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THE BOUNDARIES OF THE SOLUTIONS OF THE LINEAR VOLTERRA INTEGRAL EQUATIONS WITH CONVOLUTION KERNEL

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ABSTRACT. Some boundaries about the solution of the linear Volterra integral equations of the second type with unit source term and positive monotonically increasing convolution kernel were obtained in Ling, 1978 and 1982. A method enabling the expansion of the boundary of the solution function of an equation in this type was developed in I. Özdemir and Ö. F. Temizer, 2002.

In this paper, by using the method in Özdemir and Temizer, it is shown that the boundary of the solution function of an equation in the same form can also be expanded under different conditions than those that they used.

1. INTRODUCTION

An integral equation of the form

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\, f(\tau)\, d\tau = \phi(t) - K * f \tag{1.1}$$

is known as the second type linear Volterra integral equation with convolution kernel, where ϕ is the source term, K is the kernel which are the known functions, and f is an unknown function.

The way of obtaining a new equation equivalent to the equation of the form (1.1) is given by Theorem A, called the Equivalence Theorem by R. Ling, below.

Theorem A (Equivalence Theorem) ([2, Theorem 1.1.1]). *If*
(1) $K \in C^1[0,\infty)$,
(2) ϕ *is locally integrable,*
then the following two integral equations are equivalent:

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\ f(\tau)\, d\tau,$$
$$f(t) = \psi(t) - \int_0^t L(t-\tau)\, f(\tau)\, d\tau,$$

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where

$$\psi(t) = \phi(t) + \int_0^t g'(t-\tau)\,\phi(\tau)\,d\tau,$$
$$L(t) = g'(t) + ag(t) + \int_0^t g(t-\tau)\,K'(\tau)\,d\tau,$$

and where $a = K(0)$, g *is any function such that* $g \in C^1[0,\infty)$ *and* $g(0) = 1$.

Sufficient conditions providing the obtaining of the solution of equation (1.1) by means of $g(t)$, which is the solution of the equation

$$g(t) = 1 - \int_0^t K(t-\tau)g(\tau)d\tau = 1 - K * g, \tag{1.2}$$

is given by Theorem B, below:

Theorem B (Convolution Theorem) ([1, pp. 229–230]). *If the conditions*

(1) $\phi'(t)$ *exists for* $0 \le t \le T$, $\int_0^T |\phi'(t)|\,dt < \infty$ $(T > 0)$,

(2) $\int_0^T |K(t)|\,dt < \infty$

hold, then the solution of equation (1.1) *is given by*

$$f(t) = g(t)\,\phi(0) + \int_0^t g(t-\tau)\,\phi'(\tau)d\tau = g(t)\,\phi(0) + g * \phi' \ (0 \le t \le T), \tag{1.3}$$

where $g(t)$ *is the solution of* (1.2).

Therefore, if g is known, so is f, or if the properties of g are known, then we may be able to obtain certain properties of f by (1.3).

Some boundaries about the function f which is the solution of linear Volterra integral equation of the second type with unit source term and monotonically increasing kernel are obtained in Theorem C and Theorems 1–4, below:

Theorem C ([3, Theorem B]). *Let us consider the equation*

$$f(t) = 1 - \int_0^t K(t-\tau)\,f(\tau)d\tau = 1 - K * f. \tag{1.4}$$

If the conditions

(1) $K(t) > 0$, $K'(t) > 0$ *and* $K''(t) \le 0$ *for* $0 \le t < \infty$,

(2) $4b \le a^2$

hold, then the solution of (1.4) *satisfies the inequality* $|f(t)| \le 1$ *for* $0 \le t < \infty$, *where* $a = K(0)$ *and* $b = K'(0)$.

Theorem 1 ([3, Theorem 3.1]). *Let us consider the equation*

$$f_1(t) = 1 - \int_0^t K_1(t-\tau)\,f_1(\tau)d\tau = 1 - K_1 * f_1. \tag{1.5}$$

If

(1) $K_1(t) > 0, K_1'(t) > 0, K_1''(t) > 0$ *and* $K_1'''(t) \le 0$ *for* $0 \le t < \infty$,
(2) $a_{10}^2 < 4a_{11}$,
(3) $3a_{11} \le a_{10}^2$,
(4) $2a_{10}^3 - 9a_{10}a_{11} + 27a_{12} \le 0$,

then the solution of (1.5) *satisfies the inequality* $|f_1(t)| \le 2$ *for* $0 \le t < \infty$, *where* $a_{10} = K_1(0)$, $a_{11} = K_1'(0)$ *and* $a_{12} = K_1''(0)$.

Theorem 2 ([3, Theorem 3.3]). *Let us consider the equation*

$$f_2(t) = 1 - \int_0^t K_2(t-\tau)\, f_2(\tau) d\tau = 1 - K_2 * f_2. \tag{1.6}$$

If

(1) $K_2(t) > 0, K_2'(t) > 0, K_2''(t) > 0, K_2'''(t) > 0$ *and* $K_2^{(4)}(t) \le 0$ *for* $0 \le t < \infty$,
(2) $a_{20}^2 < 3a_{21}$,
(3) $\frac{8}{3}a_{21} \le a_{20}^2$,
(4) $a_{20}^3 - 4a_{20}a_{21} + 8a_{22} \le 0$,
(5) $-3a_{20}^4 + 16a_{20}^2a_{21} - 64a_{20}a_{22} + 256a_{23} \le 0$,

then the solution of (1.6) *satisfies the inequality* $|f_2(t)| \le 4$ *for* $0 \le t < \infty$, *where* $a_{20} = K_2(0)$, $a_{21} = K_2'(0)$, $a_{22} = K_2''(0)$ *and* $a_{23} = K_2'''(0)$.

Theorem 3. *Let us consider the equation*

$$f_3(t) = 1 - \int_0^t K_3(t-\tau)\, f_3(\tau) d\tau = 1 - K_3 * f_3. \tag{1.7}$$

If

(1) $K_3(t) > 0, K_3'(t) > 0, K_3''(t) > 0, K_3'''(t) > 0, K_3^{(4)}(t) > 0$ *and* $K_3^{(5)}(t) \le 0$ *for* $0 \le t < \infty$,
(2) $a_{30}^2 < \frac{8}{3}a_{31}$,
(3) $\frac{5}{2}a_{31} \le a_{30}^2$,
(4) $4a_{30}^3 - 15a_{30}a_{31} + 25a_{32} \le 0$,
(5) $-3a_{30}^4 + 15a_{30}^2a_{31} - 50a_{30}a_{32} + 125a_{33} \le 0$,
(6) $4a_{30}^3 - 25a_{30}a_{31} + 125a_{32} > 0$,
(7) $4a_{30}^5 - 25a_{30}^3a_{31} + 125a_{30}^2a_{32} - 625a_{30}a_{33} + 3125a_{34} \le 0$,

then $|f_3(t)| \le 8$ *for* $0 \le t < \infty$, *where* $a_{30} = K_3(0)$, $a_{31} = K_3'(0)$, $a_{32} = K_3''(0)$, $a_{33} = K_3'''(0)$ *and* $a_{34} = K_3^{(4)}(0)$, *(for the proof, see* [4]*).*

Theorem 4. *Let us consider the equation*

$$f_4(t) = 1 - \int_0^t K_4(t-\tau) f_4(\tau) d\tau = 1 - K_4 * f_4. \tag{1.8}$$

If

(1) $K_4(t) > 0, K_4'(t) > 0, K_4''(t) > 0, K_4'''(t) > 0, K_4^{(4)}(t) > 0, K_4^{(5)}(t) > 0$ *and* $K_4^{(6)}(t) \le 0$ *for* $0 \le t < \infty$,

(2) $a_{40}^2 < \dfrac{5}{2}a_{41}$,

(3) $\dfrac{12}{5}a_{41} \le a_{40}^2$,

(4) $5a_{40}^3 - 18a_{40}a_{41} + 27a_{42} \le 0$,

(5) $-5a_{40}^4 + 24_{40}^2 a_{41} - 72a_{40}a_{42} + 144a_{43} \le 0$,

(6) $7a_{40}^3 - 36a_{40}a_{41} + 108a_{42} > 0$,

(7) $a_{40}^5 - 6a_{40}^3 a_{41} + 27a_{40}^2 a_{42} - 108a_{40}a_{43} + 324a_{44} \le 0$,

(8) $5a_{40}^3 - 36a_{40}a_{41} + 216a_{42} > 0$,

(9) $-5a_{40}^4 + 36a_{40}^2 a_{41} - 216a_{40}a_{42} + 1296a_{43} > 0$,

(10) $-5a_{40}^6 + 36a_{40}^4 a_{41} - 216a_{40}^3 a_{42} + 1296a_{40}^2 a_{43} - 7776a_{40}a_{44} + 46656a_{45} \le 0$,

then $|f_4(t)| \le 16$ *for* $0 \le t < \infty$*, where* $a_{40} = K_4(0)$*,* $a_{41} = K_4'(0)$*,* $a_{42} = K_4''(0)$*,* $a_{43} = K_4'''(0)$*,* $a_{44} = K_4^{(4)}(0)$ *and* $a_{45} = K_4^{(5)}(0)$*, (for the proof, see* [4]*).*

In the proof of Theorem 1, the equivalent of (1.5) is found by first using the Equivalence Theorem, and second it is obtained that the kernel of the new equation with unit source term related to equivalent equation satisfies the conditions of Theorem C. Thus, by using Theorems C and B, respectively, a boundary $|f_1(t)| \le 2$ for the function f_1, which is the solution of equation (1.5), is obtained [3].

Also, in the proof of Theorem 2, first, the relation which is equivalent to (1.6) is found by using the Equivalence Theorem, and second, it is obtained that the kernel of the new equation with unit source term related to equivalent equation satisfies the conditions of Theorem 1, [3]. Thus, by using Theorems 1 and B, respectively, a boundary $|f_2(t)| \le 4$ is found for the function f_2, which is the solution of equation (1.6).

By the same method, in Theorem 3, by using Theorems A, 2 and B, respectively, it is obtained that a boundary is $|f_3(t)| \le 8$ for the function f_3, which is the solution of equation (1.7), [4].

Also, by using Theorems A, 3 and B, respectively, a boundary $|f_4(t)| \le 16$ is obtained for the function f_4, which is the solution of equation (1.8), [4, Theorem 4].

Furthermore, by using the method used in the proof of Theorems 1–4 successively, it is concluded that a boundary $|f_n(t)| \le 2^n$ for the function f_n, which is the solution of the equation

(1.9)
$$f_n(t) = 1 - \int_0^t K_n(t-\tau)\, f_n(\tau)d\tau = 1 - K_n * f_n \text{ (n is an integer greater than 1)},$$

with unit source term and monotonically increasing convolution kernel, could be obtained, [4]. The sufficient conditions concerning obtaining a solution of equation (1.9), as above, are given by Theorem 2.n which is the generalization of Theorem 2.

2. The main results

In the case $a_{10}^2 \geq 4a_{11}$ of Theorem 1, the new conditions enabling the validity of the consequence of Theorem 1 are given by Theorem 2.1, below, which is different than Theorem 1 of R. Ling [3].

In the present paper, it is first obtained, in the proof of Theorem 2.2 that equation (2.5) is equivalent to equation (2.4) by the Equivalence Theorem. Besides, it is obtained that the kernel L_{12} of equation (2.7) satisfies the conditions of Theorem 2.1. Thus, by using Theorems 2.1 and B, respectively, a boundary $|f_{12}(t)| \leq 4$ is obtained for the function f_{12}, which is the solution of equation (2.4).

Also, in the proof of Theorem 2.3, first, (2.15), which is equivalent to (2.14) is found by the aid of the Equivalence Theorem, and later it is obtained that the kernel L_{13} of equation (2.17) satisfies the conditions of Theorem 2.2. Thus, by using Theorems 2.2 and B, respectively, a boundary $|f_{13}(t)| \leq 8$ is found for the function f_{13} which is the solution of equation (2.14).

Besides, in the proof of Theorem 2.4, first, (2.25), which is equivalent to (2.24), is found by using the Equivalence Theorem, and later it is obtained that the kernel L_{14} of equation (2.27) satisfies the conditions of Theorem 2.3. Thus, by using Theorems 2.3 and B, respectively, a boundary $|f_{14}(t)| \leq 16$ is found for the function f_{14} which is the solution of equation (2.24).

Furthermore, by employing the method used in the proofs of Theorems 2.2–2.4, successively, it is concluded that a boundary $|f_{1n}(t)| \leq 2^n$, for the solution function f_{1n} of any equation of the form

$$f_{1n}(t) = 1 - \int_0^t K_{1n}(t-\tau)\, f_{1n}(\tau)d\tau = 1 - K_{1n} * f_{1n}$$

with unit source term and monotonically increasing convolution kernel, has been obtained in Theorem 2.n which is the generalization of Theorem 2.1.

Also, the infinitely many numbers of kernels K_{1n} of the form

$$K_{1n}(t) = \sum_{m=0}^{n+1} c_m t^{n+1-m} + c_{n+2}e^{-t}$$

satisfying the conditions of Theorem 2.n are derived by a method.

Hereafter, we assume unless stated otherwise that $t \in [0, \infty)$ and n is an arbitrary element of $\mathbb{N} = \{1, 2, 3, \dots\}$.

Theorem 2.1 ([3, Theorem 3.2]). *Let us consider the equation of the form*

$$f_{11}(t) = 1 - \int_0^t K_{11}(t-\tau)\, f_{11}(\tau)d\tau = 1 - K_{11} * f_{11}. \tag{2.1}$$

Suppose that the conditions

(1) $K_{11}(t) > 0,\ K'_{11}(t) > 0,\ K''_{11}(t) > 0$ *and* $K'''_{11}(t) \leq 0$ *for* $0 \leq t < \infty$,

(2) $a_{110}^2 < \dfrac{9}{2}a_{111}$,

(3) $4a_{111} \leq a_{110}^2$,

(4) $2a_{110}^3 - 9a_{110}a_{111} + 27a_{112} \leq 0$,

hold. Then the solution of (2.1) *satisfies the inequality* $|f_{11}(t)| \leq 2$ *for* $0 \leq t < \infty$, *where* $a_{110} = K_{11}(0)$, $a_{111} = K'_{11}(0)$, $a_{112} = K''_{11}(0)$.

We should note that condition (2) of Theorem 2.1 is equivalent to the inequality given [3]:

$$\frac{a_{110}}{3} < \frac{a_{110} - \sqrt{a_{110}^2 - 4a_{111}}}{2}.$$

Now, we can give a function K_{11} satisfying the conditions of Theorem 2.1, as follows:

Example 2.1. If there exist the numbers $a_{110},\ a_{111},\ a_{112} > 0$ satisfying conditions (2)–(4) of Theorem 2.1, then there exists at least one function K_{11} which satisfies condition (1) of Theorem 2.1, of the form

$$K_{11}(t) = \sum_{m=0}^{2} c_m t^{2-m} + c_3 e^{-t} \tag{2.2}$$

such that

$$K_{11}(0) = a_{110},\ K_{11}'(0) = a_{111},\ K_{11}''(0) = a_{112}. \tag{2.3}$$

To see the validity of this assertion, we must first show that there exist the numbers $a_{110}, a_{111}, a_{112} > 0$ satisfying conditions (2)–(4) of Theorem 2.1. Let us choose the numbers a_{110} and a_{111} satisfying conditions (2) and (3) of Theorem 2.1 and define P_{11i} for all $i \in \mathbb{N}_3$ and γ_{11} by

$$P_{11i}(\gamma) = \sum_{k=0}^{i} (-\gamma)^{i-k} a_{11(k-1)}\ (a_{11(-1)} \equiv 1, \text{ by convention) and } \gamma_{11} = \frac{a_{110}}{3}.$$

By $\mathbb{N}_k$, hereafter we mean the set of positive integers all of which are less than or equal to $k \in \mathbb{N}$, that is, $\mathbb{N}_k = \{1, 2, 3, \ldots, k\}$.

Since condition (4) of Theorem 2.1 is equivalent to

$$P_{113}(\gamma_{11}) = -\gamma_{11} P_{112}(\gamma_{11}) + a_{112} \leq 0,$$

the number $a_{112} > 0$ can be chosen as

$$0 < a_{112} \leq \gamma_{11} P_{112}(\gamma_{11}).$$

Thus, the numbers $a_{110},\ a_{111}, a_{112}$ fulfill conditions (2)–(4) of Theorem 2.1.

The solution of the system of linear equations (2.3) which is equivalent to

$$c_2 + c_3 = a_{110},\ c_1 - c_3 = a_{111},\ 2c_0 + c_3 = a_{112}$$

is obtained as

$$(c_0, c_1, c_2, c_3) = \left(\frac{a_{112} - c_3}{2}, a_{111} + c_3, a_{110} - c_3, c_3 \right),$$

where c_3 is an arbitrary constant such that

$$0 \leq c_3 < \min\{a_{110}, a_{112}\}.$$

Hence, every function K_{11} given by (2.2) also satisfies condition (1) of Theorem 2.1.

For example, if the numbers a_{110} and a_{111} are taken as $a_{110} = \frac{21}{10}$ and $a_{111} = 1$, then $\gamma_{11} = \frac{7}{10}$ and the number a_{112} satisfying the inequality

$$0 < a_{112} \leq \gamma_{11} P_{112}(\gamma_{11}) = \frac{14}{1000}$$

can be chosen as $a_{112} = \frac{1}{100}$. Also, c_3 satisfying the inequality

$$0 \le c_3 < \min\{a_{110},\ a_{112}\}$$

can be taken as $c_3 = \frac{1}{200}$. Then,

$$c_0 = \frac{1}{400},\ c_1 = \frac{201}{200},\ c_2 = \frac{419}{200}$$

and

$$K_{11}(t) = \frac{1}{400}t^2 + \frac{201}{200}t + \frac{419}{200} + \frac{1}{200}e^{-t}.$$

Theorem 2.2. *Let us consider the equation*

$$f_{12}(t) = 1 - \int_0^t K_{12}(t-\tau) f_{12}(\tau) d\tau = 1 - K_{12} * f_{12}. \tag{2.4}$$

Suppose that the conditions

(1) $K_{12}(t) > 0,\ K'_{12}(t) > 0,\ K''_{12}(t) > 0,\ K'''_{12}(t) > 0$ *and* $K^{(4)}_{12}(t) \le 0$ *for* $0 \le t < \infty$,

(2) $a_{120}^2 < \dfrac{28}{9} a_{121}$,

(3) $3a_{121} \le a_{120}^2$,

(4) $106a_{120}^3 - 405a_{120}a_{121} + 729a_{122} \le 0$,

(5) $-2a_{120}^4 + 9a_{120}^2 a_{121} - 27a_{120}a_{122} + 81a_{123} \le 0$,

hold. Then the solution of (2.4) *satisfies the inequality* $|f_{12}(t)| \le 4$ *for* $0 \le t < \infty$, *where* $a_{120} = K_{12}(0)$, $a_{121} = K'_{12}(0)$, $a_{122} = K''_{12}(0)$, $a_{123} = K'''_{12}(0)$.

Proof. If the function g is taken as $g(t) = e^{-\gamma t}$ $(\gamma \in \mathbb{R})$ in Theorem A, then it is observed that

$$f_{12}(t) = e^{-\gamma t} - L_{12} * f_{12} \tag{2.5}$$

which is equivalent to (2.4), where

$$L_{12}(t) = (a_{120} - \gamma)\, e^{-\gamma t} + K'_{12} * e^{-\gamma t}. \tag{2.6}$$

By differentiating (2.6), it is obtained by $L'_{12}(t),\ L''_{12}(t)$ and $L'''_{12}(t)$ that

$$\begin{aligned} L'_{12}(t) &= \left(\gamma^2 - a_{120}\gamma + a_{121}\right) e^{-\gamma t} + K''_{12} * e^{-\gamma t}, \\ L''_{12}(t) &= \left(-\gamma^3 + a_{120}\gamma^2 - a_{121}\gamma + a_{122}\right) e^{-\gamma t} + K'''_{12} * e^{-\gamma t} \end{aligned}$$

and

$$L'''_{12}(t) = \left(\gamma^4 - a_{120}\gamma^3 + a_{121}\gamma^2 - a_{122}\gamma + a_{123}\right) e^{-\gamma t} + K^{(4)}_{12} * e^{-\gamma t}.$$

Now, one can see that the kernel L_{12} of

$$h_{12}(t) = 1 - L_{12} * h_{12} \tag{2.7}$$

satisfies the conditions of Theorem 2.1.

The corresponding inequalities to conditions (2) and (3) of Theorem 2.1 are

$$[L_{12}(0)]^2 < \frac{9}{2} L'_{12}(0) \tag{2.8}$$

and

$$4L'_{12}(0) \le [L_{12}(0)]^2, \tag{2.9}$$

respectively. Inequalities (2.8) and (2.9) are equivalent to

$$(a_{120}-\gamma)^2 < \frac{9}{2}\left(\gamma^2 - a_{120}\gamma + a_{121}\right) \tag{2.10}$$

and

$$4\left(\gamma^2 - a_{120}\gamma + a_{121}\right) \le (a_{120}-\gamma)^2, \tag{2.11}$$

respectively. Inequality (2.10) is equivalent to

$$q_1(\gamma) = 7\gamma^2 - 5a_{120}\gamma + 9a_{121} - 2a_{120}^2 > 0.$$

The discriminant of $q_1(\gamma)=0$ is $9(9a_{120}^2 - 28a_{121})$, which is negative by condition (2) of Theorem 2.2. Hence, $q_1(\gamma) > 0$ for every $\gamma \in \mathbb{R}$.

Inequality (2.11) is equivalent to

$$q_2(\gamma) = 3\gamma^2 - 2a_{120}\gamma + 4a_{121} - a_{120}^2 \le 0.$$

The discriminant of $q_2(\gamma)=0$ is $16(a_{120}^2 - 3a_{121})$, which is nonnegative by condition (3) of Theorem 2.2. Hence, if γ is taken as $\gamma = \gamma_{12} = \frac{a_{120}}{3}$, then $q_2(\gamma) = q_2(\gamma_{12}) \le 0$.

The inequality corresponding to condition (4) of Theorem 2.1 is

$$2[L_{12}(0)]^3 - 9L_{12}(0)L'_{12}(0) + 27L''_{12}(0) \le 0$$

or

$$\begin{aligned} q_3(\gamma) = {} & 2(a_{120}-\gamma)^3 - 9(a_{120}-\gamma)(\gamma^2 - a_{120}\gamma + a_{121}) \\ & + 27(-\gamma^3 + a_{120}\gamma^2 - a_{121}\gamma + a_{122}) \le 0. \end{aligned}$$

By condition (4) of Theorem 2.2,

$$q_3(\gamma_{12}) = \frac{1}{27}(106a_{120}^3 - 405a_{120}a_{121} + 729a_{122}) \le 0.$$

The corresponding inequalities to condition (1) of Theorem 2.1 are

$$L_{12}(t),\ L'_{12}(t),\ L''_{12}(t) > 0 \text{ and } L'''_{12}(t) \le 0.$$

Since $a_{120} - \gamma_{12}$ is positive, $L_{12}(t) > 0$ and the discriminant of $\gamma^2 - a_{120}\gamma + a_{121} = 0$ is negative by condition (2) of Theorem 2.2, $\gamma_{12}^2 - a_{120}\gamma_{12} + a_{121} > 0$, and so $L'_{12}(t) > 0$.

From condition (5) of Theorem 2.2,

$$-\gamma_{12}^3 + a_{120}\gamma_{12}^2 - a_{121}\gamma_{12} + a_{122} = \frac{1}{27}\left(2a_{120}^3 - 9a_{120}a_{121} + 27a_{122}\right) > 0$$

and

$$\begin{aligned} & \gamma_{12}^4 - a_{120}\gamma_{12}^3 + a_{121}\gamma_{12}^2 - a_{122}\gamma_{12} + a_{123} \\ & \quad = \frac{1}{81}\left(-2a_{120}^4 + 9a_{120}^2a_{121} - 27a_{120}a_{122} + 81a_{123}\right) \le 0. \end{aligned}$$

Therefore, $L''_{12}(t) > 0$, $L'''_{12}(t) \le 0$. So, the solution of equation (2.7) satisfies the inequality $|h_{12}(t)| \le 2$.

By Theorem B, the solution of equation (2.5) can be expressed as

$$f_{12}(t) = h_{12}(t) - \gamma_{12}\int_0^t h_{12}(t-\tau)e^{-\gamma_{12}\tau}\,d\tau,$$

and thus,

$$|f_{12}(t)| \le 2 + 2\gamma_{12}\int_0^t e^{-\gamma_{12}\tau}\,d\tau = 2 - 2(e^{-\gamma_{12}t} - 1) \le 4,$$

which completes the proof of Theorem 2.2. $\square$

A function K_{12} satisfying conditions (1)–(5) of Theorem 2.2 can be obtained by the following method:

Example 2.2. If there exist the numbers $a_{120},\ a_{121},\ a_{122},\ a_{123} > 0$ satisfying conditions (2)–(5) of Theorem 2.2, then there exists at least one function K_{12} which satisfies condition (1) of Theorem 2.2 of the form

$$K_{12}(t) = \sum_{m=0}^{3} c_m t^{3-m} + c_4 e^{-t} \tag{2.12}$$

such that

$$K_{12}(0) = a_{120},\ K'_{12}(0) = a_{121},\ K''_{12}(0) = a_{122},\ K'''_{12}(0) = a_{123}. \tag{2.13}$$

First, let us see that there exist the numbers $a_{120},\ a_{121},\ a_{122},\ a_{123} > 0$ satisfying conditions (2)–(5) of Theorem 2.2. It is clear that the numbers a_{120} and a_{121} satisfying conditions (2) and (3) of Theorem 2.2 exist. Now, we shall define the polynomial P_{12i} for all $i \in \mathbb{N}_4$ and the number γ_{12} by

$$P_{12i}(\gamma) = \sum_{k=0}^{i} (-\gamma)^{i-k} a_{12(k-1)} \ (a_{12(-1)} \equiv 1, \text{ by convention}) \text{ and } \gamma_{12} = \frac{a_{120}}{3}.$$

Then, it can be clearly obtained by the proof of Theorem 2.2 that the numbers a_{110} and a_{111} satisfying conditions (2) and (3) of Theorem 2.1 can be defined by

$$a_{110} = P_{121}(\gamma_{12}) \text{ and } a_{111} = P_{122}(\gamma_{12}).$$

Besides, the number $a_{112} > 0$ can be found by means of a_{110} and a_{111} as it was derived in Example 2.1. Since condition (4) of Theorem 2.2 is equivalent to

$$2[P_{121}(\gamma_{12})]^3 - 9P_{121}(\gamma_{12})P_{122}(\gamma_{12}) + 27P_{123}(\gamma_{12}) \leq 0,$$

a_{122} is found as

$$a_{122} = a_{112} + \gamma_{12}P_{122}(\gamma_{12}) = a_{112} + \gamma_{12}a_{111},$$

if $P_{123}(\gamma_{12})$ is taken as

$$P_{123}(\gamma_{12}) = -\gamma_{12}P_{122}(\gamma_{12}) + a_{122} = a_{112}.$$

Furthermore, since condition (5) of Theorem 2.2 is equivalent to

$$P_{124}(\gamma_{12}) = -\gamma_{12}P_{123}(\gamma_{12}) + a_{123} \leq 0,$$

the number $a_{123} > 0$ can be chosen as

$$0 < a_{123} \leq \gamma_{12}P_{123}(\gamma_{12}) = \gamma_{12}a_{112}.$$

Clearly, the numbers $a_{120},\ a_{121},\ a_{122},\ a_{123} > 0$, obtained above, satisfy conditions (2)–(5) of Theorem 2.2. The solution of the system of linear equations (2.13) which is equivalent to

$$c_3 + c_4 = a_{120},\ c_2 - c_4 = a_{121},\ 2c_1 + c_4 = a_{122},\ 6c_0 - c_4 = a_{123}$$

is obtained as

$$(c_0, c_1, c_2, c_3, c_4) = \left(\frac{a_{123} + c_4}{6}, \frac{a_{122} - c_4}{2}, a_{121} + c_4, a_{120} - c_4, c_4 \right),$$

where c_4 is an arbitrary constant satisfying the inequality

$$\max\{-a_{121}, -a_{123}\} < c_4 \leq 0.$$

Hence, every function K_{12} of the form (2.12) also satisfies condition (1) of Theorem 2.2.

For example, if the numbers a_{120} and a_{121} are taken as $a_{120} = \frac{26}{5}$ and $a_{121} = 9$, then $\gamma_{12} = \frac{26}{15}$,

$$a_{110} = P_{121}(\gamma_{12}) = \frac{52}{15} \text{ and } a_{111} = P_{122}(\gamma_{12}) = \frac{673}{225}.$$

By using Example 2.1, $\gamma_{11} = \frac{52}{45}$ and the number $a_{112} > 0$ satisfying the inequality

$$0 < a_{112} \leq \gamma_{11} P_{112}(\gamma_{11}) = \frac{33748}{91125}$$

can be chosen as $a_{112} = \frac{3}{10}$. Hence,

$$a_{122} = a_{112} + \gamma_{12} a_{111} = \frac{37021}{6750}.$$

Furthermore, the number $a_{123} > 0$ satisfying the inequality

$$0 < a_{123} \leq \gamma_{12} a_{112} = \frac{13}{25}$$

can be taken as $a_{123} = \frac{1}{2}$. Additionally, the number c_4 satisfying the inequality

$$\max\{-a_{121}, -a_{123}\} = \max\left\{-9, -\frac{1}{2}\right\} < c_4 \leq 0$$

can be taken as $c_4 = -\frac{1}{3}$. Therefore,

$$(c_0, c_1, c_2, c_3, c_4) = \left(\frac{1}{36}, \frac{39271}{13500}, \frac{26}{3}, \frac{83}{15}, -\frac{1}{3}\right),$$

and thus,

$$K_{12}(t) = \frac{1}{36} t^3 + \frac{39271}{13500} t^2 + \frac{26}{3} t + \frac{83}{15} - \frac{1}{3} e^{-t}.$$

Theorem 2.3. *Let us consider the equation of the form*

$$f_{13}(t) = 1 - \int_0^t K_{13}(t-\tau) f_{13}(\tau) d\tau = 1 - K_{13} * f_{13} \tag{2.14}$$

under the following assumptions:

(1) $K_{13}(t) > 0, K'_{13}(t) > 0, K''_{13}(t) > 0, K'''_{13}(t) > 0, K^{(4)}_{13}(t) > 0$ *and* $K^{(5)}_{13}(t) \leq 0$ *for* $0 \leq t < \infty$,

(2) $a_{130}^2 < \dfrac{19}{7} a_{131}$,

(3) $\dfrac{8}{3} a_{131} \leq a_{130}^2$,

(4) $161 a_{130}^3 - 576 a_{130} a_{131} + 864 a_{132} \leq 0$,

(5) $-11 a_{130}^4 + 48 a_{130}^2 a_{131} - 128 a_{130} a_{132} + 256 a_{133} \leq 0$,

(6) $3 a_{130}^3 - 16 a_{130} a_{131} + 64 a_{132} > 0$,

(7) $3 a_{130}^5 - 16 a_{130}^3 a_{131} + 64 a_{130}^2 a_{132} - 256 a_{130} a_{133} + 1024 a_{134} \leq 0$,

where $a_{130} = K_{13}(0)$, $a_{131} = K'_{13}(0)$, $a_{132} = K''_{13}(0)$, $a_{133} = K'''_{13}(0)$, $a_{134} = K^{(4)}_{13}(0)$. *Then, the solution of* (2.14) *satisfies the inequality* $|f_{13}(t)| \leq 8$ *for* $0 \leq t < \infty$.

Proof. The equation (2.14) may be written in the form $f_{13}(t) = 1 - K_{13} * f_{13}$. By taking $g(t) = e^{-\gamma t}$ ($\gamma \in \mathbb{R}$) in Theorem A, the equivalent equation is obtained as

$$f_{13}(t) = e^{-\gamma t} - L_{13} * f_{13}, \tag{2.15}$$

where

$$L_{13}(t) = (a_{130} - \gamma)e^{-\gamma t} + K'_{13} * e^{-\gamma t}. \tag{2.16}$$

By differentiating (2.16), it is found by $L'_{13}(t)$, $L''_{13}(t)$, $L'''_{13}(t)$ and $L^{(4)}_{13}(t)$ that

$$\begin{aligned}
L'_{13}(t) &= \left(\gamma^2 - a_{130}\gamma + a_{131}\right) e^{-\gamma t} + K''_{13} * e^{-\gamma t}, \\
L''_{13}(t) &= \left(-\gamma^3 + a_{130}\gamma^2 - a_{131}\gamma + a_{132}\right) e^{-\gamma t} + K'''_{13} * e^{-\gamma t}, \\
L'''_{13}(t) &= \left(\gamma^4 - a_{130}\gamma^3 + a_{131}\gamma^2 - a_{132}\gamma + a_{133}\right) e^{-\gamma t} + K^{(4)}_{13} * e^{-\gamma t}
\end{aligned}$$

and

$$L^{(4)}_{13}(t) = \left(-\gamma^5 + a_{130}\gamma^4 - a_{131}\gamma^3 + a_{132}\gamma^2 - a_{133}\gamma + a_{134}\right) e^{-\gamma t} + K^{(5)}_{13} * e^{-\gamma t}.$$

Let us turn to show that the kernel L_{13} of the equation

$$h_{13}(t) = 1 - L_{13} * h_{13} \tag{2.17}$$

satisfies all the conditions of Theorem 2.2. The corresponding inequalities to conditions (2) and (3) of Theorem 2.2 are

$$[L_{13}(0)]^2 < \frac{28}{9} L'_{13}(0) \tag{2.18}$$

and

$$3L'_{13}(0) \le [L_{13}(0)]^2, \tag{2.19}$$

respectively. Inequalities (2.18) and (2.19) are equivalent to

$$(a_{130} - \gamma)^2 < \frac{28}{9}\left(\gamma^2 - a_{130}\gamma + a_{131}\right) \tag{2.20}$$

and

$$3\left(\gamma^2 - a_{130}\gamma + a_{131}\right) \le (a_{130} - \gamma)^2. \tag{2.21}$$

Since inequality (2.20) is equivalent to

$$q_1(\gamma) = 19\gamma^2 - 10a_{130}\gamma + 28a_{131} - 9a_{130}^2 > 0$$

and the discriminant of $q_1(\gamma) = 0$ is $112(7a_{130}^2 - 19a_{131})$, which is negative by condition (2) of Theorem 2.3, $q_1(\gamma) > 0$ for every $\gamma \in \mathbb{R}$.

Since inequality (2.21) is equivalent to

$$q_2(\gamma) = 2\gamma^2 - a_{130}\gamma + 3a_{131} - a_{130}^2 \le 0$$

and the discriminant of $q_2(\gamma) = 0$ is $3(3a_{130}^2 - 8a_{131})$, which is nonnegative by condition (3) of Theorem 2.3, if γ is chosen as $\gamma = \gamma_{13} = \frac{a_{130}}{4}$, then $q_2(\gamma_{13}) \le 0$.

The inequality corresponding to condition (4) of Theorem 2.2 is

$$106[L_{13}(0)]^3 - 405L_{13}(0)L'_{13}(0) + 729L''_{13}(0) \le 0,$$

which is equivalent to

$$\begin{aligned}
q_3(\gamma) = 106\left(a_{130} - \gamma\right)^3 &- 405\left(a_{130} - \gamma\right)\left(\gamma^2 - a_{130}\gamma + a_{131}\right) \\
&+ 729\left(-\gamma^3 + a_{130}\gamma^2 - a_{131}\gamma + a_{132}\right) \le 0.
\end{aligned}$$

By condition (4) of Theorem 2.3,

$$q_3(\gamma_{13}) = \frac{27}{32}\left(161a_{130}^3 - 576a_{130}a_{131} + 864a_{132}\right) \leq 0.$$

The inequality corresponding to condition (5) of Theorem 2.2 is

$$-2[L_{13}(0)]^4 + 9[L_{13}(0)]^2 L'_{13}(0) - 27L_{13}(0)L''_{13}(0) + 81L'''_{13}(0) \leq 0,$$

which is equivalent to

$$\begin{aligned} q_4(\gamma) \quad = \quad & -2\left(a_{130}-\gamma\right)^4 + 9\left(a_{130}-\gamma\right)^2\left(\gamma^2 - a_{130}\gamma + a_{131}\right) \\ & -27\left(a_{130}-\gamma\right)\left(-\gamma^3 + a_{130}\gamma^2 - a_{131}\gamma + a_{132}\right) + \\ & +81\left(\gamma^4 - a_{130}\gamma^3 + a_{131}\gamma^2 - a_{132}\gamma + a_{133}\right) \leq 0. \end{aligned}$$

By condition (5) of Theorem 2.3,

$$q_4(\gamma_{13}) = \frac{81}{256}\left(-11a_{130}^4 + 48a_{130}^2 a_{131} - 128a_{130}a_{132} + 256a_{133}\right) \leq 0.$$

Furthermore, $a_{130} - \gamma_{13}$ is positive. Thus, $L_{13}(t) > 0$. Since the discriminant of $\gamma^2 - a_{130}\gamma + a_{131} = 0$ is negative by condition (2) of Theorem 2.3, $\gamma_{13}^2 - a_{130}\gamma_{13} + a_{131} > 0$ and so, $L'_{13}(t) > 0$. Additionally, by condition (6) of Theorem 2.3,

$$-\gamma_{13}^3 + a_{130}\gamma_{13}^2 - a_{131}\gamma_{13} + a_{132} = \frac{1}{64}\left(3a_{130}^3 - 16a_{130}a_{131} + 64a_{132}\right) > 0$$

and thus, $L''_{13}(t) > 0$. By condition (7) of Theorem 2.3,

$$\begin{aligned} & \gamma_{13}^4 - a_{130}\gamma_{13}^3 + a_{131}\gamma_{13}^2 - a_{132}\gamma_{13} + a_{133} \\ & \quad = \frac{1}{256}\left(-3a_{130}^4 + 16a_{130}^2 a_{131} - 64a_{130}a_{132} + 256a_{133}\right) > 0 \end{aligned}$$

and

$$\begin{aligned} & -\gamma_{13}^5 + a_{130}\gamma_{13}^4 - a_{131}\gamma_{13}^3 + a_{132}\gamma_{13}^2 - a_{133}\gamma_{13} + a_{134} \\ & \quad = \frac{1}{1024}\left(3a_{130}^5 - 16a_{130}^3 a_{131} + 64a_{130}^2 a_{132} - 256a_{130}a_{133} + 1024a_{134}\right) \leq 0, \end{aligned}$$

and so, $L'''_{13}(t) > 0$ and $L_{13}^{(4)}(t) \leq 0$.

In conclusion, L_{13} satisfies all the conditions of Theorem 2.2 and therefore, the solution of (2.17) satisfies the inequality $|h_{13}(t)| \leq 4$. Using the Convolution Theorem, the solution of (2.15) is found as

$$f_{13}(t) = h_{13}(t) - \gamma_{13}\int_0^t h_{13}(t-\tau)e^{-\gamma_{13}\tau}d\tau,$$

and hence

$$|f_{13}(t)| \leq 4 + 4\gamma_{13}\int_0^t e^{-\gamma_{13}\tau}d\tau = 4 - 4(e^{-\gamma_{13}t} - 1) \leq 8. \qquad \square$$

At least one function K_{13} satisfying conditions (1)–(7) of Theorem 2.3 can be obtained using the following method:

Example 2.3. If there exist the numbers $a_{130},\ a_{131},\ a_{132},\ a_{133},\ a_{134} > 0$ satisfying conditions (2)–(7) of Theorem 2.3, then there exists at least one function K_{13} of the form

$$K_{13}(t) = \sum_{m=0}^{4} c_m t^{4-m} + c_5 e^{-t} \tag{2.22}$$

such that
(2.23)

$$K_{13}(0) = a_{130}, K'_{13}(0) = a_{131}, K''_{13}(0) = a_{132}, K'''_{13}(0) = a_{133}, K^{(4)}_{13}(0) = a_{134}$$

and K_{13} also satisfies condition (1) of Theorem 2.3.

To see the validity of this assertion let us first choose the numbers a_{130} and a_{131} satisfying conditions (2) and (3) of Theorem 2.3 and define the polynomial P_{13i} for all $i \in \mathbb{N}_5$ and the number γ_{13}, as follows:

$$P_{13i}(\gamma) = \sum_{k=0}^{i} (-\gamma)^{i-k} a_{13(k-1)} \ (a_{13(-1)} \equiv 1, \text{ by convention}) \text{ and } \gamma_{13} = \frac{a_{130}}{4}.$$

Then, it can be clearly seen from the proof of Theorem 2.3 that the numbers a_{120} and a_{121} defined by the equalities

$$a_{120} = P_{131}(\gamma_{13}) \text{ and } a_{121} = P_{132}(\gamma_{13})$$

satisfy conditions (2) and (3) of Theorem 2.2. It is of course that the numbers a_{122}, a_{123} can also be found by means of a_{120}, a_{121} from Example 2.2.

Besides, since condition (4) of Theorem 2.3 is equivalent to

$$106[P_{131}(\gamma_{13})]^3 - 405 P_{131}(\gamma_{13}) P_{132}(\gamma_{13}) + 729 P_{133}(\gamma_{13}) \leq 0,$$

if $P_{133}(\gamma_{13})$ is taken as

$$P_{133}(\gamma_{13}) = -\gamma_{13} P_{132}(\gamma_{13}) + a_{132} = a_{122},$$

then the number a_{132} is obtained as

$$a_{132} = a_{122} + \gamma_{13} P_{132}(\gamma_{13}) = a_{122} + \gamma_{13} a_{121} > 0.$$

Since condition (5) of Theorem 2.3 is equivalent to

$$-2[P_{131}(\gamma_{13})]^4 + 9[P_{131}(\gamma_{13})]^2 P_{132}(\gamma_{13}) - 27 P_{131}(\gamma_{13}) P_{133}(\gamma_{13}) + 81 P_{134}(\gamma_{13}) \leq 0,$$

if $P_{134}(\gamma_{13})$ is taken as

$$P_{134}(\gamma_{13}) = -\gamma_{13} P_{133}(\gamma_{13}) + a_{133} = a_{123},$$

then the number a_{133} is found as

$$a_{133} = a_{123} + \gamma_{13} P_{133}(\gamma_{13}) = a_{123} + \gamma_{13} a_{122} > 0.$$

Condition (6) of Theorem 2.3 is equivalent to $P_{133}(\gamma_{13}) > 0$ and since $P_{133}(\gamma_{13})$ is equal to a_{122} which is positive, the numbers $a_{130},\ a_{131}$ and a_{132}, found as above, satisfy condition (6) of Theorem 2.3, as well.

Finally, since condition (7) of Theorem 2.3 is equivalent to

$$P_{135}(\gamma_{13}) = -\gamma_{13} P_{134}(\gamma_{13}) + a_{134} \leq 0,$$

the number $a_{134} > 0$ can be chosen as

$$0 < a_{134} \leq \gamma_{13} P_{134}(\gamma_{13}) = \gamma_{13} a_{123}.$$

The numbers $a_{130},\ a_{131},\ a_{132},\ a_{133},\ a_{134} > 0$ obtained by using the above method satisfy conditions (2)–(7) of Theorem 2.3. The solution of the system of linear equations (2.23) which is equivalent to

$$c_4 + c_5 = a_{130},\ c_3 - c_5 = a_{131},\ 2c_2 + c_5 = a_{132},\ 6c_1 - c_5 = a_{133},\ 24c_0 + c_5 = a_{134}$$

is obtained as

$$(c_0, c_1, c_2, c_3, c_4, c_5) = \left(\frac{a_{134} - c_5}{24},\ \frac{a_{133} + c_5}{6},\ \frac{a_{132} - c_5}{2},\ a_{131} + c_5,\ a_{130} - c_5,\ c_5\right),$$

where c_5 is an arbitrary constant such that

$$0 \leq c_5 < \min\{a_{130},\ a_{132},\ a_{134}\}.$$

So, every function K_{13} of the form (2.22) also satisfies condition (1) of Theorem 2.3. For example, if the numbers a_{130} and a_{131} are chosen as $a_{130} = \frac{15}{2}$ and $a_{131} = 21$, then

$$\gamma_{13} = \frac{15}{8},\ a_{120} = P_{131}(\gamma_{13}) = \frac{45}{8},\ a_{121} = P_{132}(\gamma_{13}) = \frac{669}{64}.$$

From Example 2.2,

$$\gamma_{12} = \frac{15}{8},\ a_{110} = P_{121}(\gamma_{12}) = \frac{15}{4},\ a_{111} = P_{122}(\gamma_{12}) = \frac{219}{64}.$$

From Example 2.1, $\gamma_{11} = \frac{5}{4}$ and the number a_{112} satisfying the inequality

$$0 < a_{112} \leq \gamma_{11} P_{112}(\gamma_{11}) = \frac{95}{256}$$

can be taken as $a_{112} = \frac{3}{10}$. Thus, from Example 2.2,

$$a_{122} = a_{112} + \gamma_{12} a_{111} = \frac{32853}{10};$$

the number a_{123} satisfying the inequality

$$0 < a_{123} \leq \gamma_{12} a_{112} = \frac{9}{16}$$

can be taken as $a_{123} = \frac{1}{2}$. Hence,

$$\begin{aligned} a_{132} &= a_{122} + \gamma_{13} a_{121} = \frac{8460543}{2560}, \\ a_{133} &= a_{123} + \gamma_{13} a_{122} = \frac{98567}{16}. \end{aligned}$$

Furthermore, the number a_{134} satisfying the inequality

$$0 < a_{134} \leq \gamma_{13} a_{123} = \frac{15}{16}$$

can be chosen as $a_{134} = \frac{9}{10}$. The number c_5 satisfying the inequality

$$0 \leq c_5 < \min\{a_{130}, a_{132}, a_{134}\} = \min\left\{\frac{15}{2}, \frac{8460543}{2560}, \frac{9}{10}\right\}$$

can be chosen as $c_5 = \frac{4}{5}$. Thus, we have

$$(c_0, c_1, c_2, c_3, c_4, c_5) = \left(\frac{1}{240}, \frac{492899}{480}, \frac{1691699}{1024}, \frac{109}{5}, \frac{67}{10}, \frac{4}{5}\right)$$

and

$$K_{13}(t) = \frac{1}{240}t^4 + \frac{492899}{480}t^3 + \frac{1691699}{1024}t^2 + \frac{109}{5}t + \frac{67}{10} + \frac{4}{5}e^{-t}.$$

Theorem 2.4. *Let us consider the equation*

$$f_{14}(t) = 1 - \int_0^t K_{14}(t-\tau) f_{14}(\tau) d\tau = 1 - K_{14} * f_{14}. \tag{2.24}$$

Suppose that the conditions

(1) $K_{14}(t) > 0, K'_{14}(t) > 0, K''_{14}(t) > 0, K'''_{14}(t) > 0, K^{(4)}_{14}(t) > 0, K^{(5)}_{14}(t) > 0$ *and* $K^{(6)}_{14}(t) \leq 0$ *for* $0 \leq t < \infty$,

(2) $a_{140}^2 < \dfrac{48}{19} a_{141}$,

(3) $\dfrac{5}{2} a_{141} \leq a_{140}^2$,

(4) $718 a_{140}^3 - 2475 a_{140} a_{141} + 3375 a_{142} \leq 0$,

(5) $-7 a_{140}^4 + 30 a_{140}^2 a_{141} - 75 a_{140} a_{142} + 125 a_{143} \leq 0$,

(6) $11 a_{140}^3 - 50 a_{140} a_{141} + 125 a_{142} > 0$,

(7) $19 a_{140}^5 - 100 a_{140}^3 a_{141} + 375 a_{140}^2 a_{142} - 1250 a_{140} a_{143} + 3125 a_{144} \leq 0$,

(8) $4 a_{140}^3 - 25 a_{140} a_{141} + 125 a_{142} > 0$,

(9) $-4 a_{140}^4 + 25 a_{140}^2 a_{141} - 125 a_{140} a_{142} + 625 a_{143} > 0$,

(10) $-4 a_{140}^6 + 25 a_{140}^4 a_{141} - 125 a_{140}^3 a_{142} + 625 a_{140}^2 a_{143} - 3125 a_{140} a_{144} + 15625 a_{145} \leq 0$,

hold. Then the solution of (2.24) *satisfies the inequality* $|f_{14}(t)| \leq 16$ *for* $0 \leq t < \infty$, *where* $a_{140} = K_{14}(0)$, $a_{141} = K'_{14}(0)$, $a_{142} = K''_{14}(0)$, $a_{143} = K'''_{14}(0)$, $a_{144} = K^{(4)}_{14}(0)$, $a_{145} = K^{(5)}_{14}(0)$.

Proof. If the function g is taken as $g(t) = e^{-\gamma t}$ ($\gamma \in \mathbb{R}$) in Theorem A, then the equation which is equivalent to (2.24) is found as

$$f_{14}(t) = e^{-\gamma t} - L_{14} * f_{14}, \tag{2.25}$$

where

$$L_{14}(t) = (a_{140} - \gamma) e^{-\gamma t} + K'_{14} * e^{-\gamma t}. \tag{2.26}$$

By differentiating (2.26), it is obtained by $L'_{14}(t)$, $L''_{14}(t)$, $L'''_{14}(t)$, $L^{(4)}_{14}(t)$ and $L^{(5)}_{14}(t)$ that

$$\begin{aligned}
L'_{14}(t) &= \left(\gamma^2 - a_{140}\gamma + a_{141}\right) e^{-\gamma t} + K''_{14} * e^{-\gamma t}, \\
L''_{14}(t) &= \left(-\gamma^3 + a_{140}\gamma^2 - a_{141}\gamma + a_{142}\right) e^{-\gamma t} + K'''_{14} * e^{-\gamma t}, \\
L'''_{14}(t) &= \left(\gamma^4 - a_{140}\gamma^3 + a_{141}\gamma^2 - a_{142}\gamma + a_{143}\right) e^{-\gamma t} + K^{(4)}_{14} * e^{-\gamma t}, \\
L^{(4)}_{14}(t) &= \left(-\gamma^5 + a_{140}\gamma^4 - a_{141}\gamma^3 + a_{142}\gamma^2 - a_{143}\gamma + a_{144}\right) e^{-\gamma t} + K^{(5)}_{14} * e^{-\gamma t}
\end{aligned}$$

and

$$L^{(5)}_{14}(t) = \left(\gamma^6 - a_{140}\gamma^5 + a_{141}\gamma^4 - a_{142}\gamma^3 + a_{143}\gamma^2 - a_{144}\gamma + a_{145}\right) e^{-\gamma t} + K^{(6)}_{14} * e^{-\gamma t}.$$

Now, one can see that the kernel L_{14} of the equation

$$h_{14}(t) = 1 - L_{14} * h_{14} \tag{2.27}$$

satisfies the conditions of Theorem 2.3.

The corresponding inequalities to conditions (2) and (3) of Theorem 2.3 are

$$[L_{14}(0)]^2 < \frac{19}{7} L'_{14}(0) \tag{2.28}$$

and

$$\frac{8}{3} L'_{14}(0) \leq [L_{14}(0)]^2, \tag{2.29}$$

respectively. Inequalities (2.28) and (2.29) are equivalent to

$$(a_{140} - \gamma)^2 < \frac{19}{7} \left(\gamma^2 - a_{140}\gamma + a_{141}\right) \tag{2.30}$$

and

$$\frac{8}{3} \left(\gamma^2 - a_{140}\gamma + a_{141}\right) \leq (a_{140} - \gamma)^2 , \tag{2.31}$$

respectively. Inequality (2.30) is equivalent to

$$q_1(\gamma) = 12\gamma^2 - 5a_{140}\gamma + 19a_{141} - 7a_{140}^2 > 0.$$

The discriminant of $q_1(\gamma) = 0$ is $19(19a_{140}^2 - 48a_{141})$ which is negative by condition (2) of Theorem 2.4. Hence, $q_1(\gamma) > 0$ for every $\gamma \in \mathbb{R}$.

Inequality (2.31) is equivalent to

$$q_2(\gamma) = 5\gamma^2 - 2a_{140}\gamma + 8a_{141} - 3a_{140}^2 \leq 0.$$

The discriminant of $q_2(\gamma) = 0$ is $32(2a_{140}^2 - 5a_{141})$, which is nonnegative by condition (3) of Theorem 2.4. Hence, $q_2(\gamma_{14}) \leq 0$ whenever γ is chosen as $\gamma = \gamma_{14} = \frac{a_{140}}{5}$.

The corresponding inequalities to conditions (4)–(7) of Theorem 2.3 are

$$\begin{gathered}
161[L_{14}(0)]^3 - 576L_{14}(0)L'_{14}(0) + 864L''_{14}(0) \leq 0, \\
-11[L_{14}(0)]^4 + 48[L_{14}(0)]^2L'_{14}(0) - 128L_{14}(0)L''_{14}(0) + 256L'''_{14}(0) \leq 0, \\
3[L_{14}(0)]^3 - 16L_{14}(0)L'_{14}(0) + 64L''_{14}(0) > 0, \\
3[L_{14}(0)]^5 - 16[L_{14}(0)]^3L'_{14}(0) + 64[L_{14}(0)]^2L''_{14}(0) \\
-256L_{14}(0)L'''_{14}(0) + 1024L^{(4)}_{14}(0) \leq 0,
\end{gathered}$$

respectively. These inequalities are equivalent to

$$\begin{aligned}
q_3(\gamma) &= 161\left(a_{140} - \gamma\right)^3 - 576\left(a_{140} - \gamma\right)\left(\gamma^2 - a_{140}\gamma + a_{141}\right) \\
&\quad +864\left(-\gamma^3 + a_{140}\gamma^2 - a_{141}\gamma + a_{142}\right) \leq 0, \\
q_4(\gamma) &= -11\left(a_{140} - \gamma\right)^4 + 48\left(a_{140} - \gamma\right)^2\left(\gamma^2 - a_{140}\gamma + a_{141}\right) \\
&\quad -128\left(a_{140} - \gamma\right)\left(-\gamma^3 + a_{140}\gamma^2 - a_{141}\gamma + a_{142}\right) \\
&\quad +256\left(\gamma^4 - a_{140}\gamma^3 + a_{141}\gamma^2 - a_{142}\gamma + a_{143}\right) \leq 0, \\
q_5(\gamma) &= 3\left(a_{140} - \gamma\right)^3 - 16\left(a_{140} - \gamma\right)\left(\gamma^2 - a_{140}\gamma + a_{141}\right) \\
&\quad +64\left(-\gamma^3 + a_{140}\gamma^2 - a_{141}\gamma + a_{142}\right) > 0, \\
q_6(\gamma) &= 3\left(a_{140} - \gamma\right)^5 - 16\left(a_{140} - \gamma\right)^3\left(\gamma^2 - a_{140}\gamma + a_{141}\right) \\
&\quad +64\left(a_{140} - \gamma\right)^2\left(-\gamma^3 + a_{140}\gamma^2 - a_{141}\gamma + a_{142}\right) \\
&\quad -256\left(a_{140} - \gamma\right)\left(\gamma^4 - a_{140}\gamma^3 + a_{141}\gamma^2 - a_{142}\gamma + a_{143}\right) \\
&\quad +1024\left(-\gamma^5 + a_{140}\gamma^4 - a_{141}\gamma^3 + a_{142}\gamma^2\right. \\
&\quad \left. -a_{143}\gamma + a_{144}\right) \leq 0,
\end{aligned}$$

respectively. By conditions (4), (5), (6) and (7) of Theorem 2.4, respectively, it is obtained by $q_3(\gamma_{14})$, $q_4(\gamma_{14})$, $q_5(\gamma_{14})$ and $q_6(\gamma_{14})$ that

$$\begin{aligned} q_3(\gamma_{14}) &= \frac{32}{125}\left(718a_{140}^3 - 2475a_{140}a_{141} + 3375a_{142}\right) \le 0, \\ q_4(\gamma_{14}) &= \frac{256}{125}\left(-7a_{140}^4 + 30a_{140}^2a_{141} - 75a_{140}a_{142} + 125a_{143}\right) \le 0, \\ q_5(\gamma_{14}) &= \frac{64}{125}\left(11a_{140}^3 - 50a_{140}a_{141} + 125a_{142}\right) > 0 \end{aligned}$$

and

$$q_6(\gamma_{14}) = \frac{1024}{3125}\left(19a_{140}^5 - 100a_{140}^3a_{141} + 375a_{140}^2a_{142} - 1250a_{140}a_{143} + 3125a_{144}\right) \le 0.$$

The corresponding inequalities to condition (1) of Theorem 2.3 are $L_{14}(t)$, $L'_{14}(t)$, $L''_{14}(t)$, $L'''_{14}(t)$, $L^{(4)}_{14}(t) > 0$, $L^{(5)}_{14}(t) \le 0$.

Since $a_{140} - \gamma_{14}$ is positive, $L_{14}(t) > 0$ and the discriminant of $\gamma^2 - a_{140}\gamma + a_{141} = 0$ is negative by condition (2) of Theorem 2.4, $\gamma_{14}^2 - a_{140}\gamma_{14} + a_{141} > 0$ and so, $L'_{14}(t) > 0$.

By condition (8) of Theorem 2.4,

$$-\gamma_{14}^3 + a_{140}\gamma_{14}^2 - a_{141}\gamma_{14} + a_{142} = \frac{1}{125}\left(4a_{140}^3 - 25a_{140}a_{141} + 125a_{142}\right) > 0.$$

Therefore, $L''_{14}(t) > 0$.

By condition (9) of Theorem 2.4,

$$\begin{aligned} &\gamma_{14}^4 - a_{140}\gamma_{14}^3 + a_{141}\gamma_{14}^2 - a_{142}\gamma_{14} + a_{143} \\ &\qquad = \frac{1}{625}\left(-4a_{140}^4 + 25a_{140}^2a_{141} - 125a_{140}a_{142} + 625a_{143}\right) > 0. \end{aligned}$$

Thus, $L'''_{14}(t) > 0$.

From condition (10) of Theorem 2.4,

$$\begin{aligned} &-\gamma_{14}^5 + a_{140}\gamma_{14}^4 - a_{141}\gamma_{14}^3 + a_{142}\gamma_{14}^2 - a_{143}\gamma_{14} + a_{144} \\ &\quad = \frac{1}{3125}\left(4a_{140}^5 - 25a_{140}^3a_{141} + 125a_{140}^2a_{142} - 625a_{140}a_{143} + 3125a_{144}\right) > 0, \\ &\gamma_{14}^6 - a_{140}\gamma_{14}^5 + a_{141}\gamma_{14}^4 - a_{142}\gamma_{14}^3 + a_{143}\gamma_{14}^2 - a_{144}\gamma_{14} + a_{145} \\ &\quad = \frac{1}{15625}\left(-4a_{140}^6 + 25a_{140}^4a_{141} - 125a_{140}^3a_{142}\right. \\ &\qquad\qquad \left. + 625a_{140}^2a_{143} - 3125a_{140}a_{144} + 15625a_{145}\right) \le 0. \end{aligned}$$

So, $L^{(4)}_{14}(t) > 0$, $L^{(5)}_{14}(t) \le 0$. Hence, the solution of (2.27) satisfies the inequality $|h_{14}(t)| \le 8$ by Theorem 2.3. By Theorem B, the solution of (2.25) can be expressed as

$$f_{14}(t) = h_{14}(t) - \gamma_{14}\int_0^t h_{14}(t-\tau)e^{-\gamma_{14}\tau}d\tau$$

and thus,

$$|f_{14}(t)| \le 8 + 8\gamma_{14}\int_0^t e^{-\gamma_{14}\tau}d\tau = 8 - 8(e^{-\gamma_{14}t} - 1) \le 16,$$

which completes the proof of Theorem 2.4. □

A function K_{14} satisfying conditions (1)–(10) of Theorem 2.4 can be obtained by the following method:

Example 2.4. If there exist the numbers $a_{140},\ a_{141},\ a_{142},\ a_{143},\ a_{144},\ a_{145} > 0$ satisfying conditions (2)–(10) of Theorem 2.4, then there exists at least one function K_{14} which satisfies condition (1) of Theorem 2.4 in the form

$$K_{14}(t) = \sum_{m=0}^{5} c_m t^{5-m} + c_6 e^{-t} \tag{2.32}$$

such that

$$\begin{gathered} K_{14}(0) = a_{140}, K_{14}'(0) = a_{141}, K_{14}''(0) = a_{142}, \\ K_{14}'''(0) = a_{143}, K_{14}^{(4)}(0) = a_{144},\ K_{14}^{(5)}(0) = a_{145}. \end{gathered} \tag{2.33}$$

First, let us show that there exist the numbers $a_{140},\ a_{141},\ a_{142},\ a_{143},\ a_{144},$ $a_{145} > 0$ satisfying conditions (2)–(10) of Theorem 2.4. It is clear that the numbers a_{140} and a_{141} satisfying conditions (2) and (3) of Theorem 2.4 exist.

Now, we shall define the polynomial P_{14i} for all $i \in \mathbb{N}_6$ and the number γ_{14} by

$$P_{14i}(\gamma) = \sum_{k=0}^{i} (-\gamma)^{i-k} a_{14(k-1)}\ (a_{14(-1)} \equiv 1,\ \text{by convention) and } \gamma_{14} = \frac{a_{140}}{5}.$$

Then, it can be clearly observed from the proof of Theorem 2.4 that the numbers a_{130} and a_{131} defined by

$$a_{130} = P_{141}(\gamma_{14}) \text{ and } a_{131} = P_{142}(\gamma_{14})$$

satisfy conditions (2) and (3) of Theorem 2.3. Besides, the numbers $a_{132}, a_{133},\ a_{134}$ can also be found by means of a_{130} and a_{131} as those were found in Example 2.3. Since condition (4) of Theorem 2.4 is equivalent to

$$161[P_{141}(\gamma_{14})]^3 - 576 P_{141}(\gamma_{14}) P_{142}(\gamma_{14}) + 864 P_{143}(\gamma_{14}) \leq 0,$$

if $P_{143}(\gamma_{14})$ is taken as

$$P_{143}(\gamma_{14}) = -\gamma_{14} P_{142}(\gamma_{14}) + a_{142} = a_{132},$$

then the number a_{142} is obtained as

$$a_{142} = a_{132} + \gamma_{14} P_{142}(\gamma_{14}) = a_{132} + \gamma_{14} a_{131} > 0.$$

Since condition (5) of Theorem 2.4 is equivalent to

$$-11[P_{141}(\gamma_{14})]^4 + 48[P_{141}(\gamma_{14})]^2 P_{142}(\gamma_{14}) - 128 P_{141}(\gamma_{14}) P_{143}(\gamma_{14}) + 256 P_{144}(\gamma_{14}) \leq 0,$$

if $P_{144}(\gamma_{14})$ is taken as

$$P_{144}(\gamma_{14}) = -\gamma_{14} P_{143}(\gamma_{14}) + a_{143} = a_{133},$$

then the number a_{143} is found as

$$a_{143} = a_{133} + \gamma_{14} P_{143}(\gamma_{14}) = a_{133} + \gamma_{14} a_{132} > 0.$$

Because condition (6) of Theorem 2.4 is equivalent to

$$3[P_{141}(\gamma_{14})]^3 - 16 P_{141}(\gamma_{14}) P_{142}(\gamma_{14}) + 64 P_{143}(\gamma_{14}) = 3a_{130}^3 - 16 a_{130} a_{131} + 64 a_{132} > 0$$

and $3a_{130}^3 - 16 a_{130} a_{131} + 64 a_{132}$ is positive from Example 2.3, the numbers $a_{140},\ a_{141},$ a_{142}, found as above, satisfy condition (6) of Theorem 2.4, as well. Since condition (7) of Theorem 2.4 is equivalent to

$$\begin{gathered} 3[P_{141}(\gamma_{14})]^5 - 16[P_{141}(\gamma_{14})]^3 P_{142}(\gamma_{14}) + 64[P_{141}(\gamma_{14})]^2 P_{143}(\gamma_{14}) \\ -256 P_{141}(\gamma_{14}) P_{144}(\gamma_{14}) + 1024 P_{145}(\gamma_{14}) \leq 0, \end{gathered}$$

if $P_{145}(\gamma_{14})$ is taken as

$$P_{145}(\gamma_{14}) = -\gamma_{14}P_{144}(\gamma_{14}) + a_{144} = a_{134},$$

then the number a_{144} is found as

$$a_{144} = a_{134} + \gamma_{14}P_{144}(\gamma_{14}) = a_{134} + \gamma_{14}a_{133} > 0.$$

Since condition (8) of Theorem 2.4 is equivalent to $P_{143}(\gamma_{14}) > 0$ and $P_{143}(\gamma_{14})$ is equal to a_{132}, the numbers $a_{140}, a_{141},\ a_{142}$, obtained as above, satisfy condition (8) of Theorem 2.4, as well. Because condition (9) of Theorem 2.4 is equivalent to $P_{144}(\gamma_{14}) > 0$ and $P_{144}(\gamma_{14})$ is equal to a_{133}, the numbers $a_{140}, a_{141},\ a_{142},\ a_{143}$, found as above, satisfy condition (9) of Theorem 2.4, as well.

Finally, since condition (10) of Theorem 2.4 is equivalent to

$$P_{146}(\gamma_{14}) = -\gamma_{14}P_{145}(\gamma_{14}) + a_{145} \leq 0,$$

the number $a_{145} > 0$ which is chosen as

$$0 < a_{145} \leq \gamma_{14}P_{145}(\gamma_{14}) = \gamma_{14}a_{134}$$

and the numbers $a_{140}, a_{141},\ a_{142},\ a_{143},\ a_{144}$ satisfy condition (10) of Theorem 2.4.

Clearly, the obtained numbers $a_{140},\ a_{141},\ a_{142},\ a_{143},\ a_{144}$ and a_{145} by presented method satisfy conditions (2)–(10) of Theorem 2.4.

The solution of the system of linear equations (2.33) which is equivalent to

$$c_5 + c_6 = a_{140},\ c_4 - c_6 = a_{141},\ 2c_3 + c_6 = a_{142},$$
$$6c_2 - c_6 = a_{143},\ 24c_1 + c_6 = a_{144},\ 120c_0 - c_6 = a_{145}$$

is obtained as

$$(c_0, c_1, c_2, c_3, c_4, c_5, c_6) = \left(\frac{a_{145} + c_6}{120},\ \frac{a_{144} - c_6}{24},\ \frac{a_{143} + c_6}{6},\ \frac{a_{142} - c_6}{2},\ a_{141} + c_6,\ a_{140} - c_6,\ c_6\right),$$

where c_6 is an arbitrary constant satisfying the inequality

$$\max\{-a_{141},\ -a_{143},\ -a_{145}\} < c_6 \leq 0.$$

Thus, every function K_{14} of the form (2.32) also satisfies condition (1) of Theorem 2.4. For example, if the numbers a_{140} and a_{141} are taken as $a_{140} = \frac{173}{25}$ and $a_{141} = 19$, then $\gamma_{14},\ a_{130},\ a_{131}$ are obtained as

$$\gamma_{14} = \frac{173}{125},\ a_{130} = P_{141}(\gamma_{14}) = \frac{692}{125},\ a_{131} = P_{142}(\gamma_{14}) = \frac{177159}{15625}.$$

From Examples 2.3 and 2.2, respectively, we derive that

$$\gamma_{13} = \frac{173}{125},\ a_{120} = P_{131}(\gamma_{13}) = \frac{519}{125},\ a_{121} = P_{132}(\gamma_{13}) = \frac{87372}{15625}$$

and

$$\gamma_{12} = \frac{173}{125},\ a_{110} = P_{121}(\gamma_{12}) = \frac{346}{125},\ a_{111} = P_{122}(\gamma_{12}) = \frac{27514}{15625}.$$

From Example 2.1, $\gamma_{11} = \frac{346}{375}$, and the positive number a_{112} satisfying the inequality

$$0 < a_{112} \leq \gamma_{11}P_{112}(\gamma_{11}) = \frac{2835124}{52734375}$$

can be taken as $a_{112} = \frac{1}{25}$. Hence, one can derive from Example 2.2 that

$$a_{122} = a_{112} + \gamma_{12}a_{111} = \frac{4338047}{1953125}$$

and the number a_{123} satisfying the inequality

$$0 < a_{123} \leq \gamma_{12} a_{112} = \frac{173}{3125}$$

can be taken as $a_{123} = \frac{1}{25}$. From Example 2.3, we have that

$$\begin{aligned} a_{132} &= a_{122} + \gamma_{13} a_{121} = \frac{19953403}{1953125}, \\ a_{133} &= a_{123} + \gamma_{13} a_{122} = \frac{846747756}{244140625}. \end{aligned}$$

Furthermore, the number a_{134} satisfying the inequality

$$0 < a_{134} \leq \gamma_{13} a_{123} = \frac{173}{3125}$$

can be chosen as $a_{134} = \frac{1}{25}$. So,

$$\begin{aligned} a_{142} &= a_{132} + \gamma_{14} a_{131} = \frac{10120382}{390625}, \\ a_{143} &= a_{133} + \gamma_{14} a_{132} = \frac{171947459}{9765625}, \\ a_{144} &= a_{134} + \gamma_{14} a_{133} = \frac{147708064913}{30517578125} \end{aligned}$$

and the number a_{145} satisfying the inequality

$$0 < a_{145} \leq \gamma_{14} a_{134} = \frac{173}{3125}$$

can be chosen as $a_{145} = \frac{1}{25}$. The number c_6 satisfying the inequality

$$\max\{-a_{141}, -a_{143}, -a_{145}\} = \max\left\{-19, -\frac{171947459}{9765625}, -\frac{1}{25}\right\} < c_6 \leq 0$$

can be chosen as $c_6 = -\frac{1}{125}$. Thus, we have

$$\begin{aligned} &(c_0, c_1, c_2, c_3, c_4, c_5, c_6) \\ &\quad = \left(\frac{1}{3750}, \frac{73976102769}{366210937500}, \frac{28644889}{9765625}, \frac{10123507}{781250}, \frac{2374}{125}, \frac{866}{125}, -\frac{1}{125}\right) \end{aligned}$$

and

$$\begin{aligned} K_{14}(t) = &\frac{1}{3750} t^5 + \frac{73976102769}{366210937500} t^4 + \frac{28644889}{9765625} t^3 \\ &+ \frac{10123507}{781250} t^2 + \frac{2374}{125} t + \frac{866}{125} - \frac{1}{125} e^{-t}. \end{aligned}$$

By continuing this process for $n \in \mathbb{N}$, we have Theorem 2.n and Example 2.n which may be stated, as follows:

Theorem 2.n. *Let us consider the equation of the form*

$$f_{1n}(t) = 1 - \int_0^t K_{1n}(t-\tau)\, f_{1n}(\tau) d\tau = 1 - K_{1n} * f_{1n}. \tag{2.34}$$

Suppose that the conditions

$$(1)\quad K_{1n}(t) > 0,\ K'_{1n}(t) > 0, \ldots, K_{1n}^{(n+1)}(t) > 0 \text{ and } K_{1n}^{(n+2)}(t) \leq 0, \text{ for } 0 \leq t < \infty,$$

$$(2)\quad a_{1n0}^2 < \frac{2(5n+4)}{5n-1} a_{1n1}$$

and

$$(3)\quad \frac{2(n+1)}{n} a_{1n1} \leq a_{1n0}^2$$

hold. Furthermore, we assume that conditions $(4)-(4+t_{1n})$ *of Theorem* 2.n *are the inequalities obtained by taking*

$$P_{1n1}(\gamma_{1n}), P_{1n2}(\gamma_{1n}), \ldots, P_{1n(n+1)}(\gamma_{1n}),$$

respectively instead of the constants $a_{1(n-1)0},\ a_{1(n-1)1}, \ldots, a_{1(n-1)n}$ *in conditions* $(4)-(4+t_{1n})$ *of Theorem* 2.$(n-1)$ *for* $n \geq 2$. *Let conditions* $(5+t_{1n})-(2+n+t_{1n})$ *of Theorem 2.n be*

$$P_{1n3}(\gamma_{1n}), P_{1n4}(\gamma_{1n}), \ldots, P_{1nn}(\gamma_{1n}) > 0 \text{ for } n \geq 3,$$

respectively. Additionally, let condition $(3+n+t_{1n})$ *of Theorem* 2.*n be*

$$P_{1n(n+2)}(\gamma_{1n}) \leq 0 \text{ for } n \geq 1,$$

where

$$a_{1n0} = K_{1n}(0),\ a_{1n1} = K'_{1n}(0),\ a_{1n2} = K''_{1n}(0), \ldots, a_{1n(n+1)} = K_{1n}^{(n+1)}(0),$$

$$t_{1n} = \frac{(n-2)(n-1)}{2},$$

$$P_{1ni}(\gamma) = \sum_{k=0}^{i} (-\gamma)^{i-k} a_{1n(k-1)},\ (\ a_{1n(-1)} \equiv 1, \quad \text{by convention}) \text{ for all } i \in \mathbb{N}_{n+2}$$

and

$$\gamma_{11} = \frac{a_{110}}{3} \quad \text{for } n=1 \quad \text{and } \gamma_{1n} = \frac{a_{1n0}}{n+1} \text{ for } n \geq 2.$$

Then, the solution of (2.34) *satisfies the inequality* $|f_{1n}(t)| \leq 2^n$ *for* $0 \leq t < \infty$.

Proof. We prove the theorem by using the induction method.

The validity of Theorem 2.1 is known by [3].

Let us suppose the truth of Theorem 2.m for $m \in \mathbb{N}$. So, the solution of the equation

$$f_{1m}(t) = 1 - K_{1m} * f_{1m}$$

satisfies the inequality

$$|f_{1m}(t)| \leq 2^m. \tag{2.35}$$

Now, we should prove that Theorem 2.(m+1) is true. Namely, let us show that the solution of the equation

$$f_{1(m+1)}(t) = 1 - K_{1(m+1)} * f_{1(m+1)} \tag{2.36}$$

satisfies the condition

$$|f_{1(m+1)}(t)| \leq 2^{m+1}. \tag{2.37}$$

If the function g is taken as $g(t) = e^{-\gamma t}$ ($\gamma \in \mathbb{R}$) in Theorem A, then one can see that

$$f_{1(m+1)}(t) = e^{-\gamma t} - L_{1(m+1)} * f_{1(m+1)} \tag{2.38}$$

is equivalent to (2.36), where

$$L_{1(m+1)}(t) = (a_{1(m+1)0} - \gamma)e^{-\gamma t} + K'_{1(m+1)} * e^{-\gamma t}. \tag{2.39}$$

By differentiating (2.39), $L'_{1(m+1)}(t),\ L''_{1(m+1)}(t), \dots, L^{(m+2)}_{1(m+1)}(t)$ are found, as follows:

$$\begin{aligned}
L'_{1(m+1)}(t) &= \left(\gamma^2 - a_{1(m+1)0}\gamma + a_{1(m+1)1}\right) e^{-\gamma t} + K''_{1(m+1)} * e^{-\gamma t},\\
L''_{1(m+1)}(t) &= \left(-\gamma^3 + a_{1(m+1)0}\gamma^2 - a_{1(m+1)1}\gamma + a_{1(m+1)2}\right) e^{-\gamma t} + K'''_{1(m+1)} * e^{-\gamma t},\\
&\ \vdots\\
L^{(m)}_{1(m+1)}(t) &= \left((-1)^{m+1}\gamma^{m+1} + (-1)^m a_{1(m+1)0}\gamma^m + (-1)^{m-1} a_{1(m+1)1}\gamma^{m-1}\right.\\
&\quad \left. + \cdots - a_{1(m+1)(m-1)}\gamma + a_{1(m+1)m}\right) e^{-\gamma t} + K^{(m+1)}_{1(m+1)} * e^{-\gamma t},\\
L^{(m+1)}_{1(m+1)}(t) &= \left((-1)^{m+2}\gamma^{m+2} + (-1)^{m+1} a_{1(m+1)0}\gamma^{m+1} + (-1)^m a_{1(m+1)1}\gamma^m\right.\\
&\quad \left. + \cdots - a_{1(m+1)m}\gamma + a_{1(m+1)(m+1)}\right) e^{-\gamma t} + K^{(m+2)}_{1(m+1)} * e^{-\gamma t},\\
L^{(m+2)}_{1(m+1)}(t) &= \left((-1)^{m+3}\gamma^{m+3} + (-1)^{m+2} a_{1(m+1)0}\gamma^{m+2} + (-1)^{m+1} a_{1(m+1)1}\gamma^{m+1}\right.\\
&\quad \left. + \cdots - a_{1(m+1)(m+1)}\gamma + a_{1(m+1)(m+2)}\right) e^{-\gamma t} + K^{(m+3)}_{1(m+1)} * e^{-\gamma t}.
\end{aligned}$$

We claim that the kernel of the equation

$$h_{1(m+1)}(t) = 1 - L_{1(m+1)} * h_{1(m+1)} \tag{2.40}$$

satisfies the conditions of Theorem 2.m. That is, $L_{1(m+1)}$ satisfies all the conditions of Theorem 2.m under the assumptions of Theorem 2.(m+1).

The corresponding inequalities to conditions (2) and (3) of Theorem 2.m are

$$[L_{1(m+1)}(0)]^2 < \frac{2(5m+4)}{5m-1} L'_{1(m+1)}(0) \tag{2.41}$$

and

$$\frac{2(m+1)}{m} L'_{1(m+1)}(0) \le [L_{1(m+1)}(0)]^2, \tag{2.42}$$

respectively. Inequalities (2.41) and (2.42) are respectively equivalent to

$$\left(a_{1(m+1)0} - \gamma\right)^2 < \frac{2(5m+4)}{5m-1}\left(\gamma^2 - a_{1(m+1)0}\gamma + a_{1(m+1)1}\right) \tag{2.43}$$

and

$$\frac{2(m+1)}{m}\left(\gamma^2 - a_{1(m+1)0}\gamma + a_{1(m+1)1}\right) \le \left(a_{1(m+1)0} - \gamma\right)^2. \tag{2.44}$$

Inequality (2.43) is equivalent to

$$q_1(\gamma) = (5m+9)\gamma^2 - 10a_{1(m+1)0}\gamma + 2(5m+4)a_{1(m+1)1} - (5m-1)a^2_{1(m+1)0} > 0.$$

The discriminant of $q_1(\gamma) = 0$ is

$$4(5m+4)[(5m+4)a^2_{1(m+1)0} - 2(5m+9)a_{1(m+1)1}]$$

which is negative by condition (2) of Theorem 2.(m+1), and thus, $q_1(\gamma) > 0$ for every $\gamma \in \mathbb{R}$. Inequality (2.44) is equivalent to

$$q_2(\gamma) = (m+2)\gamma^2 - 2a_{1(m+1)0}\gamma - ma_{1(m+1)0}^2 + 2(m+1)a_{1(m+1)1} \leq 0.$$

The discriminant of $q_2(\gamma) = 0$ is

$$4(m+1)[(m+1)a_{1(m+1)0}^2 - 2(m+2)a_{1(m+1)1}],$$

which is nonnegative by condition (3) of Theorem 2.(m+1), and so $q_2(\gamma_{1(m+1)}) \leq 0$ whenever γ is chosen as $\gamma = \gamma_{1(m+1)} = \frac{a_{1(m+1)0}}{m+2}$. The corresponding inequalities to conditions $(4) - (3 + m + t_{1m})$ of Theorem 2.m are the inequalities obtained by taking

$$P_{1(m+1)1}(\gamma_{1(m+1)}),\ P_{1(m+1)2}(\gamma_{1(m+1)}), \dots, P_{1(m+1)(m+2)}(\gamma_{1(m+1)}),$$

respectively, instead of the constants $a_{1m0}, a_{1m1}, \dots, a_{1m(m+1)}$ in conditions $(4) - (3+m+t_{1m})$ of Theorem 2.m. These inequalities are conditions $(4) - (4+t_{1(m+1)})$ of Theorem 2.(m+1) for $m \geq 1$, respectively. Hence, $L_{1(m+1)}$ satisfies conditions $(4) - (4 + t_{1m})$ of Theorem 2.m. Furthermore, the corresponding inequalities to condition (1) of Theorem 2.m are

$$L_{1(m+1)}(t), L'_{1(m+1)}(t), \dots, L_{1(m+1)}^{(m+1)}(t) > 0 \text{ and } L_{1(m+1)}^{(m+2)}(t) \leq 0.$$

Since $P_{1(m+1)1}(\gamma_{1(m+1)})$ is positive, $L_{1(m+1)}(t) > 0$ and the discriminant of

$$P_{1(m+1)2}(\gamma) = 0$$

is negative by condition (2) of Theorem 2.(m+1), $P_{1(m+1)2}(\gamma_{1(m+1)}) > 0$ and so, $L'_{1(m+1)}(t) > 0$. Additionally, conditions $(5+t_{1(m+1)})-(3+m+t_{1(m+1)})$ of Theorem 2.(m+1) are

$$P_{1(m+1)3}(\gamma_{1(m+1)}),\ P_{1(m+1)4}(\gamma_{1(m+1)}), \dots, P_{1(m+1)(m+1)}(\gamma_{1(m+1)}) > 0 \text{ for } m \geq 2,$$

respectively. Therefore,

$$L''_{1(m+1)}(t),\ L'''_{1(m+1)}(t), \dots, L_{1(m+1)}^{(m)}(t) > 0 \text{ for } m \geq 2.$$

Condition $(4 + m + t_{1(m+1)})$ of Theorem 2.(m+1) is

$$\begin{aligned} P_{1(m+1)(m+3)}(\gamma_{1(m+1)}) = -\gamma_{1(m+1)}&P_{1(m+1)(m+2)}(\gamma_{1(m+1)}) \\ &+ a_{1(m+1)(m+2)} \leq 0 \text{ for } m \geq 1. \end{aligned}$$

Hence,

$$P_{1(m+1)(m+2)}(\gamma_{1(m+1)}) > 0$$

and thus,

$$L_{1(m+1)}^{(m+1)}(t) > 0,\ L_{1(m+1)}^{(m+2)}(t) \leq 0 \text{ for } m \geq 1.$$

Since $L_{1(m+1)}$ satisfies all the conditions of Theorem 2.m, the solution of the equation (2.40) satisfies (2.35). Namely, $|h_{1(m+1)}(t)| \leq 2^m$.

By Theorem B, the solution of (2.38) can be expressed in the form

$$f_{1(m+1)}(t) = h_{1(m+1)}(t) - \gamma_{1(m+1)} \int_0^t h_{1(m+1)}(t-\tau) e^{-\gamma_{1(m+1)}\tau} d\tau,$$

and so,

$$\begin{aligned}|f_{1(m+1)}(t)| &\le 2^m + 2^m\gamma_{1(m+1)}\int_0^t e^{-\gamma_{1(m+1)}\tau}d\tau \\ &= 2^m - 2^m(e^{-\gamma_{1(m+1)}t} - 1) \le 2^{m+1}.\end{aligned}$$

Thus, (2.37) is fulfilled, which completes the proof of Theorem 2.(m+1). This means that Theorem 2.n is satisfied for all $n \in \mathbb{N}$. □

A function K_{1n} satisfying conditions $(1) - (3 + n + t_{1n})$ of Theorem 2.n can be obtained by using the following method:

Example 2.n. If there exist the numbers $a_{1n0}, a_{1n1}, \ldots, a_{1n(n+1)} > 0$ satisfying conditions $(2)-(3+n+t_{1n})$ of Theorem 2.n, then there exists at least one function K_{1n} which satisfies condition (1) of Theorem 2.n of the form

$$K_{1n}(t) = \sum_{m=0}^{n+1} c_m t^{n+1-m} + c_{n+2}e^{-t} \tag{2.45}$$

such that

$$K_{1n}(0) = a_{1n0}, K'_{1n}(0) = a_{1n1}, \ldots, K_{1n}^{(n)}(0) = a_{1nn}, K_{1n}^{(n+1)}(0) = a_{1n(n+1)}. \tag{2.46}$$

To show the truth of this fact, we first choose the numbers a_{1n0}, $a_{1n1} > 0$ satisfying conditions (2) and (3) of Theorem 2.n and define the polynomial P_{1ni} for all $i \in \mathbb{N}_{n+2}$, the number γ_{1n} by

$$P_{1ni}(\gamma) = \sum_{k=0}^{i}(-\gamma)^{i-k}a_{1n(k-1)} \ (a_{1n(-1)} \equiv 1 \ \text{ by convention}),$$

$$\gamma_{11} = \frac{a_{110}}{3} \ \text{ for } n = 1 \ \text{ and } \gamma_{1n} = \frac{a_{1n0}}{n+1} \text{ for } n \ge 2.$$

So, if $n \ge 2$, then it can be clearly seen by the proof of Theorem 2.n that the numbers $a_{1(n-1)0}$ and $a_{1(n-1)1}$ defined by

$$a_{1(n-1)0} = P_{1n1}(\gamma_{1n}) \text{ and } a_{1(n-1)1} = P_{1n2}(\gamma_{1n})$$

satisfy conditions (2) and (3) of Theorem 2.(n-1). Furthermore, the numbers $a_{1(n-1)2}, a_{1(n-1)3}, \ldots, a_{1(n-1)n} > 0$ whenever $n \ge 2$ can also be found by means of the constants $a_{1(n-1)0}, a_{1(n-1)1} > 0$ as those were obtained in Example 2.(n-1). Thus, for $n \ge 2$, by taking

$$P_{1n3}(\gamma_{1n}) = a_{1(n-1)2}, \ P_{1n4}(\gamma_{1n}) = a_{1(n-1)3}, \ldots, \ P_{1n(n+1)}(\gamma_{1n}) = a_{1(n-1)n},$$

the numbers $a_{1n2}, a_{1n3}, \ldots, a_{1nn}$ are obtained as

$$\begin{aligned} a_{1n2} &= a_{1(n-1)2} + \gamma_{1n}a_{1(n-1)1} > 0, \\ a_{1n3} &= a_{1(n-1)3} + \gamma_{1n}a_{1(n-1)2} > 0, \\ &\vdots \\ a_{1nn} &= a_{1(n-1)n} + \gamma_{1n}a_{1(n-1)(n-1)} > 0. \end{aligned}$$

Besides for $n \ge 1$, since condition $(3 + n + t_{1n})$ of Theorem 2.n is

$$P_{1n(n+2)}(\gamma_{1n}) = -\gamma_{1n}P_{1n(n+1)}(\gamma_{1n}) + a_{1n(n+1)} \le 0,$$

the number $a_{112} > 0$ can be taken as

$$0 < a_{112} \le \gamma_{11}P_{112}(\gamma_{11}),$$

whenever $n = 1$, and in the case $n \geq 2$, the number $a_{1n(n+1)} > 0$ can be taken as

$$0 < a_{1n(n+1)} \leq \gamma_{1n} P_{1n(n+1)}(\gamma_{1n}) = \gamma_{1n} a_{1(n-1)n}.$$

Clearly, the numbers $a_{1n0}, a_{1n1}, \ldots, a_{1n(n+1)} > 0$ obtained by this way satisfy conditions $(2) - (3 + n + t_{1n})$ of Theorem 2.n.

The solution of the system of linear equations (2.46) which is equivalent to

$$\begin{aligned} c_{n+1} + c_{n+2} = a_{1n0}, c_n - c_{n+2} = a_{1n1}, \ldots, n!c_1 + (-1)^n c_{n+2} \\ = a_{1nn}, (n+1)!c_0 + (-1)^{n+1} c_{n+2} = a_{1n(n+1)} \end{aligned}$$

is obtained as

$$\begin{aligned} &(c_0, c_1, \ldots, c_{n+1}, c_{n+2}) \\ &= \left(\frac{a_{1n(n+1)} - (-1)^{n+1} c_{n+2}}{(n+1)!}, \frac{a_{1nn} - (-1)^n c_{n+2}}{n!}, \ldots, a_{1n0} - c_{n+2}, c_{n+2} \right), \end{aligned}$$

where c_{n+2} is an arbitrary constant satisfying the condition

$$0 \leq c_{n+2} < \min\{a_{1n0}, a_{1n2}, \ldots, a_{1n(n-1)}, a_{1n(n+1)}\}$$

if n is odd and

$$\max\{-a_{1n1}, -a_{1n3}, \ldots, -a_{1n(n-1)}, -a_{1n(n+1)}\} < c_{n+2} \leq 0$$

if n is even. Thus, every K_{1n} of the form (2.45) satisfies condition (1) of Theorem 2.n as well.

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References

[1] R. Bellman and K. L. Cooke, *Differential - Difference Equations,* Academic Press, New York, 1963. MR0147745 (26:5259)

[2] R. Ling, *Integral equations of Volterra type,* J. Math. Anal. Appl. **64** (1978), 381-397. MR0487317 (58:6964)

[3] R. Ling, *Solutions of singular integral equations,* Internat. J. Math. & Math. Sci. **5**(1) (1982), 123-131. MR0666499 (83j:45005)

[4] I. Özdemir and Ö. F. Temizer, *Expansion of the boundaries of the solutions of the linear Volterra integral equations with convolution kernel*, Integral Equations Operator Theory **43** (2002), 466-479. MR1909376 (2003b:45001)

[5] F. G. Tricomi, *Integral Equations*, Interscience Publishers, Inc., New York, 1957. MR0094665 (20:1177)

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