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COMPUTING THE ARITHMETIC GENUS OF HILBERT MODULAR FOURFOLDS

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Abstract. The Hilbert modular fourfold determined by the totally real quartic number field k is a desingularization of a natural compactification of the quotient space $\Gamma_k\backslash\mathcal{H}^4$, where $\Gamma_k = \mathrm{PSL}_2(\mathcal{O}_k)$ acts on $\mathcal{H}^4$ by fractional linear transformations via the four embeddings of k into $\mathbf{R}$. The arithmetic genus, equal to one plus the dimension of the space of Hilbert modular cusp forms of weight $(2,2,2,2)$, is a birational invariant useful in the classification of these varieties. In this work, we describe an algorithm allowing for the automated computation of the arithmetic genus and give sample results.

1. Introduction

Let k be a totally real algebraic number field of degree n over $\mathbf{Q}$ with ring of integers $\mathcal{O}_k$ and discriminant d_k. For $i = 1, \ldots, n$ and $a \in k$, let $a \mapsto a^{(i)}$ denote the ith embedding of k into the real numbers. The group $\mathrm{SL}_2(\mathcal{O}_k)$ acts on $\mathcal{H}^n$, the product of n copies of the complex upper half plane, by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z_1, z_2, \ldots, z_n) = \left(\frac{a^{(1)}z_1 + b^{(1)}}{c^{(1)}z_1 + d^{(1)}}, \frac{a^{(2)}z_2 + b^{(2)}}{c^{(2)}z_2 + d^{(2)}}, \ldots, \frac{a^{(n)}z_n + b^{(n)}}{c^{(n)}z_n + d^{(n)}}\right).$$

The *Hilbert modular group* of the field k is $\Gamma_k = \mathrm{PSL}_2(\mathcal{O}_k)$. The *Hilbert modular variety* of k is a desingularization of the natural compactification of $\Gamma_k\backslash\mathcal{H}^n$. For more detailed discussions of the construction of these varieties, see [7] and [12].

The arithmetic genus, in the sense of Hirzebruch [8], is a birational invariant useful in the classification of nonsingular varieties. Specifically, it is equal to one plus the dimension of the space of Hilbert modular cusp forms of weight $(2,2,2,2)$. By [4], the arithmetic genus of the Hilbert modular variety of k is given by

$$(-1)^n \mathrm{vol}(\Gamma_k\backslash\mathcal{H}^n) + \sum_{\eta} E(\Gamma_k, \eta) + \sum_{\kappa} L(\Gamma_k, \kappa),$$

where the first summation is over a complete set of representatives of the Γ_k-equivalence classes of fixed points of Γ_k and the second is over the set of cusps. Precise definitions of $E(\Gamma_k, \eta)$ and $L(\Gamma_k, \kappa)$ are given below.

Hirzebruch [7] provided a formula for computing the arithmetic genus in the cases where the ring $\mathcal{O}_k$ contains a unit of norm -1. This has been widely used for computing the arithmetic genus of Hilbert modular varieties of odd dimension, since -1 is a unit of norm -1 in any field of odd degree. For varieties of even

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dimension, the computation is more complicated. In the 1970's, Hirzebruch, van de Ven, and Zagier [9, 10] demonstrated how to carry out these computations for surfaces, but until recently little progress was made with higher degrees. It must be said that part of this is due to the earlier lack of computational tools. For example, a challenge as little as a decade ago, the computation of the volume term, $(-1)^n\mathrm{vol}(\Gamma_k\backslash\mathcal{H}^n)$, is all but trivialized by now-available software packages.

In [6], the current authors focused on the degree four case and determined how to compute the fixed point term, $\sum_\eta E(\Gamma_k,\eta)$. This left the cusp term, $\sum_\kappa L(\Gamma_k,\kappa)$, as the final roadblock to completing the computations. In this paper, we offer a now complete solution to the problem, providing an algorithm for computing the arithmetic genus of the Hilbert modular variety of an arbitrary quartic field k, and describing our implementation thereof.

We develop the necessary mathematics and describe our algorithm for computing the cusp term in Section 2. In Section 3, we review the needed results from [6] and describe our algorithms for computing the volume and fixed point terms. Finally, in Section 4, we describe our specific implementation and give a table of the arithmetic genera of the Hilbert modular fourfolds determined by the 210 totally real quartic fields of smallest discriminant.

2. The cusp term

In this section, we develop the tools needed for computing the cusp term,

$$\sum_\kappa L(\Gamma_k,\kappa),$$

for an arbitrary quartic field, k. We begin by defining notation.

Let $\mathcal{C}_k$ denote the group of ideal classes and $\mathcal{E}_k$ the group of units of $\mathcal{O}_k$. It is well known that the set of cusps of $\Gamma_k\backslash\mathcal{H}^4$ is in one-to-one correspondence with $\mathcal{C}_k$. Letting $[\mathfrak{a}]$ be the ideal class corresponding to the cusp κ, we have

$$L(\Gamma_k,[\mathfrak{a}]) = L(\Gamma_k,\kappa) = \frac{1}{16\pi^4}\sqrt{d(\mathfrak{a}^{-2})}\sum_{a\in\mathfrak{a}^{-2}/\mathcal{E}_k^2}\frac{\mathrm{sign}(N(a))}{|N(a)|},$$

with $d(\mathfrak{a}^{-2})$ the discriminant of $\mathfrak{a}^{-2}$ [4]. In this notation, the cusp term is given by $\sum_{[\mathfrak{a}]\in\mathcal{C}_k} L(\Gamma_k,[\mathfrak{a}])$.

We begin with a lemma proved in [6].

Lemma 1. *If $\mathcal{O}_k$ contains a unit of norm -1 or an integral ideal $\mathfrak{b}$ for which $\mathfrak{b}^2=(\beta)$ is a principal ideal with $N(\beta)<0$, then $\sum_\kappa L(\Gamma_k,\kappa)=0$.*

If the hypothesis of Lemma 1 is not satisfied by k, then there exists a real character, $\psi:\mathcal{I}_k\to\pm 1$, on the group of nonzero fractional ideals, satisfying $\psi((a))=\mathrm{sign}(N(a))$ for all nonzero $a\in\mathcal{O}_k$. It follows that $\overline{\psi}(\mathfrak{a}^2)=1$ and so, letting x vary over integral ideals in $\mathcal{O}_k$,

$$\begin{aligned}\sum_{[\mathfrak{a}]\in\mathcal{C}_k} L(\Gamma_k,[\mathfrak{a}]) &= \frac{\sqrt{d_k}}{\pi^4}\lim_{s\to 1}\sum_{[\mathfrak{a}]\in\mathcal{C}_k}\sum_{x\in[\mathfrak{a}^2]}\frac{\psi(x)}{N(x)^s}\\ &= \frac{\sqrt{d_k}}{\pi^4}\lim_{s\to 1}\sum_{[x]\in\mathcal{C}_k^2}\frac{\psi(x)}{N(x)^s}\ |\mathcal{C}_k/\mathcal{C}_k^2|.\end{aligned}$$

As in [6], let Φ be the set of characters $\phi : \mathcal{I}_k \to \pm 1$ induced by a complete set of characters on the group $\mathcal{C}_k/\mathcal{C}_k^2$. Then,

$$\begin{aligned}\sum_{[\mathfrak{a}]\in\mathcal{C}_k} L(\Gamma_k,[\mathfrak{a}]) &= \frac{\sqrt{d_k}}{\pi^4}\lim_{s\to 1}\sum_x \frac{\psi(x)}{N(x)^s}\sum_{\phi\in\Phi}\phi(x)\\ &= \frac{\sqrt{d_k}}{\pi^4}\sum_{\phi\in\Phi}\lim_{s\to 1}\sum_x \frac{\psi\phi(x)}{N(x)^s}\\ &= \frac{\sqrt{d_k}}{\pi^4}\sum_{\phi\in\Phi} L(1,\psi\phi).\end{aligned}$$

Now, since each ϕ is trivial on principal ideals, each $\psi\phi$ is a quadratic character on $\mathcal{I}_k$ satisfying $\psi\phi((a)) = \operatorname{sign}(N(a))$ for all nonzero $a \in \mathcal{O}_k$. Hence, each $\psi\phi$ corresponds, via class field theory, to a quadratic extension k' of k, unramified at each finite place and ramified at each infinite place. In fact, as ϕ ranges through Φ, the k' range through all such field extensions of k.

Since each k' is a totally complex quadratic extension of the totally real number field k, we have

$$L(1,\psi\phi) = L(1,k'/k) = \frac{\pi^4 h_{k'}R_{k'}w_k\sqrt{d_k}}{h_kR_kw_{k'}\sqrt{d_{k'}}}, \tag{1}$$

where $h_k, h_{k'}, R_k, R_{k'}, d_k, d_{k'}$ are the class numbers, regulators, and absolute values of the discriminants of $\mathcal{O}_k$ and $\mathcal{O}_{k'}$, respectively, and $w_k, w_{k'}$ are the numbers of roots of unity in these rings.

Combining the above with the fact that $w_k = 2$ yields the following theorem.

Theorem 2. *If $\mathcal{O}_k$ contains a unit of norm -1 or an integral ideal $\mathfrak{b}$ for which $\mathfrak{b}^2 = (\beta)$ is a principal ideal with $N(\beta) < 0$, then $\sum_\kappa L(\Gamma_k,\kappa) = 0$. Otherwise,*

$$\sum_\kappa L(\Gamma_k,\kappa) = \frac{2}{h_kR_k}\sum_{k'}\frac{h_{k'}R_{k'}}{w_{k'}}, \tag{2}$$

where k' ranges through the set of unramified, totally complex, quadratic extensions of k.

To compute the sum in equation (2), we use the computer algebra package KASH [3]. We determine the set of quadratic fields, k', as follows. `RayClassField` and `RayClassFieldAuto` are used to determine the narrow class field of k (that is, the field corresponding to the group of positive principal divisors). Then `OrderSubfield` is used to find all degree 8 subfields. `OrderSig` is used to determine which of these subfields are totally complex, and `OrderIsSubfield` is used to determine which of the totally complex subfields contain k. Once the appropriate quadratic extensions of k have been determined, the commands `OrderClassGroup`, `OrderReg`, and `OrderTorsionUnitRank` can be used to evaluate each summand.

This computation is carried out only when $\mathcal{O}_k$ contains no unit of norm -1 and no integral ideal $\mathfrak{b}$ for which $\mathfrak{b}^2 = (\beta)$ is a principal ideal with $N(\beta) < 0$. We check for a unit of norm -1 by using the commands `OrderUnitsFund` and `EltNorm`. If none exists, we determine whether there is an integral ideal $\mathfrak{b}$ for which $\mathfrak{b}^2 = (\beta)$ is a principal ideal with $N(\beta) < 0$, by using `OrderClassGroupCyclicFactors`. Viewing the class group as a product of cyclic groups, this command provides, for each cyclic factor, the order of the factor and an ideal in a class that generates it.

For each cyclic factor of *even* order, we raise the representative ideal to the order of the factor, thus producing a principal ideal. Finally, we use `IdealIsPrincipal` to obtain a generator for the ideal and `EltNorm` to determine whether the norm of the ideal's generator is negative.

If any of these generators is of negative norm, then setting it equal to β gives us that by Theorem 2, $\sum_\kappa L(\Gamma_k, \kappa) = 0$. We now show that if no such generator exists, then no such β exists.

Lemma 3. *If the above process fails to yield a generator of negative norm, then every generator of any principal ideal equal to the square of an integral ideal in $\mathcal{O}_k$ has positive norm.*

Proof. Suppose that $\mathfrak{b}$ is an integral ideal such that $\mathfrak{b}^2 = (\beta)$ and $N(\beta) < 0$. Since k is totally real and contains no unit of norm -1, $\mathfrak{b}$ is not principal and so is in an ideal class of order 2.

Let $\mathfrak{c}_1$, $\mathfrak{c}_2, \dots, \mathfrak{c}_n$ be representative ideals for generators for the cyclic factors of the class group of k. Let e_1, $e_2, \dots, e_n$ be minimal nonnegative integers such that $\mathfrak{c}_1^{e_1}\mathfrak{c}_2^{e_2}\cdots\mathfrak{c}_n^{e_n}$ is in the same ideal class as $\mathfrak{b}$. So we have

$$\alpha\mathfrak{c}_1^{e_1}\mathfrak{c}_2^{e_2}\cdots\mathfrak{c}_n^{e_n} = \gamma\mathfrak{b},$$

for some α, $\gamma \in \mathcal{O}_k$. Thus

$$\alpha^2\mathfrak{c}_1^{2e_1}\mathfrak{c}_2^{2e_2}\cdots\mathfrak{c}_n^{2e_n} = \gamma^2(\beta). \tag{3}$$

Since each side of equation (3) is a principal ideal, the hypothesis gives us that each $\mathfrak{c}_i^{2e_i}$ has a generator of positive norm. (Note that $e_i = 0$ for any cyclic factor of the class group of odd order, since each $\mathfrak{c}_i^{2e_i}$ is principal.) Hence $\gamma^2(\beta)$ and therefore (β) have generators of positive norm. Since $\mathcal{O}_k$ has no units of norm -1, all generators of (β) have positive norms. This contradicts our supposition and concludes the proof.

3. The volume and fixed point terms

As noted in [5], $\mathrm{vol}(\Gamma_k\backslash\mathcal{H}^n) = \frac{1}{8}\zeta_k(-1)$, which is easily computed using the command `zetak` in the software package PARI [1]. It follows from the work of Siegel [11, 13] that $15\zeta_k(-1)$ is a rational integer. Although PARI does not provide specific error-bounds for `zetak`, all of the outputs were within 10^{-25} of an integral multiple of $\frac{1}{15}$.

Computing the fixed point term is more involved. As defined in [4], letting η be the equivalence class of a fixed point z, this term is given by

$$E(\Gamma_k, [z]) = \frac{1}{|\Gamma_z|}\sum_{\substack{\overline{M}\in\Gamma_z\\ \overline{M}\neq\overline{I}}}\prod_{i=1}^{4}\left(\frac{1}{1-\zeta_i}\right), \tag{4}$$

where Γ_z is the isotropy subgroup of z and ζ is the rotation factor of $\overline{M}$. We summarize our computational formula for this term in the following theorem, which is easily derived from [6, Theorem 4.2].

For any field k', let $h_{k'}$ and $R_{k'}$ be the class number and regulator of $\mathcal{O}_{k'}$ respectively. For $r \in \{2, 3, 4, 5, 6\}$ let

$$m_r = (R_{k\left(e^{\frac{\pi i}{r}}\right)}h_{k\left(e^{\frac{\pi i}{r}}\right)})/(R_k h_k),$$

and let δ be the relative discriminant of $k(e^{\frac{\pi i}{r}})$ over k.

Theorem 4. *Let k be a totally real quartic number field.*
If $d_k = 1125$, then $\sum_\eta E(\Gamma_k, \eta) = \frac{17}{10}$.
If $d_k = 2000$, then $\sum_\eta E(\Gamma_k, \eta) = \frac{151}{60}$.
If $d_k = 2048$, then $\sum_\eta E(\Gamma_k, \eta) = \frac{91}{48}$.
If $d_k = 2304$, then $\sum_\eta E(\Gamma_k, \eta) = \frac{61}{24}$.
If $d_k \notin \{1125, 2000, 2048, 2304\}$, then

$$\sum_\eta E(\Gamma_k, \eta) = \frac{m_2}{96} f_2 + \frac{m_3}{216} f_3 + \frac{m_4}{64} f_4 + \frac{m_5}{40} f_5 + \frac{m_6}{864} f_6, \tag{5}$$

where the f_r are as given in Table 1.

The following corollary simplifies the computation.

TABLE 1. Values of f_r for $r = 2, 3, 4, 5, 6$

$\sqrt{2} \in k$, $\sqrt{3} \notin k$

$2\mathcal{O}_k$	$2\mathcal{O}_{k'}$	δ	f_2	f_4	f_6
$\mathfrak{p}^2$	-	-	6	3	0
$\mathfrak{p}^2\mathfrak{q}^2$	-	-	12	3	0
$\mathfrak{p}^4$	$\mathfrak{s}^4$	-	54	38	0
$\mathfrak{p}^4$	$\mathfrak{s}^4\mathfrak{t}^4$	-	18	20	0
$\mathfrak{p}^4$	$\mathfrak{s}^8$	-	18	9	0

$\sqrt{2} \notin k$, $\sqrt{3} \in k$

$2\mathcal{O}_k$	$2\mathcal{O}_{k'}$	δ	f_2	f_4	f_6
$\mathfrak{p}^2$	$\mathfrak{s}^2$	-	25	0	209
$\mathfrak{p}^2$	$\mathfrak{s}^2\mathfrak{t}^2$	-	15	0	209
$\mathfrak{p}^2\mathfrak{q}^2$	-	-	99	0	209
$\mathfrak{p}^4$	-	-	45	0	209

$\sqrt{2} \notin k$, $\sqrt{3} \notin k$

$2\mathcal{O}_k$	$2\mathcal{O}_{k'}$	δ	f_2	f_4	f_6
$\mathfrak{p}$	-	-	3	0	0
$\mathfrak{p}\mathfrak{q}$	-	-	3	0	0
$\mathfrak{p}\mathfrak{q}\mathfrak{r}$	-	-	3	0	0
$\mathfrak{p}\mathfrak{q}\mathfrak{r}\mathfrak{s}$	-	-	3	0	0
$\mathfrak{p}^2$	$\mathfrak{s}^2$	-	78	0	0
$\mathfrak{p}^2$	$\mathfrak{s}^2\mathfrak{t}^2$	-	48	0	0
$\mathfrak{p}^2$	$\mathfrak{s}^4$	-	15	0	0
$\mathfrak{p}^2\mathfrak{q}$	$\mathfrak{s}^2\mathfrak{t}^2$	-	30	0	0
$\mathfrak{p}^2\mathfrak{q}$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2$	-	12	0	0
$\mathfrak{p}^2\mathfrak{q}$	$\mathfrak{s}^4\mathfrak{t}^2$	-	9	0	0
$\mathfrak{p}^2\mathfrak{q}\mathfrak{r}$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2$	-	30	0	0
$\mathfrak{p}^2\mathfrak{q}\mathfrak{r}$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2\mathfrak{v}^2$	-	12	0	0
$\mathfrak{p}^2\mathfrak{q}\mathfrak{r}$	$\mathfrak{s}^4\mathfrak{t}^2\mathfrak{u}^2$	-	9	0	0
$\mathfrak{p}^2\mathfrak{q}^2$	$\mathfrak{s}^2\mathfrak{t}^2$	-	300	0	0
$\mathfrak{p}^2\mathfrak{q}^2$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2$	-	120	0	0
$\mathfrak{p}^2\mathfrak{q}^2$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2\mathfrak{v}^2$	-	48	0	0
$\mathfrak{p}^2\mathfrak{q}^2$	$\mathfrak{s}^4\mathfrak{t}^2$	-	90	0	0
$\mathfrak{p}^2\mathfrak{q}^2$	$\mathfrak{s}^4\mathfrak{t}^2\mathfrak{u}^2$	-	36	0	0
$\mathfrak{p}^2\mathfrak{q}^2$	$\mathfrak{s}^4\mathfrak{t}^4$	-	27	0	0
$\mathfrak{p}^3\mathfrak{q}$	$\mathfrak{s}^6\mathfrak{t}^2$	$\mathfrak{u}^2\mathfrak{v}^2$	21	0	0
$\mathfrak{p}^3\mathfrak{q}$	$\mathfrak{s}^6\mathfrak{t}^2$	$\mathfrak{u}^4\mathfrak{v}^2$	9	0	0
$\mathfrak{p}^4$	$\mathfrak{s}^4$	-	138	0	0
$\mathfrak{p}^4$	$\mathfrak{s}^4\mathfrak{t}^4$	-	48	0	0
$\mathfrak{p}^4$	$\mathfrak{s}^8$	$\mathfrak{u}^2$	45	0	0
$\mathfrak{p}^4$	$\mathfrak{s}^8$	$\mathfrak{u}^4$	21	0	0

$\sqrt{3} \notin k$

$3\mathcal{O}_k$	$3\mathcal{O}_{k'}$	f_3
$\mathfrak{p}$	-	9
$\mathfrak{p}\mathfrak{q}$	-	9
$\mathfrak{p}\mathfrak{q}\mathfrak{r}$	-	9
$\mathfrak{p}\mathfrak{q}\mathfrak{r}\mathfrak{s}$	-	9
$\mathfrak{p}^2$	$\mathfrak{s}^2$	54
$\mathfrak{p}^2$	$\mathfrak{s}^2\mathfrak{t}^2$	90
$\mathfrak{p}^2\mathfrak{q}$	$\mathfrak{s}^2\mathfrak{t}^2$	45
$\mathfrak{p}^2\mathfrak{q}$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2$	27
$\mathfrak{p}^2\mathfrak{q}\mathfrak{r}$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2$	45
$\mathfrak{p}^2\mathfrak{q}\mathfrak{r}$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2\mathfrak{v}^2$	27
$\mathfrak{p}^2\mathfrak{q}^2$	$\mathfrak{s}^2\mathfrak{t}^2$	234
$\mathfrak{p}^2\mathfrak{q}^2$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2$	144
$\mathfrak{p}^2\mathfrak{q}^2$	$\mathfrak{s}^2\mathfrak{t}^2\mathfrak{u}^2\mathfrak{v}^2$	90
$\mathfrak{p}^3\mathfrak{q}$	-	36
$\mathfrak{p}^4$	$\mathfrak{s}^4$	162
$\mathfrak{p}^4$	$\mathfrak{s}^4\mathfrak{t}^4$	90

$\sqrt{3} \in k$

$3\mathcal{O}_k$	$3\mathcal{O}_{k'}$	f_3
$\mathfrak{p}^2$	-	40
$\mathfrak{p}^2\mathfrak{q}^2$	-	112
$\mathfrak{p}^4$	-	76

$\sqrt{5} \in k$

$5\mathcal{O}_k$	$5\mathcal{O}_{k'}$	f_5
$\mathfrak{p}^4$	$\mathfrak{q}^4$	16
$\mathfrak{p}^4$	$\neq \mathfrak{q}^4$	12
$\neq \mathfrak{p}^4$	-	2

$\sqrt{5} \notin k$

$5\mathcal{O}_k$	$5\mathcal{O}_{k'}$	f_5
-	-	0

Corollary 5. *Let k be a totally real quartic field.*
If $\sqrt{2} \notin k$, then $f_4 = 0$.
If $\sqrt{5} \notin k$, then $f_5 = 0$.
If $\sqrt{3} \notin k$, then $f_6 = 0$.

To compute the values of $\sum_\eta E(\Gamma_k, \eta)$ we first use the KASH function `OrderDisc` to check if k is any of the four special fields listed in Theorem 4. If not, we evaluate equation (5) as follows: `EltRoot` determines which, if any, of $\sqrt{2}$, $\sqrt{5}$, and $\sqrt{3}$ are in k; then applying Corollary 5 as appropriate determines the values of some of the f_r. For the remaining f_r, we use Table 1 to determine what ideals need to be factored and use `OrderDisc` to define any needed relative discriminants. The function `Factor` is used to factor the ideals, and the exponents of the factorization are extracted for use with the table. The values of the m_r are computed directly using the KASH functions `OrderReg` and `OrderClassGroup` for the regulators and class numbers, respectively.

Table 2. Arithmetic genera of Hilbert modular fourfolds

d_k	χ	d_k	χ	d_k	χ	d_k	χ	d_k	χ	d_k	χ
725	1	7625	5	13068	7	17609	5	22221	7	26125	12
1125	2	8000	5	13448	6	17725	7	22545	9	26176	10
1600	2	8069	3	13525	6	17989	5	22592	9	26224	9
1957	1	8112	4	13625	7	18097	7	22676	9	26225	10
2000	3	8468	3	13676	6	18432	8	22784	11	26525	10
2048	2	8525	4	13725	7	18496	14	22896	8	26541	9
2225	2	8725	4	13768	5	18625	9	23252	8	26569	12
2304	3	8768	4	13824	7	18688	10	23297	7	26825	9
2525	2	8789	3	13888	6	18736	6	23301	9	26873	8
2624	2	8957	3	13968	9	19025	8	23377	7	27004	10
2777	1	9225	5	14013	5	19225	8	23525	9	27225	14
3600	4	9248	5	14197	5	19429	6	23552	9	27329	8
3981	2	9301	3	14272	5	19525	8	23600	9	27472	9
4205	2	9792	5	14336	6	19600	10	23665	7	27648	13
4225	3	9909	5	14400	8	19664	12	23724	11	27725	10
4352	4	10025	5	14656	7	19773	10	24197	9	27792	14
4400	3	10273	3	14725	6	19796	12	24336	13	28025	11
4525	3	10304	5	15125	8	19821	7	24400	12	28224	13
4752	4	10309	4	15188	5	20032	8	24417	9	28224	14
4913	3	10512	7	15317	5	20225	8	24437	8	28400	11
5125	4	10816	5	15529	5	20308	7	24525	11	28473	9
5225	3	10889	3	15952	5	20808	9	24749	7	28669	9
5725	3	11025	7	16225	7	21025	9	24832	15	28677	9
5744	3	11197	3	16317	7	21056	8	24917	8	28749	10
6125	4	11324	5	16357	5	21200	8	25088	11	29237	8
6224	3	11344	4	16400	9	21208	7	25225	10	29248	11
6809	2	11348	4	16448	7	21308	8	25488	14	29268	13
7053	3	11525	5	16448	7	21312	10	25492	9	29813	9
7056	5	11661	4	16609	5	21469	7	25525	10	29952	13
7168	4	12197	4	16997	6	21568	11	25717	7	30056	18
7225	4	12357	6	17069	5	21725	8	25808	9	30056	16
7232	4	12400	6	17417	6	21737	8	25857	14	30125	14
7488	5	12544	9	17424	10	21801	9	25893	8	30273	10
7537	2	12725	5	17428	6	21964	10	25961	8	30400	12
7600	4	13025	6	17600	7	22000	11	26032	9	30512	11

4. Implementation and results

Our implementation consists of a shell script calling two programs: a simple PARI program that produces a table of values of $\zeta_k(-1)$, and a KASH program that performs the remainder of the calculations. As input, the script takes a file of fields in a form output by PARI as found at the University of Bordeaux's number fields web site [2], with one field per line. Our PARI program creates a new file of $\zeta_k(-1)$ values which our KASH program reads along with the original input file. For each field in the input file, the KASH program computes the volume term, the fixed point term, and the cusp term as described in this paper and combines them to compute the arithmetic genus.

We ran the programs using PARI 2.1.4 and KASH 2.2 on a SUN Ultra-60, with a 450MHz Sun UltraSPARC-II processor, running the Solaris 5.8 operating system. Results for the Hilbert modular varieties defined by the 210 totally real quartic fields of smallest discriminant are given in Table 2. As an indication of the time involved in these calculations, it takes approximately 40 minutes to compute the first 100 cases, 100 minutes to compute the 210 given in the table, and 13 hours to compute the first 1000.

As a final note, running the program using KASH 2.4 resulted in errors, apparently arising from a problem with the `OrderSubfield` command.

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