

**RATE OF CONVERGENCE
OF FINITE DIFFERENCE APPROXIMATIONS
FOR DEGENERATE ORDINARY DIFFERENTIAL EQUATIONS**

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ABSTRACT. In this paper we study finite difference approximations for the following linear stationary convection-diffusion equations:

$$\frac{1}{2}\sigma^2(x)u''(x) + b(x)u'(x) - u(x) = -f(x), \quad x \in \mathbb{R},$$

where σ is allowed to be *degenerate*. We first propose a new weighted finite difference scheme, motivated by approximating the diffusion process associated with the equation in the strong sense. We show that, under certain conditions, this scheme converges with the first order rate and that such a rate is *sharp*. To the best of our knowledge, this is the first sharp result in the literature. Moreover, by using the connection between our scheme and the standard upwind finite difference scheme, we get the rate of convergence of the latter, which is also new.

1. INTRODUCTION

Numerical methods for degenerate (elliptic or parabolic) PDEs, in particular degenerate HJB equations, have been studied by many authors. Notable works include: Markov chain approximation (e.g., Kushner-Dupuis [19]); viscosity solution method (e.g., Barles-Souganidis [3]); the works by Menaldi ([20]) and Camilli-Falcone ([10]) on the so-called “control scheme”; Krylov’s “shaking the coefficients” method ([16], [17]) and its extension by Barles-Jakobsen ([1]); the recent work by Bonnans-Zidani ([4]); Karlsen’s series of works, especially on the so-called entropy weak solutions to degenerate PDEs with discontinuous coefficients (e.g., [12], [9], [15]); as well as finite element methods such as the SPUG method (e.g., Brooks-Hughes [8]) and bubbles method (e.g., Brezzi et al. [5], [7], [6]), to mention a few. After this work was submitted for publication, we also learned that some new progresses were made on the subject (see, Jakobsen [13], Barles-Jakobsen [2], Krylov [18], and Dong-Krylov [11]).

Despite all these works, the rate of convergence of numerical approximations for *degenerate* PDEs is far from being fully understood. It is somewhat surprising that even for degenerate linear ODEs, there are no sharp results in the literature. To the best of our knowledge, the best rates are $h^{\frac{1}{3}}$ for finite difference schemes (see [16] or [1]) and $h^{\frac{1}{2}}$ for control schemes (see [20]). The major difficulty lies in the

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fact that, even if all the coefficients are infinitely smooth with compact support, the solutions to degenerate linear ODEs (not to mention fully nonlinear PDEs) are in general not sufficiently smooth.

We intend to understand how much one can expect to achieve on the rate of convergence of finite difference approximations for degenerate PDEs. Instead of being ambitious on generalization, in this paper we focus on the following 1-dimensional linear stationary convection-diffusion equations:

$$(1.1) \quad \frac{1}{2}\sigma^2(x)u''(x) + b(x)u'(x) - u(x) = -f(x), \quad x \in \mathbb{R},$$

where, of course, σ is allowed to be *degenerate*.

Our goal of the paper is twofold. We first propose a weighted finite difference scheme, motivated by some probabilistic approximation. Unlike classical Markov chain approximations which approximate the diffusion process associated with (1.1) in a weak or distributional sense (in other words, it matches the transition probabilities of the Markov chain with the coefficients of the finite difference equations, see, e.g., [19]), we try to approximate the diffusion process in a strong sense. Such an idea was also used by Menaldi [20]. It turns out that these approximating processes can lead to probabilistic solutions of some finite difference equations, but with coefficients different from standard ones. We prove that, under certain conditions, our scheme converges with a rate of convergence h and that such a rate is sharp.

We next study the standard upwind finite difference schemes. We find that the standard finite difference approximation to (1.1) is the same as our new weighted finite difference approximation to a new ODE with modified coefficients σ_h, b_h . We estimate the errors of σ_h, b_h from σ, b , respectively, and then prove that this scheme converges with a rate of convergence $h^{\frac{1}{2}}$, and the rate becomes h given some stronger conditions. The latter rate h is also sharp, however, the sharpness of the general rate $h^{\frac{1}{2}}$ is still unknown.

At this point we should mention that our method takes advantage of some special properties of one dimension, and thus may be difficult to extend directly to high-dimensional equations. However, besides the fact that this is the first sharp result in the literature, it provides a benchmark on what one can (or what one cannot) expect in high dimensions.

The rest of the paper is organized as follows. In §2 we introduce the two finite difference schemes and state the main results. In §§3 and 4 we study the weighted and the standard finite difference schemes, respectively. In §5 we provide some counterexamples which show that some estimates in §2 are sharp. Finally some technical proofs are presented in the last section.

2. MAIN RESULTS

We first consider the following standard upwind finite difference approximation for (1.1):

$$\begin{aligned} \frac{1}{2}\sigma^2(x) \frac{u_h^s(x+h) + u_h^s(x-h) - 2u_h^s(x)}{h^2} + (b(x) \vee 0) \frac{u_h^s(x+h) - u_h^s(x)}{h} \\ + (b(x) \wedge 0) \frac{u_h^s(x) - u_h^s(x-h)}{h} - u_h^s(x) = -f(x). \end{aligned}$$

Here the superscript s is the abbreviation for “standard”. To simplify the presentation, throughout the paper we assume $b(x) \geq 0$. Then the above discrete equation

can be rewritten as

$$(2.1) \quad \frac{1}{2}\sigma^2(x)\frac{u_h^s(x+h) + u_h^s(x-h) - 2u_h^s(x)}{h^2} + b(x)\frac{u_h^s(x+h) - u_h^s(x)}{h} - u_h^s(x) = -f(x).$$

Our first main result is the following theorem.

Theorem 2.1. *Assume σ, b are bounded and $\sigma, b, f \in C^2$ with bounded derivatives. Assume further that*

$$(2.2) \quad L \triangleq \sup_x \left| 2b'(x) + |\sigma'(x)|^2 \right| \vee \sup_x |b'(x)| < 1.$$

Then there exists a constant C , depending on L , the bounds of σ, b , and the bounds of the derivatives, such that

$$(2.3) \quad |u_h^s(x) - u(x)| \leq Ch.$$

This rate is sharp.

We note that a condition similar to (2.2) was used by Menaldi [20]. We also note that when the solution u (not the coefficients!) is smooth enough, it is possible to obtain a much better rate.

In order to study a more general case where the coefficients are not in C^2 , we adopt the following assumption:

(A) σ, b are Lipschitz continuous and f is bounded and Hölder- α continuous for some $\alpha \in (0, 1]$.

Let L_b denote the Lipschitz constant of b , and let $\beta > 0$ be some arbitrary constant such that $\beta L_b < 1$. Throughout the paper, we use a generic constant C , which may vary from line to line, to denote upper bounds of estimates which may depend on α, β , the Lipschitz constants of σ, b , the Hölder constant of f , as well as some others specified in the context. We then have the following results.

Theorem 2.2. *Assume (A).*

(i) *If b is bounded, then*

$$(2.4) \quad |u_h^s(x) - u(x)| \leq Ch^{\frac{1}{2}(\beta \wedge \alpha)}.$$

(ii) *If $\sigma = 0$ or $\frac{b}{\sigma}$ is bounded (e.g., $b = 0$ or σ is uniformly nondegenerate), then we obtain the sharp rate*

$$(2.5) \quad |u_h^s(x) - u(x)| \leq Ch^{\beta \wedge \alpha}.$$

Remark 2.3. The sharpness of (2.4) is still unknown.

In order to prove Theorems 2.1 and 2.2, we introduce another *weighted finite difference approximation* for (1.1). This scheme is interesting in its own right, and is new to the best of our knowledge. To this end, we note that (2.1) is equivalent to

$$(2.6) \quad u_h^s(x) = p_h^s(x)u_h^s(x+h) + q_h^s(x)u_h^s(x-h) + f(x)[1 - p_h^s(x) - q_h^s(x)],$$

where

$$(2.7) \quad p_h^s \triangleq \frac{\sigma^2 + 2bh}{2[\sigma^2 + bh + h^2]}; \quad q_h^s \triangleq \frac{\sigma^2}{2[\sigma^2 + bh + h^2]}.$$

By using the superscript w to denote “weighted”, we define our new scheme as follows:

$$(2.8) \quad u_h^w(x) = p_h^w(x)u_h^w(x+h) + q_h^w(x)u_h^w(x-h) + f(x)[1 - p_h^w(x) - q_h^w(x)],$$

where

$$(2.9) \quad \begin{aligned} p_h^w &\triangleq \frac{1}{2} \exp\left(\frac{bh}{\sigma^2}\right) \left[\cosh\left(\frac{\sqrt{b^2 + 2\sigma^2}}{\sigma^2}h\right) \right]^{-1}, \\ q_h^w &\triangleq \frac{1}{2} \exp\left(-\frac{bh}{\sigma^2}\right) \left[\cosh\left(\frac{\sqrt{b^2 + 2\sigma^2}}{\sigma^2}h\right) \right]^{-1}. \end{aligned}$$

When $\sigma = 0$, we take the limits of the above expressions as the values of p_h^w and q_h^w . To be specific, we define that

$$\begin{cases} p_h^w \triangleq \exp(-\frac{h}{b}), & q_h^w \triangleq 0, & \text{if } b > 0, \\ p_h^w \triangleq 0, & q_h^w \triangleq \exp(-\frac{h}{b}), & \text{if } b < 0, \\ p_h^w \triangleq 0, & q_h^w \triangleq 0, & \text{if } b = 0. \end{cases}$$

We note that when $\sigma > 0$, it holds that $\lim_{h \rightarrow 0} \frac{p_h^w}{p_h^w} = \lim_{h \rightarrow 0} \frac{q_h^w}{q_h^w} = 1$. In this sense (2.8) and (2.6) are asymptotically equivalent, and thus one may expect that u_h^w is also an approximation of u . This is indeed true.

Theorem 2.4. *Assume (A). Then we have the following sharp estimate:*

$$(2.10) \quad |u_h^w(x) - u(x)| \leq Ch^{\beta \wedge \alpha}.$$

3. THE WEIGHTED FINITE DIFFERENCE SCHEME

In this section we shall prove (2.10) and leave the sharpness to §5. First, it is well known that the solution to (1.1) can be written as

$$(3.1) \quad u(x) = E_x \left\{ \int_0^\infty e^{-t} f(X_t) dt \right\},$$

where X is the solution to the following SDE:

$$(3.2) \quad X_t = x + \int_0^t \sigma(X_r) dW_r + \int_0^t b(X_r) dr.$$

Here the subscript x in E_x indicates the fact that $X_0 = x$, but for simplicity we shall omit it in the sequel when there is no confusion. The solution is unique in the sense that u is bounded whenever f is bounded. We refer the readers to [14] or [21] for basic theories of stochastic calculus.

The main idea for proving (2.10) is to find a similar probabilistic expression for u_h^w . We proceed by “freezing” the coefficients in (3.2). To this end, we fix h and x , and construct a sequence of increasing stopping times $\tau_n \triangleq \tau_n^{h,x}$ and a process $X^h \triangleq X^{h,x}$ as follows. First we define $\tau_0 \triangleq 0, X_{\tau_0}^h \triangleq x$. Then for $n = 1, 2, \dots$, define

$$(3.3) \quad \tau_n \triangleq \inf \{ t > \tau_{n-1} : |\sigma(X_{\tau_{n-1}}^h)[W_t - W_{\tau_{n-1}}] + b(X_{\tau_{n-1}}^h)[t - \tau_{n-1}]| = h \},$$

and for $t \in (\tau_{n-1}, \tau_n]$,

$$(3.4) \quad X_t^h \triangleq X_{\tau_{n-1}}^h + \sigma(X_{\tau_{n-1}}^h)[W_t - W_{\tau_{n-1}}] + b(X_{\tau_{n-1}}^h)[t - \tau_{n-1}].$$

We note that these processes were also used by Menaldi [20]. Since σ and b are Lipschitz continuous, hence grow at most linearly, one can easily show that $\tau_n \uparrow \infty$, almost surely. Denote $\tau(t) \triangleq \tau_{n-1}$ for $t \in [\tau_{n-1}, \tau_n)$. Then obviously it holds that

$$(3.5) \quad |X_{\tau(t)}^h - X_t^h| \leq h, \quad \text{a.s.}$$

and that

$$(3.6) \quad X_t^h = x + \int_0^t \sigma(X_{\tau(s)}^h) dW_s + \int_0^t b(X_{\tau(s)}^h) ds.$$

The following lemma gives the probabilistic solution to (2.8).

Lemma 3.1. *Assume all the conditions in Theorem 2.4 hold true. Then*

$$(3.7) \quad u_h^w(x) = E_x \left\{ \int_0^\infty e^{-t} f(X_{\tau(t)}^h) dt \right\}.$$

Proof. Let \bar{u}_h denote the right side of (3.7). Then

$$\begin{aligned} \bar{u}_h(x) &= E_x \left\{ \left(\int_0^{\tau_1} + \int_{\tau_1}^\infty \right) e^{-t} f(X_{\tau(t)}^h) dt \right\} \\ &= E_x \left\{ \int_0^{\tau_1} e^{-t} f(X_{\tau_0}^h) dt + \int_0^\infty e^{-\tau_1-s} f(X_{\tau^h, X_{\tau_1}^h}(s)} ds \right\}, \end{aligned}$$

where $\tau^h, X_{\tau_1}^h$ and $X^h, X_{\tau_1}^h$ are defined in the same way as $\tau^h (= \tau^{h,x})$ and $X^h (= X^{h,x})$, with the initial value x replaced by $X_{\tau_1}^{h,x}$. Note that $X_{\tau_0}^{h,x} = x$ and $X_{\tau_1}^{h,x} = x \pm h$. Then one can check directly that

$$(3.8) \quad \bar{u}_h(x) = \bar{p}_h(x) \bar{u}_h(x+h) + \bar{q}_h(x) \bar{u}_h(x-h) + f(x)[1 - \bar{p}_h(x) - \bar{q}_h(x)],$$

where

$$(3.9) \quad \bar{p}_h(x) \triangleq E_x \left\{ e^{-\tau_1} 1_{\{X_{\tau_1}^h = x+h\}} \right\}, \quad \bar{q}_h(x) \triangleq E_x \left\{ e^{-\tau_1} 1_{\{X_{\tau_1}^h = x-h\}} \right\}.$$

So it suffices to show that

$$(3.10) \quad \bar{p}_h(x) = p_h^w(x), \quad \bar{q}_h(x) = q_h^w(x).$$

Without loss of generality, we shall prove (3.10) only for $x = 0$.

We note that by (3.3) and (3.4) one can calculate \bar{p}_h and \bar{q}_h straightforwardly (see, e.g., [14]) and thus prove (3.10). But here we would like to provide another argument which avoids the probabilistic calculation. To this end, we consider the following ODE with constant coefficients:

$$(3.11) \quad \frac{1}{2} \sigma^2(0) \bar{u}''(x) + b(0) \bar{u}'(x) - \bar{u}(x) = -f(x).$$

By (3.1) we have

$$\bar{u}(x) = E_x \left\{ \int_0^\infty e^{-t} f(\bar{X}_t) dt \right\},$$

where \bar{X} satisfies

$$\bar{X}_t = x + \int_0^t \sigma(0)dW_r + \int_0^t b(0)dr = x + \sigma(0)W_t + b(0)t.$$

Note that $\bar{p}_h(0)$ and $\bar{q}_h(0)$ involve only $\sigma(0)$ and $b(0)$. Following the same arguments as for (3.8) we get, for the same $\bar{p}_h(0)$ and $\bar{q}_h(0)$ as in (3.9),

$$(3.12) \quad \bar{u}(0) = \bar{p}_h(0)\bar{u}(h) + \bar{q}_h(0)\bar{u}(-h) + f(0)[1 - \bar{p}_h(0) - \bar{q}_h(0)].$$

We note that (3.12) holds true for all f and the corresponding solution \bar{u} . Choose f such that $f(x) = 0$ for $|x| < h$ and $f'(h) \neq 0, f'(-h) \neq 0$. Then in $(-h, h)$, \bar{u} satisfies the homogeneous ODE:

$$\frac{1}{2}\sigma^2(0)\bar{u}''(x) + b(0)\bar{u}'(x) - \bar{u}(x) = 0.$$

Thus, for $x \in (-h, h)$,

$$\bar{u}(x) = \alpha e^{\lambda_1 x} + \beta e^{\lambda_2 x},$$

where α, β are two constants, and

$$\lambda_1 \triangleq \frac{-b(0) + \sqrt{b^2(0) + 2\sigma^2(0)}}{\sigma^2(0)}, \quad \lambda_2 \triangleq \frac{-b(0) - \sqrt{b^2(0) + 2\sigma^2(0)}}{\sigma^2(0)}.$$

Now by the continuity of \bar{u} we have

$$\bar{u}(h) = \alpha e^{\lambda_1 h} + \beta e^{\lambda_2 h}, \quad \bar{u}(-h) = \alpha e^{-\lambda_1 h} + \beta e^{-\lambda_2 h}.$$

By straightforward calculation we get the values of α, β and check that

$$(3.13) \quad \bar{u}(0) = \alpha + \beta = p_h^w(0)\bar{u}(h) + q_h^w(0)\bar{u}(-h).$$

Since $f(0) = 0$, (3.12) becomes

$$(3.14) \quad \bar{u}(0) = \bar{p}_h(0)\bar{u}(h) + \bar{q}_h(0)\bar{u}(-h).$$

Compare (3.13) and (3.14), and note that both of them hold true for arbitrary values of $\bar{u}(h)$ and $\bar{u}(-h)$; we prove (3.10) at $x = 0$, and hence the lemma. \square

To prove the theorem, we shall need another technical lemma. The arguments of the proof are mainly due to N. Krylov. We refer the readers to [14] or [21] again for preliminary materials.

Lemma 3.2. *Let X be the solution to the following linear SDE:*

$$X_t = x + \int_0^t (\alpha_s X_s + \beta_s) dW_s + \int_0^t (\gamma_s X_s + \lambda_s) ds,$$

where $|\alpha_t| \leq K_1, |\gamma_t| \leq K_2$ for some $K_1, K_2 > 0$. Then for any $\varepsilon > 0$, there exist constants C and $q > 2$, depending only on K_1, K_2 and ε , such that

$$(3.15) \quad E\{|X_t|\} \leq C e^{(K_2+\varepsilon)t} \left[|x| + E \left\{ \int_0^t [|\beta_s|^q + |\lambda_s|^q] ds \right\}^{\frac{1}{q}} \right], \quad \forall t \geq 0.$$

Moreover, if $|\beta_t|, |\lambda_t| \leq K_3$, then there exists C which may depend on K_3 as well, such that

$$(3.16) \quad E\{|X_t|\} \leq C e^{(K_2+\varepsilon)t} [|x| + K_3], \quad \forall t \geq 0.$$

Proof. Apply (3.15) with $\frac{\varepsilon}{2}$ and note that $t^{\frac{1}{q}} \leq Ce^{\frac{\varepsilon}{2}t}$; one proves (3.16).

So it remains to prove (3.15). To this end, we denote

$$d\tilde{W}_t \triangleq dW_t - \alpha_t dt, \quad M_t \triangleq \exp\left(-\int_0^t \alpha_s dW_s + \frac{1}{2} \int_0^t \alpha_s^2 ds\right), \quad \Lambda_t \triangleq \exp\left\{-\int_0^t \gamma_s ds\right\}.$$

By the Girsanov Theorem, \tilde{W} is a Brownian motion under a new probability measure \tilde{P} such that $\tilde{E}\{\xi_t\} = E\{M_t^{-1}\xi_t\}$ for any $\xi_t \in \mathcal{F}_t$. Applying Itô's formula one gets

$$d(X_t M_t \Lambda_t) = M_t \Lambda_t [\beta_t d\tilde{W}_t + \lambda_t dt].$$

Denote

$$N_t \triangleq X_t M_t \Lambda_t = x + \int_0^t M_s \Lambda_s \beta_s d\tilde{W}_s + \int_0^t M_s \Lambda_s \lambda_s ds.$$

Then

$$(3.17) \quad E\{|X_t|\} = \tilde{E}\{M_t |X_t|\} = \tilde{E}\{N_t \Lambda_t^{-1}\}.$$

By the Ito formula again we have

$$d(N_t \Lambda_t^{-1}) = \gamma_t N_t \Lambda_t^{-1} dt + M_t \beta_t d\tilde{W}_t + M_t \lambda_t dt.$$

Thus, by noting that $|\gamma_t| \leq K_2$,

$$(3.18) \quad \tilde{E}\{|N_t \Lambda_t^{-1}\} \leq |x| + K_2 \int_0^t \tilde{E}\{|N_s \Lambda_s^{-1}|\} ds + \tilde{E}\left\{\left|\int_0^t M_s \beta_s d\tilde{W}_s + \int_0^t M_s \lambda_s ds\right|\right\}.$$

Let $p \in (1, 2)$ be a constant determined later, and let $q > 2$ be its conjugate. Denote $M_t^* \triangleq \sup_{0 \leq s \leq t} |M_s|$ and $p_1 \triangleq p(1 + \frac{1}{q}) = 2p - 1$. Applying the Burkholder-Davis-Gundy inequality we have

$$(3.19) \quad \begin{aligned} & \tilde{E}\left\{\left|\int_0^t M_s \beta_s d\tilde{W}_s + \int_0^t M_s \lambda_s ds\right|\right\} \\ & \leq \tilde{E}\left\{\left(\int_0^t |M_s \beta_s|^2 ds\right)^{\frac{1}{2}}\right\} + \tilde{E}\left\{\int_0^t |M_s \lambda_s| ds\right\} \\ & \leq \tilde{E}\left\{|M_t^*|^{1+\frac{1}{q}} \left[\left(\int_0^t M_s^{-\frac{2}{q}} |\beta_s|^2 ds\right)^{\frac{1}{2}} + \int_0^t M_s^{-\frac{1}{q}} |\lambda_s| ds\right]\right\} \\ & \leq C \tilde{E}\{|M_t^*|^{p_1}\}^{\frac{1}{p}} \tilde{E}\left\{\left[\left(\int_0^t M_s^{-\frac{2}{q}} |\beta_s|^2 ds\right)^{\frac{q}{2}} + \left(\int_0^t M_s^{-\frac{1}{q}} |\lambda_s| ds\right)^q\right]^{\frac{1}{q}}\right\} \\ & \leq C \tilde{E}\{|M_t^*|^{p_1}\}^{\frac{1}{p}} \tilde{E}\left\{\int_0^t M_s^{-1} [|\beta_s|^q + |\lambda_s|^q] ds\right\}^{\frac{1}{q}} [t^{1-\frac{2}{q}} + t^{1-\frac{1}{q}}] \\ & \leq C \tilde{E}\{|M_t^*|^{p_1}\}^{\frac{1}{p}} E\left\{\int_0^t [|\beta_s|^q + |\lambda_s|^q] ds\right\}^{\frac{1}{q}} [t^{1-\frac{2}{q}} + t^{1-\frac{1}{q}}]. \end{aligned}$$

Note that $|\alpha_t| \leq K_1$ and that $M_t = \exp(-\int_0^t \alpha_s d\tilde{W}_s - \frac{1}{2} \int_0^t \alpha_s^2 ds)$ is a \tilde{P} -martingale. Then by Doob's inequality one gets

$$\begin{aligned} \tilde{E}\{|M_t^*|^{p_1}\} &\leq C\tilde{E}\{|M_t|^{p_1}\} = \tilde{E}\left\{\exp\left(-p_1 \int_0^t \alpha_s d\tilde{W}_s - \frac{p_1}{2} \int_0^t \alpha_s^2 ds\right)\right\} \\ &\leq C\tilde{E}\left\{\exp\left(-p_1 \int_0^t \alpha_s d\tilde{W}_s - \frac{p_1^2}{2} \int_0^t \alpha_s^2 ds + \frac{p_1^2 - p_1}{2} K_1^2 t\right)\right\} \\ &= C \exp\left(\frac{p_1^2 - p_1}{2} K_1^2 t\right), \end{aligned}$$

where the last equality, thanks to the fact that $\exp(-p_1 \int_0^t \alpha_s d\tilde{W}_s - \frac{p_1^2}{2} \int_0^t \alpha_s^2 ds)$, is a \tilde{P} -martingale. Plugging this into (3.19) and then into (3.18) we get

$$\tilde{E}\{|N_t \Lambda_t^{-1}\} \leq |x| + K_2 \int_0^t \tilde{E}\{|N_s \Lambda_s^{-1}\} ds + C A_t,$$

where

$$A_t \triangleq \exp\left(\frac{p_1^2 - p_1}{2p} K_1^2 t\right) E\left\{\int_0^t [|\beta_s|^q + |\lambda_s|^q] ds\right\}^{\frac{1}{q}} [t^{1-\frac{2}{q}} + t^{1-\frac{1}{q}}].$$

Note that A_t is increasing in t . The by the Gronwall inequality we have

$$(3.20) \quad \tilde{E}\{|\Lambda_t^{-1} N_t|\} \leq C e^{K_2 t} [|x| + A_t].$$

Now choose $p \triangleq 1 + \frac{\varepsilon}{\varepsilon + 6K_1^2}$. One can easily check that $\frac{p_1^2 - p_1}{2p} K_1^2 \leq \frac{\varepsilon}{2}$ and thus

$$A_t \leq C e^{\varepsilon t} E\left\{\int_0^t [|\beta_s|^q + |\lambda_s|^q] ds\right\}^{\frac{1}{q}},$$

which, combined with (3.17) and (3.20), proves (3.15) and hence the lemma. \square

Proof of Theorem 2.4. For any x , let X and X^h be defined as in (3.2) and (3.6), respectively. Denote $\Delta X_t \triangleq X_t - X_t^h$. Then

$$\Delta X_t = \int_0^t [\alpha_s \Delta X_s + \beta_s] dW_s + \int_0^t [\gamma_s \Delta X_s + \lambda_s] ds,$$

where

$$\begin{cases} \alpha_t \triangleq \frac{\sigma(X_t) - \sigma(X_t^h)}{\Delta X_t}, & \beta_t \triangleq \sigma(X_t^h) - \sigma(X_{\tau(t)}^h), \\ \gamma_t \triangleq \frac{b(X_t) - b(X_t^h)}{\Delta X_t}, & \lambda_t \triangleq b(X_t^h) - b(X_{\tau(t)}^h). \end{cases}$$

By the Lipschitz continuity of σ, b , and recalling (3.5) we have

$$|\alpha_t| \leq C, \quad |\gamma_t| \leq L_b, \quad |\beta_t| + |\lambda_t| \leq Ch.$$

Since $\beta L_b < 1$, choose $\varepsilon \triangleq \frac{\beta^{-1} - L_b}{2} > 0$. By (3.16) and the fact that $\Delta X_0 = 0$, we get

$$(3.21) \quad E|\Delta X_t| \leq Ch e^{(L_b + \varepsilon)t}.$$

Since f is bounded and Hölder- α continuous, it is also Hölder- $\alpha \wedge \beta$ continuous. By (3.5) and (3.21) we have

$$\begin{aligned} E\left\{|f(X_t) - f(X_{\tau(t)}^h)|\right\} &\leq CE\left\{|X_t - X_{\tau(t)}^h|^{\alpha \wedge \beta}\right\} \\ &\leq CE\left\{|\Delta X_t|^{\alpha \wedge \beta} + |X_t^h - X_{\tau(t)}^h|^{\alpha \wedge \beta}\right\} \leq C\left[(E\{|\Delta X_t|\})^{\alpha \wedge \beta} + h^{\alpha \wedge \beta}\right] \\ &\leq Ch^{\alpha \wedge \beta} e^{(\alpha \wedge \beta)(L_b + \varepsilon)t}. \end{aligned}$$

Then by (3.1) and (3.7) we get

$$\begin{aligned} |u(x) - u_h^w(x)| &\leq C \int_0^\infty e^{-t} E\left\{|f(X_t) - f(X_{\tau(t)}^h)|\right\} dt \\ &\leq Ch^{\alpha \wedge \beta} \int_0^\infty e^{[(\alpha \wedge \beta)(L_b + \varepsilon) - 1]t} dt = Ch^{\alpha \wedge \beta}, \end{aligned}$$

thanks to the fact that $\beta(L_b + \varepsilon) < 1$. The proof for (2.10) is now complete. \square

4. THE STANDARD FINITE DIFFERENCE SCHEME

We prove Theorems 2.1 and 2.2 in this section, but again leave the sharpness to §5. The main idea is to transform (2.6) to the form of (2.8) and thus find the probability solution to (2.6). To be specific, in light of (2.7) and (2.9), we define two functions $\sigma_h(x)$ and $b_h(x)$ by the following equations:

$$(4.1) \quad \begin{cases} \exp\left(\frac{b_h h}{\sigma_h^2}\right) \left[\cosh\left(\frac{\sqrt{b_h^2 + 2\sigma_h^2} h}{\sigma_h^2}\right) \right]^{-1} = \frac{\sigma^2 + 2b_h}{\sigma^2 + b_h + h^2}, \\ \exp\left(-\frac{b_h h}{\sigma_h^2}\right) \left[\cosh\left(\frac{\sqrt{b_h^2 + 2\sigma_h^2} h}{\sigma_h^2}\right) \right]^{-1} = \frac{\sigma^2}{\sigma^2 + b_h + h^2}. \end{cases}$$

Now by the arguments in §3 we know that

$$(4.2) \quad u_h^s(x) = E_x \left\{ \int_0^\infty e^{-t} f(\tilde{X}_{\tilde{\tau}(t)}^h) dt \right\}$$

and

$$(4.3) \quad |u(x) - u_h^s(x)| \leq \int_0^\infty e^{-t} |E\{f(X_t) - f(\tilde{X}_{\tilde{\tau}(t)}^h)\}| dt,$$

where $\tilde{\tau}$ and \tilde{X}^h are defined in the same manner as (3.3) and (3.4), with σ and b replaced by σ_h and b_h , respectively. Moreover, we have

$$(4.4) \quad |\tilde{X}_{\tilde{\tau}(t)}^h - \tilde{X}_t^h| \leq h, \quad \text{a.s.}$$

and

$$\tilde{X}_t^h = x + \int_0^t \sigma_h(\tilde{X}_{\tilde{\tau}(s)}^h) dW_s + \int_0^t b_h(\tilde{X}_{\tilde{\tau}(s)}^h) ds.$$

To prove the theorems, we need to estimate the errors of σ_h, b_h . Solving (4.1) directly we obtain

$$(4.5) \quad \sigma_h = \frac{\sqrt{2}h}{\sqrt{\log \frac{r(\sigma, b, h)}{\sigma^2} \log \frac{r(\sigma, b, h)}{\sigma^2 + 2b_h}}}, \quad b_h = \frac{h \log(1 + \frac{2b_h}{\sigma^2})}{\log \frac{r(\sigma, b, h)}{\sigma^2} \log \frac{r(\sigma, b, h)}{\sigma^2 + 2b_h}},$$

where

$$r(\sigma, b, h) \triangleq \sigma^2 + b_h + h^2 + h\sqrt{2\sigma^2 + b^2 + 2b_h + h^2}.$$

When $\sigma = 0$, we define σ_h, b_h as the limits in (4.5). That is,

$$(4.6) \quad \sigma_h = 0, \quad b_h = \frac{h}{\log(1 + \frac{h}{b})}.$$

We note that in general σ_h, b_h are *not* Lipschitz continuous in x . Following are some key estimates whose proof is quite lengthy and thus postponed to §6.

Lemma 4.1. *There exists a universal constant C such that*

$$|\sigma_h - \sigma| \leq C[1 + \frac{b}{\sqrt{\sigma^2 + bh}}]h, \quad |b_h - b| \leq Ch.$$

Consequently we have

- Corollary 4.2.** (i) *If b is bounded, then $|\sigma_h - \sigma| \leq C\sqrt{h}$.*
 (ii) *If $\frac{b}{\sigma}$ is bounded, then $|\sigma_h - \sigma| \leq Ch$.*
 (iii) *If both σ and b are bounded, then $|\sigma_h^2 - \sigma^2| \leq Ch$.*

Proof. (i) and (ii) are direct consequence of Lemma 4.1.

It remains to prove (iii). Since b is bounded, by (i) we have $|\sigma_h| \leq |\sigma| + C\sqrt{h}$, and thus

$$\begin{aligned} |\sigma_h^2 - \sigma^2| &= |\sigma_h - \sigma||\sigma_h + \sigma| \leq C[1 + \frac{b}{\sqrt{\sigma^2 + bh}}]h[2|\sigma| + C\sqrt{h}] \\ &\leq Ch\left[|\sigma| + \sqrt{h} + \frac{b|\sigma| + b\sqrt{h}}{\sqrt{\sigma^2 + bh}}\right] \leq Ch\left[1 + \frac{|\sigma| + \sqrt{bh}}{\sqrt{\sigma^2 + bh}}\right] \leq Ch. \end{aligned}$$

That proves the lemma. □

We note that all the conditions in Corollary 4.2 are necessary. Following are some counterexamples. All the calculations are straightforward, and thus we omit the proof.

Example 1. (i) (b is unbounded.)

Let $\sigma = 1, b = \frac{1}{h}$. Then $\sigma_h \sim \sqrt{\frac{2}{\log 3}}$ and thus $\sigma_h - \sigma \sim \sqrt{\frac{2}{\log 3}} - 1$.

(ii) (b is bounded, but $\frac{b}{\sigma}$ is unbounded.)

Let $\sigma = \sqrt{h}, b = 1$, then $\sigma_h \sim \sqrt{\frac{2h}{\log 3}}$, and thus $\sigma_h - \sigma \sim [\sqrt{\frac{2}{\log 3}} - 1]\sqrt{h}$.

(iii) (b is bounded, but σ is unbounded.)

Let $\sigma = \frac{1}{h}, b = 1$; then $\sigma_h^2 \sim \frac{1}{h^2} + \frac{1}{2}$, and thus $\sigma_h^2 - \sigma^2 \sim \frac{1}{2}$.

Proof of Theorem 2.2. Recall (4.3). Following the arguments in the proof of Theorem 2.4 we have

$$(4.7) \quad |u(x) - u_h^s(x)| \leq C \int_0^\infty e^{-t} E\{|X_t - \tilde{X}_t^h|^{\alpha \wedge \beta}\} dt + Ch^{\alpha \wedge \beta}.$$

Denote $\Delta \tilde{X}_t \triangleq X_t - \tilde{X}_t^h$. Then

$$\begin{aligned} d\Delta \tilde{X}_t &= [\sigma(X_t) - \sigma_h(\tilde{X}_{\tilde{\tau}(t)}^h)]dW_t + [b(X_t) - b_h(\tilde{X}_{\tilde{\tau}(t)}^h)]dt \\ &= [\alpha_t \Delta \tilde{X}_t + \beta_t]dW_t + [\gamma_t \Delta \tilde{X}_t + \lambda_t]dt, \end{aligned}$$

where

$$\begin{aligned} \alpha_t &\triangleq \frac{\sigma(X_t) - \sigma(\tilde{X}_t^h)}{\Delta \tilde{X}_t}, & \beta_t &\triangleq \sigma(\tilde{X}_t^h) - \sigma_h(\tilde{X}_{\tilde{\tau}(t)}^h), \\ \gamma_t &\triangleq \frac{b(X_t) - b(\tilde{X}_t^h)}{\Delta \tilde{X}_t}, & \lambda_t &\triangleq b(\tilde{X}_t^h) - b_h(\tilde{X}_{\tilde{\tau}(t)}^h). \end{aligned}$$

Since σ and b are Lipschitz continuous, we have $|\alpha_t| \leq C$ and $|\gamma_t| \leq L_b$. Moreover, by Corollary 4.2(i) we have $|\sigma - \sigma_h| \leq C\sqrt{h}$, therefore,

$$|\beta_t| \leq |\sigma(\tilde{X}_t^h) - \sigma(\tilde{X}_{\tilde{\tau}(t)}^h)| + |\sigma(\tilde{X}_{\tilde{\tau}(t)}^h) - \sigma_h(\tilde{X}_{\tilde{\tau}(t)}^h)| \leq C\sqrt{h}.$$

Similarly, by Lemma 4.1 we have $|\lambda_t| \leq Ch$. Now letting $0 < \varepsilon < \beta^{-1} - L_b$ and applying Lemma 3.2, we get $E\{|\Delta\tilde{X}_t|\} \leq C\sqrt{h}e^{(L_b+\varepsilon)t}$, which, combined with (4.7), proves (i).

It remains to prove (ii). We note that if $\sigma = 0$, by (4.6) we have $|\sigma_h - \sigma| = 0 \leq Ch$. If $\frac{b}{\sigma}$ is bounded, by Corollary 4.2(ii) we have $|\sigma_h - \sigma| \leq Ch$. Now (ii) follows a line-by-line analogy as above. \square

We next prove Theorem 2.1. Recall (4.3). Note that we estimate the error of $f(\tilde{X}_{\tilde{\tau}(t)}^h) - f(X_t)$ in (4.7) in the strong sense. The next lemma gives an estimate of the error in the weak sense. The idea is more or less standard (see, e.g., [22]).

Lemma 4.3. *Assume all the conditions in Theorem 2.1 hold true. Then*

$$|E\{f(\tilde{X}_T^h) - f(X_T)\}| \leq Ce^{\frac{1+L}{2}T}h, \quad \forall T > 0.$$

Proof. Fix $T > 0$. Let v be the solution to the following degenerate parabolic PDE:

$$(4.8) \quad \begin{cases} v_t + \frac{1}{2}\sigma^2(x)v_{xx} + b(x)v_x = 0, & t \in (0, T), \\ v(T, x) = f(x). \end{cases}$$

Then we have

$$(4.9) \quad E\{f(\tilde{X}_T^h)\} = E\{v(T, \tilde{X}_T^h)\}, \quad E\{f(X_T)\} = v(0, x).$$

Obviously $v \in C^{1,2}$. Applying the Ito formula one has

$$\begin{aligned} & dv(t, \tilde{X}_t^h) \\ &= \left[v_t + \frac{1}{2}\sigma_h^2(\tilde{X}_{\tilde{\tau}(t)}^h)v_{xx} + b_h(\tilde{X}_{\tilde{\tau}(t)}^h)v_x \right](t, \tilde{X}_t^h)dt + \sigma_h^2(\tilde{X}_{\tilde{\tau}(t)}^h)v_x(t, \tilde{X}_t^h)dW_t \\ &= \left[\frac{1}{2}[\sigma_h^2(\tilde{X}_{\tilde{\tau}(t)}^h) - \sigma^2(\tilde{X}_t^h)]v_{xx} + [b_h(\tilde{X}_{\tilde{\tau}(t)}^h) - b(\tilde{X}_t^h)]v_x \right](t, \tilde{X}_t^h)dt \\ &\quad + \sigma_h(\tilde{X}_{\tilde{\tau}(t)}^h)v_x(t, \tilde{X}_t^h)dW_t. \end{aligned}$$

By (4.4), Corollary 4.2(iii) and Lemma 4.1 we have

$$\begin{aligned} |\sigma_h^2(\tilde{X}_{\tilde{\tau}(t)}^h) - \sigma^2(\tilde{X}_t^h)| &\leq |\sigma_h^2(\tilde{X}_{\tilde{\tau}(t)}^h) - \sigma^2(\tilde{X}_{\tilde{\tau}(t)}^h)| + |\sigma^2(\tilde{X}_{\tilde{\tau}(t)}^h) - \sigma^2(\tilde{X}_t^h)| \leq Ch, \\ |b_h(\tilde{X}_{\tilde{\tau}(t)}^h) - b(\tilde{X}_t^h)| &\leq |b_h(\tilde{X}_{\tilde{\tau}(t)}^h) - b(\tilde{X}_{\tilde{\tau}(t)}^h)| + |b(\tilde{X}_{\tilde{\tau}(t)}^h) - b(\tilde{X}_t^h)| \leq Ch. \end{aligned}$$

We claim that

$$(4.10) \quad |v_x(t, x)| + |v_{xx}(t, x)| \leq Ce^{\frac{1+L}{2}(T-t)}, \quad \forall t \in [0, T].$$

Then

$$|E\{v(T, \tilde{X}_T^h) - v(0, x)\}| \leq Ch \int_0^T e^{\frac{1+L}{2}(T-t)} dt \leq Ce^{\frac{1+L}{2}T}h,$$

which, together with (4.9), proves the lemma.

It remains to prove (4.10). Without loss of generality, we shall prove it only at $t = 0$. To this end, we recall (4.9). Let $\nabla X, \nabla^2 X$ denote the first order and second

order derivative flows of X , differentiating with respect to x . That is,

$$\begin{aligned}
 (4.11) \quad \nabla X_t &= 1 + \int_0^t \sigma'(X_s) \nabla X_s dW_s + \int_0^t b'(X_s) \nabla X_s ds, \\
 \nabla^2 X_t &= \int_0^t [\sigma'(X_s) \nabla^2 X_s + \sigma''(X_s) |\nabla X_s|^2] dW_s \\
 (4.12) \quad &+ \int_0^t [b'(X_s) \nabla^2 X_s + b''(X_s) |\nabla X_s|^2] ds.
 \end{aligned}$$

Then we have

$$(4.13) \quad v_x(0, x) = E\{f'(X_T) \nabla X_T\}, \quad v_{xx}(0, x) = E\{f'(X_T) \nabla^2 X_T + f''(X_T) |\nabla X_T|^2\}.$$

For $\varepsilon \triangleq \frac{1-L}{2} > 0$, applying (3.16) to (4.11) we get

$$(4.14) \quad E\{|\nabla X_T|\} \leq C e^{(L_b+\varepsilon)T} \leq C e^{\frac{1+L}{2}T}.$$

Moreover, for any $q > 2$, applying the Ito formula we have

$$|\nabla X_t|^q = 1 + \int_0^t q\sigma'(X_s) |\nabla X_s|^q dW_s + \int_0^t [qb'(X_s) + \frac{q(q-1)}{2} |\sigma'(X_s)|^2] |\nabla X_s|^q ds.$$

Then letting q be close to 2 enough and applying (3.16) again one gets

$$(4.15) \quad E\{|\nabla X_T|^q\} \leq C e^{\frac{1+L}{2}T}.$$

Now applying (3.15) to (4.12) we have

$$(4.16) \quad E\{|\nabla^2 X_T|\} \leq C e^{\frac{1+L}{2}T}.$$

Now by (4.14), (4.15), (4.16), and (4.13) we prove (4.10), hence the lemma. \square

Proof of Theorem 2.1. Recalling (4.3) and (4.4), and applying Lemma 4.3 we have

$$\begin{aligned}
 |u(x) - u_h^s(x)| &\leq C \int_0^\infty e^{-t} \left| E\left\{ [f(\tilde{X}_{t(t)}^h) - f(\tilde{X}_t^h)] + [f(\tilde{X}_t^h) - f(X_t)] \right\} \right| dt \\
 &\leq C \int_0^\infty e^{-t} [h + e^{\frac{1+L}{2}t} h] dt = Ch \int_0^\infty [e^{-t} + e^{\frac{L-1}{2}t}] dt = Ch,
 \end{aligned}$$

thanks to the assumption that $L < 1$. \square

5. SHARPNESS OF THE ESTIMATES

In this section we give three examples to show that some estimates in §2 are sharp. We first consider β .

Example 2. Let $\sigma = 0, b(x) = L_b x, f(x) = 0$ for $x \in [0, 1]$. Choose $\sigma, b, f \in C_0^\infty$ appropriately outside $[0, 1]$ such that $u(1) = 1$. Then, by letting $h = \frac{1}{n}$,

$$\liminf_{n \rightarrow \infty} n^{\frac{1}{L_b}} \sup_{x \in [0,1]} |u(x) - u_h^s(x)| > 0, \quad \liminf_{n \rightarrow \infty} n^{\frac{1}{L_b}} \sup_{x \in [0,1]} |u(x) - u_h^w(x)| > 0.$$

Proof. We will only estimate $u(x) - u_h^w(x)$. The other one is similar.

First, under our assumptions on $[0, 1]$ (1.1) becomes $L_b x u'(x) - u(x) = 0$. Since $u(1) = 1$, one has

$$(5.1) \quad u(x) = x^{\frac{1}{L_b}}, \forall x \in [0, 1].$$

On the other hand, for $h = \frac{1}{n}$, and $k = 1, \dots, n-1$, by (2.8) we have $u_h^w(kh) = \exp(-\frac{1}{L_b k})u_h^w((k+1)h)$, which leads to

$$(5.2) \quad u_h^w(h) = u_h^w(1) \exp\left(-\frac{1}{L_b} \sum_{k=1}^{n-1} \frac{1}{k}\right).$$

If $|u_h^w(1) - u(1)| \geq h^{\frac{1}{L_b}}$, then we have

$$(5.3) \quad n^{\frac{1}{L_b}} |u_h^w(1) - u(1)| \geq 1.$$

Now we assume $|u_h^w(1) - 1| = |u_h^w(1) - u(1)| < h^{\frac{1}{L_b}}$. Note that

$$\sum_{k=1}^{n-1} \frac{1}{k} \geq 1 + \int_2^n \frac{dx}{x} = 1 - \log(2) + \log(n).$$

By (5.2) we have

$$u_h^w(h) \leq (1 + h^{\frac{1}{L_b}}) \exp\left(-\frac{1}{L_b} [1 - \log(2) + \log(n)]\right) = (1 + h^{\frac{1}{L_b}}) \exp\left(-\frac{1 - \log(2)}{L_b}\right) h^{\frac{1}{L_b}},$$

which, combined with (5.1), implies that

$$u(h) - u_h^w(h) \geq \left[1 - (1 + h^{\frac{1}{L_b}}) \exp\left(-\frac{1 - \log(2)}{2L_b}\right)\right] h^{\frac{1}{L_b}}.$$

Note that $1 - \log(2) > 0$. When h is small enough, we have

$$(1 + h^{\frac{1}{L_b}}) \exp\left(-\frac{1 - \log(2)}{L_b}\right) \leq \exp\left(-\frac{1 - \log(2)}{2L_b}\right) < 1.$$

Then

$$n^{\frac{1}{L_b}} [u(h) - u_h^w(h)] \geq 1 - \exp\left(-\frac{1 - \log(2)}{2L_b}\right) > 0,$$

which, combined with (5.3), proves our claim. \square

The next example shows that the first order estimate in (2.3) is sharp.

Example 3. Let $\sigma(x) = \sqrt{h}$, $b(x) = 1$, $f(x) = 0$ for $x \in [0, 1]$. Choose $\sigma, b, f \in C_0^\infty$ appropriately outside $[0, 1]$ such that $u(0) = 0$ and $u(1) = 1$. Then, by letting $h = \frac{1}{n}$,

$$\liminf_{n \rightarrow \infty} n \sup_{x \in [0, 1]} |u(x) - u_h^s(x)| > 0.$$

We note that in this example σ , and thus u , depends on h . But one may divide \mathbb{R} into countable disjoint intervals $[a_n, b_n)$. By Example 3 we can set σ, b, f on each interval such that they all vanish on a_n, b_n and $n \sup_{x \in [a_n, b_n]} |u(x) - u_{\frac{1}{n}}^s(x)| \geq c_0$ for some constant $c_0 > 0$ which is independent of n . Combining all the pieces together we get universal σ, b, f satisfying our requirement.

Proof of Example 3. First, one can easily solve (1.1):

$$u(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{e^{\lambda_1} - e^{\lambda_2}}, \quad x \in [0, 1],$$

where

$$\lambda_1 \triangleq \frac{-1 + \sqrt{1 + 2h}}{h}, \quad \lambda_2 \triangleq \frac{-1 - \sqrt{1 + 2h}}{h}.$$

On the other hand, solve (2.6) we have

$$(5.4) \quad u_h^s(kh) = c_1\gamma_1^k - c_2\gamma_2^k, \quad k = 0, 1, \dots, n,$$

for some constants c_1, c_2 , where

$$\gamma_1 \triangleq \frac{2 + h + \sqrt{1 + 4h + h^2}}{3}, \quad \gamma_2 \triangleq \frac{2 + h - \sqrt{1 + 4h + h^2}}{3}.$$

If $|u_h^s(0)| \geq \varepsilon h$ or $|u_h^s(1) - 1| \geq \varepsilon h$ where $\varepsilon > 0$ is a small number independent of h which will be determined later, then we have

$$(5.5) \quad n \sup_{x \in [0,1]} |u(x) - u_h^s(x)| \geq \varepsilon.$$

Now we assume $|u_h^s(0)| \leq \varepsilon h$ and $|u_h^s(1) - 1| \leq \varepsilon h$. By letting $k = 0$ and $k = n$ in (5.4) we get

$$c_1 = \frac{u_h^s(1) - u_h^s(0)\gamma_2^n}{\gamma_1^n - \gamma_2^n}, \quad c_2 = \frac{u_h^s(1) - u_h^s(0)\gamma_1^n}{\gamma_1^n - \gamma_2^n}.$$

Denote

$$\bar{u}_h^s(kh) \triangleq \frac{\gamma_1^k - \gamma_2^k}{\gamma_1^n - \gamma_2^n}, \quad k = 0, 1, \dots, n.$$

Note that for $\gamma_1 > 1 > \frac{1}{3} > \gamma_2 > 0$ and $\gamma_1\gamma_2 = \frac{1}{3}$, we have

$$(5.6) \quad |\bar{u}_h^s(kh) - u_h^s(kh)| = \left| \frac{[u_h^s(1) - 1][\gamma_1^k - \gamma_2^k] - u_h^s(0)[\gamma_1^k\gamma_2^n - \gamma_1^n\gamma_2^k]}{\gamma_1^n - \gamma_2^n} \right| \leq 2\varepsilon h.$$

Now we estimate $|u(kh) - \bar{u}_h^s(kh)|$. Let $x = \frac{1}{2}$, or say, $k = \frac{n}{2}$. Then we have

$$u\left(\frac{1}{2}\right) = [e^{\frac{\lambda_1}{2}} + e^{\frac{\lambda_2}{2}}]^{-1} \sim e^{-\frac{\lambda_1}{2}}, \quad \bar{u}_h^s\left(\frac{1}{2}\right) = [\gamma_1^{\frac{n}{2}} + \gamma_2^{\frac{n}{2}}]^{-1} \sim \exp\left(-\frac{n}{2} \log(\gamma_1)\right).$$

Note that $\frac{\lambda_1}{2} \sim \frac{1}{2} - \frac{h}{4}$, and

$$\frac{n}{2} \log(\gamma_1) \sim \frac{1}{2h} \log\left(1 + h - \frac{h^2}{2}\right) \sim \frac{1}{2h}(h - h^2) = \frac{1}{2} - \frac{h}{2}.$$

Then,

$$u\left(\frac{1}{2}\right) - \bar{u}_h^s\left(\frac{1}{2}\right) \sim e^{-\frac{1}{2} + \frac{h}{4}} - e^{-\frac{1}{2} + \frac{h}{2}} \sim -\frac{1}{4}e^{-\frac{1}{2}}h.$$

Now let $\varepsilon \triangleq \frac{1}{15}e^{-\frac{1}{2}} > 0$. When h is small enough, by (5.6) we have

$$|u\left(\frac{1}{2}\right) - u_h^s\left(\frac{1}{2}\right)| \geq |u\left(\frac{1}{2}\right) - \bar{u}_h^s\left(\frac{1}{2}\right)| - |u_h^s\left(\frac{1}{2}\right) - \bar{u}_h^s\left(\frac{1}{2}\right)| \geq \frac{1}{5}e^{-\frac{1}{2}}h - 2\varepsilon h \geq \varepsilon h,$$

which, combined with (5.5), proves our claim. □

Our last example shows the sharpness of α . Again we will only prove it for u_h^w .

Example 4. Assume $\alpha \in (0, 1]$. Let $\sigma \triangleq \sqrt{2}, b \triangleq 0$, and $f(x) \triangleq \sum_{m=0}^{\infty} f_m(x)$, where f_m has period 2^{-m} , and

$$(5.7) \quad f_m(x) \triangleq \begin{cases} x^\alpha, & 0 \leq x < 2^{-(m+1)}, \\ (2^{-m} - x)^\alpha, & 2^{-(m+1)} \leq x < 2^{-m}. \end{cases}$$

Then

- (i) f is bounded;
- (ii) $|f(x) - f(y)| \leq C|x - y|^\alpha \log \frac{1}{|x - y|}, \quad \forall |x - y| \leq \frac{1}{2};$

(iii) for $h = 2^{-n}$,

$$(5.8) \quad \liminf_{n \rightarrow \infty} 2^{n\alpha} |u(0) - u_h^w(0)| > 0.$$

Proof. (i) By (5.7) it obviously holds that $|f_m(x)| \leq 2^{-(m+1)\alpha}$, thus

$$|f(x)| \leq \sum_{m=0}^{\infty} 2^{-(m+1)\alpha} = \frac{1}{2^\alpha - 1} < \infty.$$

(ii) For any $|x - y| \leq \frac{1}{2}$, assuming $2^{-n} \leq |x - y| < 2^{1-n}$, we have

$$|f_m(x) - f_m(y)| \leq \begin{cases} |x - y|^\alpha, & m < n, \\ 2^{1-(m+1)\alpha}, & m \geq n. \end{cases}$$

Therefore,

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{m=0}^{\infty} |f_m(x) - f_m(y)| \leq \sum_{m=0}^{n-1} |x - y|^\alpha + \sum_{m=n}^{\infty} 2^{1-(m+1)\alpha} \\ &= n|x - y|^\alpha + 2^{1-\alpha-n\alpha} \frac{1}{1 - 2^{-\alpha}} \leq C|x - y|^\alpha \log \frac{1}{|x - y|}. \end{aligned}$$

(iii) We first note that, in our example, (1.1) and (2.8) become

$$(5.9) \quad u'' - u = -f$$

and

$$(5.10)$$

$$u_h^w(x) = \frac{1}{2 \cosh(h)} \left[u_h^w(x+h) + u_h^w(x-h) \right] + \left(1 - \frac{1}{\cosh(h)} \right) f(x),$$

respectively. The general solution to (5.9) is

$$u(x) = C_1 e^x + C_2 e^{-x} - \frac{1}{2} \int_0^x [e^{x-y} - e^{y-x}] f(y) dy.$$

Note that, for bounded f , we require u to be bounded. By letting $x \rightarrow \infty$ or $x \rightarrow -\infty$ we get

$$C_1 = \frac{1}{2} \int_0^\infty e^{-y} f(y) dy, \quad C_2 = \int_{-\infty}^0 e^y f(y) dy.$$

Thus

$$u(x) = \frac{1}{2} \int_{-\infty}^\infty e^{-|x-y|} f(y) dy.$$

Moreover, noting that f is even and $f(0) = 0$, we get

$$(5.11) \quad u(0) = \int_0^\infty e^{-x} f(x) dx.$$

Analogously, the general solution to (5.10) is

$$u_h^w(mh) = C_1 e^{mh} + C_2 e^{-mh} - \frac{\cosh(h) - 1}{\sinh(h)} \sum_{k=0}^m [e^{(m-k)h} - e^{(k-m)h}] f(kh).$$

By similar arguments we get

$$(5.12) \quad u_h^w(0) = \frac{2(\cosh(h) - 1)}{\sinh(h)} \sum_{k=1}^\infty e^{-kh} f(kh).$$

We shall use (5.11) and (5.12) to prove (5.8). The proof is quite lengthy and purely analytic. We thus postpone it to the Appendix. \square

6. APPENDIX

In this Appendix we prove (5.8) and Lemma 4.1. All the arguments are purely analytic.

Proof of (5.8). We shall use (5.11) and (5.12) to estimate $|u(0) - u_h(0)|$. First,

$$\begin{aligned} \int_0^\infty e^{-x} f_m(x) dx &= \sum_{k=0}^\infty \int_{k2^{-m}}^{(k+1)2^{-m}} e^{-x} f_m(x) dx = \sum_{k=0}^\infty e^{-k2^{-m}} \int_0^{2^{-m}} e^{-x} f_m(x) dx \\ &= \frac{1}{1 - e^{-2^{-m}}} \left[\int_0^{2^{-(m+1)}} x^\alpha e^{-x} dx + \int_{2^{-(m+1)}}^{2^{-m}} (2^{-m} - x)^\alpha e^{-x} dx \right] \\ &= \frac{1}{1 - e^{-2^{-m}}} \int_0^{2^{-(m+1)}} x^\alpha [e^{-x} + e^{x-2^{-m}}] dx. \end{aligned}$$

By (5.11) we have

$$\begin{aligned} u(0) &= \sum_{m=0}^{n-1} \frac{1}{1 - e^{-2^{-m}}} \int_0^{2^{-(m+1)}} x^\alpha [e^{-x} + e^{x-2^{-m}}] dx \\ &\quad + \sum_{m=n}^\infty \frac{1}{1 - e^{-2^{-m}}} \int_0^{2^{-(m+1)}} x^\alpha [e^{-x} + e^{x-2^{-m}}] dx. \end{aligned}$$

Using substitution $m = n - m' - 1$ in the first summation and $m = n + m'$ in the second summation, we get

$$\begin{aligned} u(0) &= \sum_{m=0}^{n-1} \frac{1}{1 - e^{-2^{m-n+1}}} \int_0^{2^{m-n}} x^\alpha [e^{-x} + e^{x-2^{m-n+1}}] dx \\ &\quad + \sum_{m=0}^\infty \frac{1}{1 - e^{-2^{-m-n}}} \int_0^{2^{-(m+n+1)}} x^\alpha [e^{-x} + e^{x-2^{-m-n}}] dx. \end{aligned}$$

Then by letting $x = hx' = 2^{-n}x'$, we have

$$\begin{aligned} u(0) &= h^\alpha \sum_{m=0}^{n-1} \frac{h}{1 - e^{-2^{m+1}h}} \int_0^{2^m} x^\alpha [e^{-xh} + e^{xh-2^{m+1}h}] dx \\ (6.1) \quad &+ h^\alpha \sum_{m=0}^\infty \frac{h}{1 - e^{-2^{-m}h}} \int_0^{2^{-(m+1)}} x^\alpha [e^{-xh} + e^{xh-2^{-m}h}] dx. \end{aligned}$$

On the other hand, note that $f_m(kh) = 0$ for $m \geq n$ and f_m is periodic with period 2^{-m} . So by (5.12) we have

$$\begin{aligned} u_h^w(0) &= \frac{2(\cosh(h) - 1)}{\sinh(h)} \sum_{k=1}^{\infty} e^{-kh} \sum_{m=0}^{n-1} f_m(kh) \\ &= \frac{2(\cosh(h) - 1)}{\sinh(h)} \sum_{m=0}^{n-1} \sum_{k=1}^{2^{n-m}-1} e^{-kh} f_m(kh) \sum_{l=0}^{\infty} e^{-2^{n-m}lh} \\ &= \frac{2(\cosh(h) - 1)}{\sinh(h)} \sum_{m=0}^{n-1} \frac{1}{1 - e^{-2^{n-m}h}} \\ &\quad \times \left[\sum_{k=1}^{2^{n-m}-1} |kh|^{\alpha} (e^{-kh} + e^{(k-2^{n-m})h}) + 2^{-(m+1)\alpha} e^{-2^{n-m-1}h} \right]. \end{aligned}$$

Using the substitution $m = n - m' - 1$ again we have

$$(6.2) \quad \begin{aligned} u_h^w(0) &= h^{\alpha} \frac{2(\cosh(h) - 1)}{\sinh(h)} \sum_{m=0}^{n-1} \frac{1}{1 - e^{-2^{m+1}h}} \\ &\quad \times \left[\sum_{k=1}^{2^m-1} k^{\alpha} (e^{-kh} + e^{(k-2^{m+1})h}) + 2^{m\alpha} e^{-2^m h} \right]. \end{aligned}$$

Subtracting (6.1) from (6.2), and dividing by $h^{\alpha} = 2^{-n\alpha}$, we get

$$(6.3) \quad \begin{aligned} &2^{n\alpha} [u_h^w(0) - u(0)] \\ &= - \sum_{m=0}^{\infty} \frac{h}{1 - e^{-2^{-m}h}} \int_0^{2^{-(m+1)}} x^{\alpha} [e^{-xh} + e^{xh-2^{-m}h}] dx \\ &\quad + \sum_{m=0}^{n-1} \left\{ \frac{2(\cosh(h) - 1)}{\sinh(h)(1 - e^{-2^{m+1}h})} \right. \\ &\quad \times \left[\sum_{k=1}^{2^m-1} k^{\alpha} (e^{-kh} + e^{(k-2^{m+1})h}) + 2^{m\alpha} e^{-2^m h} \right] \\ &\quad \left. - \frac{h}{1 - e^{-2^{m+1}h}} \int_0^{2^m} x^{\alpha} [e^{-xh} + e^{xh-2^{m+1}h}] dx \right\} \\ &= -I(h) + \sum_{m=0}^{n-1} I_m(h), \end{aligned}$$

where I and I_m are defined in an obvious way. Now let $n \rightarrow \infty$, or equivalently, $h \rightarrow 0$. We shall study the limits of the terms in (6.3).

First,

$$\begin{aligned} &\frac{h}{1 - e^{-2^{-m}h}} \int_0^{2^{-(m+1)}} x^{\alpha} [e^{-xh} + e^{xh-2^{-m}h}] dx \\ &\leq \frac{1}{1 - e^{-2^{-m}h}} 2^{-(m+1)\alpha} \left[(1 - e^{-2^{-(m+1)}h}) + e^{-2^{-m}h} (e^{2^{-(m+1)}h} - 1) \right] = 2^{-(m+1)\alpha}. \end{aligned}$$

Since $\sum_{m=1}^{\infty} 2^{-(m+1)\alpha} < \infty$, by the Dominated Convergence Theorem one gets that

$$\begin{aligned} \lim_{h \rightarrow 0} I(h) &= \sum_{m=0}^{\infty} \lim_{h \rightarrow 0} \frac{h}{1 - e^{-2^{-m}h}} \int_0^{2^{-(m+1)}} x^\alpha [e^{-xh} + e^{xh-2^{-m}h}] dx \\ &= \sum_{m=0}^{\infty} 2^m \int_0^{2^{-(m+1)}} x^\alpha 2 dx \\ &= \sum_{m=0}^{\infty} \frac{1}{(1 + \alpha)2^{(m+1)\alpha}} = \frac{1}{(1 + \alpha)(2^\alpha - 1)} \triangleq I. \end{aligned}$$

To estimate $I_m(h)$, we discuss two cases. First, if $\alpha = 1$, one can check directly that

$$\sum_{k=1}^{2^m-1} kx^k = \frac{x}{(1-x)^2}(1-x^{2^m}) - \frac{x}{1-x} 2^m x^{2^m-1},$$

and that

$$\int_0^{2^m} x e^{ax} dx = [a^{-1}2^m - a^{-2}]e^{2^m a} + a^{-2}.$$

Then by straightforward calculation we have

$$I_m(h) = \left[\frac{1}{\sinh(h)} - \frac{1}{h} \right] \frac{1 - e^{-2^m h}}{1 + e^{-2^m h}} \leq 0.$$

Thus

$$\limsup_{h \rightarrow 0} 2^n [u_h^w(0) - u(0)] \leq -I < 0,$$

which obviously proves (5.8) in the case that $\alpha = 1$.

Now we assume $\alpha \in (0, 1)$. Note that

$$I_m(h) = I_m^1(h) + I_m^2(h),$$

where

$$\begin{cases} I_m^1(h) \triangleq \frac{1}{1 - e^{-2^{m+1}h}} \left[\frac{2(\cosh(h) - 1)}{\sinh(h)} - h \right] \int_0^{2^m} x^\alpha [e^{-xh} + e^{xh-2^{m+1}h}] dx, \\ I_m^2(h) \triangleq \frac{2(\cosh(h) - 1)}{\sinh(h)(1 - e^{-2^{m+1}h})} \\ \times \left\{ \sum_{k=1}^{2^m} \int_{k-1}^k [(k^\alpha e^{-kh} - x^\alpha e^{-kh}) + e^{-2^{m+1}h}(k^\alpha e^{kh} - x^\alpha e^{xh})] dx - 2^{m\alpha} e^{-2^m h} \right\}. \end{cases}$$

Now for $m < n$, we have

$$(6.4) \quad |I_m^1(h)| \leq \frac{Ch^2}{1 - e^{-2^{m+1}h}} 2^{m\alpha} \int_0^{2^m} [e^{-xh} + e^{xh-2^{m+1}h}] dx = Ch2^{m\alpha} \leq C2^{m(\alpha-1)}.$$

Moreover, noting that

$$|g(k) - g(x)| \leq \int_x^k |g'(y)| dy$$

for any differentiable function g and any $x \in [k-1, k]$, we have

$$\begin{aligned} & (1 - e^{-2^{m+1}h}) |I_m^2(h)| \\ & \leq Ch \left\{ \sum_{k=1}^{2^m} \int_{k-1}^k \left[|k^\alpha e^{-kh} - x^\alpha e^{-xh}| \right. \right. \\ & \quad \left. \left. + e^{-2^{m+1}h} |k^\alpha e^{kh} - x^\alpha e^{xh}| \right] dx + 2^{m\alpha} e^{-2^m h} \right\} \\ & \leq Ch \sum_{k=1}^{2^m} \int_{k-1}^k \left[(\alpha x^{\alpha-1} e^{-xh} + hx^\alpha e^{-xh}) \right. \\ & \quad \left. + e^{-2^{m+1}h} (\alpha x^{\alpha-1} e^{xh} + hx^\alpha e^{xh}) \right] dx + Ch 2^{m\alpha} \\ & \leq C\alpha h \int_0^{2^m} x^{\alpha-1} dx \\ & \quad + Ch^2 2^{m\alpha} \int_0^{2^m} [e^{-xh} + e^{(x-2^{m+1})h}] dx + Ch 2^{m\alpha} \\ & = Ch 2^{m\alpha} + Ch 2^{m\alpha} [1 - e^{-2^{m+1}h}] + Ch 2^{m\alpha} \leq Ch 2^{m\alpha}. \end{aligned}$$

Thus

$$(6.5) \quad |I_m^2(h)| \leq \frac{Ch 2^{m\alpha}}{1 - e^{-2^{m+1}h}} = \frac{C 2^{m+1}h}{1 - e^{-2^{m+1}h}} 2^{m(\alpha-1)} \leq C 2^{m(\alpha-1)}.$$

Combining (6.4) and (6.5), one gets $|I_m(h)| \leq C 2^{m(\alpha-1)}$, which obviously implies that

$$\lim_{n \rightarrow \infty} \sum_{m=0}^{n-1} I_m(h) = \sum_{m=0}^{\infty} I_m,$$

where

$$I_m \triangleq \lim_{h \rightarrow 0} I_m(h) = 2^{-m} \left[\sum_{k=1}^{2^m-1} k^\alpha + 2^{m\alpha-1} - \int_0^{2^m} x^\alpha dx \right].$$

Note that $\sum_{m=0}^{\infty} I_m$ and I are two numbers, so it suffices to show that

$$(6.6) \quad \sum_{m=0}^{\infty} I_m < I.$$

To this end, we note that

$$\begin{aligned}
 2^m I_m &= \sum_{k=1}^{2^m-1} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} (k^\alpha - x^\alpha) dx + \int_{2^{m-\frac{1}{2}}}^{2^m} (2^{m\alpha} - x^\alpha) dx - \int_0^{\frac{1}{2}} x^\alpha dx \\
 &= \sum_{k=1}^{2^m-1} \int_0^{\frac{1}{2}} [2k^\alpha - (k+x)^\alpha - (k-x)^\alpha] dx + \int_0^{\frac{1}{2}} [2^{m\alpha} - (2^m-x)^\alpha] dx - \int_0^{\frac{1}{2}} x^\alpha dx \\
 &= \alpha \sum_{k=1}^{2^m-1} \int_0^{\frac{1}{2}} \int_0^x [(k-y)^{\alpha-1} - (k+y)^{\alpha-1}] dy dx \\
 &\quad + \alpha \int_0^{\frac{1}{2}} \int_0^x [(2^m-y)^{\alpha-1} - 2^{m(\alpha-1)} + 2^{m(\alpha-1)}] dy dx - \frac{1}{(1+\alpha)2^{1+\alpha}} \\
 &= \alpha(1-\alpha) \sum_{k=1}^{2^m-1} \int_0^{\frac{1}{2}} \int_0^x \int_{-y}^y (k+z)^{\alpha-2} dz dy dx \\
 &\quad + \alpha(1-\alpha) \int_0^{\frac{1}{2}} \int_0^x \int_0^y (2^m-z)^{\alpha-2} dz dy dx + \frac{\alpha}{8} 2^{m(\alpha-1)} - \frac{1}{(1+\alpha)2^{1+\alpha}} \\
 &\leq \frac{\alpha(1-\alpha)}{8} \left[\sum_{k=1}^{2^m-1} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} z^{\alpha-2} dz + \int_{2^{m-\frac{1}{2}}}^{2^m} z^{\alpha-2} dz \right] + \frac{\alpha}{8} 2^{m(\alpha-1)} - \frac{1}{(1+\alpha)2^{1+\alpha}} \\
 &\leq \frac{\alpha(1-\alpha)}{8} \int_{\frac{1}{2}}^\infty z^{\alpha-2} dz + \frac{\alpha}{8} 2^{m(\alpha-1)} - \frac{1}{(1+\alpha)2^{1+\alpha}} \\
 &= \frac{\alpha}{8} 2^{1-\alpha} + \frac{\alpha}{8} 2^{m(\alpha-1)} - \frac{1}{(1+\alpha)2^{1+\alpha}}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \sum_{m=0}^\infty I_m &\leq \sum_{m=0}^\infty 2^{-m} \left[\frac{\alpha}{8} 2^{1-\alpha} + \frac{\alpha}{8} 2^{m(\alpha-1)} - \frac{1}{(1+\alpha)2^{1+\alpha}} \right] \\
 &= \frac{\alpha}{2^{1+\alpha}} + \frac{\alpha}{2(4-2^\alpha)} - \frac{1}{(1+\alpha)2^\alpha} \\
 &\leq \frac{1}{2^{1+\alpha}} + \frac{1}{4} - \frac{1}{(1+1)2^\alpha} = \frac{1}{4} < \frac{1}{2} \leq \frac{1}{(1+\alpha)(2^\alpha-1)} = I.
 \end{aligned}$$

That proves (6.6), whence (5.8), for $\alpha < 1$. □

Proof of Lemma 4.1. Recall (4.5). Let, by abusing the notation r ,

$$x \triangleq \frac{h^2}{\sigma^2 + bh}, \quad y \triangleq \frac{bh}{\sigma^2 + bh}, \quad r(x, y) \triangleq x + \sqrt{x^2 + 2x + y^2}.$$

Then $y \leq 1$,

$$\begin{aligned}
 \frac{\sqrt{\sigma^2 + bh}}{\sqrt{\sigma^2 + bh} + b} \left[\frac{\sigma_h}{h} - \frac{\sigma}{h} \right] &= \frac{\sqrt{x}}{\sqrt{x} + y} \left[\frac{\sqrt{2}}{\sqrt{\log \frac{1+r(x,y)}{1-y} \log \frac{1+r(x,y)}{1+y}}} - \sqrt{\frac{1-y}{x}} \right] \\
 &\triangleq g_1(x, y),
 \end{aligned}$$

and

$$\frac{b_h}{h} - \frac{b}{h} = \frac{\log \frac{1+y}{1-y}}{\log \frac{1+r(x,y)}{1-y} \log \frac{1+r(x,y)}{1+y}} - \frac{y}{x} \triangleq g_2(x, y).$$

We want to show that g_1 and g_2 are bounded. Obviously both functions are continuous in $(0, \infty) \times (0, 1)$. So it suffices to estimate them on the boundaries. First, it is easy to check that

$$(6.7) \quad \begin{cases} g_1(\infty, y) = g_2(\infty, y) = 0, & \forall y \in (0, 1), \\ g_1(x, 0) = \frac{\sqrt{2}}{\log(1+x+\sqrt{x^2+2x})} - \frac{1}{\sqrt{x}}, & g_2(x, 0) = 0, & \forall x \in (0, \infty), \\ g_1(x, 1) = 0, & g_2(x, 1) = \frac{1}{\log(1+x)} - \frac{1}{x}, & \forall x \in (0, \infty), \\ \lim_{x \rightarrow \infty, y \rightarrow 0} g_1(x, y) = \lim_{x \rightarrow \infty, y \rightarrow 0} g_2(x, y) = 0, \\ \lim_{x \rightarrow \infty, y \rightarrow 1} g_1(x, y) = \lim_{x \rightarrow \infty, y \rightarrow 1} g_2(x, y) = 0. \end{cases}$$

Next, for fixed $y \in (0, 1]$ and small $x > 0$, we have

$$r(x, y) = y + \frac{1+y}{y}x - \frac{1-y^2}{2y^3}x^2 + o(x^2),$$

where $o(x^2)$ denotes variables with higher order than x^2 as $x \rightarrow 0$. Then

$$\log \frac{1+r(x, y)}{1+y} = \log\left(1 + \frac{r-y}{1+y}\right) = \frac{x}{y} - \frac{x^2}{2y^3} + o(x^2).$$

Therefore, for $\forall y \in (0, 1)$, we have

$$(6.8) \quad \begin{cases} g_1(0, y) = \lim_{x \rightarrow 0} \frac{\sqrt{2x}}{y \sqrt{[\log \frac{1+y}{1-y} + o(1)][\frac{x}{y} + o(x)]}} - \frac{\sqrt{1-y}}{y} = \frac{\sqrt{2}}{\sqrt{y \log \frac{1+y}{1-y}}} - \frac{\sqrt{1-y}}{y}, \\ g_2(0, y) = \lim_{x \rightarrow 0} \left[\frac{\log \frac{1+y}{1-y}}{[\log \frac{1+y}{1-y} + \frac{x}{y} + o(x)][\frac{x}{y} - \frac{x^2}{2y^3} + o(x^2)]} - \frac{y}{x} \right] = \frac{1}{2y} - \frac{1}{\log \frac{1+y}{1-y}}. \end{cases}$$

Moreover, one can easily prove that

$$(6.9) \quad \lim_{x \rightarrow 0, y \rightarrow 1} g_1(x, y) = 0, \quad \lim_{x \rightarrow 0, y \rightarrow 1} g_2(x, y) = \frac{1}{2}.$$

Finally, we assume $(x, y) \rightarrow (0, 0)$. Let

$$\tilde{x} \triangleq \frac{2y}{1-y}, \quad \tilde{y} \triangleq \frac{r-y}{1+y}, \quad \tilde{z} \triangleq \tilde{x} + \tilde{y} + \tilde{x}\tilde{y}.$$

Then $\tilde{x}, \tilde{y}, \tilde{z} > 0$, $(\tilde{x}, \tilde{y}, \tilde{z}) \rightarrow (0, 0, 0)$, and

$$x = \frac{\tilde{y}\tilde{z}}{(2+\tilde{x})(1+\tilde{y})}, \quad y = \frac{\tilde{x}}{2+\tilde{x}}.$$

We shall use Taylor expansions to estimate g_1 and g_2 . Note that

$$\frac{1+y}{1-y} = 1 + \tilde{x}, \quad \frac{1+r}{1+y} = 1 + \tilde{y}, \quad \frac{1+r}{1-y} = 1 + \tilde{z}.$$

One can calculate straightforwardly

$$\begin{aligned} g_1(x, y) &= \frac{1}{\sqrt{x+y}} \left[\frac{\sqrt{2x}}{\sqrt{\log \frac{1+r}{1-y} \log \frac{1+r}{1+y}}} - \sqrt{1-y} \right] \\ &= \frac{1}{\sqrt{x+y}} \left[\frac{\sqrt{2\tilde{y}\tilde{z}}}{\sqrt{(2+\tilde{x})(1+\tilde{y}) \log(1+\tilde{z}) \log(1+\tilde{y})}} - \frac{\sqrt{2}}{\sqrt{2+\tilde{x}}} \right] \\ &= \frac{\sqrt{2}}{\sqrt{2+\tilde{x}}} \frac{1}{\sqrt{x+y}} \left[\frac{\sqrt{\tilde{y}\tilde{z}}}{\sqrt{(1+\tilde{y}) \log(1+\tilde{z}) \log(1+\tilde{y})}} - 1 \right] \\ &= \frac{1+o(1)}{\sqrt{x+y}} \left[\frac{\tilde{y}\tilde{z}}{(1+\tilde{y}) \log(1+\tilde{z}) \log(1+\tilde{y})} - 1 \right] \\ &\quad \div \left[\frac{\sqrt{\tilde{y}\tilde{z}}}{\sqrt{(1+\tilde{y}) \log(1+\tilde{z}) \log(1+\tilde{y})}} + 1 \right] \\ &= \frac{1+o(1)}{2(\sqrt{x+y})} \frac{\tilde{y}\tilde{z} - (1+\tilde{y}) \log(1+\tilde{z}) \log(1+\tilde{y})}{(1+\tilde{y}) \log(1+\tilde{z}) \log(1+\tilde{y})} \\ &= \frac{1+o(1)}{2(\sqrt{x+y})} \frac{\tilde{y}\tilde{z} - (1+\tilde{y})\tilde{z}[1 - \frac{1}{2}\tilde{z} + o(\tilde{z})]\tilde{y}[1 - \frac{1}{2}\tilde{y} + o(\tilde{y})]}{\tilde{y}\tilde{z}[1+o(1)]} \\ &= \frac{1+o(1)}{2} \frac{\frac{1}{2}(\tilde{z}-\tilde{y}) + o(\tilde{y}+\tilde{z})}{\sqrt{\frac{\tilde{y}\tilde{z}}{2} + \frac{\tilde{x}}{2}}} = \frac{1+o(1)}{2} \frac{\tilde{x} + o(\tilde{x}+\tilde{y})}{\sqrt{2\tilde{y}\tilde{z} + \tilde{x}}}. \end{aligned}$$

Since $\frac{\tilde{x}+\tilde{y}}{\sqrt{2\tilde{y}\tilde{z}+\tilde{x}}} \leq 1$, we get, for x, y small enough,

$$(6.10) \quad |g_1(x, y)| \leq \frac{1+o(1)}{2} \leq 1.$$

Similarly, we have

$$\begin{aligned} (6.11) \quad g_2(x, y) &= \frac{\log(1+\tilde{x})}{\log(1+\tilde{z}) \log(1+\tilde{y})} - \frac{\tilde{x}(1+\tilde{y})}{\tilde{y}\tilde{z}} \\ &= \frac{\tilde{y}\tilde{z} \log(1+\tilde{x}) - \tilde{x}(1+\tilde{y}) \log(1+\tilde{y}) [\log(1+\tilde{x}) + \log(1+\tilde{y})]}{\tilde{y}\tilde{z} \log(1+\tilde{z}) \log(1+\tilde{y})} \\ &= \frac{[\tilde{y}\tilde{z} - \tilde{x}(1+\tilde{y}) \log(1+\tilde{y})] \log(1+\tilde{x}) - \tilde{x}(1+\tilde{y}) \log^2(1+\tilde{y})}{\tilde{y}\tilde{z} \log(1+\tilde{z}) \log(1+\tilde{y})} \\ &= \frac{[\tilde{y}\tilde{z} - \tilde{x}\tilde{y}(1 + \frac{1}{2}\tilde{y} + o(\tilde{y}))]\tilde{x}(1 - \frac{1}{2}\tilde{x} + o(\tilde{x})) - \tilde{x}(1+\tilde{y})\tilde{y}^2(1 - \tilde{y} + o(\tilde{y}))}{\tilde{y}^2\tilde{z}^2(1+o(1))} \\ &= \frac{\tilde{x}\tilde{y}^2(1+o(\tilde{x})) - \tilde{x}\tilde{y}^2(1+o(\tilde{y}))}{\tilde{y}^2\tilde{z}^2(1+o(1))} = \frac{\tilde{x}[o(\tilde{x}) + o(\tilde{y})]}{\tilde{z}^2(1+o(1))} = o(1). \end{aligned}$$

By (6.7)–(6.11), we prove that g_1 and g_2 are bounded. That completes the proof. □

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