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LOWER BOUNDS FOR THE CONDITION NUMBER OF A REAL CONFLUENT VANDERMONDE MATRIX

REN-CANG LI

ABSTRACT. Lower bounds on the condition number $\kappa_p(V_{\rm c})$ of a real confluent Vandermonde matrix $V_{\rm c}$ are established in terms of the dimension n, or n and the largest absolute value among all nodes that define the confluent Vandermonde matrix and the interval that contains the nodes. In particular, it is proved that for any modest $k_{\max}$ (the largest multiplicity of distinct nodes), $\kappa_p(V_{\rm c})$ behaves no smaller than $\mathcal{O}_n((1+\sqrt{2}\,)^n)$, or than $\mathcal{O}_n((1+\sqrt{2}\,)^{2n})$ if all nodes are nonnegative. It is not clear whether those bounds are asymptotically sharp for modest $k_{\max}$.

1. Introduction

Given n numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ called *nodes*, the associated *Vandermonde matrix* is defined as

$$V \stackrel{\rm def}{=} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}. \tag{1.1}$$

It, for example, arises from polynomial interpolation and others [3]. V is invertible if all nodes α_j are distinct, i.e., $\alpha_i \neq \alpha_j$ for $i \neq j$, but it becomes singular whenever $\alpha_i = \alpha_j$ for some $i \neq j$. A generalization of V for nodes not all of which are distinct is the so-called *confluent Vandermonde matrices*, e.g.,

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ \alpha_1 & 1 & 0 & \alpha_4 & \alpha_5 & 1 \\ \alpha_1^2 & 2\alpha_1 & 2 & \alpha_4^2 & \alpha_5^2 & 2\alpha_5 \\ \alpha_1^3 & 3\alpha_1^2 & 6\alpha_1 & \alpha_4^3 & \alpha_5^3 & 3\alpha_5^2 \\ \alpha_1^4 & 4\alpha_1^3 & 12\alpha_1^2 & \alpha_4^4 & \alpha_5^4 & 4\alpha_5^3 \\ \alpha_1^5 & 5\alpha_1^4 & 20\alpha_1^3 & \alpha_4^5 & \alpha_5^5 & 5\alpha_5^4 \end{pmatrix},$$

where $\alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_5 = \alpha_6$. The second, third, and sixth columns are obtained by "differentiating" the previous column. Confluent Vandermonde matrices

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arise in Hermite interpolation [4], for example. Adopting the formulation in [8], we define the *confluent Vandermonde matrix* $V_{\rm c}$ as follows. First

$$\boxed{\begin{array}{l}\{\alpha_j\}_{j=1}^n \text{ are ordered so that equal nodes are contiguous, i.e.,}\\ \qquad\qquad \alpha_i=\alpha_j \quad (i<j) \quad \Rightarrow \quad \alpha_i=\alpha_{i+1}=\cdots=\alpha_j.\end{array}} \tag{1.2}$$

Define

$$V_{\rm c} = (f_1(\alpha_1)\ f_2(\alpha_2)\ \cdots\ f_n(\alpha_n)), \tag{1.3}$$

where the vector function $f_j(t)$ is defined recursively by

$$f_j(t) = \begin{cases} (1\ t\ \cdots\ t^{n-1})^T, & \text{if } j=1 \text{ or } \alpha_j \neq \alpha_{j-1},\\ \frac{d}{dx} f_{j-1}(t), & \text{otherwise,}\end{cases} \tag{1.4}$$

where "$\cdot^T$" is the transpose of a vector or matrix. As far as defining $V_{\rm c}$ is concerned, α_j can be real or complex. But in this paper, we shall focus on real α_j. In what follows, α_j and $V_{\rm c}$, as well as

$$\alpha_{\max} \stackrel{\rm def}{=} \max_j |\alpha_j|,$$

are reserved for their assignments here.

(Optimal) condition numbers for real Vandermonde matrices have been systematically studied by Gautschi and his coauthor (see [7] and references therein), and more recently by Tyrtyshnikov [12], Beckermann [2], and Li [10]. In this paper, we shall establish three lower bounds on the ℓ_p-condition number $\kappa_p(V_{\rm c}) \equiv \|V_{\rm c}\|_p \|V_{\rm c}^{-1}\|_p$ in terms of n, or n and $\alpha_{\max}$ and the interval $[\alpha,\beta]$ that contains all nodes. In particular, we will show that for fixed $k_{\max}$ (the largest multiplicity of distinct nodes), $\kappa_p(V_{\rm c})$ behaves no smaller than $\mathcal{O}_n((1+\sqrt{2}\,)^n)$, where notation $a_n = \mathcal{O}_n(b_n)$ means $c_1 n^{d_1} \le a_n/b_n \le c_2 n^{d_2}$ for some constants c_1, c_2, d_1, and d_2.

Optimally conditioned confluent Vandermonde matrices can be much worse ill-conditioned than optimally conditioned Vandermonde matrices. One extreme example would be that all nodes are equal $\alpha_1 = \cdots = \alpha_n$ for which $V_{\rm c}$ is lower triangular, and thus

$$\kappa_p(V_{\rm c}) \ge (n-1)! \sim \sqrt{2\pi}\, n^{n-1/2}\, e^{-n}$$

by Stirling's asymptotic formula [1, Page 18], and it becomes an equality for $\alpha_1 = \cdots = \alpha_n = 0$. While for optimally conditioned Vandermonde matrices, $\kappa_p(V)$ goes to ∞ as fast as $(1+\sqrt{2}\,)^n$ modulo a factor n^d for $|d| \le 1$ [2, 10].

The rest of this paper is organized as follows. A general lower bound on $\kappa_p(V_{\rm c})$ is established in Section 2, but it is not uniform. Uniform bounds for $p=\infty$ are obtained in Section 3 for all real $V_{\rm c}$ and for $V_{\rm c}$ with nonnegative nodes. Finally we present our concluding remarks in Section 4.

2. A general lower bound

Given $1 \le p \le \infty$, the ℓ_p-norm of vector $u = (\mu_1\ \mu_2\ \cdots \mu_n)^T$ is defined as

$$\|u\|_p = \left(\sum_{j=1}^n |\mu_j|^p \right)^{1/p},$$

and $\|u\|_\infty = \lim_{p\to\infty}\|u\|_p = \max_j |\mu_j|$. The associated ℓ_p-operator norm of the $m \times n$ matrix A is defined as

$$\|A\|_p = \max_{u\neq 0} \frac{\|Au\|_p}{\|u\|_p}. \tag{2.1}$$

It can be proved that $\|A\|_p = \|A^T\|_{p'}$, upon noticing

$$\|A\|_p = \max_{u\neq 0, v\neq 0} \frac{|v^T Au|}{\|v\|_{p'}\|u\|_p},$$

where $1/p + 1/p' = 1$ (see also [9]).

Let $[\alpha, \beta]$ be the interval in which all α_j lie.

$$\begin{aligned} T_n(t) &= \cos(n \arccos t) && \text{for } |t| \le 1, && (2.2)\\ &= \frac{1}{2}\left(t + \sqrt{t^2-1}\right)^n + \frac{1}{2}\left(t - \sqrt{t^2-1}\right)^n && \text{for } |t| \ge 1 && (2.3)\end{aligned}$$

is the nth Chebyshev polynomial of the first kind. Define the nth translated Chebyshev polynomial $T_n(x;\omega,\tau) \stackrel{\text{def}}{=} T_n(x/\omega + \tau)$, where

$$\omega = \frac{\beta-\alpha}{2} > 0, \qquad \tau = -\frac{\beta+\alpha}{\beta-\alpha}.$$

Let $a_{jn} \equiv a_{jn}(\omega,\tau)$ be the coefficient of x^j in $T_n(x;\omega,\tau)$, i.e.,

$$T_n(x;\omega,\tau) = a_{nn}x^n + a_{n-1\,n}x^{n-1} + \cdots + a_{1n}x + a_{0n}. \tag{2.4}$$

Define [10]

$$S_{n,p}(\omega,\tau) = \left(\sum_{j=0}^{n} |a_{jn}|^p\right)^{1/p}.$$

Now we are ready to state our main theorem for the section.

Theorem 2.1. *Assume that there are ℓ distinct nodes α_j, and let $k_{\max}$ be the largest multiplicity of the distinct nodes. Then*

$$\kappa_p(V_c) \ge \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \omega^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\} \frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}. \tag{2.5}$$

Proof. Inequality (2.5) is a consequence of Lemmas 2.1 and 2.3 below. □

For $k_{\max} = 1$, i.e., $\ell = n$ and $k_1 = \cdots = k_n = 1$ (and thus $V_c = V$), (2.5) becomes one of the lower bounds for $\kappa_p(V)$ in [10]. The right-hand side of (2.5) entails the explicit computation of $S_{n,p'}(\omega,\tau)$. It can also be estimated fairly well, too, by

$$\begin{aligned} n^{-1/p}S_{n-1,1}(\omega,\tau) \le{}& S_{n-1,p'}(\omega,\tau) \le S_{n-1,1}(\omega,\tau), && (2.6)\\ \lceil n/2\rceil^{-1/p}S_{n-1,1}(\omega,0) \le{}& S_{n-1,p'}(\omega,0) \le S_{n-1,1}(\omega,0), && (2.7)\end{aligned}$$

in connection with the explicit formulas for $S_{n-1,1}(\omega,\tau)$ for $\tau = 0$ or $|\tau| \ge 1$ in [10]. Here $\lceil\xi\rceil$ is the smallest integer that is larger than ξ. The formulas are

$$S_{n-1,1}(\omega,0) = T_{n-1}(\iota/\omega) \sim \frac{1}{2}\left(\frac{1}{\omega} + \sqrt{1+\frac{1}{\omega^2}}\right)^{n-1}, \tag{2.8}$$

where $\iota = \sqrt{-1}$, and for $\alpha \geq 0$ (for which $\tau \leq -1$),

$$(2.9)\quad S_{n-1,1}(\omega,\tau) = T_{n-1}(|\tau| + 1/\omega) \sim \frac{1}{2}\left[\left(\frac{1}{\omega} + |\tau|\right) + \sqrt{\left(\frac{1}{\omega} + |\tau|\right)^2 - 1}\right]^{n-1}.$$

Lemma 2.1. *Assume that there are ℓ distinct nodes α_j. Then*

$$(2.10)\qquad \|V_{\rm c}\|_p \;\geq\; \max\left\{\ell^{1/p'}, \alpha_{\max}^{n-1}\right\},$$

$$(2.11)\qquad \|V_{\rm c}\|_p \;\geq\; \left(\sum_{j=1}^{n} \alpha_{\max}^{(j-1)p}\right)^{1/p}.$$

Proof. Let e_j be the jth column of the $n \times n$ identity matrix I_n (or simply I if n is clear from the context). Use $\|V_{\rm c}\|_p \geq \|V_{\rm c}^T e_1\|_{p'}$ and $\|V_{\rm c}\|_p \geq \|V_{\rm c}^T e_n\|_{p'}$ to get (2.10), and use $\|V_{\rm c}\|_p \geq \max_j \|V_{\rm c}^T e_j\|_p$ to get (2.11). □

Lemma 2.2. *For $0 \leq k \leq n$,*

$$(2.12)\qquad \left|\frac{d}{dx^k} T_n(x;\omega,\tau)\right| \leq \frac{[n(n-1)\cdots(n-k+1)]^2}{\omega^k} \quad \textit{for } x \in [\alpha,\beta].$$

Proof. It follows from $T_n(x;\omega,\tau) = T_n(x/\omega + \tau) \equiv T_n(t)$ that

$$\frac{d^k}{dx^k} T_n(x;\omega,\tau) = \frac{1}{\omega^k} T_n^{(k)}(t),$$

where $t \equiv t(x) = x/\omega + \tau$. It suffices to show that $|T_n^{(k)}(t)| \leq [n(n-1)\cdots(n-k+1)]^2$ for $t \in [-1,1]$ since $t(x)$ maps $x \in [\alpha,\beta]$ to $t \in [-1,1]$. By Markov's inequality [5, Page 233],

$$\begin{aligned}
\max_{t\in[-1,1]} |T_n^{(k)}(t)| &\leq (n-k+1)^2 \max_{t\in[-1,1]} |T_n^{(k-1)}(t)| \\
&\leq \cdots \\
&\leq [n(n-1)\cdots(n-k+1)]^2 \max_{t\in[-1,1]} |T_n(t)| \\
&= [n(n-1)\cdots(n-k+1)]^2,
\end{aligned}$$

as expected. □

Lemma 2.3. *Under the conditions of* Theorem 2.1*,*

$$(2.13)\qquad \|V_{\rm c}^{-1}\|_p \geq \min_{1\leq k\leq k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \omega^{k-1} \times \frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}}.$$

Proof. For the sake of this proof, let the ℓ distinct nodes have multiplicities $k_1, k_2, \ldots, k_\ell$, respectively, where $k_1 + k_2 + \cdots + k_\ell = n$, and the first k_1 α_j's are equal, the next k_2 α_j's are equal, and so on. Let v be the vector of the coefficients of the translated Chebyshev polynomial $T_{n-1}(x;\omega,\tau)$, i.e., $v = (a_{0\,n-1}\ a_{1\,n-1}\ \cdots\ a_{n-1\,n-1})^T$. Then

$$V_{\rm c}^T v = \left(T_{n-1}(\alpha_1;\omega,\tau)\ T_{n-1}'(\alpha_1;\omega,\tau)\ \cdots\ T_{n-1}^{(k_1-1)}(\alpha_1;\omega,\tau)\ \cdots\ \cdots\right)^T,$$

which yields, by Lemma 2.2, for $1 \le p' < \infty$

$$
\begin{aligned}
(2.14)\quad \|V_{\mathrm{c}}^T v\|_{p'}^{p'} &\le \sum_{j=1}^{\ell} \left(1^{p'} + \left[\frac{(n-1)^2}{\omega}\right]^{p'} \right. \\
&\qquad \left. + \cdots + \left[\frac{[(n-1)(n-2)\cdots(n-k_j+1)]^2}{\omega^{k_j-1}}\right]^{p'} \right) \\
&\le \sum_{j=1}^{\ell} k_j \times \left(\max_{1\le k\le k_j} \left[\frac{(n-1)!}{(n-k)!}\right]^2 \frac{1}{\omega^{k-1}} \right)^{p'} \\
(2.15)\quad &\le n \times \left(\max_{1\le k\le k_{\max}} \left[\frac{(n-1)!}{(n-k)!}\right]^2 \frac{1}{\omega^{k-1}} \right)^{p'},
\end{aligned}
$$

which gives

$$
\|V_{\mathrm{c}}^T v\|_{p'} \le n^{1/p'} \times \max_{1\le k\le k_{\max}} \left[\frac{(n-1)!}{(n-k)!}\right]^2 \frac{1}{\omega^{k-1}}. \tag{2.16}
$$

This is proved so far for $1 \le p' < \infty$, but it can be verified that (2.16) holds for $p' = \infty$, too. Therefore, we have

$$
\begin{aligned}
\|V_{\mathrm{c}}^{-T}\|_{p'} &= \max_u \frac{\|u\|_{p'}}{\|V_{\mathrm{c}}^T u\|_{p'}} \ge \frac{\|v\|_{p'}}{\|V_{\mathrm{c}}^T v\|_{p'}} \\
&\ge \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \omega^{k-1} \times \frac{S_{n-1,p'}(\omega,\tau)}{n^{1/p'}},
\end{aligned}
$$

as was to be shown. □

In general, we may use (2.14), instead of (2.15), in estimating $\|V_{\mathrm{c}}^{-1}\|_p$. Doing so, however, will lead to a more complicated lower bound on $\kappa_p(V_{\mathrm{c}})$.

Remark 2.1. Lemma 2.3 is made possible by Lemma 2.2 which is proved with the help of Markov's inequality. Another classical inequality for the same purpose is Bernstein's inequality [5, Page 233], using which we can obtain the following. *For* $0 \le k \le n$, *if* $\alpha < a \stackrel{\mathrm{def}}{=} \min_j \alpha_j < b \stackrel{\mathrm{def}}{=} \max_j \alpha_j < \beta$, *then*

$$
\left| \frac{d}{dx^k} T_n(x;\omega,\tau) \right| \le \frac{n(n-1)\cdots(n-k+1)}{\left[\omega\sqrt{1 - \left(\frac{\max\{\beta-b,\, a-\alpha\}}{\omega}\right)^2}\right]^k} \quad \text{for } x \in [\alpha,\beta]. \tag{2.17}
$$

This inequality improves (2.12) in the numerator part but has complications in the denominator, and also it requires the interval $[\alpha,\beta]$ to be (slightly) larger than the smallest interval containing all nodes. This can be bad because larger $[\alpha,\beta]$ will weaken the effectiveness of $S_{n,p'}(\omega,\tau)$ in the later bounds on $\kappa_p(V_{\mathrm{c}})$; for example $S_{n,p'}(\omega,\tau)$ is decreasing in ω [10].

3. Two uniform bounds

We present two theorems here, one for any real V_{c} and one for V_{c} with nonnegative nodes. Their proofs will be given later after two lemmas. Again let $1 \le p \le \infty$ and $1/p + 1/p' = 1$.

Theorem 3.1. *Under the conditions of* Theorem 2.1, *if*

$$k_{\max} - 1 \le \frac{n-1}{\sqrt{2}}\left[1 - (1+\sqrt{2}\,)^{-2n+2}\right] \sim \frac{n-1}{\sqrt{2}}, \tag{3.1}$$

then

$$\begin{aligned}\kappa_p(V_{\rm c}\,) &\ge \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{S_{n-1,1}(1,0)}{n^{1/p'}\,\lceil n/2\rceil^{1/p}}\\ &\sim \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{\left[1+\sqrt{2}\right]^{n-1}}{n^{1/p'}\,\lceil n/2\rceil^{1/p}}.\end{aligned}$$

Theorem 3.2. *Under the conditions of* Theorem 2.1, *if all $\alpha_i \ge 0$ and*

$$k_{\max} - 1 \le \frac{n-1}{\sqrt{2}}\left[1 - (1+\sqrt{2}\,)^{-4(n-1)}\right]^{-1} \sim \frac{n-1}{\sqrt{2}}, \tag{3.2}$$

then

$$\begin{aligned}\kappa_p(V_{\rm c}\,) &\ge \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{1}{2^{k_{\max}-1}}\frac{S_{n-1,1}(1/2,1)}{n}\\ &\sim \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{1}{2^{k_{\max}-1}}\frac{\left[1+\sqrt{2}\right]^{2(n-1)}}{n}.\end{aligned}$$

Lemma 3.1. *Let $j \ge 0$ and $m \ge 1$. $\rho^j S_{m,1}(\rho,0)$ is decreasing in ρ for $0 \le \rho \le 1$ if*

$$j \le \frac{m}{\sqrt{2}}\left[1 - (1+\sqrt{2}\,)^{-2m}\right] \sim \frac{m}{\sqrt{2}}. \tag{3.3}$$

Proof. We claim that under inequality (3.3), $\frac{d}{d\rho}\rho^j S_{m,1}(\rho,0) \le 0$ for $0 \le \rho \le 1$. To this end, we notice that

$$\frac{d}{d\rho}\rho^j S_{m,1}(\rho,0) = j\rho^{j-1}S_{m,1}(\rho,0) + \rho^j\,\frac{d}{d\rho}S_{m,1}(\rho,0).$$

Now for $0 \le \rho \le 1$ and by (2.8), we have

$$\begin{aligned}S_{m,1}(\rho,0) &\le \frac{1}{2}\left[\frac{1}{\rho} + \sqrt{1+\frac{1}{\rho^2}}\right]^m \left[1+\epsilon^{-2m}\right],\\ -\frac{d}{d\rho}S_{m,1}(\rho,0) &\ge \frac{m}{2}\left[\frac{1}{\rho} + \sqrt{1+\frac{1}{\rho^2}}\right]^{m-1}\left[1-\delta^{-2m}\right]\\ &\quad\times\left[\frac{1}{\rho^2} + \frac{1}{\rho^2\sqrt{1+\rho^2}}\right],\end{aligned}$$

where $\epsilon = 1+\sqrt{2}$ and $\delta = 0$ for even m, and $\epsilon = 0$ and $\delta = 1+\sqrt{2}$ for odd m. Therefore, for $\rho \le 1$,

$$\begin{aligned}
\frac{\frac{d}{d\rho}\rho^j S_{m,1}(\rho,0)}{m\,\rho^{j-1}S_{m,1}(\rho,0)} &= \frac{j}{m} + \frac{\rho\frac{d}{d\rho}S_{m,1}(\rho,0)}{m\,S_{m,1}(\rho,0)} \\
&\le \frac{j}{m} - \frac{\rho\left[\frac{1}{\rho^2}+\frac{1}{\rho^2\sqrt{1+\rho^2}}\right]}{\frac{1}{\rho}+\sqrt{1+\frac{1}{\rho^2}}}\,\frac{1-\delta^{-2m}}{1+\epsilon^{-2m}} \\
&= \frac{j}{m} - \frac{1}{\sqrt{1+\rho^2}}\,\frac{1-\delta^{-2m}}{1+\epsilon^{-2m}} \\
&\le \frac{j}{m} - \frac{1}{\sqrt{2}}\,\frac{1-\delta^{-2m}}{1+\epsilon^{-2m}} \\
&\le \frac{j}{m} - \frac{1}{\sqrt{2}}\left[1-(1+\sqrt{2}\,)^{-2m}\right] \\
&\le 0
\end{aligned}$$

upon using (3.3). $\square$

Lemma 3.2. *Let $j \ge 0$, $\gamma \ge 1$, and $m \ge 1$. For j satisfying* (3.3) *and $\rho > 0$,*

$$\rho^j \,\max\{\gamma,\rho^m\}\,S_{m,1}(\rho,0) \ge S_{m,1}(1,0).$$

Proof. Let $\Phi_1 = \rho^j \times \gamma\, S_{m,1}(\rho,0)$ and $\Phi_2 = \rho^j \times \rho^m\, S_{m,1}(\rho,0)$. Then $\max\{\Phi_1,\Phi_2\}$ is Φ_1 for $\rho \le \gamma^{1/m}$ and Φ_2 for $\rho \ge \gamma^{1/m}$. Φ_2 is increasing in ρ for $\rho > 0$ because $\rho^m S_{m,1}(\rho,0)$ is a polynomial in ρ with nonnegative coefficients and thus increasing in ρ for $\rho > 0$. So

$$\max\{\Phi_1,\Phi_2\} \ge \Phi_2 \ge S_{m,1}(1,0) \quad \text{for } \rho \ge 1.$$

For $0 \le \rho \le 1$, Φ_1 is decreasing in ρ by Lemma 3.1, and thus

$$\max\{\Phi_1,\Phi_2\} \ge \Phi_1 \ge S_{m,1}(1,0) \quad \text{for } \rho \le 1.$$

This completes the proof. $\square$

Proof of Theorem 3.1. Setting $-\alpha = \beta = \alpha_{\max}$ in (2.5), we have, upon using (2.7),

$$\begin{aligned}
\kappa_p(V_{\rm c}) &\ge \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \alpha_{\max}^{k-1} \times \max\{\ell^{1/p'},\alpha_{\max}^{n-1}\}\,\frac{S_{n-1,1}(\alpha_{\max},0)}{n^{1/p'}\,\lceil n/2\rceil^{1/p}} \\
&\ge \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{1}{n^{1/p'}\,\lceil n/2\rceil^{1/p}} \min_{1\le k\le k_{\max}} \widetilde{\Phi}, \qquad (3.4)
\end{aligned}$$

where $\widetilde{\Phi} = \alpha_{\max}^{k-1} \times \max\{\ell^{1/p'},\alpha_{\max}^{n-1}\}\,S_{n-1,1}(\alpha_{\max},0)$. Apply Lemma 3.2 with $j = k-1$, $m = n-1$, $\gamma = \ell^{1/p'}$, and $\rho = \alpha_{\max}$ to get $\widetilde{\Phi} \ge S_{n-1,1}(1,0)$, as needed. $\square$

Proof of Theorem 3.2. Setting $0 = \alpha < \beta = \alpha_{\max}$ in (2.5), we have, upon using (2.6),

$$\begin{aligned}
\kappa_p(V_{\rm c}) &\ge \min_{1\le k\le k_{\max}} \left[\frac{(n-k)!}{(n-1)!}\right]^2 \left[\frac{\alpha_{\max}}{2}\right]^{k-1} \times \max\{\ell^{1/p'},\alpha_{\max}^{n-1}\}\,\frac{S_{n-1,1}(\alpha_{\max}/2,1)}{n} \\
&\ge \left[\frac{(n-k_{\max})!}{(n-1)!}\right]^2 \frac{1}{n2^{k_{\max}-1}} \min_{1\le k\le k_{\max}} \widetilde{\Psi},
\end{aligned}$$

where

$$\widetilde{\Psi} = \alpha_{\max}^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\}\, S_{n-1,1}(\alpha_{\max}/2, 1).$$

It can be verified by (2.3), (2.8), and (2.9) that

$$S_{n-1,1}(\alpha_{\max}/2, 1) = S_{2(n-1),1}(\sqrt{\alpha_{\max}}, 0).$$

Therefore

$$\begin{aligned}\widetilde{\Psi} &= (\sqrt{\alpha_{\max}})^{2(k-1)} \times \max\left\{\ell^{1/p'}, (\sqrt{\alpha_{\max}})^{2(n-1)}\right\} S_{2(n-1),1}(\sqrt{\alpha_{\max}}, 0)\\ &\geq S_{2(n-1),1}(1, 0),\end{aligned}$$

upon using Lemma 3.2 with $j = 2(k-1)$, $m = 2(n-1)$, $\gamma = \ell^{1/p'}$, and $\rho = \sqrt{\alpha_{\max}}$. □

4. Concluding remarks

We have obtained three lower bounds on the condition number $\kappa_p(V_c)$ of a real confluent Vandermonde matrix V_c. Two of them are uniform in the sense that they depend on n, the dimension of V_c only, while the other one is more general, as is the function of n and $\alpha_{\max}$ and the interval $[\alpha, \beta]$ that contains all α_j. These bounds grow exponentially for any fixed $k_{\max}$, much as expected. While it is not clear in general if (any of) our bounds are asymptotically optimal, in contrast to those for Vandermonde matrices by Beckermann [2] and recently by the author [10], our bounds are unlikely to be asymptotically optimal if $k_{\max}$ also grows, e.g., linearly in n. This is illustrated by the extreme example $k_{\max} = n$, as we commented in Section 1.

We have focused on real confluent Vandermonde matrices here. It is conceivable that there would be much better conditioned complex confluent Vandermonde matrices or confluent Vandermonde-like matrices. This is partly an intuition one might get from that although real Vandermonde matrices are very ill-conditioned [7, 2, 10, 12], there exist very well-conditioned complex Vandermonde matrices and Vandermonde-like matrices [6, 11]. We plan to investigate this issue in future work.

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Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506
E-mail address: `rcli@ms.uky.edu`
URL: `http://www.ms.uky.edu/~rcli`