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CONTINUOUS-TIME KREISS RESOLVENT CONDITION ON INFINITE-DIMENSIONAL SPACES

TATJANA EISNER AND HANS ZWART

Dedicated to M.N. Spijker on the occasion of his 65th birthday.

Abstract. Given the infinitesimal generator A of a C_0-semigroup on the Banach space X which satisfies the Kreiss resolvent condition, i.e., there exists an $M > 0$ such that $\|(sI - A)^{-1}\| \leq \frac{M}{\mathrm{Re}(s)}$ for all complex s with positive real part, we show that for general Banach spaces this condition does not give any information on the growth of the associated C_0-semigroup. For Hilbert spaces the situation is less dramatic. In particular, we show that the semigroup can grow at most like t. Furthermore, we show that for every $\gamma \in (0, 1)$ there exists an infinitesimal generator satisfying the Kreiss resolvent condition, but whose semigroup grows at least like t^γ. As a consequence, we find that for $\mathbb{R}^N$ with the standard Euclidian norm the estimate $\|\exp(At)\| \leq M_1 \min(N, t)$ cannot be replaced by a lower power of N or t.

1. Introduction

Let us begin by introducing some notation: $(T(t))_{t\geq 0}$ will denote a C_0-semigroup on the Banach space X, and A its infinitesimal generator. The celebrated Hille–Yosida theorem states that $\|T(t)\| \leq M$ for all $t \geq 0$ if and only if $\|\mathrm{Re}(s)^n (sI - A)^{-n}\| \leq M$ for all $s \in \mathbb{C}_0^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ and $n \in \mathbb{N}$. Unfortunately, this theorem can be very hard to check. Hence, people have tried to find conditions that are easier to check. One of the conditions that has been proposed is the Kreiss resolvent condition, originally stated in Kreiss [11] for A being a matrix. This condition corresponds precisely to the first condition in the Hille–Yosida theorem; i.e.,

$$\|(sI - A)^{-1}\| \leq \frac{M}{\mathrm{Re}(s)}, \qquad s \in \mathbb{C}_0^+. \tag{1}$$

From the Hille–Yosida theorem it is clear that if the semigroup is bounded, then (1) holds. Furthermore, it is easy to see that if $M = 1$, then (1) is equivalent to the Hille–Yosida conditions. For Banach spaces and $M > 1$ it is known that the Kreiss resolvent condition does not imply the boundedness of the semigroup; see, e.g., Engel and Nagel [8, section V.1.b.]. For a finite-dimensional space, i.e., $X = \mathbb{R}^N$ it was shown in Dorsselaer et al. [6] that (1) implies that $\|\exp(At)\| \leq eMN$; see also [13]. Furthermore, if the norm on $\mathbb{R}^N$ is the maximum norm, then there exists

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an A such that $\sup_{t\geq 0}\|\exp(At)\| \geq \frac{2N-1}{\pi+1}M$; see Kraaijevanger [10]. For discrete time, $\dim(X) = N$, it is known that $\|A^k\| \leq eM_d \min\{(N+1), k\}$, where M_d is the constant in the Kreiss resolvent estimate for the unit disc; see [6]. However, we were not able to find a continuous-time counterpart of this result in the literature. Using a scaled version of the example in [10], one can construct an example satisfying the Kreiss resolvent condition (1) on $\mathbb{R}^N$, but $\|\exp(At)\| \geq c\min\{N, t\}$ for some constant c independent of t and N. For a nice overview of these and related results and for historic remarks, we refer the reader to [6]. We remark that the discrete-time counterpart of the Kreiss resolvent estimate has attracted more attention than the continuous-time version which is the subject of this article. A cited reference showed that there are 29 citations to the original Kreiss paper [11] on the continuous-time version, whereas there are 68 citations to [12] discussing the discrete-time version. The reason for this lies in the fact that in numerical schemes one normally discretizes with respect to time, and hence obtains a difference equation. However, for a partial differential equation, one could only discretize with respect to the spatial variable, and thus obtain a differential equation on a finite-dimensional space. It is important to know the properties of this equation. The Kreiss resolvent condition (1) will be much easier to check than the actual growth. For more discussion we refer to Section 5 of [6]. For an overview on the Kreiss resolvent condition and many more references, we refer to [1, 6, 19].

On the basis of the discrete-time result and the above mentioned example, one might hope that the Kreiss resolvent condition (1) implies that the semigroup grows at most like t. In Section 2 we show that this indeed holds for Hilbert spaces. Unfortunately, for a general Banach space, exponential growth is possible; see Section 3. We show that for any $M > 1$ and $\alpha > 0$ there exists an infinitesimal generator which satisfies (1) with this M, but whose corresponding semigroup grows like $\exp(\alpha t)$. In Section 4 for every $\gamma \in [0, 1)$ we construct a Hilbert space and an infinitesimal generator A satisfying the Kreiss resolvent condition on this Hilbert space, but the corresponding semigroup grows at least like t^γ.

This Hilbert space example leads to a finite-dimensional example, showing that if an $N \times N$ matrix A satisfies (1), then the supremum of $\exp(At)$ over $t > 0$ can be of the order N^γ.

2. An upperbound on the growth

In this section we show that for every Hilbert space, the Kreiss resolvent condition implies that the semigroup grows at most like t. The proof uses the following lemma of Eisner [7], which holds on a general Banach space.

Lemma 2.1. *Assume that A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and let $s_0(A)$ be the pseudospectral bound; i.e.,*

$$s_0(A) = \inf\{r \in \mathbb{R} \mid (sI - A)^{-1} \textit{is uniformly bounded on } \mathrm{Re}(s) > r\}.$$

If there exists an $r > s_0(A)$ such that for all $x \in X$ and $y \in X^*$

$$\int_{-\infty}^{\infty} |\langle((r + i\omega)I - A)^{-2}x, y\rangle| d\omega < \infty, \tag{2}$$

then for all $x \in X$ and $t > 0$ the following equality holds:

$$T(t)x = \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{(r+i\omega)t}((r + i\omega)I - A)^{-2}x d\omega. \tag{3}$$

Note that since the left-hand side does not depend on r, the right-hand side should give the same answer for all such $r > s_0(A)$. A consequence of this lemma is the following result.

Theorem 2.2. *Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and let $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$. Assume that $\mathbb{C}_0^+$ is contained in the resolvent set of A and that the following conditions hold:*

1. *A satisfies the Kreiss estimate (1);*
2. *There exists a $\rho > 0$ such that for all $x \in X$ the function $\omega \mapsto ((\rho + i\omega)I - A)^{-1}x$ belongs to $L^p(\mathbb{R}, X)$;*
3. *There exists a $\tilde{\rho} > 0$ such that for all $y \in X^*$ the function $\omega \mapsto ((\tilde{\rho} + i\omega)I - A^*)^{-1}y$ belongs to $L^q(\mathbb{R}, X)$.*

Then there exists a $K > 0$ such that $\|T(t)\| \leq K(1 + t)$ for all $t \geq 0$.

Proof. The proof consists of several steps. We first show that items (2) and (3) above hold for all $\rho, \tilde{\rho} > 0$. Secondly, we estimate the value of the integral in equation (2). Finally, we apply Lemma 2.1 to show the assertion.

Step 1. Since the Kreiss estimate holds, we have that $s_0(A) \leq 0$; see Lemma 2.1.

Step 2. Since the function $\omega \mapsto ((\rho + i\omega)I - A)^{-1}x$ is an element of $L^p(\mathbb{R}, X)$ for all $x \in X$, we conclude from the uniform boundedness theorem that there exists a constant $M_0 > 0$ such that

$$\|((\rho + i\cdot)I - A)^{-1}x\|_{L^p} \leq M_0\|x\| \tag{4}$$

for all $x \in X$. Analogously, there exists a constant $\tilde{M}_0 > 0$ such that

$$\|((\rho + i\cdot)I - A^*)^{-1}y\|_{L^q} \leq \tilde{M}_0\|y\| \tag{5}$$

holds for all $y \in X^*$.

Step 3. Since $\mathbb{C}_0^+$ is contained in the resolvent set of A, we obtain by the resolvent identity that for every $r > 0$ and $x \in X$

$$((r + i\omega)I - A)^{-1}x = \left[I + (\rho - r)((r + i\omega)I - A)^{-1}\right]((\rho + i\omega)I - A)^{-1}x,$$

and hence

$$\begin{aligned}\|((r + i\cdot)I - A)^{-1}x\|_{L^p} &\leq \left[1 + |\rho - r|\frac{M}{r}\right]\|((\rho + i\cdot)I - A)^{-1}x\|_{L^p}\\ &\leq \left[1 + |\rho - r|\frac{M}{r}\right] M_0\|x\|,\end{aligned}$$

where we have used (4) and the Kreiss estimate (1). Analogously

$$\|((r + i\cdot)I - A^*)^{-1}y\|_{L^q} \leq \left[1 + |\tilde{\rho} - r|\frac{M}{r}\right]\tilde{M}_0\|y\|$$

holds for every $y \in X^*$.

By the Cauchy–Schwarz inequality

$$\begin{aligned}\|\langle((r + i\cdot)I - A)^{-2}x, y\rangle\|_{L^1} \leq& \|((r + i\cdot)I - A)^{-1}x\|_{L^p}\|((r + i\cdot)I - A^*)^{-1}y\|_{L^q}\\ \leq& \left[1 + |\rho - r|\frac{M}{r}\right]\left[1 + |\tilde{\rho} - r|\frac{M}{r}\right] M_0\tilde{M}_0\|x\|\|y\|\\ \leq& M_1\left[1 + \frac{1}{r}\right]^2\|x\|\|y\| \qquad (6)\end{aligned}$$

holds for some constant M_1 and all $r > 0$.

Step 4. Combining equation (3) with (6), we see that

$$\begin{aligned} |\langle T(t)x, y\rangle| &\leq \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{rt}|\langle((r+i\omega)I - A)^{-2}x, y\rangle| d\omega \\ &\leq \frac{1}{2\pi t}e^{rt}M_1\left[1+\frac{1}{r}\right]^2 \|x\|\|y\|. \end{aligned} \tag{7}$$

Since $s_0(A) \leq 0$, estimate (7) holds for all $r > 0$ and we may choose $r := 1/t$. This gives

$$|\langle T(t)x, y\rangle| \leq \frac{1}{2\pi t}eM_1\left[1+t\right]^2 \|x\|\|y\|. \tag{8}$$

So for large t the norm of the semigroup is bounded by $\frac{eM_1}{\pi}t$. Since any C_0-semigroup is uniformly bounded on a compact time interval, the result follows. □

As it is clear from the proof of the above theorem, the relation between the constants M (the constant in the Kreiss resolvent condition) and K (the constant in the growth) involves other constants as well. Namely, $K = \frac{eM_0\tilde{M}_0}{\pi}\max\{\rho M, \frac{1+(1-\rho)M}{2}\}$. Hence one does not have that K is a universal constant times M.

As a corollary of Theorem 2.2 we obtain the following result for Hilbert spaces; see also Zwart [20].

Corollary 2.3. *If A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space H, and if A satisfies the Kreiss estimate* (1), *then there exists a $K > 0$ such that $\|T(t)\| \leq K[1+t]$ for all $t \geq 0$.*

Proof. We show that conditions (2) and (3) in Theorem 2.2 are automatically satisfied with $p = q = 2$ for generators on Hilbert spaces. For a Hilbert space the growth bound is equal to the pseudospectral bound $s_0(A)$; see Proposition 2 of [17]. Hence for $\rho > 0$, the function $t \mapsto e^{-\rho t}T(t)x$ is square integrable for every $x \in H$. By the Paley–Wiener theorem this implies that the function $\omega \mapsto ((\rho + i\omega)I - A)^{-1}x$ is square integrable. Using a similar argument for the dual semigroup, we see that the function $\omega \mapsto ((\rho+i\omega)I - A^*)^{-1}y$ is also square integrable for every $y \in H$. By Theorem 2.2 there exists a $K > 0$ such that $\|T(t)\| \leq K[1+t]$ for all $t \geq 0$. □

3. Worst growth on a Banach space

In this section we construct a Banach space and a generator satisfying the Kreiss estimate, but the corresponding semigroup has exponential growth. The example shows that this construction is possible for all $M > 1$; see (1). The example is based on the counterexample in Engel and Nagel [8, p. 254].

Example 3.1. Let α be a positive number. As state space we choose $X_\alpha := C_0([0,\infty)) \cap L_1((0,\infty), e^s ds)$ with norm

$$\|f\|_{X_\alpha} = \alpha \sup_{\eta\geq 0}|f(\eta)| + \int_0^\infty |f(\eta)|e^\eta d\eta.$$

With this norm X_α becomes a Banach space, and this space is similar to X_1. On the space X_α we define the operator

$$A_0 f = \dot{f}$$

with $D(A_0) = \{f \in X \mid f \in C^1([0,\infty)), \dot{f} \in X_\alpha\}$. In Engel and Nagel it is shown that the resolvent set of A_0 on X_1 contains every complex number with real part greater than -1. Since X_α is similar to X_1, the same assertion holds for the resolvent set of A_0 on X_α. It is easy to see that A_0 is the infinitesimal generator of the C_0-semigroup $(T_0(t))_{t\geq 0}$ with $(T_0(t)f)(\eta) = f(t+\eta)$. From this expression one easily derives that $\|T_0(t)\| = 1$ for all $t \geq 0$, and that the inverse of $sI - A_0$ is given by

$$\left((sI - A_0)^{-1} f\right)(\eta) = \int_\eta^\infty e^{s(\eta-\xi)} f(\xi) d\xi \tag{9}$$

for $\mathrm{Re}(s) > -1$. Using this expression, one can show that

$$\|(sI - A_0)^{-1} f\|_{X_\alpha} = \|(\mathrm{Re}(s)I - A_0)^{-1}\left(e^{-i\mathrm{Im}(s)\cdot} f(\cdot)\right)\|_{X_\alpha}.$$

Since $\|e^{-i\mathrm{Im}(s)\cdot} f(\cdot)\|_{X_\alpha} = \|f\|_{X_\alpha}$, we find that

$$\|(sI - A_0)^{-1}\| = \|(\mathrm{Re}(s)I - A_0)^{-1}\|. \tag{10}$$

Using equation (9) and the definition of the norm, we find for $r > -1$ that

$$\begin{aligned}
&\|(rI - A_0)^{-1} f\|_{X_\alpha} \\
&\quad = \alpha \sup_{\eta\geq 0} \left| \int_\eta^\infty e^{r(\eta-\xi)} f(\xi) d\xi \right| + \int_0^\infty \left| \int_\eta^\infty e^{r(\eta-\xi)} f(\xi) d\xi \right| e^\eta d\eta \\
&\quad = \alpha \sup_{\eta\geq 0} \left| \int_\eta^\infty e^{(r+1)(\eta-\xi)} e^{-\eta} e^{\xi} f(\xi) d\xi \right| \\
&\quad\quad + \int_0^\infty \left| \int_\eta^\infty e^{(r+1)(\eta-\xi)} e^{\xi} f(\xi) d\xi \right| d\eta \\
&\quad \leq \alpha \sup_{\eta\geq 0} \int_\eta^\infty |e^\xi f(\xi)| d\xi + \int_0^\infty e^{-(r+1)\xi} d\xi \int_0^\infty |e^\xi f(\xi)| d\xi \\
&\quad \leq \alpha \|f\|_X + \frac{1}{r+1} \|f\|_X. \qquad (11)
\end{aligned}$$

Since A_0 is the infinitesimal generator of a contraction semigroup, we know that

$$\|(rI - A_0)^{-1}\| \leq \frac{1}{r}$$

for all $r > 0$. Combining this with (11) and (10), we find that

$$\|(sI - A_0)^{-1}\| \leq \begin{cases} \alpha + \frac{1}{\mathrm{Re}(s)+1}, & \mathrm{Re}(s) \in (-1, 0], \\ \min\{\frac{1}{\mathrm{Re}(s)}, \alpha + \frac{1}{\mathrm{Re}(s)+1}\}, & \mathrm{Re}(s) > 0. \end{cases} \tag{12}$$

Choose an $\varepsilon > 0$, then on $[1/\varepsilon, \infty)$ we have that $r^{-1} \leq (1+\varepsilon)(r+1)^{-1}$ and for $\alpha = \varepsilon^2/(1+\varepsilon)$ we have that $\alpha + (r+1)^{-1} \leq (1+\varepsilon)(r+1)^{-1}$ for $r \in (-1, 1/\varepsilon]$. Thus for any $\varepsilon > 0$ we can find an α such that

$$\|(sI - A_0)^{-1}\| \leq \frac{1+\varepsilon}{\mathrm{Re}(s)+1}, \qquad \mathrm{Re}(s) > -1. \tag{13}$$

Yet we construct the infinitesimal generator with exponential growth. Define for $\gamma > 0$

$$\mathcal{A}_\gamma := \gamma A_0 + \gamma I.$$

For s with positive real part we have

$$\begin{aligned} \|(sI-\mathcal{A}_\gamma)^{-1}\| &= \|((s-\gamma)I-\gamma A_0)^{-1}\| \\ &= \frac{1}{\gamma}\left\|\left(\left(\frac{s}{\gamma}-1\right)I-A_0\right)^{-1}\right\| \leq \frac{1+\varepsilon}{\mathrm{Re}(s)}, \end{aligned} \tag{14}$$

where we have used (13). Thus $\mathcal{A}_\gamma$ satisfies the Kreiss estimate. Since A_0 is the infinitesimal generator, we have that $\mathcal{A}_\gamma$ is it too. The corresponding semigroup is given by

$$\mathcal{T}_\gamma(t) = e^{\gamma t}T_0(\gamma t).$$

Since $\|T_0(t)\| = 1$ for all $t \geq 0$, we find that

$$\|\mathcal{T}_\gamma(t)\| = e^{\gamma t}. \tag{15}$$

Thus we have constructed an infinitesimal generator satisfying the Kreiss estimate (1), but having exponential growth.

4. Worst growth on a Hilbert space

In the previous section we have seen that exponential growth is possible on a Banach space. From Corollary 2.3 we know that on a Hilbert space the growth is at most like t. In this section we construct for every $\gamma \in (0,1)$ an infinitesimal generator on a Hilbert space that satisfies the Kreiss resolvent condition (1), but whose semigroup grows at least like t^γ. It turns out that the generator is a bounded operator. As a consequence of this construction, we find $N \times N$ matrices Q_N satisfying the Kreiss resolvent condition for the same constant, and the supremum of $e^{Q_N t}$ is at least of the order N^γ.

The idea of this example is based on the papers by Spijker, Tracogna, and Welfert [18], Borovykh and Spijker [3], and on the one page note by Kalton and Montgomery-Smith [16]. Note that the basis of the idea is already in the paper by McCarthy and Schwartz [15] from 1965.

Let w be a positive measurable function from the interval $(-\pi,\pi)$ to $\mathbb{R}$. By $L_2((-\pi,\pi),w)$ we denote the set of all measurable functions from $(-\pi,\pi)$ to $\mathbb{C}$ for which $\int_{-\pi}^{\pi}|f(x)|^2w(x)dx < \infty$. This space is a Hilbert space with the inner product

$$\langle f,g\rangle_w = \int_{-\pi}^{\pi} f(x)\overline{g(x)}w(x)dx. \tag{16}$$

We denote the finite span by $\mathrm{span}_{k\in\mathbb{Z}}\{e^{ikx}\}$; i.e., $f \in \mathrm{span}_{k\in\mathbb{Z}}\{e^{ikx}\}$ can be written as $f = \sum_{k\in\mathbb{Z}}\alpha_k e^{ikx}$ with all but finitely many α_k's equal to zero. Using a result by Hunt, Muckenhoupt, and Wheeden [9] it is not hard to show the following.

Lemma 4.1. *For $n \in \mathbb{Z}$ and $f \in \mathrm{span}_{k\in\mathbb{Z}}\{e^{ikx}\}$, we define*

$$(P_nf)(x) = \sum_{k=-\infty}^{n}\alpha_k e^{ikx}. \tag{17}$$

Let w satisfy the condition

$$\sup_{I\subset[-\pi,\pi]}\frac{1}{|I|^2}\int_I w(x)dx\int_I w(x)^{-1}dx < \infty, \tag{18}$$

where I is an interval and $|I|$ is the length of this interval. Then the following holds.

(1) *The P_n's are projections on $L_2((-\pi,\pi),w)$, and they are uniformly bounded; i.e.,*

$$\|P_n f\|_w \le c_w \|f\|_w \tag{19}$$

for all $f \in L_2((-\pi,\pi),w)$, where c_w does not depend on f and n.

(2) *For all $f \in L_2((-\pi,\pi),w)$ we have that*

$$\lim_{n\to-\infty} P_n f = 0, \qquad \lim_{n\to\infty} P_n f = f, \tag{20}$$

The above results imply that the set $\{\ldots, e^{-i2x}, e^{-ix}, 1, e^{ix}, e^{i2x}, \ldots\}$ is a conditional basis on $L_2((-\pi,\pi),w)$.

Proof. The proof consists of several steps. In the first two steps we show that (19) is satisfied, and in the last step we prove (20).

Step 1. Define the mapping

$$\left(M_n f\right)(x) = e^{inx} f(x)$$

Since e^{inx} has absolute value one, it is easy to see that M_n is a bounded linear mapping on $L_2((-\pi,\pi),w)$ with norm one.

The conjugate function $\tilde{f}$ is defined as

$$\tilde{f}(y) := \frac{1}{2\pi} \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \le |y| \le \pi} \frac{f(x-y)}{\tan(\frac{y}{2})} dy. \tag{21}$$

We denote $Hf := \tilde{f}$. It is not hard to see (e.g., using induction) that

$$\left(He^{inx}\right)(y) = \begin{cases} -ie^{iny}, & n > 0, \\ 0, & n = 0, \\ ie^{iny}, & n < 0. \end{cases} \tag{22}$$

Theorem 1 of [9] shows that H is a bounded linear mapping on $L_2((-\pi,\pi),w)$ whenever w satisfies condition (18).

Step 2. For $f = \sum_k \alpha_k e^{ik\cdot} \in \mathrm{span}_{k\in\mathbb{Z}}\{e^{ikx}\}$ we consider the mapping

$$\begin{aligned} -iM_n H M_{-n} f + f + \alpha_n e^{inx} &= -iM_n H\left(\sum_k \alpha_k e^{i(k-n)x}\right) + f + \alpha_n e^{inx} \\ &= 2\sum_{k\le n} \alpha_k e^{ikx} = 2P_n f, \end{aligned}$$

where we have used (22). So we see that the right combination of M's and H gives the projection P_n. Using the bounds on M_n, M_{-n}, H from Step 1, we see that this projection is uniformly bounded if and only if the norm of $\alpha_n e^{inx}$ is bounded by some constant (independent of n and f) times the norm of f. Using (18), we see that

$$\begin{aligned} \|\alpha_n e^{inx}\|_w^2 &= |\alpha_n|^2 \int_{-\pi}^{\pi} |e^{inx}|^2 w(x)dx \\ &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right|^2 \int_{-\pi}^{\pi} w(x)dx \\ &\le \frac{1}{4\pi^2}\int_{-\pi}^{\pi} |f(x)|^2 w(x)dx \int_{-\pi}^{\pi} |e^{-inx}|^2 w(x)^{-1}dx \int_{-\pi}^{\pi} w(x)dx \\ &= c_1\|f\|_w^2, \end{aligned}$$

where $c_1 = \frac{1}{4\pi^2}\int_{-\pi}^{\pi} w(x)^{-1}dx \int_{-\pi}^{\pi} w(x)dx$, which is finite by (18). So we have shown that

$$\|P_n f\|_w \le c_w \|f\|_w$$

for all $f \in \operatorname{span}_{k\in\mathbb{Z}}\{e^{ikx}\}$. Since by [9, Thm. 8] this span is dense in $L_2((-\pi,\pi),w)$, we have proved (19).

Step 3. As in the previous step we first choose $f \in \operatorname{span}_{k\in\mathbb{Z}}\{e^{ikx}\}$. Since $f = \sum_{k\in\mathbb{Z}} \alpha_k e^{ikx}$ with all but finitely many α_k's equal to zero, it is easy to see that (20) holds. Since the projections are uniformly bounded and since the finite span is dense in $L_2((-\pi,\pi),w)$, we have that (20) holds for all $f \in L_2((-\pi,\pi),w)$. $\square$

Since the exponential function will be used a lot, we simplify notation a little bit. For $n \in \mathbb{Z}$ we define $\phi_n(x) = e^{inx}$, $x \in [-\pi,\pi]$. A consequence of the above lemma is the following result.

Lemma 4.2. *Assume that w is a positive weight satisfying condition* (18) *and that $\{\beta_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is a sequence with $\beta_n \le \beta_{n+1}$ for all $n \in \mathbb{Z}$.*

(1) *If $\sup|\beta_n| < \infty$, then the operator Q defined as $Q\phi_n = i\beta_n\phi_n$, $n \in \mathbb{N}$ extends to a bounded linear operator on $L_2((-\pi,\pi),w)$. Furthermore, we have that*

$$\|Q\| \le c_w \left[\lim_{n\to\infty}[\beta_n - \beta_{-n}]\right] + \lim_{n\to\infty}|\beta_n|. \tag{23}$$

(2) *The operator $R(s,Q)$ defined as $R(s,Q)\phi_n = (s - i\beta_n)^{-1}\phi_n$, $n \in \mathbb{N}$ extends for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) \neq 0$ to a bounded linear operator on $L_2((-\pi,\pi),w)$. Furthermore, we have that*

$$\|R(s,Q)\| \le \frac{1+\pi c_w}{|\operatorname{Re}(s)|} \quad \textit{for } \operatorname{Re}(s) \neq 0. \tag{24}$$

(3) *If $\sup|\beta_n| < \infty$, then every $s \in \mathbb{C}$ with $\operatorname{Re}(s) \neq 0$ is in the resolvent set of Q, and $R(s,Q) = (sI - Q)^{-1}$.*

In both estimates c_w is the constant from (19).

Proof. 1. Let f be an element in the span of ϕ_n. Then $f = \sum_{n=-N}^{N} \alpha_n\phi_n$ for some $N > 0$. By the linearity of Q we have

$$\begin{aligned} Q\left(\sum_{n=-N}^{N}\alpha_n\phi_n\right) &= \sum_{n=-N}^{N}\alpha_n Q\phi_n = \sum_{n=-N}^{N}\alpha_n i\beta_n\phi_n \\ &= \sum_{n=-N}^{N} i\beta_n\left(P_n f - P_{n-1}f\right) \\ &= \sum_{n=-N}^{N-1}\left[i\beta_n - i\beta_{n+1}\right]P_n f + i\beta_N f, \end{aligned} \tag{25}$$

where we have used that for our f, $P_N f = f$, and $P_{-N-1} f = 0$. Thus

$$\begin{aligned}\left\| Q\left(\sum_{n=-N}^{N} \alpha_n \phi_n\right)\right\|_w &\leq \sum_{n=-N}^{N-1} |i\beta_n - i\beta_{n+1}| \|P_n f\|_w + |\beta_N| \|f\|_w \\ &= \sum_{n=-N}^{N-1} [\beta_{n+1} - \beta_n] \|P_n f\|_w + |\beta_N| \|f\|_w \\ &\leq c_w \sum_{n=-N}^{N-1} [\beta_{n+1} - \beta_n] \|f\|_w + |\beta_N| \|f\|_w \\ &\leq \left[c_w \lim_{N\to\infty} [\beta_N - \beta_{-N}] + \lim_{N\to\infty} |\beta_N|\right] \|f\|_w,\end{aligned}$$

where we have used twice that $\beta_n \leq \beta_{n+1}$. Since the expression between the square brackets is finite, and since the span of the ϕ_n is dense, we have that Q is a bounded operator. Furthermore, we get that $\|Q\|$ is bounded by the expression within the square brackets, or equivalently that (23) holds.

2. Similar to (25) we have for $f = \sum_{n=-N}^{N} \alpha_n \phi_n$ that

$$R(s,Q)f = \sum_{n=-N}^{N-1} \left[(s - i\beta_n)^{-1} - (s - i\beta_{n+1})^{-1}\right] P_n f + (s - i\beta_N)^{-1} f.$$

Since for every real β we have that $|(s - i\beta)^{-1}| \leq |\mathrm{Re}(s)|^{-1}$, we obtain by (19) that

$$\|R(s,Q)f\|_w \leq \left[c_w \sum_{n=-N}^{N-1} \left|\frac{1}{(s - i\beta_n)} - \frac{1}{(s - i\beta_{n+1})}\right| + \frac{1}{|\mathrm{Re}(s)|}\right] \|f\|_w. \tag{26}$$

So it remains to show that $|\mathrm{Re}(s)| \sum_{n=-N}^{N-1} \left|\frac{1}{(s-i\beta_n)} - \frac{1}{(s-i\beta_{n+1})}\right|$ is bounded. For this we write $s = a + ib$, with a and b real, $a \neq 0$.

$$\begin{aligned}&|a| \sum_{n=-N}^{N-1} \left|\frac{1}{(a + ib - i\beta_n)} - \frac{1}{(a + ib - i\beta_{n+1})}\right| \\ &\quad= \sum_{n=-N}^{N-1} \left|\frac{1}{1 + i\frac{b-\beta_n}{a}} - \frac{1}{1 + i\frac{b-\beta_{n+1}}{a}}\right| \\ &\quad= \sum_{n=-N}^{N-1} \left|\int_{\frac{b-\beta_{n+1}}{|a|}}^{\frac{b-\beta_n}{|a|}} \frac{-i}{(1+i\eta)^2} d\eta\right| \\ &\quad\leq \int_{\frac{b-\beta_N}{|a|}}^{\frac{b-\beta_{-N}}{|a|}} \frac{1}{1+\eta^2} d\eta \leq \pi, \qquad (27)\end{aligned}$$

where we have used the monotonicity of β_n. Combining (26) with (27), we conclude that (24) holds.

3. Since $R(s,Q)$ is the inverse of Q on the basis elements and since $R(s,Q)$ is bounded, the assertion follows immediately. □

Note that the above proof is an adaptation of Lemma 3.2.5 of Benamara and Nikolski [2] which gives a bound on diagonal operators on a conditional basis.

Let w be a weight that satisfies condition (18). On $L_2((-\pi,\pi),w)$ we introduce the operators that will be used for our counterexample. For $N>0$ we define A_N as

$$A_N\phi_n := \begin{cases} \frac{in}{N}\phi_n, & |n|\le N,\\ -i\phi_n, & n<-N,\\ i\phi_n, & n>N. \end{cases} \tag{28}$$

From Lemma 4.2 the following properties are immediate:

- The A_N extend to linear bounded operators on $L_2((-\pi,\pi),w)$ and the norm of these operators is uniformly bounded by $2c_w+1$.
- For each $s\in\mathbb{C}$ with nonzero real part we have that $sI-A$ is boundedly invertible on $L_2((-\pi,\pi),w)$ and

$$\|(sI-A_N)^{-1}\| \le \frac{1+\pi c_w}{|\mathrm{Re}(s)|} \qquad \text{for } \mathrm{Re}(s)\neq 0. \tag{29}$$

Hence the operators A_N satisfy the Kreiss estimate for the same constant.

Since A_N is a bounded operator, it generates the C_0-group $\left(e^{A_N t}\right)_{t\in\mathbb{R}}$. This group has the following property:

Lemma 4.3. *For the operator A_N as defined in* (28) *we have that*

$$e^{A_N t}\phi_n = \begin{cases} e^{i\frac{n}{N}t}\phi_n, & |n|\le N,\\ e^{-it}\phi_n, & n<-N,\\ e^{it}\phi_n, & n>N, \end{cases} \tag{30}$$

and

$$e^{A_N N\pi}\left(\frac{\sin((N+1/2)x)}{\sin(x/2)}\right) = (-1)^N\frac{\cos((N+1/2)x)}{\cos(x/2)}. \tag{31}$$

Proof. Using the fact that A_N is diagonal, it is not hard to show that (30) holds. So we concentrate on the other equality. First we remark that

$$\frac{\sin((N+1/2)x)}{\sin(x/2)} = \sum_{|n|\le N} e^{inx} = \sum_{|n|\le N}\phi_n(x). \tag{32}$$

So by equation (30) we have that

$$\begin{aligned} e^{A_N N\pi}\frac{\sin((N+1/2)x)}{\sin(x/2)} &= e^{A_N N\pi}\sum_{|n|\le N}\phi_n \\ &= \sum_{|n|\le N} e^{A_N N\pi}\phi_n = \sum_{|n|\le N} e^{in\pi}\phi_n \\ &= \sum_{|n|\le N}(-1)^n e^{inx} = (-1)^N\frac{\cos((N+1/2)x)}{\cos(x/2)}. \end{aligned}$$

Hence we have shown (31). □

With these lemmas we can now construct our example.

Example 4.4. Let γ be a positive number less than 1. We construct a bounded operator $\mathcal{A}_\gamma$ such that $e^{\mathcal{A}_\gamma t}$ behaves at least like $|t|^\gamma$.

We begin by choosing the weight function. Let $0<\gamma<1$ be fixed and

$$w(x)=\begin{cases}|x|^\gamma, & -\frac{\pi}{2}\le x\le\frac{\pi}{2},\\ (\pi-|x|)^{-\gamma}, & \frac{\pi}{2}<|x|<\pi.\end{cases} \tag{33}$$

We proceed as follows. In Step 1 we show that this weight satisfies condition (18), and in Step 2 we prove that the induced operator norm on $L_2((-\pi,\pi),w)$ of $e^{A_N N\pi}$ is larger than N^γ. In the last step we construct $\mathcal{A}_\gamma$.

Step 1. To prove (18), let us consider first $I=[a,b]$, where $0<a<b\le\frac{\pi}{2}$. Then

$$\begin{aligned}K_I &:= \frac{1}{|I|^2}\int_I w(x)dx\int_I w(x)^{-1}dx=\frac{1}{(b-a)^2}\int_a^b x^\gamma dx\int_a^b x^{-\gamma}dx\\ &= \frac{1}{(1-\gamma^2)(b-a)^2}(b^{1+\gamma}-a^{1+\gamma})(b^{1-\gamma}-a^{1-\gamma})=[z:=\frac{b}{a}]\\ &= \frac{1}{(1-\gamma^2)(z-1)^2}(z^{1+\gamma}-1)(z^{1-\gamma}-1)\le\frac{1}{1-\gamma^2}.\end{aligned}$$

Since this estimate is independent of a, it holds also for intervals of the form $I=[a,b]$, where $0\le a<b\le\frac{\pi}{2}$.

Let I be now an arbitrary interval. Notice that if $|I|\ge\delta$ for some $\delta>0$, then we have immediately that $K_I\le C\delta^{-2}$, where $C:=\int_{-\pi}^{\pi}w(s)ds\cdot\int_{-\pi}^{\pi}w(s)^{-1}ds$. So we can assume that $|I|\le\frac{\pi}{2}$. By symmetry it suffices to consider intervals of the form $J=[-a,b]$ for $0\le a<b\le\frac{\pi}{2}$. For such intervals we obtain

$$\begin{aligned}(a+b)^2K_{[-a,b]} &= a^2K_{[-a,0]}+b^2K_{[0,b]}\\ &\quad+\int_{-a}^0 w(x)dx\cdot\int_0^b w(x)^{-1}dx+\int_0^b w(x)dx\cdot\int_{-a}^0 w(x)^{-1}dx\\ &\le \frac{a^2+b^2}{1-\gamma^2}+2\int_0^b w(x)dx\cdot\int_0^b w(x)^{-1}dx\\ &\le \frac{a^2+3b^2}{1-\gamma^2}\le\frac{3}{1-\gamma^2}(a+b)^2.\end{aligned}$$

So the weight w satisfies condition (18).

Step 2. In this step we show that

$$\|e^{A_N N\pi}f_N\|_w^2\ge N^{2\gamma}\|f_N\|_w^2, \tag{34}$$

where

$$f_N(x)=\frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

First we estimate the left-hand side of (34). For this purpose we need the following trigonometrical facts which are easy to prove:

$$\cos(x/2)\le\frac{\pi-x}{2},\qquad x\le\pi;$$
$$|\cos((N+1/2)x)|\ge\frac{2}{\pi}(N+1/2)(\pi-x),\qquad \frac{2N\pi}{2N+1}\le x\le\pi.$$

Using these inequalities and Lemma 4.3, we have

$$\begin{aligned}\|e^{A_N N\pi} f_N\|_w^2 &= 2\int_0^\pi \left|\frac{\cos((N+1/2)x)}{\cos(x/2)}\right|^2 w(x)dx \\ &\geq 2\int_{2N\pi/(2N+1)}^\pi \left|\frac{\cos((N+1/2)x)}{\cos(x/2)}\right|^2 \frac{dx}{(\pi-x)^\gamma} \\ &\geq \frac{8(2N+1)^2}{\pi^2}\int_{2N\pi/(2N+1)}^\pi \frac{dx}{(\pi-x)^\gamma} \\ &= \frac{8(2N+1)^2}{\pi^2}\frac{\pi^{1-\gamma}}{1-\gamma}\left(1-\frac{2N}{2N+1}\right)^{1-\gamma} \\ &= \frac{8}{(1-\gamma)\pi^{1+\gamma}}(2N+1)^{1+\gamma} \geq \frac{2^{4+\gamma}}{(1-\gamma)\pi^2}N^{1+\gamma}.\end{aligned}$$

To estimate the right-hand side of (34), we need

$$|f_N(x)| \leq \frac{\pi}{x}, \qquad 0 < x \leq \pi, \tag{35}$$

$$|f_N(x)| \leq 2N+1, \qquad 0 \leq x \leq \pi. \tag{36}$$

For $x \in (0, \pi]$ we have

$$\left|\frac{\sin((N+1/2)x)}{\sin(x/2)}\right| \leq \frac{1}{\sin(x/2)} \leq \frac{\pi}{x}.$$

The inequality (36) follows directly from (32).

Using the inequalities (35) and (36), we obtain the following estimate for the left-hand side of (34):

$$\begin{aligned}\|f_N\|_w^2 &= 2\int_0^\pi |f_N(x)|^2 w(x)dx \\ &\leq 2\int_0^{1/N}(2N+1)^2 x^\gamma dx + 2\int_{1/N}^{\pi/2}\frac{\pi^2}{x^2}x^\gamma dx + 2\int_{\pi/2}^\pi \frac{\pi^2}{x^2}\frac{dx}{(\pi-x)^\gamma} \\ &\leq 2\int_0^{1/N}(2N+1)^2 x^\gamma dx + 2\pi^2\int_{1/N}^\infty x^{\gamma-2}dx + 8\int_{\pi/2}^\pi \frac{dx}{(\pi-x)^\gamma} \\ &= 2(2N+1)^2\frac{1}{N^{1+\gamma}(1+\gamma)} - \frac{2\pi^2}{\gamma-1}\frac{1}{N^{\gamma-1}} + \frac{8}{1-\gamma}\left(\frac{\pi}{2}\right)^{1-\gamma} \\ &\leq \frac{18}{\gamma+1}N^{1-\gamma} + \frac{2\pi^2}{1-\gamma}N^{1-\gamma} + \frac{8}{1-\gamma}\left(\frac{\pi}{2}\right)^{1-\gamma} \\ &\leq \frac{4\pi^2}{1-\gamma^2}N^{1-\gamma} + \frac{8}{1-\gamma}\left(\frac{\pi}{2}\right)^{1-\gamma}N^{1-\gamma} \leq \frac{16\pi^2}{1-\gamma^2}N^{1-\gamma}.\end{aligned}$$

So we see that

$$\frac{\|e^{A_N N\pi} f_N\|_w^2}{\|f_N\|_w^2} \geq 2^\gamma(1+\gamma)N^{2\gamma} \geq N^{2\gamma},$$

and therefore $\|e^{A_N N\pi}\| \geq N^\gamma$ holds for all $N \in \mathbb{N}$. Note that the operator $-A$ has the same form as A, so by an analogous construction we obtain that the group satisfies

$$\|e^{\pm A_N N\pi}\| \geq N^\gamma. \tag{37}$$

Step 3. Let $0 < \gamma < 1$. Consider the Hilbert space $H := l^2\left(L_2((-\pi,\pi),w)\right)$, where w is given by (33). The inner product on this space is given by

$$\langle (x_n), (y_n)\rangle = \sum_{n=1}^{\infty} \langle x_n, y_n\rangle_w.$$

For the (block) diagonal operator $Q = \mathrm{diag}(Q_n)$ the norm is given by

$$\|Q\| = \sup_n \|Q_n\|. \tag{38}$$

On H we define $\mathcal{A}_\gamma := \mathrm{diag}(A_n)$. By (29) and (38), $\mathcal{A}_\gamma$ is a bounded operator on H and it satisfies the Kreiss resolvent condition

$$\|R(s, \mathcal{A}_\gamma)\| \le \frac{1 + \pi c_w}{|\mathrm{Re}(s)|}$$

for all s with $\mathrm{Re}(s) \neq 0$. Moreover, by estimate (37) and (38) the group generated by $\mathcal{A}_\gamma$ satisfies

$$\|e^{\mathcal{A}_\gamma N\pi}\| \ge \|e^{A_N N\pi}\| \ge N^\gamma,$$

and the same holds at $t = -N\pi$. So we conclude that $e^{\mathcal{A}_\gamma t}$ grows at least as $|t|^\gamma$.

Remark 4.5. The operator $\mathcal{A}_\gamma$ constructed in the above example is the infinitesimal generator of an unbounded group. If the group on positive time would be bounded, then the Kreiss resolvent condition on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$ implies that the group is bounded on all time; see [4].

Note that it is not clear if there exists an infinitesimal generator on a Hilbert space that satisfies the Kreiss resolvent condition and whose semigroup grows exactly like t.

We end this section with the investigation of the Kreiss resolvent condition for matrices. For every $\gamma \in (0,1)$ and for every N we construct a (stable) $N \times N$ matrix Q_N that satisfies the Kreiss resolvent condition for a constant M independent of N and γ. Furthermore, for some t the norm of the exponential function $\exp(Q_N t)$ becomes at least N^γ. The example is more or less present in the previous example, but in order to clarify the construction, we shall present the details. Before we do so, we first present a simple lemma.

Lemma 4.6. *Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let W be bounded, linear operator on H which is positive and boundedly invertible. With this W we define a new norm on H,*

$$\|f\|_W^2 = \langle f, Wf\rangle.$$

Then for any bounded linear mapping Q on H we have that

$$\|Q\|_W := \sup_{f\neq 0} \frac{\|Qf\|_W}{\|f\|_W} = \|W^{\frac{1}{2}} Q W^{-\frac{1}{2}}\| := \sup_{f \neq 0} \frac{\|W^{\frac{1}{2}} Q W^{-\frac{1}{2}} f\|}{\|f\|}.$$

Example 4.7. Let $\mathcal{V}_N$ be the $(2N{+}1)$-dimensional linear subspace of $L_2((-\pi,\pi),w)$ which is spanned by $\phi_k(\cdot) = e^{ik\cdot}$, $k = -N, \dots, N$.

If $f(\cdot) = \sum_{k=-N}^{N} \alpha_k e^{ik\cdot}$, then

$$\|f\|_w^2 = \langle (\alpha_k), W(\alpha_k)\rangle_{\mathbb{C}^{2N+1}}, \tag{39}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^{2N+1}}$ is the standard inner-product on $\mathbb{C}^{2N+1}$, and

$$W = (W_{kl})_{k,l=1,\ldots,2N+1} \quad \text{with} \tag{40}$$

$$W_{kl} = \int_{-\pi}^{\pi} e^{i(k-1-N)x} e^{-i(l-1-N)x} w(x) dx, \tag{41}$$

where w is given by (33). With this W we define a new inner product on $\mathbb{C}^{2N+1}$, namely

$$\|(\alpha_k)\|_W^2 = \langle (\alpha_k), W(\alpha_k) \rangle_{\mathbb{C}^{2N+1}}. \tag{42}$$

Furthermore, the $2N+1$ by $2N+1$ matrix Q_N is given by

$$Q_N = W^{\frac{1}{2}} \operatorname{diag}\left(i \frac{k}{N} \right) W^{-\frac{1}{2}}. \tag{43}$$

This matrix satisfies the Kreiss condition and $\sup_{t>0} \| \exp(Q_N t) \| \geq N^{\gamma}$.

We start by showing that this matrix satisfies the Kreiss condition with a constant independent of N. It is easy to see that

$$(sI - Q_N)^{-1} = W^{\frac{1}{2}} \operatorname{diag}\left((s - i\frac{k}{N})^{-1} \right) W^{-\frac{1}{2}}.$$

Using Lemma 4.6, we find that

$$\begin{aligned} \|(sI - Q_N)^{-1}\|_{\mathbb{C}^{2N+1}} &= \|\operatorname{diag}\left((s - i\frac{k}{N})^{-1} \right) \|_W \\ &= \sup_{(\alpha_k)\neq 0} \frac{\| \left(\operatorname{diag}\left((s - i\frac{k}{N})^{-1} \right) \right) (\alpha_k) \|_W}{\|(\alpha_k)\|_W} \\ &= \sup_{f \in \mathcal{V}_N \setminus \{0\}} \frac{\|(sI - A_N)^{-1} f\|_w}{\|f\|_w}, \end{aligned}$$

where we have used (28) and (39). Since $\mathcal{V}_N$ is a subspace of $L_2((-\pi,\pi), w)$, we have by (29) that

$$\|(sI - Q_N)^{-1}\|_{\mathbb{C}^{2N+1}} \leq \frac{1 + \pi c_w}{|\mathrm{Re}(s)|} \tag{44}$$

for $\mathrm{Re}(s) \neq 0$. Similarly one can show that

$$\|e^{Q_N N \pi}\|_{\mathbb{C}^{2N+1}} \geq N^{\gamma}, \tag{45}$$

thus completing the finite-dimensional example.

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Arbeitsbereich Funktionalanalysis, Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany

E-mail address: talo@fa.uni-tuebingen.de

Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

E-mail address: h.j.zwart@math.utwente.nl