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A TWO-GRID DISCRETIZATION METHOD FOR DECOUPLING SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we propose a two-grid finite element method for solving coupled partial differential equations, e.g., the Schrödinger-type equation. With this method, the solution of the coupled equations on a fine grid is reduced to the solution of coupled equations on a much coarser grid together with the solution of decoupled equations on the fine grid. It is shown, both theoretically and numerically, that the resulting solution still achieves asymptotically optimal accuracy.

1. Introduction

The idea of the two-grid finite element method was originally proposed by Xu in [19, 20, 21, 22] for discretizing nonsymmetric and indefinite partial differential equations. By employing two finite element spaces of different scales, one coarse space and one fine space, this method was first used for symmetrization of nonsymmetric problems, which reduces the solution of a nonsymmetric problem on a fine grid to the solution of a corresponding (but much smaller) nonsymmetric problem discretized on the coarse grid and the solution of a symmetric positive definite problem on the fine grid. This method was also used for linearization for nonlinear problems [12, 19, 20], for localization and parallelization for solving a large class of partial differential equations [16, 17, 18]. There are also many other authors who have used this method for many different applications. See, for example, Axelsson et al. [2, 3, 4], Girault and Lions [7], Layton et al. [9, 10, 11], and Utnes [15].

In this paper, we explore the two-grid idea in a new direction, namely we will use the two-grid discretization method to decouple the systems of partial differential equations. For clarity, we will use a simple model problem of the Schrödinger equation which arises from quantum mechanics to illustrate our idea. Similar to the two-grid discretization method for symmetrization, linearization, localization and parallelization as mentioned above, we use the two-grid method to decouple a system of partial differential equations by first discretizing the original systems of partial differential equations on the coarse grid and then discretizing a decoupled

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system on the fine grid. As a result, the computational complexity of solving, say, a model Schrödinger equation is comparable to solving two decoupled Poisson equations on the same fine grid.

Similar to other applications, the two-grid discretization method for decoupling a system of partial differential equations is not only an efficient numerical method by itself for such applications, but its analysis should provide some insights on how a multiscale idea can be applied for systems of partial differential equations.

The rest of this paper is organized as follows. In section 2, we introduce a model Schrödinger equation used to demonstrate our method. In section 3, we propose the two-grid finite element algorithms and analyze the convergence. Section 4 is devoted to the presentation of numerical examples showing the effectiveness of our method.

2. A model Schrödinger equation

The Schrödinger equation is the fundamental equation in quantum mechanics. It also arises in mathematically modelling underwater acoustics, where the Helmholtz equation for the acoustic pressure is transformed into an equation of the same form by applying the so-called "parabolic approximation" [14]. For simplification, here we consider the following boundary value problem of the Schrödinger type:

$$-\Delta\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) = f(\mathbf{x}), \qquad \forall\ \mathbf{x} \in \Omega, \tag{2.1}$$

$$\psi(\mathbf{x}) = 0, \qquad \forall\ \mathbf{x} \in \partial\Omega, \tag{2.2}$$

where $\Omega \subset \mathbf{R}^2$ is a polygonal domain which, for simplicity of exposition, will be assumed to be convex. In general, $f(\mathbf{x})$, the potential function $V(\mathbf{x})$ and unknown function $\psi(\mathbf{x})$ are complex valued.

For any complex-valued function $w(\mathbf{x})$, we denote its real part by $w_1(\mathbf{x})$, the imaginary part by $w_2(\mathbf{x})$, and the vector function $(w_1(\mathbf{x}), w_2(\mathbf{x}))$ by $\mathbf{w}(\mathbf{x})$. Then problem (2.1)–(2.2) is equivalent to the following coupled equations:

$$-\Delta\psi_1(\mathbf{x}) + V_1(\mathbf{x})\psi_1(\mathbf{x}) - V_2(\mathbf{x})\psi_2(\mathbf{x}) = f_1(\mathbf{x}), \qquad \forall\ \mathbf{x} \in \Omega, \tag{2.3}$$

$$-\Delta\psi_2(\mathbf{x}) + V_1(\mathbf{x})\psi_2(\mathbf{x}) + V_2(\mathbf{x})\psi_1(\mathbf{x}) = f_2(\mathbf{x}), \qquad \forall\ \mathbf{x} \in \Omega, \tag{2.4}$$

$$\psi_j(\mathbf{x}) = 0, \quad j = 1, 2, \qquad \forall\ \mathbf{x} \in \partial\Omega. \tag{2.5}$$

Let $L^2(\Omega)$ be the inner product space with the inner product given by

$$(u,\ v) = \int_\Omega u(\mathbf{x})v(\mathbf{x})d\mathbf{x}$$

for real-valued and Lebesgue square integrable functions $u(\mathbf{x})$ and $v(\mathbf{x})$, let $H^m(\Omega)$ be the standard Sobolev space with a norm given by $\|u\|_m^2 = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^2(\Omega)}^2$ for a real-valued function $u(\mathbf{x})$, and let $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ consisting of functions with vanishing trace on $\partial\Omega$. Then the equivalent variational form of (2.3)–(2.5) is defined as follows.

$$\begin{gathered}\text{Find } \boldsymbol{\psi} \in H_0^1(\Omega) \times H_0^1(\Omega) \text{ such that} \\ a(\boldsymbol{\psi}, \mathbf{w}) = (\mathbf{f}, \mathbf{w}), \qquad \forall\ \mathbf{w} \in H_0^1(\Omega) \times H_0^1(\Omega),\end{gathered} \tag{2.6}$$

where

$$(\mathbf{f}, \mathbf{w}) = (f_1,\ w_1) + (f_2,\ w_2)\,, \qquad a(\boldsymbol{\psi}, \mathbf{w}) = \widehat{a}(\boldsymbol{\psi}, \mathbf{w}) + N(\boldsymbol{\psi}, \mathbf{w})$$

with

$$\begin{aligned}\widehat{a}(\boldsymbol{\psi},\mathbf{w}) &= (\nabla\psi_1,\ \nabla w_1)+(\nabla\psi_2,\ \nabla w_2)\,,\\ N(\boldsymbol{\psi},\mathbf{w}) &= (V_1\psi_1,\ w_1)-(V_2\psi_2,\ w_1)+(V_1\psi_2,\ w_2)+(V_2\psi_1,\ w_2)\,.\end{aligned}$$

Let the notation "$\lesssim$" be equivalent to "$\leq C$" for some positive constant C and $\|\mathbf{w}\|_m$ denote $\sqrt{\|w_1\|_m^2+\|w_2\|_m^2}$ for any vector function $\mathbf{w}(\mathbf{x})$. Let us first state a simple regularity result.

Theorem 1. *Assume that*

$$\mathbf{f}\in L^2(\Omega)\times L^2(\Omega),\ \ \mathbf{V}\in L^\infty(\Omega)\times L^\infty(\Omega),\ \ V_1(\mathbf{x})\geq 0 \text{ in } \Omega. \tag{2.7}$$

Then the variational problem (2.6) *has a unique solution* $\boldsymbol{\psi}\in H^2(\Omega)\times H^2(\Omega)$, *and*

$$\|\boldsymbol{\psi}\|_2\lesssim\|\mathbf{f}\|_0\,. \tag{2.8}$$

Proof. From (2.7), we can easily check that

$$\begin{aligned}|a(\mathbf{u},\mathbf{w})| &\lesssim \|\mathbf{u}\|_1\,\|\mathbf{w}\|_1\,,\ \forall\ \mathbf{u},\mathbf{w}\in H_0^1(\Omega)\times H_0^1(\Omega),\\ \|\mathbf{w}\|_1^2 &\lesssim a(\mathbf{w},\mathbf{w}),\qquad \forall\ \mathbf{w}\in H_0^1(\Omega)\times H_0^1(\Omega).\end{aligned}$$

Therefore, by the Lax-Milgram theorem, the variation problem (2.6) has a unique solution $\boldsymbol{\psi}\in H_0^1(\Omega)\times H_0^1(\Omega)$. Note that $\psi_1(\mathbf{x}),\ \psi_2(\mathbf{x})$ are the weak solutions of problem (2.3)–(2.5). By the regularity theory for elliptic boundary value problems [6], we have

$$\begin{aligned}\|\psi_1\|_2\lesssim\|f_1+V_2\psi_2\|_0\lesssim\|\psi_2\|_0+\|f_1\|_0\,,\\ \|\psi_2\|_2\lesssim\|f_2-V_2\psi_1\|_0\lesssim\|\psi_1\|_0+\|f_2\|_0\,.\end{aligned}$$

The above two inequalities imply that

$$\|\boldsymbol{\psi}\|_2\lesssim\|\boldsymbol{\psi}\|_0+\|\mathbf{f}\|_0\,. \tag{2.9}$$

From (2.6) we have

$$\|\boldsymbol{\psi}\|_1^2\lesssim a(\boldsymbol{\psi},\boldsymbol{\psi})=(\mathbf{f},\boldsymbol{\psi})\lesssim\|\boldsymbol{\psi}\|_0\,\|\mathbf{f}\|_0\,,$$

and then

$$\|\boldsymbol{\psi}\|_1\lesssim\|\mathbf{f}\|_0\,.$$

Therefore, (2.8) follows from (2.9) and the above inequality. □

Let T_h be a quasi-uniform triangulation of Ω with mesh size $h>0$, and let $S_0^h\subset H_0^1(\Omega)$ be the corresponding piecewise linear polynomial space. Then the finite element approximation of problem (2.6) is defined as follows.

$$\begin{aligned}&\text{Find } \boldsymbol{\psi}_h\in S_0^h\times S_0^h \text{ such that}\\ &a(\boldsymbol{\psi}_h,\mathbf{w}_h)=(\mathbf{f},\mathbf{w}_h),\qquad\forall\ \mathbf{w}_h\in S_0^h\times S_0^h.\end{aligned} \tag{2.10}$$

As shown in the following theory, the error analysis of the above finite element discretization can be obtained by standard techniques.

Theorem 2. *Under the assumption* (2.7), $\boldsymbol{\psi}_h$ *has the error estimate*

$$\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_s\lesssim h^{2-s}\,\|\boldsymbol{\psi}\|_2\,,\quad s=0,1. \tag{2.11}$$

Proof. Let $\mathbf{e}_h = \boldsymbol{\psi} - \boldsymbol{\psi}_h$. Then it follows from (2.6) and (2.10) that

$$a(\mathbf{e}_h, \mathbf{w}_h) = 0, \quad \forall\ \mathbf{w}_h \in S_0^h \times S_0^h. \tag{2.12}$$

Let $\boldsymbol{\psi}^I \in S_0^h \times S_0^h$ be the interpolation of $\boldsymbol{\psi}$. Then

$$\|\mathbf{e}_h\|_1^2 \lesssim a(\mathbf{e}_h, \mathbf{e}_h) = a(\mathbf{e}_h, \boldsymbol{\psi} - \boldsymbol{\psi}^I) \lesssim \|\mathbf{e}_h\|_1 \left\|\boldsymbol{\psi} - \boldsymbol{\psi}^I\right\|_1,$$

which implies that

$$\|\mathbf{e}_h\|_1 \lesssim \left\|\boldsymbol{\psi} - \boldsymbol{\psi}^I\right\|_1 \lesssim h \|\boldsymbol{\psi}\|_2. \tag{2.13}$$

We consider the auxiliary problem of (2.3)–(2.5):

$$-\Delta u_1(\mathbf{x}) + V_1(\mathbf{x})u_1(\mathbf{x}) + V_2(\mathbf{x})u_2(\mathbf{x}) = g_1(\mathbf{x}), \quad \forall\ \mathbf{x} \in \Omega, \tag{2.14}$$

$$-\Delta u_2(\mathbf{x}) + V_1(\mathbf{x})u_2(\mathbf{x}) - V_2(\mathbf{x})u_1(\mathbf{x}) = g_2(\mathbf{x}), \quad \forall\ \mathbf{x} \in \Omega, \tag{2.15}$$

$$u_j(\mathbf{x}) = 0, \quad j = 1, 2, \qquad \forall\ \mathbf{x} \in \partial\Omega. \tag{2.16}$$

Then similar to Theorem 1, for any $\mathbf{g} \in L^2(\Omega)\times L^2(\Omega)$ there exists a unique solution $\mathbf{u} \in \left(H^2(\Omega) \bigcap H_0^1(\Omega)\right) \times \left(H^2(\Omega) \bigcap H_0^1(\Omega)\right)$ such that

$$a(\mathbf{w}, \mathbf{u}) = (\mathbf{g}, \mathbf{w}), \quad \forall\ \mathbf{w} \in H_0^1(\Omega) \times H_0^1(\Omega), \tag{2.17}$$

and

$$\|\mathbf{u}\|_2 \lesssim \|\mathbf{g}\|_0. \tag{2.18}$$

Take $\mathbf{g} = \mathbf{e}_h$ in (2.17) and let $\mathbf{u}^I \in S_0^h \times S_0^h$ be the interpolation of $\mathbf{u}$. Then we have

$$\begin{aligned} \|\mathbf{e}_h\|_0^2 &= a(\mathbf{e}_h, \mathbf{u}) = a(\mathbf{e}_h, \mathbf{u} - \mathbf{u}^I) \\ &\lesssim \|\mathbf{e}_h\|_1 \left\|\mathbf{u} - \mathbf{u}^I\right\|_1 \lesssim h \|\mathbf{e}_h\|_1 \|\mathbf{u}\|_2 \lesssim h \|\mathbf{e}_h\|_1 \|\mathbf{e}_h\|_0, \end{aligned}$$

which implies that

$$\|\mathbf{e}_h\|_0 \lesssim h \|\mathbf{e}_h\|_1. \tag{2.19}$$

Therefore, (2.11) follows from (2.12) and (2.19). ☐

3. A new two-grid finite element method

The finite element discretization (2.10) apparently corresponds to a coupled system of equations in the general case that the potential function $V(\mathbf{x})$ given in (2.1) is complex valued. In order to reduce the computational cost, following Xu [12, 19, 20], we introduce another finite element space $S_0^H (\subset S_0^h \subset H_0^1(\Omega))$ defined on a coarser quasi-uniform triangulation (with meshsize $H > h$) of Ω, and propose the following algorithm.

Algorithm A1.

Step 1. *Find* $\boldsymbol{\psi}_H \in S_0^H \times S_0^H$ *such that*

$$a(\boldsymbol{\psi}_H, \boldsymbol{\chi}) = (\mathbf{f}, \boldsymbol{\chi}), \quad \forall\ \boldsymbol{\chi} \in S_0^H \times S_0^H. \tag{3.1}$$

Step 2. *Find* $\boldsymbol{\psi}_h^* \in S_0^h \times S_0^h$ *such that*

$$\widehat{a}(\boldsymbol{\psi}_h^*, \mathbf{w}_h) = (\mathbf{f}, \mathbf{w}_h) - N(\boldsymbol{\psi}_H, \mathbf{w}_h), \quad \forall\ \mathbf{w}_h \in S_0^h \times S_0^h. \tag{3.2}$$

We note that the linear system in Step 2 is a decoupled system which involves only two separate Poisson equations, and only on the coarser space a coupled system needs to be solved in Step 1. As the following theorem shows, $\boldsymbol{\psi}_h^*$ can reach the optimal accuracy in H^1-norm if the coarse meshsize H is taken to be $\sqrt{h}$. Because the dimension of S_0^H is much smaller than the dimension of S_0^h, the efficiency of the algorithm is then evident.

Theorem 3. *Under the assumption* (2.7), $\boldsymbol{\psi}_h^*$ *hass the following error estimate:*

$$\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^*\|_1 \lesssim H^2. \tag{3.3}$$

Consequently,

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^*\|_1 \lesssim h + H^2, \tag{3.4}$$

namely, $\boldsymbol{\psi}_h^*$ *has the same accuracy as* $\boldsymbol{\psi}_h$ *in* H^1*-norm if* $H = \sqrt{h}$.

Proof. Let $\mathbf{e}_h = \boldsymbol{\psi} - \boldsymbol{\psi}_h,\ \widehat{\mathbf{e}}_h = \boldsymbol{\psi}_h - \boldsymbol{\psi}_h^*$. Then from (2.10) and (3.2) we get

$$\widehat{a}\left(\widehat{\mathbf{e}}_h,\ \mathbf{w}_h\right) + N(\boldsymbol{\psi}_h - \boldsymbol{\psi}_H, \mathbf{w}_h) = 0, \forall\ \mathbf{w}_h \in S_0^h \times S_0^h. \tag{3.5}$$

By taking $\mathbf{w}_h = \widehat{\mathbf{e}}_h$ in the above equality, we have

$$\|\widehat{\mathbf{e}}_h\|_1^2 \lesssim \widehat{a}\left(\widehat{\mathbf{e}}_h,\ \widehat{\mathbf{e}}_h\right) \lesssim \|\boldsymbol{\psi}_h - \boldsymbol{\psi}_H\|_0 \|\widehat{\mathbf{e}}_h\|_0\,,$$

and then

$$\|\widehat{\mathbf{e}}_h\|_1 \lesssim \|\boldsymbol{\psi}_h - \boldsymbol{\psi}_H\|_0\,. \tag{3.6}$$

From Theorem 2 we get

$$\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_H\|_0 \quad \leq \quad \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_0 + \|\boldsymbol{\psi} - \boldsymbol{\psi}_H\|_0 \lesssim h^2 + H^2.$$

Therefore, (3.3) follows from (3.6) and the above inequality. Also, (3.4) follows from (2.11), (3.3) and the following inequality:

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^*\|_1 \leq \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_1 + \|\widehat{\mathbf{e}}_h\|_1\,. \qquad \square$$

Algorithm A1 can be improved in a successive fashion as follows.

Algorithm A2. *Let* $\boldsymbol{\psi}_h^0 = \mathbf{0}$. *Assume that* $\boldsymbol{\psi}_h^k \in S_0^h \times S_0^h$ *has been obtained, then* $\boldsymbol{\psi}_h^{k+1} \in S_0^h \times S_0^h$ *is defined as follows:*

Step 1. *Find* $\mathbf{e}_H \in S_0^H \times S_0^H$ *such that*

$$a(\mathbf{e}_H, \boldsymbol{\chi}) = (\mathbf{f}, \boldsymbol{\chi}) - a(\boldsymbol{\psi}_h^k, \boldsymbol{\chi}), \qquad \forall\ \boldsymbol{\chi} \in S_0^H \times S_0^H. \tag{3.7}$$

Step 2. *Find* $\boldsymbol{\psi}_h^{k+1} \in S_0^h \times S_0^h$ *such that*

$$\widehat{a}(\boldsymbol{\psi}_h^{k+1}, \mathbf{w}_h) = (\mathbf{f}, \mathbf{w}_h) - N(\boldsymbol{\psi}_h^k + \mathbf{e}_H, \mathbf{w}_h), \qquad \forall\ \mathbf{w}_h \in S_0^h \times S_0^h. \tag{3.8}$$

Theorem 4. *Under the assumption* (2.7), $\boldsymbol{\psi}_h^k$ *admits the following error estimate:*

$$\left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\right\|_1 \lesssim H^{k+1}, \quad k \geq 1. \tag{3.9}$$

Consequently,

$$\left\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^k\right\|_1 \lesssim h + H^{k+1}, \quad k \geq 1, \tag{3.10}$$

namely, $\boldsymbol{\psi}_h^k,\ k \geq 1$, *has the same accuracy as* $\boldsymbol{\psi}_h$ *in* H^1*-norm if* $H = h^{\frac{1}{k+1}}$.

Proof. From (2.10) and (3.8), we have

$$\widehat{a}\left(\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^{k+1}, \mathbf{w}_h\right) = -N\left(\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H), \mathbf{w}_h\right), \ \forall\ \mathbf{w}_h \in S_0^h \times S_0^h, \tag{3.11}$$

which, by taking $\mathbf{w}_h = \boldsymbol{\psi}_h - \boldsymbol{\psi}_h^{k+1}$, gives

$$\left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^{k+1}\right\|_1^2 \lesssim \widehat{a}\left(\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^{k+1}, \boldsymbol{\psi}_h - \boldsymbol{\psi}_h^{k+1}\right) \lesssim \left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_0 \left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^{k+1}\right\|_0,$$

and then

$$\left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^{k+1}\right\|_1 \lesssim \left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_0. \tag{3.12}$$

It follows from (2.10) and (3.7) that

$$a\left(\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H), \boldsymbol{\chi}\right) = 0, \ \forall\ \boldsymbol{\chi} \in S_0^H \times S_0^H. \tag{3.13}$$

Thus

$$\begin{aligned}
\left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_1^2 &\lesssim a\left(\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H), \boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right) \\
&= a\left(\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H), \boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\right) \\
&\lesssim \left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_1 \left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\right\|_1,
\end{aligned}$$

and then

$$\left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_1 \lesssim \left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\right\|_1. \tag{3.14}$$

Let $\mathbf{u}$ be the solution of problem (2.17) with $\mathbf{g} = \boldsymbol{\psi}_h - \left(\boldsymbol{\psi}_h^k + \mathbf{e}_H\right)$ and let $\mathbf{u}^I \in S_0^H \times S_0^H$ be the interpolation of $\mathbf{u}$. Then according to (2.18) and (3.13), we have

$$\begin{aligned}
\left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_0^2 &= a\left(\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H), \mathbf{u}\right) = a\left(\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H), \mathbf{u} - \mathbf{u}^I\right) \\
&\lesssim \left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_1 \|\mathbf{u} - \mathbf{u}^I\|_1 \\
&\lesssim H \left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_1 \|\mathbf{u}\|_2 \\
&\lesssim H \left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_1 \left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_0,
\end{aligned}$$

which implies that

$$\left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_0 \lesssim H \left\|\boldsymbol{\psi}_h - (\boldsymbol{\psi}_h^k + \mathbf{e}_H)\right\|_1. \tag{3.15}$$

Therefore, from (3.12), (3.15), and (3.14), we have

$$\left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\right\|_1 \lesssim H \left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^{k-1}\right\|_1 \lesssim H^{k-1} \left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^1\right\|_1, \quad k \geq 1. \tag{3.16}$$

Note that $\boldsymbol{\psi}_h^1$ is the solution $\boldsymbol{\psi}_h^*$ obtained by Algorithm A1, thus, (3.9) follows from (3.16) and (3.3). Additionally, (3.10) follows from (3.9), (2.11), and the following inequality:

$$\left\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^k\right\|_1 \leq \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_1 + \left\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\right\|_1. \qquad \square$$

According to Theorem 4, it suffices to take $H = h^{\frac{1}{k+1}}$ to obtain the optimal approximation in H^1-norm. Therefore, the dimension of S_0^H can be much smaller than the dimension of S_0^h, and thus the dominated part of the work in Algorithm A2 is to solve two separate Laplacian systems in Step 2, which is much easier solved than the coupled system in (2.10).

4. Numerical examples

In this section we will demonstrate the efficiency of our algorithms proposed in section 3 by two numerical examples. Here, we consider the following boundary value problem of the Schrödinger type:

$$-\Delta\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) = f(\mathbf{x}), \qquad \forall\ \mathbf{x} \in \Omega, \tag{4.1}$$

$$\psi(\mathbf{x}) = 0, \qquad \forall\ \mathbf{x} \in \partial\Omega, \tag{4.2}$$

where $\Omega = (0,1)^2$, $V(\mathbf{x}) = 1 + i$.

Example 1. $f(\mathbf{x})$ is so chosen that $\psi(\mathbf{x}) = (0.5 + i)\sin(\pi x)\sin(\pi y)$ is the exact solution.

The domain Ω is uniformly divided by two nested triangulations of mesh size H and h, respectively, S_0^H and S_0^h are the corresponding piecewise linear finite element spaces. The standard finite element solution $\boldsymbol{\psi}_h$ on different meshes is first computed by (2.10), and the numerical results are listed in Table 1, which shows that $\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_s \approx O(h^{2-s}), s = 0, 1$. For $H = 1/4$, $1/8$, and $h = H^2$, $\boldsymbol{\psi}_h^*$ is computed by Algorithm A1, and the numerical results are listed in Table 2. We can see that $\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^*\|_1 \approx O(H^2)$, which coincides with the theoretical result obtained in Theorem 3. For $H = 1/4$ and $h = H^3$, $\boldsymbol{\psi}_h^2$ is computed by Algorithm A2, and its error together with the errors of $\boldsymbol{\psi}_h^1$, $\boldsymbol{\psi}_h^3$ and the standard finite element solution $\boldsymbol{\psi}_h$ are listed in Table 3. Just as Theorem 4 shows, in case that $h = H^3$, $\boldsymbol{\psi}_h^2$ has the same accuracy as $\boldsymbol{\psi}_h$ in H^1-norm. Finally, for fixed

Table 1. Errors of $\boldsymbol{\psi}_h$ in H^1 and L^2 norms.

Mesh	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_1$	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_0$
$h = 1/16$	2.43D-01	5.78D-03
$h = 1/32$	1.22D-01	1.45D-03
$h = 1/64$	6.09D-02	3.63D-04

Table 2. Errors of $\boldsymbol{\psi}_h^*$ with $h = H^2$.

Mesh	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^*\|_1$	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h^*\|_0$
$H = 1/4,\ h = 1/16$	2.44D-01	4.70D-03
$H = 1/8,\ h = 1/64$	6.13D-02	1.22D-03

Table 3. Errors of $\boldsymbol{\psi}_h^2, \boldsymbol{\psi}_h^1$ and $\boldsymbol{\psi}_h$; $H = 1/4, h = H^3$.

	H^1-norm	L^2-norm
$\boldsymbol{\psi} - \boldsymbol{\psi}_H$	9.42D-01	8.56D-02
$\boldsymbol{\psi} - \boldsymbol{\psi}_h^1$	6.60D-02	5.24D-03
$\boldsymbol{\psi} - \boldsymbol{\psi}_h^2$	6.09D-02	3.70D-04
$\boldsymbol{\psi} - \boldsymbol{\psi}_h^3$	6.09D-02	3.63D-04
$\boldsymbol{\psi} - \boldsymbol{\psi}_h$	6.09D-02	3.63D-04

TABLE 4. Errors between $\boldsymbol{\psi}_h$ and $\boldsymbol{\psi}_h^k$, $k = 1, 2, 3$. $h = 1/64$ is fixed.

H	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^1\\|_1$	Ratio	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^2\\|_1$	Ratio	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^3\\|_1$	Ratio
1/4	2.52D-02		3.10D-04		4.00D-06	
1/8	6.55D-03	3.8	2.15D-05	14.4	7.58D-08	52.8
1/16	1.58D-03	4.1	1.27D-06	16.9	1.11D-09	68.3
1/32	3.17D-04	5.0	5.38D-08	23.6	1.19D-11	93.3

TABLE 5. Errors between $\boldsymbol{\psi}_h$ and $\boldsymbol{\psi}_h^k$, $k = 1, 2, 3$. $h = 1/64$ is fixed.

H	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^1\\|_0$	Ratio	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^2\\|_0$	Ratio	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^3\\|_0$	Ratio
1/4	5.46D-03		6.65D-05		8.50D-07	
1/8	1.42D-03	3.8	4.60D-06	14.5	1.60D-08	53.1
1/16	3.43D-04	4.1	2.72D-07	16.9	2.33D-10	68.7
1/32	6.89D-05	5.0	1.15D-08	23.7	2.54D-12	91.7

$h = 1/64$ and different $H = 1/4,\ 1/8,\ 1/16,\ 1/32$, $\boldsymbol{\psi}_h^k$, $k = 1, 2, 3$, are computed by Algorithm A2. The errors between the standard finite element solution $\boldsymbol{\psi}_h$ and $\boldsymbol{\psi}_h^k$ are listed in Table 4 for H^1-norm and Table 5 for L^2-norm, and the ratios of the errors are also listed. From Table 4, we can see that $\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^1\|_1 \approx O(H^2)$, which verifies the theoretical results (3.9) with $k = 1$ in Theorem 4. Note that $\boldsymbol{\psi}_h^1$ is just $\boldsymbol{\psi}_h^*$ computed by Algorithm A1, so the theoretical results (3.3) in Theorem 3 are also valid. However, for $k \geq 2$, $\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\|_1$ is decreasing rather faster than $O(H^{k+1})$; it seems as if $\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\|_1 / \|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^{k-1}\|_1 \approx O(H^2)$, therefore, $\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\|_1 \approx O(H^{2k})$. This suggests that the error bound obtained in (3.9) may not be optimal, but this error bound is the best one we are able to obtain so far. We plan to make further theoretical investigation in a future work to see if further improvements can be made on this type of error estimate. As to the errors in L^2-norm, Table 5 shows that $\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\|_0$ has the same order of convergence as $\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\|_1$. Therefore, both Algorithms A1 and A2 are not optimal in L^2-norm, which is a typical behavior of this kind of two-grid method.

In order to make more observations about the behavior of $\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^k\|_s$, $s = 0, 1$, we consider another example as follows.

TABLE 6. Errors between $\boldsymbol{\psi}_h$ and $\boldsymbol{\psi}_h^k$, $k = 1, 2, 3$. $h = 1/64$ is fixed.

H	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^1\\|_1$	Ratio	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^2\\|_1$	Ratio	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^3\\|_1$	Ratio
1/4	1.17D-01		1.44D-03		1.84D-05	
1/8	3.03D-02	3.9	9.83D-05	14.7	3.42D-07	53.8
1/16	7.30D-03	4.2	5.79D-06	17.0	4.96D-09	69.0
1/32	1.46D-03	5.0	2.45D-07	23.6	4.98D-11	99.6

TABLE 7. Errors between $\boldsymbol{\psi}_h$ and $\boldsymbol{\psi}_h^k$, $k = 1, 2, 3$. $h = 1/64$ is fixed.

H	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^1\\|_0$	Ratio	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^2\\|_0$	Ratio	$\\|\boldsymbol{\psi}_h - \boldsymbol{\psi}_h^3\\|_0$	Ratio
1/4	2.54D-02		3.09D-04		3.92D-06	
1/8	6.58D-03	3.9	2.11D-05	14.6	7.24D-08	54.1
1/16	1.59D-03	4.1	1.24D-06	17.1	1.05D-09	69.0
1/32	3.18D-04	5.0	5.25D-08	23.6	1.09D-11	96.3

Example 2. $f(\mathbf{x})$ is so chosen that $\psi(\mathbf{x}) = (1+20i)x(1-x)\sin(\pi y)$ is the exact solution.

Similar to Example 1, for fixed $h = 1/64$ and different $H = 1/4,\ 1/8,\ 1/16,\ 1/32$, $\boldsymbol{\psi}_h^k,\ k = 1,2,3$, are computed by Algorithm A2, the errors between the standard finite element solution $\boldsymbol{\psi}_h$ and $\boldsymbol{\psi}_h^k$ are listed in Tables 6 and 7. From these numerical results, we can also have the same conclusions about $\boldsymbol{\psi}_h^k$ as in Example 1.

5. Concluding remarks

Using the Schrödinger equation as an illustration, we presented in this paper a new two-grid discretization technique to decouple systems of partial differential equations. This is a new application of the two-grid idea. This two-grid decoupling technique can obviously be extended in many different ways, for example for different discretizations such as finite volume and finite difference methods, for other types of systems of partial differential equations. We hope this short paper will trigger some subsequent works on this new idea.

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