

TABLE OF INTEGRALS
OF SQUARED JACOBIAN ELLIPTIC FUNCTIONS
AND REDUCTIONS
OF RELATED HYPERGEOMETRIC R -FUNCTIONS

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ABSTRACT. Any product of real powers of Jacobian elliptic functions can be written in the form $cs^{m_1}(u, k) ds^{m_2}(u, k) ns^{m_3}(u, k)$. If all three m 's are even integers, the indefinite integral of this product with respect to u is a constant times a multivariate hypergeometric function $R_{-a}(b_1, b_2, b_3; x, y, z)$ with half-odd-integral b 's and $-a + b_1 + b_2 + b_3 = 1$, showing it to be an incomplete elliptic integral of the second kind unless all three m 's are 0. Permutations of c , d , and n in the integrand produce the same permutations of the variables $\{x, y, z\} = \{cs^2, ds^2, ns^2\}$, allowing as many as six integrals to take a unified form. Thirty R -functions of the type specified, incorporating 136 integrals, are reduced to a new choice of standard elliptic integrals obtained by permuting x , y , and z in $R_D(x, y, z) = R_{-3/2}(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x, y, z)$, which is symmetric in its first two variables and has an efficient algorithm for numerical computation.

1. INTRODUCTION

We shall suppress the variables (u, k) of the twelve Jacobian elliptic functions for brevity, and the integrals will be indefinite integrals with respect to u . Any one of the functions can be expressed in terms of the three that Neville [Ne, pp. 61, 178] called the primitive functions: cs , ds , and ns . For example, $sn(u, k) = sn = 1/ns$, $cn = cs/ns$, and $dn = ds/ns$. Consequently, any product of real powers of Jacobian functions can be written as $cs^{m_1}ds^{m_2}ns^{m_3}$. Unless the sum of the m 's is an odd positive integer, Theorem 1 will express the integral of this product as a constant times a hypergeometric R -function of the variables cs^2 , ds^2 , and ns^2 .

The hypergeometric R -function (of three variables in the present context),

$$(1.1) \quad R_{-a}(b_1, b_2, b_3; z_1, z_2, z_3),$$

is defined [Ca, § 5.1, § 5.2, and (5.9-1)] as the integral average of z^{-a} over the convex hull of $\{z_1, z_2, z_3\}$ with respect to a Dirichlet distribution with parameters (b_1, b_2, b_3) . This representation can be used if the b 's and z 's have positive real parts. R_{-a} is homogeneous of degree $-a$ in the z 's and is unchanged by simultaneous permutation of the b 's and z 's. More briefly, it is symmetric in the subscripts 1, 2, and 3. By [Ca, (5.9-2)] it satisfies a multivariate generalization of

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$$dz^{-a}/dz = -a z^{-a-1}:$$

$$(1.2) \quad DR_{-a} = -a R_{-a-1}, \quad D = \sum_{i=1}^3 \partial/\partial z_i.$$

It has also an important single-integral representation [Ca, (6.8-6)] for z 's lying in the complex plane cut along the nonpositive real axis

$$(1.3) \quad R_{-a}(b_1, b_2, b_3; z_1, z_2, z_3) = \frac{1}{B(a, a')} \int_0^\infty t^{a'-1} \prod_{i=1}^3 (t + z_i)^{-b_i} dt,$$

which converges if a and a' are positive, where $a+a' = c \equiv b_1+b_2+b_3$. If all three z_i take a common value z , the integral reduces to z^{-a} times the beta function $B(a, a')$, and R_{-a} reduces to z^{-a} . The R -function is defined only if $c \neq 0, -1, -2, \dots$ (see [Ca, §6.8]).

The cases with $a' = 1$ (whence $-a = 1 - c$) are relatively simple in several respects. In particular the derivative is then an elementary function by (1.2) and [Ca, (6.6-5)]:

$$(1.4) \quad DR_{1-c} = (1-c)R_{-c} = (1-c) \prod_{i=1}^3 z_i^{-b_i}.$$

It is (1.4) that allows the integral of $cs^{m_1} ds^{m_2} ns^{m_3}$, $m_1 + m_2 + m_3 \neq 1, 3, 5, \dots$, to be represented as a function of type R_{1-c} , as will be proved in Theorem 1.

If m_1, m_2 , and m_3 are even integers, not all 0, we shall find that the integral of $cs^{m_1} ds^{m_2} ns^{m_3}$ is an elliptic integral of the second kind. It can be expressed in terms of standard integrals (and rational functions of cs, ds , and ns), and the conventional choice, as in [BF, 361.01-361.39], consists of

$$(1.5) \quad u = R_{-1/2}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; cs^2, ds^2, ns^2), \quad E(u) = R_{-1/2}(\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}; cs^2, ds^2, ns^2).$$

Both functions obviously have $a' = 1$ since a' is the sum of all the visible parameters. Although u , an integral of the first kind, has equal b -parameters and is therefore symmetric in c, d , and n , this symmetry is absent in $E(u)$ where the b 's are all different. Therefore $E(u)$ is not suitable for obtaining reduction formulas that remain valid when c, d , and n are permuted. The same remark applies if $E(u)$ is replaced by another integral of the second kind with $a' = 1$:

$$(1.6) \quad R_D(u) = R_{-3/2}(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; cs^2, ds^2, ns^2),$$

which has two equal b 's and so is symmetric in c and d but not in n .

Replacement of $E(u)$ by the completely symmetric integral of the second kind,

$$(1.7) \quad R_G(u) = R_{1/2}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; cs^2, ds^2, ns^2),$$

is the choice made in a different context in [Ca, Table 9.3-1] and also in [NC, Table II], a table that reduces nine integrals of the second kind with $a' = 1$ among other cases. Unfortunately R_G , with $a' = 2$, has more complicated properties and less satisfactory methods of numerical computation than R_D . Although reductions to u and $R_G(u)$ remain valid under permutation of c, d , and n , their coefficients in the formulas tend to be rather cumbersome and are not always functions of k alone.

A novel and perhaps more satisfactory choice is to replace both u and $E(u)$ by three integrals of the second kind: $R_D(u)$ and the two functions obtained from it

by exchanging n with c or d . That is to say, we define

$$(1.8) \quad R_D(x, y, z) = R_{-3/2}(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x, y, z),$$

which has an efficient algorithm for numerical computation as a function of real or complex $x, y,$ and z (see [num, p.20]), and use as standard integrals

$$(1.9) \quad R_D(y, z, x), \quad R_D(z, x, y), \quad R_D(x, y, z), \quad \{x, y, z\} = \{cs^2, ds^2, ns^2\}.$$

This choice permits reduction formulas that unify integrals differing only by permutations of $c, d,$ and n . Also, the coefficient of each R_D depends only on k . For checking against other tables, it is useful to note that

$$(1.10) \quad \frac{k^2}{3} R_D(cs^2, ds^2, ns^2) = u - E(u),$$

$$(1.11) \quad \frac{k^2 k'^2}{3} R_D(ns^2, cs^2, ds^2) = -k'^2 u + E(u) - \frac{k^2 cs}{ds ns}, \quad k'^2 = 1 - k^2,$$

$$(1.12) \quad \frac{k'^2}{3} R_D(ds^2, ns^2, cs^2) = -E(u) + \frac{ds}{cs ns},$$

$$(1.13) \quad 1/\sqrt{xyz} = R_{-3/2}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z) = \frac{1}{3} \sum R_D(y, z, x),$$

$$(1.14) \quad u = R_{-1/2}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z) = R_F(x, y, z) = \frac{1}{3} \sum x R_D(y, z, x),$$

$$(1.15) \quad R_G(u) = R_{1/2}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z) = R_G(x, y, z) = \frac{1}{6} \sum x(y+z) R_D(y, z, x),$$

where R_F is the symmetric incomplete elliptic integral of the first kind and the summations extend over the three cyclic permutations of $x, y,$ and z .

To prove (1.10), (1.11), and (1.12), exchange x and y (and b_1 and b_2) in the first line of [Ca, Table 9.3-1] to get $E(u)$. Then eliminate R_G between this and each of the three forms of the third line obtained by permuting $x, y,$ and z (and the b 's) to get the functions in (1.9). Equations (1.13) and (1.14) are the special cases $\mathbf{b} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ of (4.7) and (4.8), respectively; (1.13) shows why the choice (1.9) of standard integrals does not lead to unique reduction formulas but can be used to choose a reduction that shows any symmetry resulting from equalities between $m_1, m_2,$ and m_3 . Equation (1.15) will be proved in §3.4.

2. TABLE OF REDUCTIONS

Table 1 lists reductions of some hypergeometric R -functions with $a' = 1$:

$$(2.1) \quad R_{1-c}(b_1, b_2, b_3; x, y, z), \quad c = \sum_{i=1}^3 b_i \neq 0, -1, -2, \dots$$

Because the parameters a, b_1, b_2, b_3 are half-odd-integers in each case while a' is a positive integer and (1.14) is omitted, the functions are incomplete elliptic integrals of the second kind (see [Ca, pp. 259-262]). By Theorem 1 below, this table serves to reduce many integrals of products of Jacobian elliptic functions. To unify integrals that differ only by permutations of $c, d,$ and n , we let

$$(2.2) \quad \{p, q, r\} = \{c, d, n\}$$

so that the table needs to include only integrands $ps^{m_1}qs^{m_2}rs^{m_3}$ with $m_1 \geq m_2 \geq m_3$. If two m 's are equal, then $ps^2qs^2rs^{-4}$, for example, can be made to represent $ds^2ns^2cs^{-4}$ or $ns^2cs^2ds^{-4}$ or $cs^2ds^2ns^{-4}$ by choosing (p, q, r) to be (d, n, c) or (n, c, d)

or (c,d,n), respectively. If the m 's are all different, six integrands can be represented by one. Although Table 1 lists only cases where the m 's are even integers, we begin with a theorem that is less restrictive.

Theorem 1. *Let $m_1, m_2,$ and m_3 be real numbers, $M = \sum_{i=1}^3 m_i \neq 1, 3, 5, 7, \dots,$ $b_i = (1 - m_i)/2$ for $i = 1, 2, 3,$ and $c = \sum_{i=1}^3 b_i.$ Then*

$$(2.3) \quad \int ps^{m_1} qs^{m_2} rs^{m_3} du = \frac{1}{1-M} R_{1-c}(b_1, b_2, b_3; ps^2, qs^2, rs^2).$$

Remark. As noted earlier, the R -function is not defined if $c = 0, -1, -2, \dots$. Since $c = (3 - M)/2$ we require $M \neq 3, 5, 7, \dots$.

Proof. Define

$$J = R_{1-c}(b_1, b_2, b_3; ps^2, qs^2, rs^2).$$

Since $cs^2 = ns^2 - 1$ and $ds^2 = ns^2 - k^2$, it follows that $ps^2, qs^2,$ and rs^2 differ by constants and have the same derivative [cdn, (2.3)] with respect to u :

$$(2.4) \quad \frac{d}{du} ps^2 = \frac{d}{du} qs^2 = \frac{d}{du} rs^2 = -2(ps)(qs)(rs).$$

Then (1.2) and (1.4) with $(z_1, z_2, z_3) = (ps^2, qs^2, rs^2)$ imply

$$\begin{aligned} \frac{dJ}{du} &= (DJ) \frac{d}{du} ps^2 \\ &= -2(ps)(qs)(rs)(1-c)(ps)^{-2b_1}(qs)^{-2b_2}(rs)^{-2b_3} \\ &= (2c-2)ps^{m_1}qs^{m_2}rs^{m_3}. \end{aligned}$$

Since $2c - 2 = 1 - M$, integration proves (2.3). □

2.1. Summary of notation for Table 1.

$$(2.5) \quad (1-M) \int ps^{m_1} qs^{m_2} rs^{m_3} du = R_{1-c}(b_1, b_2, b_3; x, y, z), \quad M \neq 1, 3, 5, 7, \dots,$$

$$(2.6)$$

$$M = \sum_{i=1}^3 m_i, \quad c = \sum_{i=1}^3 b_i, \quad m_i + 2b_i = 1, \quad M + 2c = 3, \quad 1 - M = 2(c - 1).$$

In Table 1 the column labeled **m** shows $m_1, m_2, m_3,$ and the column labeled **2b** shows $2b_1, 2b_2, 2b_3.$

$$(2.7) \quad (x, y, z) = (ps^2, qs^2, rs^2), \quad \{p, q, r\} = \{c, d, n\},$$

$$(2.8) \quad \begin{aligned} C_0 R_{1-c}(b_1, b_2, b_3; x, y, z) \\ = C_x R_D(y, z, x) + C_y R_D(z, x, y) + C_z R_D(x, y, z) + C_a / \sqrt{xyz}, \end{aligned}$$

$$(2.9) \quad P = x - y = ps^2 - qs^2 = \Delta(p, q), \quad \Delta(n, c) = -\Delta(c, n) = 1,$$

$$(2.10) \quad Q = y - z = qs^2 - rs^2 = \Delta(q, r), \quad \Delta(n, d) = -\Delta(d, n) = k^2,$$

$$(2.11) \quad R = z - x = rs^2 - ps^2 = \Delta(r, p), \quad \Delta(d, c) = -\Delta(c, d) = k'^2 = 1 - k^2.$$

$$(2.12) \quad P + Q + R = 0, \quad zP + xQ + yR = 0.$$

TABLE 1. Reductions of $(1 - M) \int \text{ps}^{m_1} \text{qs}^{m_2} \text{rs}^{m_3} du = R_{1-c}(b_1, b_2, b_3; x, y, z)$. For notation see Section 2.1.

m	1-M	2b	1-c	C_0	C_x	C_y	C_z	C_a
-2,-2,-2	7	3,3,3	-7/2	$-3PQR/7$	Q	R	P	0
0,-2,-4	7	1,3,5	-7/2	$9Q^2R/7$	$-Q$	$4R$	$Q - 4R$	$3QR/z$
0,0,-6	7	1,1,7	-7/2	$45Q^2R^2/7$	$QR - 4Q^2$	$QR - 4R^2$	$4P^2 - 10QR$	$3QR(zQ - zR + 3QR)/z^2$
0,-2,-2	5	1,3,3	-5/2	$3Q/5$	0	-1	1	0
0,0,-4	5	1,1,5	-5/2	$9QR/5$	$-Q$	R	$Q - R$	$3QR/z$
2,-2,-4	5	-1,3,5	-5/2	$9Q^2/5$	0	$4P$	$3R - P$	$3xQ/z$
2,0,-6	5	-1,1,7	-5/2	$9Q^2R/P$	Q	$-4R$	$4R - Q$	$3QR(3xQ - zP)/z^2P$
4,-2,-6	5	-3,3,7	-5/2	$3Q^3/P$	0	$R - 7P$	$P - 7R$	$3xQ(-zP + xQ + zR)/z^2P$
0,0,-2	3	1,1,3	-3/2	1	0	0	1	0
2,-2,-2	3	-1,3,3	-3/2	Q	0	$-P$	$-R$	0
2,0,-4	3	-1,1,5	-3/2	$3Q$	0	P	$-P$	$3xQ/z$
2,2,-6	3	-1,-1,7	-3/2	$15QR/P$	Q	R	P	$3QR(5xy - xQ + yR)/z^2P$
4,0,-6	3	-3,1,7	-3/2	$15Q^2/P$	0	$2Q - 4P$	$Q + 4P$	$3xQ(5xQ - zP + 2QR)/z^2P$
4,-2,-4	3	-3,3,5	-3/2	$3Q^2/P$	0	$3P - R$	$4R$	$-3xQR/zP$
2,0,-2	1	-1,1,3	-1/2	3	0	$-P$	0	$3x$
2,2,-4	1	-1,-1,5	-1/2	9	P	$-P$	0	$3(3xy - xQ + yR)/z$
4,-2,-2	1	-3,3,3	-1/2	$3Q$	0	PR	$-PR$	$3xQ$
4,0,-4	1	-3,1,5	-1/2	$9Q/P$	0	$P - 2Q$	R	$3xQ(3z - R)/zP$
4,2,-6	1	-3,-1,7	-1/2	$45Q/P$	$4Q$	$P - 4Q$	$-P$	$3Q(15xyz - 5xzQ - zPR + 3yR^2)/z^2P$
6,0,-6	1	-5,1,7	-1/2	$45Q^2/P$	0	$3PQ - 4P^2 - 8Q^2$	$4P^2 - 4Q^2$	$3xQ(Q(15z^2 - 10xR - 7R^2) + zPR)/z^2P$
2,0,0	-1	-1,1,1	1/2	3	PR	0	0	$3yz$
2,2,-2	-1	-1,-1,3	1/2	3	PR	PQ	0	$3(z^2 + QR)$
4,0,-2	-1	-3,1,3	1/2	3	PR	$-PR$	0	$3(yz + xR)$
4,2,-4	-1	-3,-1,5	1/2	$9/P$	$4R$	$3Q - R$	0	$3(3z^3 + 3xzR - zPR + 4zQR - QR^2)/zP$
6,-2,-2	-1	-5,3,3	1/2	$3Q/PR$	Q	R	P	$3Q(yz - xP + xR)/PR$
6,0,-4	-1	-5,1,5	1/2	$9Q/PR$	$4Q$	$P - 4Q$	$-P$	$3Q(4yz^2 - x^3 - xzQ + 3xzR)/zPR$
2,2,0	-3	-1,-1,1	3/2	$3/QR$	P	$-P$	0	$3z(xy + yQ - xR)/QR$
4,0,0	-3	-3,1,1	3/2	$3/PR$	0	$2R - P$	$R - 2P$	$3x(x^2 + yP - zR - (P - R)^2)/PR$
4,2,-2	-3	-3,-1,3	3/2	$3/PR$	$Q - 3R$	$-4Q$	0	$3(x^2y - yzP - 4z^2R - 4QR^2)/PR$
6,0,-2	-3	-5,1,3	3/2	$3/PR$	$2P - 4R$	$P + 4R$	0	$3(x^2y + 2yzP - 4z^2R + 3xPR - 4QR^2)/PR$

Without reference to Jacobian elliptic functions, Table 1 also provides a table of reductions of R -functions with $a' = 1$ and variables x, y, z . Functions with $a' = 2, 3, 4, \dots$ can be obtained by successive applications of [Ca, (5.9-7)] (see §3.4). Functions with $a' = 0, -1, -2, \dots$ are elementary and can be obtained directly from [Ca, (6.8-15) and (6.2-1)].

3. EXAMPLES

3.1. Example: Integral of any single squared Jacobian function. For $(m_1, m_2, m_3) = (0, 0, -2), (2, 0, -2),$ and $(2, 0, 0)$, Table 1 gives, respectively,

$$(3.1) \quad \int sr^2 du = \frac{1}{3} R_D(ps^2, qs^2, rs^2),$$

$$(3.2) \quad \int pr^2 du = \frac{1}{3} \Delta(q, p) R_D(rs^2, ps^2, qs^2) + \frac{ps}{qs rs},$$

$$(3.3) \quad \int ps^2 du = \frac{1}{3} \Delta(p, q) \Delta(p, r) R_D(qs^2, rs^2, ps^2) - \frac{qs rs}{ps},$$

where $\{p, q, r\} = \{c, d, n\}$, the Δ 's are functions of k^2 given by (2.9)–(2.11), and R_D is symmetric in its first two variables. We illustrate the derivation of these equations by considering $(m_1, m_2, m_3) = (2, 0, 0)$. Table 1 gives

$$(-1) \int ps^2 qs^0 rs^0 du = \frac{1}{3} [PR R_D(y, z, x) + 3yz/\sqrt{xyz}],$$

$$\int ps^2 du = -\frac{1}{3} [\Delta(p, q) \Delta(r, p) R_D(qs^2, rs^2, ps^2) + 3qs rs/ps],$$

in agreement with (3.3).

The three cases of (3.1), for $r=n, d,$ or c , are simple and similar to one another, while the corresponding reductions to $E(u)$ and u , which follow by inspection from (1.10)–(1.12), are quite dissimilar. They are given in [BF, (310.02), (318.02), and (316.02)]; note that [BF] uses tn in place of sc . Likewise (3.3), with $p=n, d,$ or c and with Δ 's taken from (2.9)–(2.11), agrees with and can replace [BF, (311.02), (319.02), and (317.02)]. Finally (3.2), with $pr = cn, nc, dn, nd, cd,$ or dc , agrees with and can replace six integrals in [BF, 3XX.02, XX=12,13,14,15,20, and 21].

3.2. Example: Integral of a product of squared Jacobian functions. Since $sr^2 pr^2 = ps^2 qs^0 rs^{-4}$, Table 1 gives

$$3 \int sr^2 pr^2 du = R_{-3/2}(-\frac{1}{2}, \frac{1}{2}, \frac{5}{2}; ps^2, qs^2, rs^2)$$

$$= \frac{1}{3Q} \left[P R_D(rs^2, ps^2, qs^2) - P R_D(ps^2, qs^2, rs^2) + \frac{3xQ}{z\sqrt{xyz}} \right],$$

$$(3.4) \quad \int sr^2 pr^2 du = \frac{\Delta(p, q)}{9\Delta(q, r)} [R_D(rs^2, ps^2, qs^2) - R_D(ps^2, qs^2, rs^2)] + \frac{ps}{3qs rs^3}.$$

This formula can replace [BF, 361.01, .02, .07, .15, .19, and .27]. For instance, if $(p, q, r) = (c, d, n)$ it becomes

$$\int sn^2 cn^2 du = \frac{k'^2}{9k^2} [R_D(ns^2, cs^2, ds^2) - R_D(cs^2, ds^2, ns^2)] + \frac{cs}{3ds ns^3},$$

which agrees with [BF, 361.01] after substitution of (1.11) and (1.10).

3.3. Example: Reducing an R -function with $a' = 1$. If the objective is to reduce an R -function with $a' = 1$, the variables x, y, z should be used throughout. For instance, the row of Table 1 with $(2b_1, 2b_2, 2b_3) = (-1, 1, 1)$ gives

$$(3.5) \quad 3R_{1/2}(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z) = (x - y)(z - x) R_D(y, z, x) + 3yz/\sqrt{xyz}.$$

This reduction is used in the next example. Because $b_2 = b_3$ we expect symmetry of (3.5) in y and z , which is indeed evident.

3.4. Example: Reducing an R -function with $a' > 1$. As mentioned in the last sentence of Section 2.1, functions with $a' = 2, 3, 4, \dots$ can be obtained by successive applications of [Ca, (5.9-7)] with parameters $t = -a$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{z} = (x, y, z) = (z_1, z_2, z_3)$; \mathbf{e}_i is a triple with 1 in the i th place and 0's elsewhere. Replacing \mathbf{b} by $\mathbf{b} - \mathbf{e}_i$ (and c by $c - 1$), we have

$$(3.6) \quad (a' - 1)R_{-a}(\mathbf{b}, \mathbf{z}) = (c - 1)R_{-a}(\mathbf{b} - \mathbf{e}_i, \mathbf{z}) - az_i R_{-a-1}(\mathbf{b}, \mathbf{z}).$$

For instance, choosing $a' = 2$, $\mathbf{b} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $i = 1$, we find $a = -\frac{1}{2}$ and

$$R_{1/2}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z) = \frac{1}{2} R_{1/2}(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z) + \frac{1}{2} x R_{-1/2}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z).$$

The left side is $R_G(x, y, z)$ by (1.15). On the right side the first term is reduced by (3.5) and the second is a multiple of R_F :

$$(3.7) \quad 6R_G(x, y, z) = (x - y)(z - x) R_D(y, z, x) + 3yz/\sqrt{xyz} + 3x R_F(x, y, z).$$

To make the right side look symmetric in x, y, z , like the left side, we substitute (1.13) and (1.14) and collect terms to get

$$6R_G(x, y, z) = [(x - y)(z - x) + yz + x^2]R_D(y, z, x) + (yz + xy)R_D(z, x, y) + (yz + xz)R_D(x, y, z).$$

The expression in square brackets reduces to $xy + xz$, proving (1.15).

4. METHOD OF MAKING TABLE 1

The entries in Table 1 were found and checked by using relations between associated R -functions (those whose corresponding parameters differ by integers). Relations useful for present purposes involve only R -functions with $a' = 1$ or 0, where a' is the sum of the exhibited parameters. If $a' = 0$ the function is algebraic in its variables by [Ca, (6.6-5)] because the b -parameters are half-odd integers. If $a' = 1$, the representation by a Dirichlet average [Ca, (5.9-1)] would be excluded by a negative b -parameter and the single-integral representation (1.3) would be excluded if $1 - c > 0$. Relations that are useful in making or extending Table 1 are given in the following lemma.

Lemma 1. *Let $\mathbf{b} = (b_1, b_2, b_3)$, $c = \sum_{i=1}^3 b_i$, and $\mathbf{z} = (z_1, z_2, z_3)$. Let $\{i, j, k\} = \{1, 2, 3\}$ and let \mathbf{e}_i be a triple with 1 in the i th place and 0's in the j th and k th places. (For example, $\mathbf{b} - \mathbf{e}_2 = (b_1, b_2 - 1, b_3)$.) Then the R -function of three variables satisfies*

$$(4.1) \quad (1 - b_k)R_{1-c}(\mathbf{b}, \mathbf{z}) = b_i R_{1-c}(\mathbf{b} + \mathbf{e}_i - \mathbf{e}_k, \mathbf{z}) + b_j R_{1-c}(\mathbf{b} + \mathbf{e}_j - \mathbf{e}_k, \mathbf{z}) + (1 - c)z_k \prod_{s=1}^3 z_s^{-b_s},$$

(4.2)

$$cR_{1-c}(\mathbf{b}, \mathbf{z}) = b_j(z_j - z_i)R_{-c}(\mathbf{b} + \mathbf{e}_j, \mathbf{z}) + b_k(z_k - z_i)R_{-c}(\mathbf{b} + \mathbf{e}_k, \mathbf{z}) + cz_i \prod_{s=1}^3 z_s^{-b_s},$$

$$(4.3) \quad (c-2)(z_i - z_j)R_{1-c}(\mathbf{b}, \mathbf{z}) = (c-1)[R_{2-c}(\mathbf{b} - \mathbf{e}_i, \mathbf{z}) - R_{2-c}(\mathbf{b} - \mathbf{e}_j, \mathbf{z})],$$

$$(4.4) \quad cR_{1-c}(\mathbf{b}, \mathbf{z}) = b_j(z_j - z_i)R_{-c}(\mathbf{b} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, \mathbf{z}) - (b_i + b_j)(z_k - z_i)R_{-c}(\mathbf{b} + \mathbf{e}_i, \mathbf{z}) + cz_k \prod_{s=1}^3 z_s^{-b_s},$$

$$(4.5) \quad (b_k - 1)cR_{1-c}(\mathbf{b}, \mathbf{z}) = (c-1)b_j(z_j - z_i)R_{-c}(\mathbf{b} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, \mathbf{z}) - c(b_i + b_j)R_{1-c}(\mathbf{b} + \mathbf{e}_i - \mathbf{e}_k, \mathbf{z}) + c(c-1)z_k \prod_{s=1}^3 z_s^{-b_s}.$$

If $b_i + b_j = 0$, then $c = b_k$ and

$$(4.6) \quad b_k R_{1-b_k}(\mathbf{b}, \mathbf{z}) = b_j(z_j - z_i)R_{-b_k}(\mathbf{b} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, \mathbf{z}) + b_k(z_j/z_i)^{b_i} z_k^{1-b_k}.$$

Proof. Choose $t = -c$ in [Ca, (5.9-5)] and use [Ca, (6.6-5)] to get

$$(4.7) \quad c \prod_{s=1}^3 z_s^{-b_s} = \sum_{s=1}^3 b_s R_{-c}(\mathbf{b} + \mathbf{e}_s, \mathbf{z}).$$

Then replace \mathbf{b} by $\mathbf{b} - \mathbf{e}_k$ (and hence c by $c-1$ and b_k by b_k-1) to prove (4.1).

To prove (4.2) multiply both sides of (4.7) by z_i and subtract it from the case $t = -c$ of [Ca, (5.9-6)]:

$$(4.8) \quad cR_{1-c}(\mathbf{b}, \mathbf{z}) = \sum_{s=1}^3 b_s z_s R_{-c}(\mathbf{b} + \mathbf{e}_s, \mathbf{z}).$$

Equation (4.3) is the first equation in [Ca, Ex. 5.9-6 and p. 305] with $t = 2 - c$.

To prove (4.4) and (4.5), first replace \mathbf{b} by $\mathbf{b} + \mathbf{e}_i - \mathbf{e}_k$ (and hence b_i by $b_i + 1$ and b_k by $b_k - 1$) in (4.2) to get

$$(4.9) \quad cR_{1-c}(\mathbf{b} + \mathbf{e}_i - \mathbf{e}_k, \mathbf{z}) = b_j(z_j - z_i)R_{-c}(\mathbf{b} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, \mathbf{z}) + (b_k - 1)(z_k - z_i)R_{-c}(\mathbf{b} + \mathbf{e}_i, \mathbf{z}) + cz_k \prod_{s=1}^3 z_s^{-b_s}.$$

Next, exchange j and k in (4.3) and then replace \mathbf{b} by $\mathbf{b} + \mathbf{e}_i$ (and hence c by $c+1$) in (4.3) to get

$$(4.10) \quad (c-1)(z_i - z_k)R_{-c}(\mathbf{b} + \mathbf{e}_i, \mathbf{z}) = c[R_{1-c}(\mathbf{b}, \mathbf{z}) - R_{1-c}(\mathbf{b} + \mathbf{e}_i - \mathbf{e}_k, \mathbf{z})].$$

Elimination of $R_{1-c}(\mathbf{b} + \mathbf{e}_i - \mathbf{e}_k, \mathbf{z})$ or of $R_{-c}(\mathbf{b} + \mathbf{e}_i, \mathbf{z})$ between (4.9) and (4.10) produces (4.4) or (4.5). Both equations have (4.6) as a special case. \square

Remark. A section of the table that contains R -functions with the same value of c will be called a level of the table. One level is higher than another if its value of c is the larger of the two. For example the highest level shown consists of the first three rows of the table, in which $c = 9/2$. Then (4.1) connects three functions in the same level, and (4.3) expresses a function in terms of two functions in the next lower level. If two b -parameters have sum 0, (4.6) expresses the function in terms of a single function in the next higher level, but otherwise (4.2) and (4.4) express it in terms of two functions in the next higher level. Finally, (4.5) expresses it in terms of another function in the same level and one in the next higher level. These observations are helpful to keep in mind while making, checking, or extending the table.

When working with specific numerical values of the parameters, it is convenient to define

$$(4.11) \quad [2b_1, 2b_2, 2b_3] = R_{1-c}(\mathbf{b}, \mathbf{z}).$$

For example $[1, 1, 3] = R_{-3/2}(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x, y, z)$. Since this function is $R_D(x, y, z)$, one of the three standard functions used in Table 1, it is a natural starting point in making the table, where it occupies the first line in the third level (with $1 - c = -3/2$). We shall illustrate the procedure and the use of (4.1)–(4.6). To get the first line in the next higher (second) level, we use (4.3) with $(i, j, k) = (2, 3, 1)$ to find $\frac{3}{2}(y - z)[1, 3, 3] = \frac{5}{2}([1, 1, 3] - [1, 3, 1])$. By the invariance of the R -function under simultaneous permutations of the b -parameters and the variables, $[1, 3, 1] = R_D(z, x, y)$, and hence

$$(3Q/5)[1, 3, 3] = R_D(x, y, z) - R_D(x, z, y).$$

From this result we can get the third line in the starting (third) level by using (4.6) with $(i, j, k) = (1, 2, 3)$ to find $\frac{5}{2}[-1, 1, 5] = \frac{1}{2}(y - x)[1, 3, 3] + \frac{5}{2}\sqrt{x/y}z^{-3/2}$. Multiplication by $6Q/5$ gives

$$3Q[-1, 1, 5] = PR_D(x, z, y) - PR_D(x, y, z) + 3Q\sqrt{x/y}z^{-3/2}.$$

An alternative path from the starting point uses (4.6) to express $[-1, 1, 3]$ on the next lower (fourth) level in terms of $[1, 3, 1] = R_D(z, x, y)$ and then uses (4.3) to return to the third level by expressing $[-1, 3, 3]$ in terms of $[-1, 1, 3]$ and $[-1, 3, 1]$, which are related by permutation.

We now outline one way of obtaining the remaining three lines shown in the third level.

- Function $[1, 1, 5]$ (second level) follows from $[-1, 1, 5]$ and $[1, 1, 3]$ by (4.3) with $(i, j, k) = (1, 3, 2)$.
- Function $[-1, -1, 7]$ (third level) follows from $[1, 1, 5]$ and $[-1, 1, 5]$ by (4.5) with $(i, j, k) = (2, 1, 3)$.
- Function $[-1, 3, 5]$ (second level) follows from $[-1, 1, 5]$ and $[-1, 3, 3]$ by (4.3) with $(i, j, k) = (2, 3, 1)$.
- Function $[-3, 3, 5]$ (third level) follows from $[-1, 3, 5]$ and $[-1, 5, 3]$ (differing only by permutation) by (4.4) with $(i, j, k) = (1, 2, 3)$.
- Function $[-3, 1, 7]$ (third level) follows from $[-3, 3, 5]$ and $[-1, 1, 5]$ by (4.1) with $(i, j, k) = (2, 1, 3)$.

As more lines in the table are established, more ways of checking for errors become available. For example, $[-1, 1, 3]$ follows alternatively from $[1, 1, 3]$ and $[-1, 3, 3]$ by (4.2) with $(i, j, k) = (1, 2, 3)$, and also from $[1, 1, 3]$ and $[-1, 1, 5]$ by (4.2) with

$(i, j, k) = (1, 3, 2)$. For another example, $[-1, 3, 3]$ follows alternatively from $[1, 1, 5]$ and $[1, 3, 3]$ by (4.4) with $(i, j, k) = (1, 3, 2)$. The result can then be simplified by using (1.13) to eliminate $R_D(y, z, x)$ and show the expected symmetry in y and z .

For integrals of Jacobian functions the symbols P , Q , and R should be used as much as feasible in the coefficient C_a of $1/\sqrt{xyz}$ because they are simple functions of the modulus. For other purposes it is straightforward to rewrite C_a in terms of x , y , and z by using (2.9)–(2.11).

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