bounds for h_K^-

To begin with we prove the following,

Lemma 7. *Let K be a CM-field of degree $2n \geq 2m$ with maximal real subfield k, $t \geq 3$ a given positive integer, and let $\varepsilon \geq 0.05$ be a positive real number. Then we have*

(1) $D_K \geq \exp\big(2nB(m)\big)\prod_{i=2}^{t-1}\big(\exp(4l_i)\big)^{g_i}$,

(2) $\frac{D_K{}^{\varepsilon}}{\log\log D_K} \geq \frac{\exp(2\varepsilon nB(m))}{\log(2nB(m))}\prod_{i=2}^{t-1}\Big(\exp(4\varepsilon l_i)/\big(1+\frac{2l_i}{mB(m)}\big)\Big)^{g_i}$ *if* $m \geq 5$,

(3) $\frac{D_K{}^{\varepsilon}}{\log D_K} \geq \frac{\exp(2\varepsilon nB(m))}{2nB(m)}\prod_{i=2}^{t-1}\Big(\exp(4\varepsilon l_i)/\big(1+\frac{2l_i}{mB(m)}\big)\Big)^{g_i}$ *if* $m \geq 6$,

(4) $\frac{D_K{}^{\varepsilon}}{(\log D_K)^{1/2}} \geq \frac{\exp(2\varepsilon nB(m))}{(2nB(m))^{1/2}}\prod_{i=2}^{t-1}\Big(\exp(4\varepsilon l_i)/\big(1+\frac{l_i}{mB(m)}\big)\Big)^{g_i}$ *if* $m \geq 5$.

Proof. For (1), by Proposition 2 and since $n \mapsto B(n)$ increases with n, we have

$$\log D_K \geq 2\log D_k \geq 2nB(n) + 4\sum_{i\geq 2} g_i l_i \geq 2nB(m) + 4\sum_{i=2}^{t-1} g_i l_i. \tag{5.1}$$

(2) Now, using $A^{1+\delta} \geq Ae^{\delta} \geq A(1+\delta)$ for $A \geq e$ and $\delta \geq 0$, we have

$$\log\Big(A + 4\sum_{i=2}^{t-1} g_i l_i\Big) \leq \Big(1+\sum_{i=2}^{t-1}\frac{4g_i l_i}{A}\Big)\log A \leq \Big(\prod_{i=2}^{t-1}\big(1+\frac{4l_i}{A}\big)^{g_i}\Big)\log A \tag{5.2}$$

for $A = 2nB(m) \geq e$. Since $2n \geq 2m \geq 10$ implies $A = 2nB(m) \geq 10B(m) \geq 10B(5) \geq 10 \geq e$ (see the end of Section 2) and since $x \mapsto x^\varepsilon / \log\log x$ is increasing for $x \geq e^{10}$ and $\varepsilon \geq 0.05$, using (5.1) we obtain

$$\frac{D_K^\varepsilon}{\log\log D_K} \geq \frac{\exp\left(\varepsilon(A + 4\sum_{i=2}^{t-1} g_i l_i)\right)}{\log(A + 4\sum_{i=2}^{t-1} g_i l_i)},$$

and (2) follows, by (5.2). The proofs of (3) and (4) are similar. For (3) we use the fact that $x \mapsto x^\varepsilon / \log x$ is increasing if $x \geq e^{20}$ and $\varepsilon \geq 0.05$. $\square$

Combining Propositions 5 and 6 and Lemma 7 we get lower bounds for h_K^- as follows.

Proposition 8. *Let $m \geq 1$ and $t \geq 3$ be given integers and let $b > 0$ be given. Set*

$$C_7(n, b, t) = (2\pi)^{-1}\big(C_5(n, b, t)\exp(-B(n))\big)^{c_4(n,b,t)} \exp(B(n)/2).$$

Suppose that $c_4(n, b, t) \leq 0.4$ for every $n \geq 5$.

(1) *Assume the Generalized Riemann Hypothesis. Let $\rho > e$ be given. For any normal CM-field K of degree $2n \geq 2m \geq 10$ and root discriminant $\rho_K \geq \rho$ we have*

(5.3)
$$h_K^- \geq \frac{2}{c_6(2m, \rho)} \frac{C_7(m, b, t)^n}{\log(2nB(m))} \prod_{i=2}^{t-1} \left(\frac{\exp((1 - 2c_4(m, b, t))l_i)}{(1 + \frac{2l_i}{mB(m)})(1 - i^{-2})\exp(2L_i(m, b, t))} \right)^{g_i}.$$

(2) *Set $c = (2 + \sqrt{3})/4 = 0.93301\cdots$. For any normal CM-field K of degree $2n \geq 2m \geq 100$ we have unconditionally*

$$h_K^- \geq \frac{C_7(m, b, t)^n}{ce^{1/(2c)} nB(m)} \prod_{i=2}^{t-1} \left(\frac{\exp((1 - 2c_4(m, b, t) - \frac{2}{m})l_i)}{\exp(2L_i(m, b, t))} \right)^{g_i}. \tag{5.4}$$

(3) *Assume the Generalized Riemann Hypothesis. For any CM-field K (not necessarily normal over $\mathbb{Q}$) of degree $2n \geq 2m \geq 10$ we have*

(5.5)
$$h_K^- \geq \frac{2\,C_7(m, b, t)^n}{e^{3/2}(2nB(m))^{1/2}\big(\exp(1/\sqrt{2B(m)}\,)\big)^{1/\sqrt{n}}} \prod_{i=2}^{t-1} \left(\frac{\exp((1 - 2c_4(m, b, t))l_i)}{(1 + \frac{l_i}{mB(m)})\exp(2L_i(m, b, t))} \right)^{g_i}.$$

Proof. According to (1.4) we have $h_K^- \geq 2\big(D_K^{1/4}/(2\pi)^n\big)(\kappa_K/\kappa_k)$.
(1) By Propositions 5 and 6 point (1), and since $D_K \geq {D_k}^2$,

(5.6)
$$h_K^- \geq \frac{2}{c_6(2m, \rho)} \left(\frac{C_5(m, b, t)^{c_4(m,b,t)}}{2\pi} \right)^n \frac{D_K^{\frac{1}{4} - \frac{1}{2}c_4(m,b,t)}}{\log\log D_K} \prod_{i=2}^{t-1} \left(\frac{1}{(1 - i^{-2})\exp(2L_i(m, b, t))} \right)^{g_i}.$$

By applying Lemma 7 point (2) with $\varepsilon = \frac{1}{4} - \frac{1}{2}c_4(m, b, t)$, we get (5.3).
(2) Set $I = [1 - 1/(c\log D_K), 1)$. We consider the following three cases.

(i) If $\zeta_k(s)$ has a zero $\beta \in I$, then $\zeta_K(\beta) = 0$. By Propositions 5 and 6 point (2), and since $(1 - \beta)/E_\sigma = \sigma - \beta > \sigma - 1$,

(5.7)
$$h_K^- \geq \frac{(\sigma - 1)}{e^{1/(2c)}} \left(\frac{C_5(m, b, t)^{c_4(m,b,t)}}{2\pi} \right)^n D_K^{\frac{1}{4} - \frac{1}{2}c_4(m,b,t)} \prod_{i=2}^{t-1} \left(\frac{1}{\exp(2L_i(m, b, t))} \right)^{g_i}.$$

By applying Lemma 7 point (1), we have

$$h_K^- \geq \frac{(\sigma-1)}{e^{1/(2c)}}\, C_7(m,b,t)^n \prod_{i=2}^{t-1}\Big(\frac{\exp((1-2c_4(m,b,t))l_i)}{\exp(2L_i(m,b,t))}\Big)^{g_i}.$$

Here, σ was given in Proposition 5.

(ii) If $\zeta_k(s)$ has no zero β in I and $\zeta_K(s)$ has a simple zero in I, then by [S, Theorem 3] and [B, Theorem 17] there exists an imaginary quadratic subfield F of K such that $\zeta_F(\beta)=0$ and

$$1-\beta > \frac{6}{\pi}\frac{1}{\sqrt{D_F}} > \frac{6}{\pi}\frac{1}{D_K^{1/(2n)}}.$$

We have

$$\begin{aligned} h_K^- &\geq 2\tfrac{D_K^{1/4}}{(2\pi)^n}\tfrac{1}{E_\sigma}\Big(\tfrac{C_5(m,b,t)^n}{D_k}\Big)^{c_4(m,b,t)}\tfrac{1-\beta}{2e^{1/(2c)}}\prod_{i=2}^{t-1}\Big(\tfrac{1}{\exp(2L_i(m,b,t))}\Big)^{g_i} \\ &\geq \tfrac{6}{\pi e^{1/(2c)}}\Big(\tfrac{C_5(m,b,t)^{c_4(m,b,t)}}{2\pi}\Big)^n D_K^{\frac{1}{4}-\frac{1}{2}c_4(m,b,t)-\frac{1}{2n}}\prod_{i=2}^{t-1}\Big(\tfrac{1}{\exp(2L_i(m,b,t))}\Big)^{g_i}. \end{aligned} \tag{5.8}$$

By applying Lemma 7 point (1), we have

$$h_K^- \geq \frac{6}{\pi}\frac{C_7(m,b,t)^n}{e^{1/(2c)}e^{B(m)}}\prod_{i=2}^{t-1}\Big(\frac{\exp((1-2c_4(m,b,t)-2/m)l_i)}{\exp(2L_i(m,b,t))}\Big)^{g_i}.$$

(iii) If $\zeta_k(s)$ has no zero in I and $\zeta_K(s)$ has no simple zero in I, then either $\zeta_K(s)$ has no zero at all in I or $\zeta_K(s)$ has a double zero in I. This is because $\zeta_K(s)$ has at most two zeros with multiplicity in I by [LLO, Lemma 15]. Then $\zeta_K(1-1/(c\log D_K))\leq 0$ and

$$h_K^- \geq \frac{2}{ce^{1/(2c)}}\Big(\frac{C_5(m,b,t)^{c_4(m,b,t)}}{2\pi}\Big)^n\frac{D_K^{\frac{1}{4}-\frac{1}{2}c_4(m,b,t)}}{\log D_K}\prod_{i=2}^{t-1}\Big(\frac{1}{\exp(2L_i(m,b,t))}\Big)^{g_i} \tag{5.9}$$

by Propositions 5 and 6 point (2).

By applying Lemma 7 point (3) with $\varepsilon=\frac{1}{4}-\frac{1}{2}c_4(m,b,t)$, we have

$$h_K^- \geq \frac{C_7(m,b,t)^n}{ce^{1/(2c)}nB(m)}\prod_{i=2}^{t-1}\Big(\frac{\exp((1-2c_4(m,b,t))l_i)}{(1+\frac{2l_i}{mB(m)})\exp(2L_i(m,b,t))}\Big)^{g_i}.$$

Now, we compare the following three terms:

$$(\sigma-1),\qquad \frac{6}{\pi e^{B(m)}}\prod_{i=2}^{t-1}\Big(\frac{1}{\exp(2l_i/m)}\Big)^{g_i},\qquad \frac{1}{cnB(m)}\prod_{i=2}^{t-1}\left(\frac{1}{1+\frac{2l_i}{mB(m)}}\right)^{g_i}.$$

Notice that $\sigma-1>1/100$ (see Proposition 5 and Table 1), that $5.372\cdots = \log(8\pi e^{\gamma})+\frac{\pi}{2}\geq B(m)\geq B_{50}(F_{T,b_T})=3.524\cdots>3$ with $b_T=2.44976$, and that

$$\frac{6}{\pi e^{B(m)}} > \frac{6}{\pi e^{5.38}} > \frac{1}{120} > \frac{1}{135} = \frac{1}{0.9\cdot 50\cdot 3} > \frac{1}{cnB(m)}$$

for $m\geq 50$. Since $\exp(2l_i/m)\geq 1+2l_i/m\geq 1+2l_i/(mB(m))$, it follows that all three terms above are greater than or equal to

$$\frac{1}{cnB(m)}\prod_{i=2}^{t-1}\Big(\frac{1}{\exp(2l_i/m)}\Big)^{g_i}.$$

The result follows.

(3) By Propositions 5 and 6 point (3),

$$h_K^- \geq \frac{2}{e^{3/2}}\Big(\frac{C_5(m,b,t)^{c_4(m,b,t)}}{2\pi}\Big)^n \frac{D_K^{\frac{1}{4}-\frac{1}{2}c_4(m,b,t)}}{(\log D_K)^{1/2}} \times \exp\Big(\frac{-1}{(\log D_K)^{1/2}}\Big)\prod_{i=2}^{t-1}\Big(\frac{1}{\exp(2L_i(m,b,t))}\Big)^{g_i}. \tag{5.10}$$

By Lemma 7 point (4) with $\varepsilon = \frac{1}{4}-\frac{1}{2}c_4(m,b,t)$, we have

$$\frac{D_K^{\frac{1}{4}-\frac{1}{2}c_4(m,b,t)}}{(\log D_K)^{1/2}} \geq \frac{\exp\Big(2nB(m)(\frac{1}{4}-\frac{1}{2}c_4(m,b,t))\Big)}{(2nB(m))^{1/2}}\prod_{i=2}^{t-1}\left(\frac{\exp\big((1-2c_4(m,b,t))l_i\big)}{1+\frac{l_i}{mB(m)}}\right)^{g_i},$$

and by Lemma 7 point (1), we have

$$\exp\left(\frac{-1}{(\log D_K)^{1/2}}\right) \geq \exp\left(\frac{-1}{(2nB(m))^{1/2}}\right).$$

The result follows. □

6. Proof of Theorem 1

6.1. **Proof of Theorem 1 point (1).** Let K be a normal CM-field of degree $2n \geq 2m \geq 10$. For $i < t$, we set

$$M(m,i,b,t) = \frac{\exp\big((1-2c_4(m,b,t))l_i\big)}{\big(1+\frac{2l_i}{mB(m)}\big)(1-i^{-2})\exp(2L_i(m,b,t))}$$

if i is a power of a prime, $M(m,i,b,t)=1$ otherwise. If for a given m we have $M(m,i,b,t) \geq 1$ for $2 \leq i < t$, then, according to (5.3), for any normal CM-field K of degree $2n \geq 2m \geq 10$ with $\rho_K \geq \rho$ we have

$$h_K^- \geq \frac{2}{c_6(2m,\rho)}\frac{C_7(m,b,t)^n}{\log(2nB(m))}.$$

To find favorable values for b and t that maximize this right-hand side we proceed as follows. First, we fix m, say $m = 83$. Using Proposition 2 with $F(x) = F_{O,b_O}(x)$ we get $\rho_K \geq \rho := 54.88741$. We get $B(83) = 4.00528\cdots$ with $F(x) = F_{O,b_O}(x)$ and $c_6(166,\rho) = 6.43518\cdots$. Second, we fix b, say $b = 2$. Third, we let $t = 3$ and choose σ and $\tilde{\sigma}$ as in the proof of Proposition 5. We compute $c_4(83,2,3) = 0.37137\cdots < 0.4$, $C_5(83,2,3) = 28.47193\cdots$, and $C_7(83,2,3) = 0.92404\cdots$ and verify that $M(83,2,2,3) = 1.53445\cdots \geq 1$. This yields

$$h_K^- \geq \frac{2}{6.43519}\frac{(0.92404)^n}{\log(8.01057n)} \geq 0.00006.$$

Now we repeat the third step with $b = 2$ for all t with $4 \leq t \leq 101$ and observe the following.

(a) For all t with $3 \leq t \leq 101$ we always have $c_4(83,2,t) \leq 0.4$.
(b) For any given t with $3 \leq t \leq 101$ we have $M(83,i,2,t) \geq 1$ for every i with $2 \leq i < t$.
(c) $C_7(83,2,t)$ is increasing for $3 \leq t \leq 54$ and $1.0196596\cdots = C_7(83,2,54) > C_7(83,2,55) = 1.0196589\cdots$.

Hence we get

$$h_K^- \geq \frac{2}{6.43519} \frac{(1.01965)^n}{\log(8.01057n)} \geq 0.24045.$$

We add 1 to b and repeat the third step with $b = 3$ for all t with $3 \leq t \leq 101$. We verify that (a) and (b) are satisfied, and $C_7(83, 3, t)$ is increasing for $3 \leq t \leq 87$ and $1.1152586 \cdots = C_7(83, 3, 87) > C_7(83, 3, 88) = 1.1152514 \cdots$. At this time we get

$$h_K^- \geq \frac{2}{6.43519} \frac{(1.11525)^n}{\log(8.01057n)} \geq 408.768.$$

We repeat the second and the third step by adding 1 to b until $b = 20$. For a given b, if $C_7(83, b, t)$ is increasing for all t with $3 \leq t \leq 101$, then we take $C_7(83, b, 101)$. It happens when $b \in \{13, 14, 15, 16, 17, 18, 19, 20\}$. We verify that the value $C_7(83, 12, 101) = 1.18138 \cdots$ is the largest among all computed $C_7(83, b, t)$'s. We continue this process for all $b = 12 + \frac{1}{10}k$ with integers k with $-10 < k < 10$. We verify that the value $C_7(83, 12.1, 101) = 1.18144 \cdots$ is the largest among all computed $C_7(83, b, t)$'s. For $b = 12.1 + \frac{1}{100}k$ with $-10 < k < 10$, we repeat the computations and obtain the largest value $C_7(83, 12.10, 101) = 1.18144 \cdots$. Similarly we get the value $C_7(83, 12.103, 101) = 1.18145 \cdots$, which is the largest for all $b = 12.10 + \frac{1}{1000}k$ with $-10 < k < 10$. We did not compute $C_7(83, b, t)$ with $b = 12.103 + 10^{-4}k$ with $-10 < k < 10$. This is because there are negligible changes in $C_7(83, b, t)$'s, $3 \leq t \leq 101$ for a given b with the precision of 10^{-3}. We conclude that for any normal CM-field K of degree $2n \geq 2 \cdot 83$,

$$h_K^- \geq \frac{2}{6.43519} \frac{(1.18145)^n}{\log(8.01057n)} \geq 48984.$$

We do the same computations for every m with $49 \leq m \leq 83$. Our computational results are summarized in Tables 3, 4, and 5. The values b, t, and $C_7(n, b, t)$ given in Table 5 are the most favorable among our computational results.

Table 3. Lower bounds for h_K^- when $n = 83$ and $b = 12.103$

t	$C_7(83, 12.103, t)$	$h_K^- \geq$	t	$C_7(83, 12.103, t)$	$h_K^- \geq$
2	0.95408	0.00096	40	1.17115	28682
3	1.03532	0.85265	60	1.17639	34303
4	1.07183	15.136	80	1.17938	42349
10	1.13531	1795.0	100	1.18136	48675
20	1.15796	9250.1	101	1.18145	48984

Table 4. Lower bounds for h_K^- when $n = 50$ and $b = 8.678$

t	$C_7(50, 8.678, t)$	$h_K^- \geq$	t	$C_7(50, 8.678, t)$	$h_K^- \geq$
2	0.88222	0.00009	43	1.06685	1.27972
3	0.95360	0.00468	44	1.06708	1.29359
4	0.98514	0.02381	45	1.06730	1.30699
10	1.03835	0.33047	46	1.06715	1.29784
20	1.05627	0.77749	47	1.06701	1.28935
30	1.06268	1.05213	48	1.06687	1.28092
40	1.06611	1.23608	49	1.06673	1.27254

TABLE 5. Lower bounds for h_K^-

n	$\rho:\rho_K \geq \rho$	$B(n)$	$c_6(2n,\rho)$	b	t	$C_7(n,b,t)$	$h(n): h_K^- \geq h(n)$
83	54.887	4.0053	6.4352	12.103	101	1.1814	48984
82	54.608	4.0002	6.4408	12.103	101	1.1788	34577
81	54.327	3.9950	6.4465	12.103	101	1.1761	24457
80	54.041	3.9898	6.4523	12.103	101	1.1734	17311
79	53.752	3.9844	6.4582	12.103	101	1.1707	12279
78	53.460	3.9789	6.4642	12.103	101	1.1679	8719.0
77	53.164	3.9734	6.4703	12.103	101	1.1650	6201.3
76	52.864	3.9677	6.4765	12.103	101	1.1622	4415.7
75	52.561	3.9620	6.4829	12.103	101	1.1592	3150.2
74	52.253	3.9561	6.4893	12.103	101	1.1563	2250.6
73	51.942	3.9501	6.4959	12.103	101	1.1533	1611.3
72	51.626	3.9440	6.5026	12.103	101	1.1502	1154.7
71	51.306	3.9378	6.5095	12.103	101	1.1471	828.99
70	50.982	3.9315	6.5165	12.103	101	1.1439	596.26
69	50.654	3.9250	6.5237	12.103	101	1.1407	429.46
68	50.321	3.9184	6.5310	12.019	99	1.1374	309.80
67	49.983	3.9117	6.5384	12.019	99	1.1340	223.99
66	49.640	3.9048	6.5460	12.019	99	1.1306	162.25
65	49.293	3.8978	6.5538	11.890	96	1.1272	117.69
64	48.940	3.8906	6.5618	11.890	96	1.1237	85.605
63	48.582	3.8833	6.5699	11.757	93	1.1201	62.408
62	48.219	3.8758	6.5783	10.853	75	1.1164	45.633
61	47.851	3.8681	6.5868	10.740	73	1.1127	33.435
60	47.477	3.8602	6.5956	10.740	73	1.1090	24.566
59	47.097	3.8522	6.6045	10.440	68	1.1052	18.092
58	46.711	3.8440	6.6137	10.440	68	1.1013	13.359
57	46.318	3.8356	6.6231	9.839	59	1.0973	9.8912
56	45.920	3.8269	6.6328	9.839	59	1.0932	7.3451
55	45.515	3.8181	6.6427	9.693	57	1.0891	5.4688
54	45.103	3.8090	6.6530	9.383	53	1.0849	4.0813
53	44.685	3.7996	6.6634	9.218	51	1.0806	3.0566
52	44.259	3.7901	6.6742	9.218	51	1.0763	2.2956
51	43.825	3.7802	6.6853	9.218	51	1.0718	1.7285
50	43.385	3.7701	6.6968	8.678	45	1.0673	1.3069
49	42.936	3.7597	6.7085	8.678	45	1.0626	0.9912

Since the complex conjugation is in the center of the Galois group $G(K/\mathbb{Q})$, every normal CM-field of degree 98 is an imaginary abelian number field. Hence its relative class number is greater than one. (See [CK1].) Consequently it follows that $h_K^- > 1$ if $n \geq 49$. For $n \geq 50$ we get (1.1). (In the tables the values $C_7(n,b,t)$ and $h(n)$ (resp. $B(n)$ and $c_6(2n,\rho)$) are rounded down (resp. up) after four decimal places.)

Set

$$N(n,i,b,t) = \frac{1}{(1-i^{-2})\exp(2L_i(n,b,t))}$$

TABLE 6. Upper bounds for root discriminants

n	$\rho : \rho \leq \rho_K$	b	t	$c_4(n,b,t)$	$C_5(n,b,t)$	$\alpha(n)$	$c_6(2n,\alpha(n)) \leq$	$Bess(n)$
48	42.479	5.105	4	0.21615	37.899	50.71	6.6778	125.2
40	38.491	4.921	4	0.21817	36.167	54.98	6.7432	204.0
32	33.770	4.721	4	0.22059	33.900	61.90	6.8245	505.6
30	32.444	4.669	4	0.22126	33.211	64.37	6.8482	729.3
24	28.017	4.494	4	0.22368	30.725	75.08	6.9303	5252
20	24.608	4.370	4	0.22554	28.554	87.36	6.9972	6499
18	22.733	4.301	4	0.22664	27.250	96.55	7.0355	
16	20.726	4.231	4	0.22780	25.741	109.3	7.0779	
15	19.668	4.195	4	0.22842	24.893	117.7	7.1008	6875
10	13.744	3.996	4	0.23209	19.296	211.0	7.2384	7653
8	11.042	3.903	4	0.23397	16.106	324.7	7.3076	
5	6.644	3.752	4	0.23728	9.5580	1163	7.4320	10250

if i is a power of a prime, and $N(n,i,b,t) = 1$ otherwise. If $N(n,i,b,t) \geq 1$ for $2 \leq i \leq t-1$, then, according to (5.6),

$$h_K^- \geq \frac{2}{c_6(2n,\rho)} \Big(\frac{C_5(n,b,t)^{c_4(n,b,t)}}{2\pi} \Big)^n \frac{D_K^{\frac{1}{4} - \frac{1}{2}c_4(n,b,t)}}{\log\log D_K}.$$

In a similar fashion as above, for $5 \leq n \leq 48$ we find optimal values of b and t that minimize $\alpha(n)$ so that $\rho_K \geq \alpha(n)$ implies $h_K^- > 1$. We summarize our computational results in Table 6. Here $Bess(n)$ means Bessassi's upper bounds for root discriminants in [B].

6.2. **Proof of Theorem 1 point (2).** Set $C_8(m) = ce^{1/2c}B(m)$ and

$$P(m,i,b,t) = \frac{\exp((1 - 2c_4(m,b,t) - \frac{2}{m})l_i)}{\exp(2L_i(m,b,t))}$$

if i is a power of a prime, $P(m,i,b,t) = 1$ otherwise. For given m, b, and t, if $P(m,i,b,t) \geq 1$ for all i with $2 \leq i < t$ and $c_4(m,b,t) \leq 0.4$, then, according to (5.4), for every normal CM-field of degree $2n \geq 2m \geq 100$ we have

$$h_K^- \geq \frac{C_7(m,b,t)^n}{nC_8(m)}.$$

In a similar fashion as (1) we get the results in Table 7

The results in Theorem 1 point (2) follow immediately.

6.3. **Proof of Theorem 1 point (3).** Set $C_9(m) = e^{3/2}(2B(m))^{1/2}/2$, $C_{10}(m) = \exp(1/\sqrt{2B(m)})$, and

$$R(m,i,b,t) = \frac{\exp((1 - 2c_4(m,b,t))l_i)}{(1 + \frac{l_i}{mB(m)})\exp(2L_i(m,b,t))}$$

if i is a power of a prime, $R(m,i,b,t) = 1$ otherwise. For given m, b, and t, if $c_4(m,b,t) \leq 0.4$ and $R(m,i,b,t) \geq 1$ for all i with $i < t$, then, according to (5.5), for any CM-field K of degree $2n \geq 2m \geq 10$ we have

$$h_K^- \geq \frac{C_7(m,b,t)^n}{\sqrt{n}C_9(m)C_{10}(m)^{1/\sqrt{n}}}.$$

TABLE 7. Lower bounds for h_K^-

n	$\rho_K \geq$	$C_8(n)$	b	t	$C_7(n,b,t)$	$h(n) : h_K^- \geq h(n)$
134	44.636	6.0568	8.866	47	1.0813	43.703
133	44.569	6.0544	8.866	47	1.0806	37.421
132	44.501	6.0520	8.774	46	1.0799	32.044
131	44.432	6.0495	8.678	45	1.0792	27.442
130	44.362	6.0470	8.678	45	1.0785	23.532
129	44.292	6.0445	8.678	45	1.0777	20.181
128	44.221	6.0419	8.678	45	1.0770	17.311
127	44.149	6.0394	8.678	45	1.0763	14.851
126	44.076	6.0367	8.678	45	1.0755	12.758
125	44.003	6.0341	8.678	45	1.0748	10.950
124	43.928	6.0314	8.678	45	1.0740	9.4016
123	43.853	6.0286	8.480	43	1.0732	8.0832
122	43.777	6.0259	8.480	43	1.0725	6.9517
121	43.700	6.0230	8.480	43	1.0717	5.9807
120	43.622	6.0202	8.480	43	1.0709	5.1414
119	43.543	6.0173	8.480	43	1.0700	4.4266
118	43.463	6.0144	8.480	43	1.0692	3.8127
117	43.382	6.0114	8.378	42	1.0684	3.2819
116	43.301	6.0084	8.378	42	1.0675	2.8294
115	43.218	6.0054	8.378	42	1.0667	2.4380
114	43.134	6.0023	8.378	42	1.0658	2.1041
113	43.049	5.9991	8.272	41	1.0649	1.8150
112	42.963	5.9960	8.272	41	1.0641	1.5666
111	42.876	5.9927	8.272	41	1.0632	1.3530
110	42.788	5.9894	8.163	40	1.0622	1.1694
109	42.699	5.9861	8.163	40	1.0613	1.0103
108	42.609	5.9827	8.163	40	1.0604	0.8735

Similarly as (1) we obtain favorable values for $C_9(m)$, $C_{10}(m)$, $C_7(m,b,t)$, and lower bounds for h_K^- in Table 8.

The results in Theorem 1 point (3) follow immediately.

Remark 3. There is no theoretical reason to take b with $2 \leq b \leq 20$ and t with $t \leq 101$. If we compute $C_7(m,b,t)$ with $t > 101$ for given m and b, we can get better lower bounds for h_K^-. In fact, when $n = 83$, varying b between 2 and 20 as in the proof of Theorem 1 we have computed $C_7(83,b,t)$ for all $t \geq 3$ until we find α such that $C_7(83,b,t)$ is increasing for $3 \leq t \leq \alpha$ and $C_7(83,b,\alpha) > C_7(83,b,\alpha+1)$. Among all computed $C_7(83,b,t)$'s, $C_7(83,18.8,494) = 1.18243\cdots$ is the largest one. We can hence say that for any normal CM-field K of degree $2n \geq 2 \cdot 83$,

$$h_K^- \geq \frac{2}{6.43519} \frac{(1.18243)^n}{\log(8.01057n)} \geq 52473$$

if we assume the Generalized Riemann Hypothesis.

Remark 4. Our upper bounds for ρ_K for a normal CM-field K with $h_K^- = 1$ in Table 6 are sharper than Bessassi's bounds. In the cases of $5 \leq n \leq 23$, Bessassi has used

$$\kappa_k \leq e\Big(\frac{e \log D_k}{2n}\Big)^{n-1} \qquad \text{([Lou2, Theorem 1])}$$

TABLE 8.

n	$\rho_K \geq$	$B(n)$	$C_9(n)$	$C_{10}(n)$	b	t	$C_7(n,b,t)$	$h(n) : h_K^- \geq h(n)$
88	56.232	4.0295	6.3614	1.4223	12.103	101	1.1940	96586
87	55.969	4.0248	6.3577	1.4226	12.103	101	1.1915	68066
86	55.703	4.0200	6.3540	1.4229	12.103	101	1.1890	48046
85	55.434	4.0152	6.3502	1.4232	12.103	101	1.1865	33972
84	55.162	4.0103	6.3463	1.4235	12.103	101	1.1840	24048
83	54.887	4.0053	6.3423	1.4238	12.103	101	1.1814	17054
82	54.608	4.0002	6.3383	1.4242	12.103	101	1.1788	12102
81	54.327	3.9950	6.3342	1.4245	12.103	101	1.1761	8605.5
80	54.041	3.9898	6.3300	1.4248	12.103	101	1.1734	6123.9
79	53.752	3.9844	6.3257	1.4252	12.103	101	1.1707	4367.8
78	53.460	3.9789	6.3214	1.4255	12.103	101	1.1679	3118.3
77	53.164	3.9734	6.3170	1.4259	12.103	101	1.1650	2230.3
76	52.864	3.9677	6.3125	1.4262	12.103	101	1.1622	1597.1
75	52.561	3.9620	6.3079	1.4266	12.103	101	1.1592	1145.9
74	52.253	3.9561	6.3033	1.4270	12.103	101	1.1563	823.47
73	51.942	3.9501	6.2985	1.4273	12.103	101	1.1533	593.03
72	51.626	3.9440	6.2936	1.4277	12.103	101	1.1502	427.54
71	51.306	3.9378	6.2887	1.4281	12.103	101	1.1471	308.80
70	50.982	3.9315	6.2836	1.4285	12.103	101	1.1439	223.47
69	50.654	3.9250	6.2784	1.4290	12.103	101	1.1407	161.96
68	50.321	3.9184	6.2732	1.4294	12.019	99	1.1374	117.57
67	49.983	3.9117	6.2678	1.4298	12.019	99	1.1340	85.559
66	49.640	3.9048	6.2623	1.4303	12.019	99	1.1306	62.379
65	49.293	3.8978	6.2566	1.4307	11.890	96	1.1272	45.548
64	48.940	3.8906	6.2508	1.4312	11.890	96	1.1237	33.353
63	48.582	3.8833	6.2450	1.4317	11.757	93	1.1201	24.481
62	48.219	3.8758	6.2389	1.4322	10.853	75	1.1164	18.025
61	47.851	3.8681	6.2327	1.4327	10.740	73	1.1127	13.300
60	47.477	3.8602	6.2264	1.4332	10.740	73	1.1090	9.8425
59	47.097	3.8522	6.2199	1.4338	10.440	68	1.1052	7.3018
58	46.711	3.8440	6.2133	1.4343	10.440	68	1.1013	5.4316
57	46.318	3.8356	6.2065	1.4349	9.839	59	1.0973	4.0520
56	45.920	3.8269	6.1995	1.4355	9.839	59	1.0932	3.0321
55	45.515	3.8181	6.1923	1.4361	9.693	57	1.0891	2.2753
54	45.103	3.8090	6.1849	1.4367	9.383	53	1.0849	1.7115
53	44.685	3.7996	6.1773	1.4373	9.218	51	1.0806	1.2923
52	44.259	3.7901	6.1695	1.4380	9.218	51	1.0763	0.97862

to obtain upper bounds ρ_K, however, even though in these cases we obtained better bounds for ρ_K using Proposition 5.

Remark 5. In Theorem 1, points (2) and (3), we cannot obtain lower bounds $\alpha(n)$ for ρ_K such that $h_K^- = 1$ would imply $\rho_K \leq \alpha(n)$. For $\frac{1}{\exp(2L_i(m,b,t))} < 1$ we can not remove the term $\prod_{i=2}^{t-1} \left(\frac{1}{\exp(2L_i(m,b,t))}\right)^{g_i}$ in the inequalities (5.7) through (5.10).

ACKNOWLEDGMENT

This work was supported by KRF-Grant 2002-041-C00002.

References

[B] S. Bessassi, *Bounds for the degrees of CM-fields of class number one*, Acta Arith., 106.3(2003), 213-245. MR1957106 (2003m:11183)

[CK1] K.-Y. Chang and S.-H. Kwon, *Class numbers of imaginary abelian number fields*, Proc. Amer. Math. Soc.,128(2000), 2517-2528. MR1707511 (2000m:11108)

[CK2] K.-Y. Chang and S.-H. Kwon, *The non-abelian normal CM-fields of degree* 36 *with class number one*, Acta Arith., 101(2002), 53-61. MR1879845 (2003e:11119)

[CK3] K.-Y. Chang and S.-H. Kwon, *The class number one problem for some non-abelian normal CM-fields of degree* 48, Math. Comp., 72(2003), 1003-1017. MR1954981 (2003m:11185)

[H] J. Hoffstein, *Some analytic bounds for zeta functions and class numbers*, Invent. Math., 55(1979), 37-47. MR0553994 (80k:12019)

[HS] M. Hall, Jr. and J. K. Senior, *The groups of order* 2^n $(n \leq 6)$, Macmillan, NY, 1964. MR0168631 (29:5889)

[JNO] R. James, M. F. Newman, and E. A. O'Brien, *The groups of order* 128, J. Algebra, 129(1990), 136-158. MR1037398 (90j:20050)

[Lef] Y. Lefeuvre, *Corps diédraux à multiplication complexe principaux*, Ann. Inst. Fourier (Grenoble), 50(2000), 67-103. MR1762338 (2001g:11166)

[LLO] F. Lemmermeyer, S. Louboutin, and R. Okazaki, *The class number one problem for some non-abelian normal CM-fields of degree* 24, J. Théor. Nombres Bordeaux, 11(1999), 387-406. MR1745886 (2001j:11104)

[LO1] S. Louboutin and R. Okazaki, *Determination of all non-normal quartic CM-fields and of all non-abelian normal octic CM-fields with class number one*, Acta Arith., 67.1(1994), 47-92. MR1292520 (95g:11107)

[LO2] S. Louboutin and R. Okazaki, *The class number one problem for some non-abelian normal CM-fields of* 2*-power degrees*, Proceedings London Math. Soc., Vol. 76, Part 3(1998), 523-548. MR1616805 (99c:11138)

[LOO] S. Louboutin, R. Okazaki, and M. Olivier, *The class number one problem for some non-abelian normal CM-fields*, Trans. Amer. Math. Soc., 349(1997), 3657-3678. MR1390044 (97k:11149)

[Lou1] S. Louboutin, *The class number one problem for the non-abelian normal CM-fields of degree* 16, Acta Arith., 82.2(1997), 173-196. MR1477509 (98j:11097)

[Lou2] S. Louboutin, *Explicit bounds for residues of Dedekind zeta functions, values of L-functions at* $s = 1$ *and relative class numbers*, J. Number Theory, 85(2000), 263-282. MR1802716 (2002i:11111)

[Lou3] S. Louboutin, *Explicit lower bounds for residues at* $s = 1$ *of Dedekind zeta functions and relative class numbers of CM-fields*, Trans. Amer. Math. Soc., 355(2003), 3079-3098. MR1974676 (2004f:11134)

[LPCK] S. Louboutin, Y.-H. Park, K.-Y. Chang, and S.-H. Kwon, *The class number one problem for the non-abelian normal CM-fields of degree* $2pq$, Preprint.

[M] G. A. Miller, *Determination of all the groups of order* 64, Amer. J. Math., 52(1930), 617-634. MR1507920

[O1] A. M. Odlyzko, *Some analytic extimates of class numbers and discriminants*, Invent. Math., 29(1975), 275-286. MR0376613 (51:12788)

[O2] A. M. Odlyzko, *Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: A Survey of recent results*, Sém. Théor. Nombres Bordeaux 2(1990), 119-141. MR1061762 (91i:11154)

[Ob] E. A. O'Brien, *The groups of order dividing* 256, Bull. Austral. Math. Soc., 39(1989), 159-160.

[P] Y.-H. Park, *The class number one problem for the non-abelian normal CM-fields of degree* 24 *and* 40, Acta Arith., 101.1(2002), 63-80. MR1879844 (2002k:11200)

[Poi1] G. Poitou, *Minorations de discriminants*, Sém. Bourbaki, 28e année (1975/76), 136-153. MR0435033 (55:7995)

[Poi2] G. Poitou, *Sur les petits discriminants*, Sém. Delange-Pisot-Poitou, 18e année (1976/77), 6-01 - 6-17. MR0551335 (81i:12007)

[PK] S.-M. Park and S.-H. Kwon, *Class number one problem for normal CM-fields*, Preprint.

[PYK] S.-M. Park, H.-S. Yang, and S.-H. Kwon, *Class number one problem for the normal CM-fields of degree* 32, Preprint.

[S] H.M. Stark, *Some effective cases of the Brauer-Siegel theorem*, Invent. Math., 23(1974), 135-152. MR0342472 (49:7218)

[TW] A. D. Thomas and G. V. Wood, *Group Tables*, Shiva Mathematics series 2, 1980. MR0572793 (81d:20002)

[W] L. C. Washington, *Introduction to cyclotomic fields*, 2nd ed. (1997), Grad. Texts in Math. 83, Springer-Verlag. MR1421575 (97h:11130)

[Wo] *http://mathworld.wolfram.com/Finitegoup.html.*

[Y] K. Yamamura, *The determination of the imaginary abelian number fields with class number one*, Math. Comp., 62(1994), 899-921. MR1218347 (94g:11096)

Department of Mathematics Education, Korea University, 136-701, Seoul, Korea
E-mail address: thisknow@korea.ac.kr

Department of Mathematics Education, Korea University, 136-701, Seoul, Korea
E-mail address: sounhikwon@korea.ac.kr