ON IRREGULAR PRIME POWER DIVISORS OF THE BERNOULLI NUMBERS

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ABSTRACT. Let \( B_n \) (\( n = 0, 1, 2, \ldots \)) denote the usual \( n \)th Bernoulli number. Let \( l \) be a positive even integer where \( l = 12 \) or \( l \geq 16 \). It is well known that the numerator of the reduced quotient \( |B_l/l| \) is a product of powers of irregular primes. Let \((p, l)\) be an irregular pair with \( B_l/l \not\equiv B_{l+p-1}/(l+p-1) \mod p^2 \). We show that for every \( r \geq 1 \) the congruence \( B_{m_r}/m_r \equiv 0 \mod p^r \) has a unique solution \( m_r \) where \( m_r \equiv l \mod p-1 \) and \( l \leq m_r < (p-1)p^{r-1} \). The sequence \((m_r)_{r \geq 1} \) defines a \( p \)-adic integer \( \chi(p, l) \) which is a zero of a certain \( p \)-adic zeta function \( \zeta_{p, l} \) originally defined by T. Kubota and H. W. Leopoldt. We show some properties of these functions and give some applications. Subsequently we give several computations of the (truncated) \( p \)-adic expansion of \( \chi(p, l) \) for irregular pairs \((p, l)\) with \( p \) below 1000.

1. Introduction

The classical Bernoulli numbers \( B_n \) are defined by
\[
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi.
\]
These numbers are rational where \( B_n = 0 \) for odd \( n > 1 \) and \((-1)^{\frac{n+1}{2}}B_{n} > 0 \) for even \( n > 0 \). The Bernoulli numbers are connected with the Riemann zeta function \( \zeta(s) \) on the positive real axis by Euler’s formula
\[
(1.1) \quad \zeta(n) = -\frac{1}{2} \frac{(2\pi)^n}{n!} B_n, \quad n \in \mathbb{N}, \quad 2 \nmid n;
\]
the functional equation of \( \zeta(s) \) leads to
\[
(1.2) \quad \zeta(1-n) = -\frac{B_n}{n} \quad \text{for } n \in \mathbb{N}, \quad n \geq 2.
\]

Let, as usual, \( \varphi \) denote Euler’s totient function. The Kummer congruences state for \( n, m, p, r \in \mathbb{N}, \) \( m, n \) even, \( p \) prime with \( p-1 \nmid n \) that
\[
(1.3) \quad (1-p^{n-1})B_n/n \equiv (1-p^{m-1})B_m/m \pmod{p^r}
\]
when \( n \equiv m \pmod{\varphi(p^r)} \); see [9, Thm. 5, p. 239].

In 1850 E. E. Kummer [14] introduced two classes of odd primes, later called regular and irregular. An odd prime \( p \) is called regular if \( p \) does not divide the
class number of the cyclotomic field \( \mathbb{Q}(\mu_p) \) where \( \mu_p \) is the set of \( p \)th roots of unity; otherwise irregular. Kummer proved that Fermat’s last theorem (FLT) is true if the exponent is a regular prime. Kummer also gave an equivalent definition of irregularity concerning Bernoulli numbers. We recall the usual definition from [9, pp. 233–234].

**Definition 1.1.** Let \( p \) be an odd prime. The pair \((p, l)\) is called an irregular pair if \( p \) divides the numerator of \( B_l \) where \( l \) is even and \( 2 \leq l \leq p - 3 \). The index of irregularity of \( p \) is defined to be

\[
i(p) := \# \{(p, l) \text{ is an irregular pair} : l = 2, 4, \ldots, p - 3\}.
\]

The prime \( p \) is called a regular prime if \( i(p) = 0 \); otherwise, an irregular prime.

We introduce the following notation for rational numbers. If \( q \) is rational, then we use the representation \( q = N/D \) where \((N,D) = 1\) and \( D > 0 \). We define \( \text{denom}(q) = D \), resp. \( \text{numerator}(q) = N \), for the denominator, resp. the numerator of \( q \). The notation \( m \mid q \) where \( m \) is a positive integer means that \( m \mid \text{numerator}(q) \); we shall also write \( q \equiv 0 \pmod{m} \) in this case.

For now, let \( n \) be an even positive integer. An elementary property of the Bernoulli numbers, independently discovered by T. Clausen [6] and K. G. C. von Staudt [20] in 1840, is the following. The structure of the denominator of \( B_n \) is given by

\[
B_n + \sum_{p-1 \mid n} \frac{1}{p} \in \mathbb{Z} \quad \text{which implies} \quad \text{denom}(B_n) = \prod_{p-1 \mid n} p.
\]

A further result, often associated with the name of J. C. Adams (see [9, Prop. 15.2.4, p. 238] and Section 8 below) is that \( B_n/n \) is a \( p \)-integer for all primes \( p \) with \( p - 1 \nmid n \). Therefore

\[
\prod_{p-1 \mid n} p^\text{ord}_p n
\]

divides \( \text{numerator}(B_n) \); since this factor is cancelled in the numerator of \( B_n/n \), we shall call it the trivial factor of \( B_n \). By \( |B_n| > 2(n/2\pi e)^n \) (see [9, Eq. (8), p. 232], and Table A.1), one can easily show that the numerator of \( |B_n/n| \) equals 1 for \( 2 \leq n \leq 10 \) and \( n = 14 \). Otherwise this numerator is a product of powers of irregular primes; this is a consequence of the Kummer congruences. The determination of irregular primes, resp. irregular pairs, is still a difficult task; see [2]. One can easily show that infinitely many irregular primes exist; for a short proof see Carlitz [4]. Unfortunately, the more difficult question of whether infinitely many regular primes exist is still open. However, calculations in [2] show that about 60% of all primes less than 12 million are regular, which agree with an expected distribution proposed by Siegel [18].

The values of \( B_n \) and \( B_n/n \) for \( n \leq 20 \) are given in Table A.1 irregular pairs with \( p < 1000 \) in Table A.3. For brevity we write \( \hat{B}(n) = B_n/n \); these are called the divided Bernoulli numbers. Throughout this paper all indices concerning Bernoulli numbers are even and \( p \) denotes an odd prime.

### 2. Preliminaries

The definition of irregular pairs can be extended to irregular prime powers as was first proposed by the author in [11, Section 2.5].
Remark

Therefore one has

\[ \text{irregular pair of order } n \text{ if } p^n \mid \hat{B}(l) \text{ where } l \text{ is even and } 2 \leq l < \varphi(p^n). \]

Define

\[ \Psi_{n}^{\text{irr}} := \{(p, l) : p^n \mid \hat{B}(l), p \text{ is an odd prime, } 2 \leq l < \varphi(p^n), 2 \mid l\} \]

as the set of irregular pairs of order \( n \). For a prime \( p \) the index of irregular pairs of order \( n \) is defined by

\[ i_n(p) := \#\{(p, l) : (p, l) \in \Psi_{n}^{\text{irr}}\}. \]

Define the map

\[ \lambda_n : \Psi_{n+1}^{\text{irr}} \to \Psi_{n}^{\text{irr}}, \quad (p, l) \mapsto (p, l \mod \varphi(p^n)), \]

where \( x \mod y \) gives the least nonnegative residue \( x' \) with \( 0 \leq x' < y \) for positive integers \( x \) and \( y \). Let \((p, l_n) \in \Psi_{n}^{\text{irr}} \) and \((p, l_m) \in \Psi_{m}^{\text{irr}} \) be irregular pairs of order \( n \), resp. \( m \), where \( n > m \geq 1 \). We say that \((p, l_n)\) is related to \((p, l_m)\) if \( l_n \equiv l_m \pmod{\varphi(p^n)} \) holds. Note that “related” is not a symmetric relation.

Remark 2.2. This definition includes the older Definition 1.1 in the case \( n = 1 \). Therefore one has \( i_1(p) = i(p) \). Let \((p, l) \in \Psi_{n}^{\text{irr}} \) with \( n \geq 1 \). The Kummer congruences (1.4) imply that \( p^n \mid \hat{B}(l + \nu \varphi(p^n)) \) for all \( \nu \in \mathbb{N}_0 \). Conversely, if \( p^n \mid \hat{B}(m) \) for some even integer \( m \), then there exists an irregular pair \((p, l) \in \Psi_{n}^{\text{irr}} \) where \( l \equiv m \pmod{\varphi(p^n)} \) with \( l \leq m \) holds.

The map \( \lambda_n \) is well defined by the properties mentioned above. Let \((p, l_n) \in \Psi_{n}^{\text{irr}} \) and \((p, l_m) \in \Psi_{m}^{\text{irr}} \) where \( n > m \geq 1 \) and \((p, l_n)\) is related to \((p, l_m)\). By applying the maps \( \lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_m \) one derives a chain of related irregular pairs of descending order

\[ (2.1) \quad (p, l_n) \in \Psi_{n}^{\text{irr}}, \quad (p, l_{n-1}) \in \Psi_{n-1}^{\text{irr}}, \quad (p, l_{n-2}) \in \Psi_{n-2}^{\text{irr}}, \ldots, \quad (p, l_m) \in \Psi_{m}^{\text{irr}}, \]

where

\[ l_n \geq l_{n-1} \geq l_{n-2} \geq \cdots \geq l_m. \]

Definition 2.3. For \((p, l) \in \Psi_{n}^{\text{irr}}, n \geq 1 \), define

\[ \Delta(p, l) \equiv p^{-n} \left( \hat{B}(l + \varphi(p^n)) - \hat{B}(l) \right) \pmod{p} \]

with \( 0 \leq \Delta(p, l) < p \). When \( \Delta(p, l) = 0 \) we call \( \Delta(p, l) \) singular. For an irregular prime \( p \) set

\[ \Delta(p) := \left\{ \begin{array}{ll} 1, & \text{if } \Delta \neq 0, \\ 0, & \text{if } \Delta = 0, \end{array} \right. \]

with

\[ \Delta_p = \prod_{\nu=1}^{i(p)} \Delta(p, l_{\nu}), \quad (p, l_{\nu}) \in \Psi_{1}^{\text{irr}}. \]

Then \( \Delta(p) = 1 \) if and only if all \( \Delta(p, l_{\nu}) \) are nonsingular.

We need a generalized form of the Kummer congruences which allows us to obtain most of the later results; see Carlitz [4, Thm. 3, p. 425] and especially Fresnel [7, Cor. 6, p. 319].
Theorem 2.4 (Carlitz). Let \( k, m, n, p, r, \omega \in \mathbb{N} \), \( m \) even, \( p \) prime with \( p - 1 \nmid m \), and \( \omega = k\varphi(p^{\nu}) \). Then

\[
\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^{\nu}(1 - p^{m+\nu\omega-1}) \tilde{B}(m + \nu\omega) \equiv 0 \pmod{p^{nr}}.
\]

Here we need a special version without Euler factors which \( p \)-adically shows the periodic behavior of the divided Bernoulli numbers.

Corollary 2.5. Let \((p, l) \in \Psi^{\text{irr}}_n\), \( n \geq 1 \). Let \( k, m, n, r \in \mathbb{N} \), \( r > 1 \), and \( \omega = k\varphi(p^{\nu}) \). For \( m = l + j\varphi(p^{\nu}) \) with \( j \geq 0 \), we have

\[
\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^{\nu} p^{-\nu} \tilde{B}(n + \nu\omega) \equiv 0 \pmod{(p^{m-1}, p^{\varphi(n(r-1)})�).
\]

Proof. Since \((p, l) \in \Psi^{\text{irr}}_n\), we know that \( p^n \mid \tilde{B}(l + j\varphi(p^{\nu})) \) for all \( j \geq 0 \). Hence, we can reduce congruence (2.2) to (mod \( p^{\varphi(n(r-1))} \) when multiplying it by \( p^{-\nu} \). One easily sees that all Euler factors in the sum of (2.2) vanish (mod \( p^{m-1} \)).

Proposition 2.7 below shows how to find irregular pairs of higher order. Beginning from an irregular pair of order \( n \), we can characterize related irregular pairs of order \( n + 1 \) if they exist. First we need a lemma.

Lemma 2.6. Let \( n \) be a positive integer and \( p \) be an odd prime. Let \((\alpha_{\nu})_{\nu \geq 0} \) be a sequence of \( p \)-integers \( \alpha_{\nu} \in \mathbb{Q} \) for all \( \nu \in \mathbb{N}_0 \). If one has

\[
\alpha_{\nu} - 2\alpha_{\nu+1} + \alpha_{\nu+2} \equiv 0 \pmod{p^n},
\]

then the sequence is equidistant (mod \( p^n \)). For \( \alpha_0 \not\equiv \alpha_1 \pmod{p} \) the elements \( \alpha_0 \) up to \( \alpha_{p-1} \) run through all residues (mod \( p^n \)). Then exactly one element \( \alpha_s \equiv 0 \pmod{p^n} \) exists with \( 0 \leq s < p^n \) where \( s \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^n} \).

Proof. We rewrite congruence (2.3) to

\[
\alpha_{\nu} - \alpha_{\nu+1} \equiv \alpha_{\nu+1} - \alpha_{\nu+2} \pmod{p^n}, \quad \nu \in \mathbb{N}_0.
\]

One easily observes by induction on \( \nu \) that all elements \( \alpha_{\nu} \) are equidistant (mod \( p^n \)). Let \( \delta \equiv \alpha_1 - \alpha_0 \pmod{p^n} \), then we obtain

\[
\alpha_{\nu} \equiv \alpha_0 + \delta \nu \pmod{p^n}.
\]

The case \( \alpha_0 \not\equiv \alpha_1 \pmod{p} \) provides that \( \alpha_0 + \delta \nu \), resp. \( \alpha_{\nu} \), runs through all residues (mod \( p^n \)) for \( 0 \leq \nu < p^n \), since \( \delta \) is invertible (mod \( p^n \)). Then exactly one element \( \alpha_s \) exists with \( 0 \equiv \alpha_s \equiv \alpha_0 + \delta s \pmod{p^n} \) and \( 0 \leq s < p^n \). This finally gives \( s \equiv -\alpha_0 \delta^{-1} \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^n} \).

Proposition 2.7. Let \((p, l) \in \Psi^{\text{irr}}_n\), \( n \geq 1 \). Let the sequence \((\alpha_{\nu})_{\nu \geq 0} \) satisfy

\[
\alpha_{\nu} \equiv p^{-\nu} \tilde{B}(l + \nu\varphi(p^{\nu})) \pmod{p}.
\]

For \( \Delta_{(p, l)} \equiv \alpha_1 - \alpha_0 \pmod{p} \) where \( 0 \leq \Delta_{(p, l)} < p \) there exist three cases:

1. If \( \Delta_{(p, l)} = 0 \) and \( \alpha_0 \not\equiv 0 \pmod{p} \), then there are no related irregular pairs of order \( n + 1 \) and higher.
2. If \( \Delta_{(p, l)} = 0 \) and \( \alpha_0 \equiv 0 \pmod{p} \), then all \((p, l + \nu\varphi(p^{\nu})) \in \Psi^{\text{irr}}_{n+1} \) are related irregular pairs of order \( n + 1 \) for \( \nu = 0, \ldots, p - 1 \).
3. If \( \Delta_{(p, l)} \not\equiv 0 \), then exactly one related irregular pair of order \( n + 1 \) exists.

One has \((p, l + s\varphi(p^{\nu})) \in \Psi^{\text{irr}}_{n+1} \) with \( 0 \leq s < p \) where \( s \equiv -\alpha_0 \Delta_{(p, l)}^{-1} \pmod{p} \).
Proof. Let \( j \geq 0 \). Using Corollary \( 2.5 \) with \( r = 2, \omega = \varphi(p^n) \), and \( m = l + j\varphi(p^n) \geq 2 \), we get
\[
\sum_{\nu=0}^{2} \binom{2}{\nu} (-1)^{\nu} p^{-\nu} \hat{B}(m + \nu \omega) \equiv 0 \pmod{p},
\]
which is
\[
\alpha_j - 2\alpha_{j+1} + \alpha_{j+2} \equiv 0 \pmod{p}.
\]
This satisfies the conditions of Lemma \( 2.6 \). We obtain three cases:

**Case (1).** We have \( \alpha_0 \equiv \alpha_1 \pmod{p} \) and \( \alpha_0 \neq 0 \pmod{p} \). One observes that \( \alpha_j \equiv 0 \pmod{p} \), resp. \( p^{n+1} \mid \hat{B}(l + j\varphi(p^n)) \), for all \( j \geq 0 \). Therefore, there cannot exist related irregular pairs of order \( n+1 \). Also there cannot exist related irregular pairs of order \( r > n+1 \); otherwise, we would get a contradiction to \( (2.1) \).

**Case (2).** We have \( \alpha_0 \equiv \alpha_1 \pmod{p} \) and \( \alpha_0 \equiv 0 \pmod{p} \). This yields \( \alpha_j \equiv 0 \pmod{p} \), resp. \( p^{n+1} \mid \hat{B}(l + j\varphi(p^n)) \), for all \( j \geq 0 \). Hence, \( p \) related irregular pairs of order \( n+1 \) exist where \( (p, l + \nu \varphi(p^n)) \in \Psi_{n+1}^{\text{irr}} \) for \( \nu = 0, \ldots, p-1 \).

**Case (3).** We have \( \alpha_0 \neq \alpha_1 \pmod{p} \). Lemma \( 2.6 \) provides exactly one element \( \alpha_s \equiv 0 \pmod{p} \) with the desired properties. Hence, \( (p, l + s\varphi(p^n)) \) is the only related irregular pair of order \( n+1 \).

**Remark 2.8.** Vandiver \( 19 \) describes the result of the previous proposition for the case \( n = 1 \) and only for the first irregular primes 37, 59, and 67. For these primes Pollaczek \( 16 \) has calculated the indices \( s \) of the now-called irregular pair of order two, but case \( p = 67 \) with \( s = 2 \) is incorrect; see column \( s_2 \) of Table \( 4.3 \). This error was already noticed by Johnson \( 10 \) who had also determined all irregular pairs \( (p, l') \) of order two with \( p \) below 8000. Wagstaff \( 21 \) has extended calculations of irregular pairs, indices \( s \), and associated cyclotomic invariants up to \( p < 125 000 \). He also checked that FLT is true for all such exponents \( p \) in that range. Finally, Buhler, Crandall, Eruvall, Metsänkylä, and Shokrollahi \( 2 \) have extended calculations of irregular pairs and associated cyclotomic invariants up to \( p < 12 000 000 \). For all these irregular pairs \( (p, l) \) in that range, \( \Delta_{(p, l)} \neq 0 \) is always valid which ensures that each time there is only one related irregular pair \( (p, l') \) of order two. Hence \( i_2(p) = i(p) \) for these irregular primes \( p \). One has to notice that always \( (p, l) \neq (p, l') \). So far, no irregular pair \( (p, l) \) has been found with \( p^2 \mid \hat{B}(l) \).

Using Proposition \( 2.7 \) one can successively find irregular pairs of higher order. We can easily extend the result starting from an irregular pair \( (p, l) \in \Psi_n^{\text{irr}} \) and require that \( l > n \) to obtain a related irregular pair \( (p, l') \in \Psi_{2n}^{\text{irr}} \).

**Proposition 2.9.** Let \( (p, l) \in \Psi_n^{\text{irr}}, n \geq 1 \). Suppose that \( l > n \). Let the sequence \( (\alpha_j)_{j \geq 0} \) satisfy
\[
\alpha_j \equiv p^{-n} \hat{B}(l + j\varphi(p^n)) \pmod{p^n}.
\]
If \( \Delta_{(p, l)} \neq 0 \), then there is exactly one related irregular pair of order \( 2n \)
\[
(p, l + s\varphi(p^n)) \in \Psi_{2n}^{\text{irr}}
\]
with \( 0 \leq s < p^n \) where \( s \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^n} \). Correspondingly, there also exists exactly one related irregular pair of order \( n + \nu \)
\[
(p, l + s\varphi(p^n)) \in \Psi_{n+\nu}^{\text{irr}}
\]
for each \( \nu = 1, \ldots, n-1 \) with \( 0 \leq s_\nu < p^\nu \) where \( s_\nu \equiv s \pmod{p^\nu} \).
Proof. Let \( j \geq 0 \). Using Corollary 2.5 again with \( r = 2, \omega = \varphi(p^n) \), and \( m = l + j\varphi(p^n) > n \) yields
\[
\alpha_j - 2\alpha_{j+1} + \alpha_{j+2} \equiv 0 \pmod{p^n}.
\]
If \( \Delta_{(p, l)} \neq 0 \), then Lemma 2.6 provides exactly one element \( \alpha_s \equiv 0 \pmod{p^n} \) with \( 0 \leq s < p^n \) where \( s \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^n} \). Therefore, \((p, l + s\varphi(p^n))\) is the only related irregular pair of order \( 2n \). Similarly, regarding the congruences above (mod \( p^n \)) instead of (mod \( p^n \)) for \( \nu = 1, \ldots, n-1 \) yields the proposed properties. \( \square \)

Finally, we start from an irregular pair \((p, l) \in \Psi_{nir}^n \) where we have to suppose that \( l > (r-1)n \) with \( r \geq 2 \) to obtain a related irregular pair \((p, l') \in \Psi_{rin}^n \).

**Proposition 2.10.** Let \((p, l) \in \Psi_{nir}^n, n \geq 1 \). Let \( r > 1 \) be an integer and suppose that \( l > (r-1)n \). Let the sequence \((\alpha_j)_{j \geq 0} \) satisfy
\[
\alpha_j \equiv p^{-\nu}B(l + j\varphi(p^n)) \pmod{p^{(r-1)n}}.
\]
Then this sequence satisfies
\[
\sum_{\nu=0}^{r} \binom{r}{\nu}(-1)^\nu \alpha_{\nu+j} \equiv 0 \pmod{p^{(r-1)n}}
\]
for all \( j \geq 0 \). The elements \( \alpha_0 \) up to \( \alpha_{r-1} \) induce the entire sequence \((\alpha_j)_{j \geq 0} \). Elements with \( \alpha_s \equiv 0 \pmod{p^{(r-1)n}} \) where \( 0 \leq s < p^{(r-1)n} \) provide related irregular pairs of order \( rn \) with \((p, l + s\varphi(p^n)) \in \Psi_{rin}^n \). If \( \Delta_{(p, l)} \neq 0 \) and the elements \( \alpha_0 \) up to \( \alpha_{r-1} \) are equidistant (mod \( p^{(r-1)n} \)), then there is exactly one related irregular pair of order \( rn \)
\[
(p, l + s\varphi(p^n)) \in \Psi_{rin}^n
\]
with \( 0 \leq s < p^{(r-1)n} \) where \( s \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^{(r-1)n}} \). Correspondingly, there exists exactly one related irregular pair of order \( n + k \)
\[
(p, l + s_k\varphi(p^n)) \in \Psi_{rin}^{n+k}
\]
for each \( k = 1, \ldots, (r-1)n-1 \) with \( 0 \leq s_k < p^k \) where \( s_k \equiv s \pmod{p^k} \).

**Proof.** Let \( j \geq 0 \). Clearly, by Corollary 2.5 with \( r > 1, \omega = \varphi(p^n), \) and \( m = l + j\varphi(p^n) > (r-1)n \), we have
\[
\sum_{\nu=0}^{r} \binom{r}{\nu}(-1)^\nu \alpha_{\nu+j} \equiv 0 \pmod{p^{(r-1)n}}.
\]
This induces the whole sequence \((\alpha_j)_{j \geq 0} \) by
\[
(2.4) \quad \alpha_{r+j} \equiv (-1)^{r+1} \sum_{\nu=0}^{r-1} \binom{r}{\nu}(-1)^\nu \alpha_{\nu+j} \pmod{p^{(r-1)n}}.
\]
Among all elements \( \alpha_s \) with \( 0 \leq s < p^{(r-1)n} \), an element \( \alpha_s \equiv 0 \pmod{p^{(r-1)n}} \) provides a related irregular pair \((p, l + s\varphi(p^n)) \in \Psi_{rin}^n \) of order \( rn \).

Now we assume that \( \Delta_{(p, l)} \neq 0 \) and the first elements \( \alpha_0 \) up to \( \alpha_{r-1} \) are equidistant (mod \( p^{(r-1)n} \)). We show that this property transfers to the entire sequence. Let \( \gamma, \delta \) be integers. It is easily seen for \( r > 1 \) that
\[
(2.5) \quad \sum_{\nu=0}^{r} \binom{r}{\nu}(-1)^\nu(\gamma + \delta\nu) = 0.
\]
Consider \( \gamma \equiv \alpha_0 \pmod{p^{(r-1)n}} \) and \( \delta \equiv \alpha_1 - \alpha_0 \pmod{p^{(r-1)n}} \) where \( p \nmid \delta \) by assumption. Combining (2.4) and (2.5) yields

\[
(2.6) \quad (-1)^{r+1} \sum_{\nu=0}^{r-1} \binom{r}{\nu} (-1)^\nu (\alpha_0 + \delta \nu) \equiv \alpha_0 + \delta r \equiv \alpha_r \pmod{p^{(r-1)n}},
\]

which shows inductively that all successive elements \( \alpha_j \) with \( j \geq r \) are equidistant (mod \( p^{(r-1)n} \)). Since \( \delta \) is invertible (mod \( p^{(r-1)n} \)), exactly one solution exists with

\[
0 \equiv \alpha_s \equiv \alpha_0 + \delta s \pmod{p^{(r-1)n}} \quad \text{where} \quad s \equiv -\alpha_0 (\alpha_1 - \alpha_0)^{-1} \pmod{p^{(r-1)n}}.
\]

Using similar arguments, Congruences (2.4) and (2.6) are also valid (mod \( p^k \)) for \( k = 1, \ldots, (r-1)n - 1 \) which provides for each \( k \) a unique solution \( s_k \) with \( 0 \leq s_k < p^k \)

where \( s_k \equiv s \pmod{p^k} \) holds. \( \square \)

In [11, pp. 125–130] several examples and calculations are given which use the previous propositions. These results are reprinted in Table A.4 and below. It will turn out later that calculations can be further simplified. Because of the rapid growth of indices, it is useful to write indices of irregular pairs of higher order \( p \)-adically.

**Definition 2.11.** Let \( (p, l) \in \Psi_{n}^{\text{irr}}, n \geq 1 \). We write

\[
(p, s_1, s_2, \ldots, s_n) \in \widehat{\Psi}_{n}^{\text{irr}} \quad \text{where} \quad l = \sum_{\nu=1}^{n} s_\nu \varphi(p^{\nu-1})
\]

for the \( p \)-adic notation of \( (p, l) \) with \( 0 \leq s_\nu < p \) for \( \nu = 1, \ldots, n \) and \( 2 \leq s_1 \leq p - 3, 2 \mid s_1 \). The corresponding set is denoted as \( \widehat{\Psi}_{n}^{\text{irr}} \), the map corresponding to \( \lambda_n \) is given by

\[
\widehat{\lambda}_{n} : \widehat{\Psi}_{n+1}^{\text{irr}} \rightarrow \widehat{\Psi}_{n}^{\text{irr}}, \quad (p, s_1, s_2, \ldots, s_n, s_{n+1}) \mapsto (p, s_1, s_2, \ldots, s_n).
\]

The pair \( (p, l) \) and the element \( (p, s_1, s_2, \ldots, s_n) \) are called associated.

**Remark 2.12.** The definition of \( \widehat{\Psi}_{n}^{\text{irr}} \) means that we have \( \Psi_{1}^{\text{irr}} = \widehat{\Psi}_{1}^{\text{irr}} \) for \( n = 1 \). For \( n \geq 2 \) we can define a map \( \Psi_{n}^{\text{irr}} \rightarrow \widehat{\Psi}_{n}^{\text{irr}}, \ (p, l) \mapsto (p, s_1, \ldots, s_n) \) where the \( s_k \) are uniquely determined by the \( p \)-adic representation

\[
l = s_1 + (p-1)\delta, \quad \delta = \sum_{\nu=0}^{n-2} s_{\nu+2} p^\nu, \quad 0 \leq s_{\nu+2} < p
\]

and by \( s_1 \equiv l \pmod{p-1} \) with \( 2 \leq s_1 \leq p - 3 \). If \( s_k = 0 \) with \( k \geq 2 \), then there is an irregular pair \( (p, l_k) \) of order \( k \) with

\[
(p, l_k) \in \Psi_{k}^{\text{irr}} \quad \text{and} \quad (p, l_k) \in \Psi_{k-1}^{\text{irr}}.
\]

Note that \( (p, s_1, s_2, \ldots, s_n) \) is also called an irregular pair with \( (s_1, s_2, \ldots, s_n) \) as the second parameter given \( p \)-adically.

### 3. Main results

**Theorem 3.1.** Let \( (p, l_1) \in \Psi_{1}^{\text{irr}} \). If \( \Delta_{(p, l_1)} \neq 0 \), then for each \( n > 1 \) there exists exactly one related irregular pair of order \( n \). There is a unique sequence \( (l_n)_{n \geq 1} \), resp. \( (s_n)_{n \geq 1} \), with

\[
(p, l_n) \in \Psi_{n}^{\text{irr}}, \quad \text{resp.} \ (p, s_1, \ldots, s_n) \in \widehat{\Psi}_{n}^{\text{irr}},
\]
and
\[ l_1 \leq l_2 \leq l_3 \leq \cdots, \quad \lim_{n \to \infty} l_n = \infty. \]

Moreover one has
\[ \Delta(p, l_1) = \Delta(p, l_2) = \Delta(p, l_3) = \cdots. \]

If \( \Delta(p) = 1 \), then
\[ i(p) = i_2(p) = i_3(p) = \cdots. \]

**Theorem 3.2.** Let \( (p, l_n) \in \Psi^\text{irr}_n, n \geq 1 \) with \( \Delta(p, l_n) = 0 \). Then there are two cases:

1. \( (p, l_n) \notin \Psi^\text{irr}_{n+1} \): There are no related irregular pairs of order \( n+1 \) and higher.
2. \( (p, l_n) \in \Psi^\text{irr}_{n+1} \): There exist \( p \) related irregular pairs of order \( n+1 \) where \( (p, l_{n+1}, j) \in \Psi^\text{irr}_{n+1} \) with \( \Delta(p, l_{n+1}, j) = 0 \) and \( l_{n+1} = l_n + j\varphi(p^n) \) for \( j = 0, \ldots, p-1 \).

The property of \( \Delta(p, l) \), whether \( \Delta(p, l) \) vanishes or not, is passed on to all related irregular pairs of higher order. The case of a singular \( \Delta(p, l) \) would possibly imply a strange behavior without any regularity. By calculation in [2] up to \( p < 12000000 \), no such \( \Delta(p, l) \) was found. The following diagram illustrates both cases.

**Diagram 3.3.**

\[
\begin{array}{c}
\Psi_1^\text{irr} \quad \Delta(p, l_1) = 0 \\
\Psi_2^\text{irr} \quad \Psi_3^\text{irr} \quad \Psi_4^\text{irr} \\
l_1 \quad l_2 \quad l_3 \\
\end{array}
\quad
\begin{array}{c}
\Psi_1^\text{irr} \quad \Delta(p, l_1) \neq 0 \\
\Psi_2^\text{irr} \quad \Psi_3^\text{irr} \quad \Psi_4^\text{irr} \\
l_1 \quad l_2, l_3 \quad l_4 \\
\end{array}
\]

Here a vertical line indicates that \( (p, l_n) \in \Psi^\text{irr}_n \cap \Psi^\text{irr}_{n+1} \) happens. On the left-hand side we then have \( p \) related irregular pairs of order \( n+1 \) which are represented by branches. In this case the corresponding Bernoulli number \( \hat{B}(l_n) \) decides whether there exist further branches or they stop. Instead of \( n \) the order of the \( p \)-power must be at least \( n+1 \). This also means that an associated irregular pair \( (p, s_1, \ldots, s_{n+1}) \in \Psi^\text{irr}_{n+1} \) must have a zero \( s_{n+1} = 0 \) each time. In contrast the right-hand side shows that in the case of \( \Delta(p, l_1) \neq 0 \) there is only one related irregular pair of each higher order. If \( \Delta(p) = 1 \), then higher powers \( p^\nu \) are equally distributed among the numerators of \( \hat{B}(n) \). For each irregular pair considered, there exists exactly one index \( n_{k, \nu} \) with \( n_{k, \nu} = n_{0, \nu} + k\varphi(p^n) \), \( k \in \mathbb{N}_0 \) in the disjoint intervals
\[ (k \varphi(p^n), (k + 1) \varphi(p^n)) \]
for which \( p^\nu \mid \hat{B}(n_{k, \nu}) \) is valid.

In [11] pp. 128–130 irregular pairs of order 10 were calculated for all irregular primes \( p < 1000 \). These results are reprinted in Table 3.3. In this table only one irregular pair has a zero in its \( p \)-adic notation:
\[ (157, 62, 40, 145, 67, 29, 69, 0, 87, 89, 21) \in \hat{\Psi}^\text{irr}_{10}. \]
Hence, one has with a relatively small index that

\[(157, 6557686520486) \in \Psi_6^{\text{irr}} \cap \Psi_7^{\text{irr}}.\]

It seems that these zeros can be viewed as exceptional; see also Table A.2. It would be of interest to investigate in which regions such indices may occur. This could explain why no irregular pair \((p, l) \in \Psi_1^{\text{irr}} \cap \Psi_2^{\text{irr}}\) has yet been found, because these regions are beyond present calculations. Here we have index 1200000 in \([2]\) against index 6557686520486. Because of the rare occurrence of zeros one can expect that \((p, l) \in \Psi_1^{\text{irr}} \cap \Psi_2^{\text{irr}}, \text{resp. } p^2 \mid B(l),\) will not happen often.

Another phenomenon is the occurrence of huge irregular prime factors. Wagstaff \([22]\) has completely factored the numerators of the Bernoulli numbers \(p, l \in \Psi_n^{\text{irr}} \cap \Psi_{n+1}^{\text{irr}}\) up to index 6557686520486. Because of the rare occurrence of zeros one can expect that \((p, l) \in \Psi_1^{\text{irr}} \cap \Psi_2^{\text{irr}}, \text{resp. } p^2 \mid B(l),\) will not happen often.

Finally, summarizing all facts together, the property \(\Delta(p) = 1\) can be viewed as a structural property of the Bernoulli numbers. This leads us to the following conjecture named by the author \(\Delta\text{-Conjecture.}\)

**Conjecture 3.4 (\(\Delta\text{-Conjecture).** For all irregular primes \(p\) the following properties, which are equivalent, hold:

1. \(\Delta(p, l)\) is nonsingular for all irregular pairs \((p, l) \in \Psi_1^{\text{irr}},\)
2. \(\Delta(p) = 1,\)
3. \(i(p) = i_2(p) = i_3(p) = \cdots.\)

Assuming the \(\Delta\text{-Conjecture} one can also prove the existence of infinitely many irregular primes using only information about the numerators of divided Bernoulli numbers; see \([11]\) Satz 2.8.2, p. 87]. Now we give the proofs of the theorems above.

**Proposition 3.5.** Let \((p, l_n) \in \Psi_n^{\text{irr}}, n \geq 1 \text{ with } \Delta(p, l_n) \neq 0. Then there is exactly one related irregular pair \((p, l_{n+1}) \in \Psi_{n+1}^{\text{irr}}\) with \(\Delta(p, l_{n+1}) = \Delta(p, l_n).\)

**Proof.** We write \(\Delta_n = \Delta(p, l_n).\) Note that \(l_n > 2\) and \(p > 3.\) Define the sequence \((\alpha_j)_{j \geq 0}\) by

\[
\alpha_j \equiv p^{-n}B(l_n + j \varphi(p^n)) \pmod{p^2}.
\]

Using Corollary 2.5 with \(r = 3, \omega = \varphi(p^n),\) and \(m = l_n + j \varphi(p^n) > 2,\) we have for \(n(r-1) \geq 2\) that

\[
\alpha_j - 3\alpha_{j+1} + 3\alpha_{j+2} - \alpha_{j+3} \equiv 0 \pmod{p^2}.
\]

Taking differences with \(\beta_j = \alpha_{j+1} - \alpha_j\) yields

\[
\beta_j - 2\beta_{j+1} + \beta_{j+2} \equiv 0 \pmod{p^2}.
\]

By Lemma 2.6 the sequence \((\beta_j)_{j \geq 0}\) is equidistant \(\pmod{p^2}.\) Proposition 2.7 shows that the sequence \((\alpha_j)_{j \geq 0}\) is equidistant \(\pmod{p}\). By definition we have \(\beta_j \equiv \Delta_n \pmod{p}.\) Therefore, we can choose suitable \(\gamma, \delta \in \mathbb{Z}\) so that

\[
\alpha_{j+1} - \alpha_j \equiv \beta_j \equiv \Delta_n + p(\gamma + j\delta) \pmod{p^2}.
\]

This yields

\[
\alpha_j \equiv \alpha_0 + \sum_{\nu=0}^{j-1} \left(\Delta_n + p(\gamma + \nu\delta)\right)
\]

\[(3.1) \quad \equiv \alpha_0 + j\Delta_n + jp\gamma + \left(\begin{array}{c} j \\ 2 \end{array}\right)p\delta \pmod{p^2}.
\]
Finally, we obtain the proposed equation

\[ \Delta \]

with exactly one related irregular pair of order \( n \). As a consequence of Lemma \[2.6\] we observe that

\[ \alpha_{s+jp} \equiv 0 \pmod{p}. \]

Thus we obtain a sequence \( (\alpha_j')_{j \geq 0} \) defined by

\[ \alpha_j' \equiv \alpha_{s+jp}/p \equiv p^{-(n+1)} \widehat{B}(l_{n+1} + j\varphi(p^{n+1})) \pmod{p} \]

which we can use to determine related irregular pairs of order \( n+2 \) using Proposition \[2.7\]. By definition we have

\[ \Delta_{n+1} \equiv \alpha_1' - \alpha_0' \pmod{p}. \]

It follows from \[3.1\] that

\[ p \Delta_{n+1} \equiv p(\alpha_1' - \alpha_0') \equiv \alpha_{s+p} - \alpha_s \]

\[ \equiv p \Delta_n + p^2 \gamma + p^2 \left( \frac{s+p}{2} - \frac{s}{2} \right) \]

\[ \equiv p \Delta_n \pmod{p^2}, \]

since

\[ \left( \frac{s+p}{2} - \frac{s}{2} \right) = \frac{1}{2} p(p + 2s - 1). \]

Finally, we obtain the proposed equation \( \Delta_{n+1} = \Delta_n \). \( \square \)

**Proof of Theorem 3.1** Using Proposition \[3.5\] with induction on \( n \) provides

\[ \Delta_{(p,l_1)} = \Delta_{(p,l_2)} = \Delta_{(p,l_3)} = \cdots \]

with exactly one related irregular pair of order \( n \)

\( (p, l_n) \in \Psi_{n}^{\text{irr}}, \) resp. \( (p, s_1, \ldots, s_n) \in \tilde{\Psi}_{n}^{\text{irr}}. \)

The latter pair is given by Definition \[2.11\]. Proposition \[2.7\] shows for each step \( n \) that

\[ l_{n+1} = l_n + s_n \varphi(p^n), \quad 0 \leq s_n < p, \]

with a suitable integer \( s_n \). This ensures that \( l_1 \leq l_2 \leq l_3 \leq \cdots \) as an increasing sequence \( (l_j)_{j \geq 1}. \) Clearly, this sequence is not eventually constant, because \( p^n \mid \widehat{B}(l_n) \) with \( 0 < |\widehat{B}(l_n)| < \infty. \) Therefore \( \lim_{n \to \infty} l_n = \infty. \)

Starting with \( (p, l_1) \in \Psi_{1}^{\text{irr}} \) this provides a unique sequence \( (l_j)_{j \geq 1}. \) If we have another irregular pair \( (p, l'_1) \in \Psi_{1}^{\text{irr}} \) with \( \Delta_{(p,l'_1)} \neq 0 \) and \( (p, l_1) \neq (p, l'_1), \) then

\[ l'_j \neq l_k \quad \text{for all} \ j, k \in \mathbb{N}, \]

because \( l'_j \not\equiv l_1 \pmod{\varphi(p)} \) and \( l'_1 \equiv l'_j \pmod{\varphi(p)}, \) resp. \( l_1 \equiv l_k \pmod{\varphi(p)}, \) by \[3.3\].

If \( \Delta(p) = 1, \) then for each of the \( i(p) \) irregular pairs \( (p, l_{1,\nu}), \nu = 1, \ldots, i(p) \) there exists exactly one related irregular pair of higher order. Finally, it follows that \( i(p) = i_2(p) = i_3(p) = \cdots \) and so on. \( \square \)
Theorem 4.1. Clearly, the (non)existence of related irregular pairs in case (1), resp. (2), is given by Proposition 2.7 case (1), resp. (2). Hence, we only have to show the remaining part of case (2). In this case we have $p$ related irregular pairs $(p, l_{n+1}, \nu) = (p, l_n + \nu \varphi(p^n)) \in \Psi_{n+1}^{\text{irr}}$ of order $n+1$ with $\nu = 0, \ldots, p-1$. Although $\Delta_{(p, l_n)} = 0$ we can use Proposition 3.5 by modifying essential steps. We then have

$$\alpha_j \equiv \beta_j \equiv 0 \pmod{p}.$$

Congruence (3.2) must be replaced by

(3.2) \hfill $s = 0, \ldots, p - 1$

since $\Delta_{(p, l_n)} = 0$ yields $p$ values of $s$. It follows that

$$\Delta_{(p, l_n)} = \Delta_{(p, l_{n+1}, \nu)} = 0 \quad \text{for} \quad \nu = 0, \ldots, p - 1. \quad \square$$

4. A $p$-adic view

Let $\mathbb{Z}_p$ be the ring of $p$-adic integers and $\mathbb{Q}_p$ be the field of $p$-adic numbers. The ultrametric absolute value $| \cdot |_p$ is defined by $|x|_p = p^{-\text{ord}_p x}$ on $\mathbb{Q}_p$. Define $| \cdot |_\infty$ as the standard norm on $\mathbb{Q}$. For $n \geq 1$ we define $\psi_n : \mathbb{Z}_p \to \mathbb{Z}$ giving the projection of $\mathbb{Z}_p$ onto the set $[0, p^n) \cap \mathbb{Z}$ so that for $x \in \mathbb{Z}_p$ we have $x - \psi_n(x) \in p^n \mathbb{Z}_p$ where $0 \leq \psi_n(x) < p^n$. We denote $\mathbb{P}$ as the set of the rational primes. Now, we shall use (1.2) to reformulate our results. Let $(p, l) \in \Psi_1^{\text{irr}}$ then

$$\Delta_{(p, l)} \equiv \frac{\zeta(1-l) - \zeta(1-(l+p-1))}{p} \pmod{p}$$

with $0 \leq \Delta_{(p, l)} < p$.

Theorem 4.1. Let $(p, l) \in \Psi_1^{\text{irr}}$ with $\Delta_{(p, l)} \neq 0$. We define the sequence $(l_n)_{n \geq 1}$ recursively by $l_1 = l$ and, for $n \geq 1$, by

$$l_{n+1} = l_n + \varphi(p) \psi_1 \left( \zeta(1-l_n) \right) = l_n + \varphi(p^n) \psi_1 \left( \frac{\zeta(1-l_n)}{p^n \Delta_{(p, l)}} \right).$$

Then we have $\zeta(1-l_n) \in p^n \mathbb{Z}_p$ and consequently

$$\lim_{n \to \infty} |\zeta(1-l_n)|_p = 0 \quad \text{with} \quad l_n \to \infty.$$

Proof. We rewrite our results using $\zeta(1-l_n) = -\tilde{B}(l_n)$. Theorem 3.1 provides one and only one sequence $(l_n)_{n \geq 1}$ with $l_1 = l$ and $(p, l_n) \in \Psi_n^{\text{irr}}$ with $l_n \to \infty$. This implies the $p$-adic convergence $\zeta(1-l_n) \to 0$. Additionally, from Proposition 2.7 we have

$$l_{n+1} = l_n + s \varphi(p^n)$$

for each step where

$$s \equiv -p^{-n} \tilde{B}(l_n) \Delta_{(p, l)}^{-1} \pmod{p}, \quad 0 \leq s < p.$$

Rewriting the last congruence yields

$$s = \psi_1 \left( \frac{\zeta(1-l_n)}{p^n \Delta_{(p, l)}} \right).$$

The rest follows by $\psi_n(a p^{n-1}) = p^{n-1} \psi_1(a)$ for $a \in \mathbb{Z}_p$. \quad \square

These results can be also applied to the so-called $p$-adic zeta functions which were originally defined by T. Kubota and H. W. Leopoldt [13] in 1964; for a detailed theory see Koblitz [12, Chapter II]. We introduce some definitions.
Definition 4.2. Let $p$ be a prime. For $n \geq 1$ define
\[
\zeta_p(1-n) := (1-p^{n-1}) \zeta(1-n) = -(1-p^{n-1}) B(n).
\]
Define the $p$-adic zeta function for $p \geq 5$ and a fixed $s_1 \in \{2, 4, \ldots, p-3\}$ by
\[
\zeta_{p,s_1} : \mathbb{Z}_p \to \mathbb{Z}_p, \quad \zeta_{p,s_1}(s) := \lim_{t \to s} \zeta_p(1-(s_1+(p-1)t)),
\]
resp. for $p \geq 2$ and $s_1 = 1$ by
\[
\zeta_{p,0} : \mathbb{Z}_p \setminus \{0\} \to \mathbb{Q}_p, \quad \zeta_{p,0}(s) := \lim_{t \to s} \zeta_p(1-(p-1)t),
\]
for $p$-adic integers $s$ by taking any sequence $(t_n)_{n \geq 1}$ of nonnegative integers, resp. positive integers in the case $s_1 = 0$, which $p$-adically converges to $s$.

Remark 4.3. Let $p \geq 2$ and $s_1 \geq 0$. The $p$-adic zeta function $\zeta_{p,s_1}(s)$ interpolates the zeta function $\zeta_p(1-n)$ at nonnegative integer values $s$ by
\[
\zeta_{p,s_1}(s) = \zeta_p(1-n),
\]
where $n \equiv s_1 \pmod{p-1}$ and $n = s_1 + (p-1)s$.

Let $p \geq 5$ and $s_1 \neq 0$. The Kummer congruences $[13]$ state for $r \geq 1$ that
\[
\zeta_{p,s_1}(s) \equiv \zeta_{p,s_1}(s') \pmod{p^r}
\]
when $s \equiv s' \pmod{p^{r-1}}$ for nonnegative integers $s$ and $s'$. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, the $p$-adic zeta function $\zeta_{p,s_1}$ extends uniquely, by means of the interpolation property and the Kummer congruences, to a continuous function on $\mathbb{Z}_p$; see [12] Thm. 8, p. 46.

Definition 4.4. Let $(p,l) \in \Psi_{1r}^\text{irr}$ with $\Delta_{(p,l)} \neq 0$. Define a characteristic $p$-adic integer which contains all information about irregular pairs of higher order by
\[
\chi_{(p,l)} = \sum_{\nu \geq 0} s_{\nu} + 2 p^{\nu} \in \mathbb{Z}_p,
\]
where $(s_\nu)_{\nu \geq 1}$ is the sequence defined by Theorem 3.1 and $l = s_1$.

Lemma 4.5. Let $(p,l) \in \Psi_{1r}^\text{irr}$ with $\Delta_{(p,l)} \neq 0$. Let $r = \text{ord}_p \hat{B}(l)$ and $(p,s_1,\ldots, s_{r+1}) \in \hat{\Psi}_{r+1}^\text{irr}$ be the related irregular pair of order $r + 1$. Then
\[
s_{r+1} \Delta_{(p,l)} \equiv -p^{-r} \hat{B}(l) \pmod{p}
\]
with $s_1 = l$, $s_\nu = 0$ for $\nu = 2,\ldots, r$, $s_{r+1} \neq 0$. If $r = 1$, then $\chi_{(p,l)} \in \mathbb{Z}_p^*$; otherwise, $\chi_{(p,l)} \in p^{-1} \mathbb{Z}_p$ for $r \geq 2$.

Proof. Since $r = \text{ord}_p \hat{B}(l) \geq 1$, we have $(p,l) \in \Psi_{1r}^\text{irr}$ for all $\nu = 1,\ldots, r$. Then Proposition 2.7 and Theorem 3.1 provide
\[
s \equiv -p^{-r} \hat{B}(l) \Delta_{(p,l)}^{-1} \pmod{p}, \quad (p,l+s \varphi(p^r)) \in \Psi_{r+1}^\text{irr}
\]
with $0 \leq s < p$. We have $s \neq 0$ since $\text{ord}_p (p^{-r} \hat{B}(l)) = 0$. Let $(p,s_1,\ldots, s_{r+1}) \in \hat{\Psi}_{r+1}^\text{irr}$ be the related irregular pair of order $r + 1$. Then we see that $s_1 = l$, $s_{r+1} = s \neq 0$, and $s_\nu = 0$ for $\nu = 2,\ldots, r$. By Definition 4.3 we have
\[
\chi_{(p,l)} = s_2 + s_3 p + \cdots + s_{r+1} p^{r-1} + \cdots.
\]
Case $r = 1$ yields $s_2 \neq 0$ and $\chi_{(p,l)} \in \mathbb{Z}_p^*$; otherwise, case $r \geq 2$ implies that $\text{ord}_p \chi_{(p,l)} = r - 1$ and $\chi_{(p,l)} \in p^{r-1} \mathbb{Z}_p$. 

\[\square\]
Theorem 4.6. Let \((p, l) \in \Psi^\text{irr}_r\) with \(\Delta_{(p, l)} \neq 0\). The \(p\)-adic zeta function \(\zeta_{p,l}(s)\) has a unique zero at \(s = \chi(p,l)\).

Proof. From Theorem 3.1 we have a sequence \((l_n)_{n \geq 1}\) with \(l = l_1\). In view of Definition 4.2, Theorem 4.1 also states that

\[
\lim_{n \to \infty} |\zeta_p(1 - l_n)|_p = 0 \quad \text{with} \quad l_n \to \infty.
\]

We can transfer this result to the \(p\)-adic zeta function \(\zeta_{p,l}\) by the interpolation property. We see that \(p\)-adically

\[
\lim_{n \to \infty} l_n = l + (p - 1)\chi(p,l).
\]

Since the function \(\zeta_{p,l}\) is continuous, the \(p\)-adic integer \(\chi(p,l)\) is a zero of \(\zeta_{p,l}\). We shall show that this zero is unique. Assume that \(\zeta_{p,l}(\xi) = 0\) with some \(\xi \in \mathbb{Z}_p\).

We can use the arguments given above in the opposite direction. Since \(\zeta_{p,l}\) is continuous, there exists a sequence \((l'_n)_{n \geq 1}\) of positive integers with

\[
\lim_{n \to \infty} l'_n = l + (p - 1)\xi \quad \text{and} \quad \lim_{n \to \infty} |\zeta_p(1 - l'_n)|_p = 0.
\]

We can choose a subsequence \((l''_n)_{n \geq 1}\) of \((l'_n)_{n \geq 1}\) such that \(\zeta_p(1 - l''_n) \in p^n\mathbb{Z}_p\). By use of the Kummer congruences, we construct the sequence \((\tilde{l}_n)_{n \geq 1}\) where \(\tilde{l}_n = l''_n \mod \varphi(p^n))\) with \(l \leq \tilde{l}_n < \varphi(p^n)\). Now we have \((p, \tilde{l}_n) \in \Psi^\text{irr}_r\) for all \(n \geq 1\). Since \(l = l_1 = \tilde{l}_1\), Theorem 3.1 shows that \((l_n)_{n \geq 1} = (\tilde{l}_n)_{n \geq 1}\). This implies that \(\xi = \chi(p,l)\).

Remark 4.7. Let \((p,l) \in \Psi^\text{irr}_r\) with \(\Delta_{(p,l)} \neq 0\). The related irregular pair \((p, s_1, \ldots, s_r) \in \hat{\Psi}^\text{irr}_r\) is a \(p\)-adic approximation of the zero \(\chi(p,l)\) of the \(p\)-adic zeta function \(\zeta_{p,l}\). For the first irregular primes 37, 59, and 67 elements of \(\Psi^\text{irr}_{100}\) were calculated in [11] pp. 127–128]. These results are reprinted in Table A.2. By Lemma 4.5 the statement \(p^2 \nmid B_1\) is equivalent to the fact that \(\chi(p,l)\) is a unit in \(\mathbb{Z}_p\).

From now on, we shall assume the \(\Delta\)-Conjecture. We shall see that the zeros \(\chi(p,l)\) play an important role in the representation of the Riemann zeta function at odd negative integer arguments.

Theorem 4.8. Let \(n\) be an even positive integer. Under the assumption of the \(\Delta\)-Conjecture we have

\[
\zeta(1 - n) = (-1)^{\frac{n}{2}} \prod_{p \sim 1|n} \frac{|n|_p}{p} \prod_{\substack{p, l \in \Psi^\text{irr}_r \mod \varphi(p^n) \equiv 0 \mod p - 1}} \frac{p}{|\chi(p,l) - \frac{n - 1}{p - 1}|_p}.
\]

Proof. Since both products above have only positive terms, the sign follows by (1.2).

The first product describes the denominator of \(\zeta(1 - n)\) which is a consequence of (1.4) and (1.5). We have to show that the second product describes the unsigned numerator of \(\zeta(1 - n)\) which only consists of powers of irregular primes. Let \(p\) be an irregular prime divisor of \(\zeta(1 - n)\). From Remark 2.2 we have

\[
\text{ord}_p B(n) = r \implies n \equiv l_r \mod \varphi(p^r) \quad \text{with} \quad (p,l_r) \in \Psi^\text{irr}_r.
\]

The irregular pair \((p,l_r)\) of order \(r\) is related to some irregular pair \((p,l) \in \Psi^\text{irr}_r\) with \(\Delta_{(p,l)} \neq 0\), where \(l \equiv l_r \equiv n \mod \varphi(p)\). We also have by Definition 4.3 that

\[
n \equiv l_r \equiv l + (p - 1)\chi(p,l) \mod (p - 1)p^{r-1}\mathbb{Z}_p.
\]
and equally by reduction that
\[
\frac{n - l}{p - 1} \equiv \chi(p, l) \pmod{p^{r-1}Z_p}.
\]
The last congruence is not valid \((\bmod p^rZ_p)\) by construction. Therefore
\[
\left| \chi(p, l) - \frac{n - l}{p - 1} \right|_p = p^{-(r-1)}
\]
which provides, with an additional factor \(p\), the desired \(p\)-power in the second product. Considering all irregular primes \(p\) which can appear, the second product equals the numerator of \(\zeta(1 - n)\) without sign.

With some technical definitions we can combine both products of the theorem above. This yields a more accessible representation of the Riemann zeta function by means of \(p\)-adic analysis.

**Theorem 4.9.** Define \(\Psi_0 = \Psi irrit \cup (\mathbb{P} \times \{0\})\) and set \(\chi(p, 0) = 0\) for all \(p \in \mathbb{P}\). Define \(\rho(l) = 1 - 2\text{sign}(l) = \pm 1\) for \(l \geq 1\). Let \(n\) be an even positive integer. Under the assumption of the \(\Delta\)-Conjecture we have
\[
\zeta(1 - n) = (-1)^\frac{n}{2} \prod_{\substack{(p, l) \not\in \Psi_0 \setminus \{0\} \mod{p-1} \atop l \equiv n \mod{p-1}}} \left( \frac{|\chi(p, l) - \frac{n - l}{p - 1}|_p}{p} \right)^{\rho(l)}.
\]

**Proof.** We only have to consider case \(l = 0\). Then we have \(p - 1 \nmid n\), \(\rho(0) = 1\), and \(|\chi(p, 0) - \frac{n - l}{p - 1}|_p = |n|_p\). The other case \(l > 0\) is already covered by Theorem 4.8. \(\Box\)

We shall give an interpretation of this formula above in Remark 4.17 by means of \(p\)-adic zeta functions. This generalization shows the significance of proving the \(\Delta\)-Conjecture at all.

**Theorem 4.10.** Let \((p, l) \in \Psi irrit\) with \(\Delta(p, l) \neq 0\). Let \(s, t \in \mathbb{Z}_p\). Then a strong version of the Kummer congruences holds that
\[|\zeta_{p, l}(s) - \zeta_{p, l}(t)|_p = |p(s - t)|_p.\]
Moreover
\[
\frac{\zeta_{p, l}(s) - \zeta_{p, l}(t)}{p(s - t)} \equiv -\Delta_{(p, l)} \pmod{p\mathbb{Z}_p} \quad \text{for} \quad s \neq t
\]
and
\[
\zeta'_{p, l}(s) \equiv -p \Delta_{(p, l)} \pmod{p^2\mathbb{Z}_p}.
\]
Thus, \(\Delta_{(p, l)}\) is closely associated with the \(p\)-adic zeta function \(\zeta_{p, l}\) in the non-singular case \(\Delta_{(p, l)} \neq 0\). We will prove this theorem later.

**Corollary 4.11.** Let \((p, l) \in \Psi irrit\) with \(\Delta(p, l) \neq 0\). The \(p\)-adic zeta function \(\zeta_{p, l}(s)\) has a simple zero at \(s = \chi(p, l)\). Moreover, for \(s \in \mathbb{Z}_p\),
\[
\zeta_{p, l}(s) = p(s - \chi(p, l)) \zeta'_{p, l}(s)
\]
where \(\zeta'_{p, l}(s)\) is a continuous function on \(\mathbb{Z}_p\) with \(\zeta'_{p, l}(s) \equiv -\Delta_{(p, l)} \pmod{p\mathbb{Z}_p}\). Consequently,
\[
|\zeta_{p, l}(s)|_p = |p(s - \chi(p, l))|_p.
\]
Proof. Theorem 4.10 shows that \( \zeta_{p, \ell}(s) \) has a unique zero at \( s = \chi_{(p, \ell)} \). We can use Theorem 4.10 to define

\[
\zeta_{p, \ell}^*(s) = \begin{cases} 
\frac{\zeta_{p, \ell}(s)}{p(s - \chi_{(p, \ell)})}, & s \neq \chi_{(p, \ell)}, \\
\frac{\zeta'_{p, \ell}(s)}{p}, & s = \chi_{(p, \ell)}.
\end{cases}
\]

Theorem 4.10 implies that \( \zeta_{p, \ell}^*(s) \equiv -\Delta_{(p, \ell)} \pmod{p\mathbb{Z}_p} \) for all \( s \in \mathbb{Z}_p \). Hence \( \zeta_{p, \ell}(s) \) has no zeros and consequently \( \zeta_{p, \ell}(s) \) has a simple zero at \( s = \chi_{(p, \ell)} \). Since \( \zeta_{p, \ell}(s) \) is continuous on \( \mathbb{Z}_p \) and \( \zeta_{p, \ell}'(s) \) exists at \( s = \chi_{(p, \ell)} \), the function \( \zeta_{p, \ell}^*(s) \) is also continuous on \( \mathbb{Z}_p \). Finally we obtain \( \zeta_{p, \ell}(s) = p(s - \chi_{(p, \ell)}) \zeta_{p, \ell}^*(s) \) and \( |\zeta_{p, \ell}(s)|_p = |p(s - \chi_{(p, \ell)})|_p |\zeta_{p, \ell}^*(s)|_p = |p(s - \chi_{(p, \ell)})|_p \).

\( \square \)

Definition 4.12. Let \( p \) be a prime. Define

\[
\zeta_{p, 0}^* : \mathbb{Z}_p \to \mathbb{Z}_p, \quad \zeta_{p, 0}^*(s) := -\lim_{t \to s} \left( 1 - p^{t(p-1)} \right)^{-1} \frac{B_{t(p-1)}}{p-1}
\]

for \( p \)-adic integers \( s \) by taking any sequence \( (t_\nu)_{\nu \geq 1} \) of nonnegative integers which \( p \)-adically converges to \( s \).

Proposition 4.13. Let \( p \) be a prime. The function \( \zeta_{p, 0}^*(s) \) is continuous on \( \mathbb{Z}_p \setminus \{0\} \) and satisfies the properties

\[
\zeta_{p, 0}^*(0) = -1, \quad \zeta_{p, 0}^*(s) = ps \zeta_{p, 0}(s), \quad s \in \mathbb{Z}_p \setminus \{0\}
\]

and

\[
\lim_{s \to 0} |ps \zeta_{p, 0}(s) - \zeta_{p, 0}^*(0)|_p < 1.
\]

Moreover \( \zeta_{p, 0}^*(s) \equiv -1 \pmod{p\mathbb{Z}_p} \) for \( s \in \mathbb{Z}_p \) in case \( p > 2 \) and \( s \in p\mathbb{Z}_p \) in case \( p = 2 \). Additionally, if \( p = 2 \), then \( \zeta_{p, 0}^*(s) = 0 \) for \( s \equiv 1 \pmod{p\mathbb{Z}_p} \).

Proof. From Definition 4.12 and Definition 4.12 we have for \( s \in \mathbb{Z}_p \setminus \{0\} \) and any sequence \( (t_\nu)_{\nu \geq 1} \) of positive integers which \( p \)-adically converges to \( s \) that

\[
ps \zeta_{p, 0}(s) = -\lim_{t \to s} \left( 1 - p^{t(p-1)} \right) \frac{B_{t(p-1)}}{t(p-1)}
\]

and

\[
-\lim_{t \to s} \left( 1 - p^{t(p-1)} \right) \frac{B_{t(p-1)}}{t(p-1)} = \zeta_{p, 0}^*(s).
\]

Moreover we have

\[
\zeta_{p, 0}^*(0) = -(1 - p^{-1}) B_0 / (p - 1) = -1.
\]

By Clausen–von Staudt \([1, 4]\) we obtain \( \zeta_{p, 0}^*(s) \equiv -1 \pmod{p\mathbb{Z}_p} \) for \( s \in \mathbb{Z}_p \) in case \( p > 2 \) and \( s \in p\mathbb{Z}_p \) in case \( p = 2 \). Additionally we have \( \zeta_{p, 0}^*(s) = 0 \) for \( s \in \mathbb{Z}_p \setminus \{0\} \) and \( B_n = 0 \) for all odd integers \( n > 1 \). We use the fact that \( \zeta_{p, 0} : \mathbb{Z}_p \to \mathbb{Q}_p \) is a continuous function on \( \mathbb{Z}_p \setminus \{0\} \); see \([12]\) Thm. 8, p. 46]. Hence \( \zeta_{p, 0}^*(s) = ps \zeta_{p, 0}(s) \) is also continuous on \( \mathbb{Z}_p \setminus \{0\} \). It remains to show that

\[
\lim_{s \to 0} |ps \zeta_{p, 0}(s) - \zeta_{p, 0}^*(0)|_p < 1.
\]

We can choose a zero sequence \( (s_\nu)_{\nu \geq 1} \) where its elements are arbitrarily close to 0, say \( 0 < |s_\nu|_p < p^{-r} \) with some positive integer \( r \). Then we deduce that

\[
ps \zeta_{p, 0}(s_\nu) - \zeta_{p, 0}^*(0) \equiv \zeta_{p, 0}(s_\nu) + 1 \equiv 0 \pmod{p\mathbb{Z}_p}.
\]
for all $\nu \geq 1$. This implies, together with the continuity of $p^s \zeta_{p,0}(s)$, the estimate above.

**Corollary 4.14.** Let $p$ be a prime. The $p$-adic zeta function $\zeta_{p,0}(s)$ has a simple pole at $s = 0$. Moreover, for $s \in \mathbb{Z}_p \setminus \{0\}$,

$$\zeta_{p,0}(s) = \zeta_{p,0}^*(s)/(ps).$$

Consequently,

$$|\zeta_{p,0}(s)|_p = |ps|_p^{-1}$$

for $s \in \mathbb{Z}_p \setminus \{0\}$ in case $p > 2$ and $s \in p\mathbb{Z}_p \setminus \{0\}$ in case $p = 2$.

**Proof.** By Definition 4.2 we have $|\zeta_{p,0}(s)|_p = |\zeta_{p,0}^*(s)/(ps)|$ for $s \in \mathbb{Z}_p \setminus \{0\}$. Moreover, $\lim_{s \to 0} ps \zeta_{p,0}(s) = \zeta$ with $|\zeta - \zeta_{p,0}(0)|_p < 1$. This implies that the limit $\lim_{s \to 0} \zeta_{p,0}(s)$ does not exist and that $\zeta_{p,0}(s)$ has a simple pole at $s = 0$. We have $|\zeta_{p,0}^*(s)|_p = 1$ for $s \in \mathbb{Z}_p \setminus \{0\}$ in case $p > 2$ and $s \in p\mathbb{Z}_p \setminus \{0\}$ in case $p = 2$. In these cases we then obtain $|\zeta_{p,0}(s)|_p = |ps|_p^{-1}$.

**Remark 4.15.** One can show even further that $\zeta_{p,0}^*(s)$ is continuous on $\mathbb{Z}_p$ where $\lim_{s \to 0} \zeta_{p,0}^*(s) = \zeta_{p,0}^*(0)$; moreover $\zeta_{p,0}^*(s)$ satisfies the Kummer congruences. This can be derived, e.g., by means of $p$-adic integration (see Koblitz [12, pp. 42–46]) or by using certain congruences of Carlitz [3].

**Theorem 4.16.** Define $\Psi_0 = \Psi_1^\ir \cup (\mathbb{P} \times \{0\})$ and set $\chi(p,0) = 0$ for all $p \in \mathbb{P}$. Define $s_{p,l}(n) = (n - l)/(p - 1)$. Let $n$ be an even positive integer. Then

$$|\zeta(1-n)|_\infty = \prod_{l \equiv n \pmod{p-1}} |\zeta_{p,l}(s_{p,l}(n))|_p^{-1} = \prod_{l \equiv n \pmod{p-1}} |\zeta_{p,l}(s_{p,l}(n))|_p^{-1}. \quad (1)$$

**Proof.** Since $n$ is an even positive integer, the product formula states that

$$\prod_{p \in \mathbb{P} \cup \{\infty\}} |\zeta(1-n)|_p = 1.$$

By Definition 4.2 we have $|\zeta(1-n)|_p = |\zeta_{p,l}(s)|_p$ where $l \equiv n \pmod{p-1}$ with $0 \leq l < p - 1$ and $s = s_{p,l}(n) = (n - l)/(p - 1)$. Thus

$$|\zeta(1-n)|_\infty^2 = \prod_{p \in \mathbb{P}} |\zeta(1-n)|_p = \prod_{l \equiv n \pmod{p-1}} |\zeta_{p,l}(s_{p,l}(n))|_p.$$

We have $|\zeta_{p,l}(s)|_p = 1$ for $s \in \mathbb{Z}_p$ if $(p,l) \notin \Psi_1^\ir$ and $l \neq 0$. Hence

$$\prod_{l \equiv n \pmod{p-1}} |\zeta_{p,l}(s_{p,l}(n))|_p = \prod_{l \equiv n \pmod{p-1}} |\zeta_{p,l}(s_{p,l}(n))|_p.$$

Next we assume the $\Delta$-Conjecture. By Corollary 4.11 we then have for $(p,l) \in \Psi_1^\ir$ that $|\zeta_{p,l}(s)|_p = |p(s - \chi(p,l))|_p$ for $s \in \mathbb{Z}_p$. Without any assumption, Corollary
Theorem 4.13 shows that $|ζ_{p,0}(s)|_p = |ps|_p^{-1} = |p(s - χ_{(p,0)})|_p^{-1}$ for $s \in \mathbb{Z}_p \setminus \{0\}$ in case $p > 2$ and $s \in p\mathbb{Z}_p \setminus \{0\}$ in case $p = 2$. Since $n$ is even and $s_{2,0}(n) = n$, we finally obtain
\[
\prod_{(p,l) \in \Psi_0 \ (l \equiv n \ (\mod \ p-1))} \prod_{s_{p,l}(n) > 0} |ζ_{p,l}(s_{p,l}(n))|_p = \prod_{(p,l) \in \Psi_{\text{irr}} \ (l \equiv n \ (\mod \ p-1))} |p(s_{p,l}(n) - χ_{(p,l)})|_p \prod_{p-1|n} |p(s_{p,0}(n) - χ_{(p,0)})|_p^{-1}.
\]

□

Remark 4.17. Assuming the $∆$-Conjecture, the numerator of $|ζ(1 - n)|_∞$ with even $n > 0$ is essentially described by simple zeros $χ_{(p,l)}$ of $p$-adic zeta functions $ζ_{p,l}(s)$ where $l \equiv n \ (\mod \ p-1)$ and $(p,l) \in Ψ_{\text{irr}}$. More precisely, by the variable substitution $s = (n - l)/(p - 1)$, the term
\[
|p(s - χ_{(p,l)})|_p^{-1}
\]
is equal to $p^r$ for some suitable $r > 0$ which is the $p$ power factor in the numerator of $|ζ(1 - n)|_∞$. The term above is induced by
\[
|ζ_{p,l}(s)|_p = |p(s - χ_{(p,l)})|_p = p^{-1}|(s - χ_{(p,l)})|_p.
\]
Since $χ_{(p,l)}$ is a simple zero of $ζ_{p,l}(s)$, $|ζ_{p,l}(s)|_p$ is determined mainly by a linear factor that one can also interpret as a distance between $s$ and $χ_{(p,l)}$.

Similar arguments can be applied to the denominator of $|ζ(1 - n)|_∞$. Without any assumption, the denominator of $|ζ(1 - n)|_∞$ is essentially described by simple poles $χ_{(p,0)}$ of $p$-adic zeta functions $ζ_{p,0}(s)$ where $p - 1 | n$. Again, the term
\[
|p(s - χ_{(p,0)})|_p^{-1}
\]
is equal to $p^r$ for some suitable $r > 0$ which is the $p$ power factor in the denominator of $|ζ(1 - n)|_∞$. This term is induced by
\[
|ζ_{p,0}(s)|_p = |p(s - χ_{(p,0)})|_p^{-1} = p |(s - χ_{(p,0)})|_p^{-1},
\]
where $ζ_{p,0}(s)$ has a simple pole at $s = χ_{(p,0)}$.

Now we shall make some preparations to give later a proof of Theorem 4.10

Definition 4.18. Define the linear finite-difference operator $D$ and its powers by
\[
D^r f(s) = \sum_{\nu = 0}^{\infty} \binom{r}{\nu} (-1)^\nu f(s + \nu)
\]
for $r \geq 0$ and any function $f : \mathbb{Z}_p \to \mathbb{Z}_p$. The series
\[
f(s) = \sum_{\nu \geq 0} a_\nu \binom{s}{\nu}
\]
with coefficients $a_\nu \in \mathbb{Z}_p$ where $|a_\nu|_p \to 0$ is called a Mahler series which defines a continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$.

The following theorem of Mahler shows that the converse also holds; see [17 Thm. 1, p. 173]. Note that the sign $(-1)^\nu$ depends on the definition of $D$.

Theorem 4.19 (Mahler). Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a continuous function. Then $f$ has a Mahler series
\[
f(s) = \sum_{\nu \geq 0} a_\nu \binom{s}{\nu},
\]
where the coefficients $a_\nu$ are given by $a_\nu = (-1)^\nu D^\nu f(0)$ and $|a_\nu|_p \to 0$ holds.
Definition 4.20. Let \((p, l) \in \Psi_1^{irr}\) with \(\Delta_{(p, l)} \neq 0\). For \(n \geq 1\) define the \(p\)-adic zeta function of order \(n\) by
\[
\zeta_{p, l, n}(s) = p^{-n} \zeta_{p, l} \left( \psi_{n-1}(\chi_{(p, l)}) + p^{n-1}s \right), \quad s \in \mathbb{Z}_p,
\]
which for \(s = 0\) corresponds to the related irregular pair of order \(n\).

Proposition 4.21. Let \((p, l) \in \Psi_1^{irr}\) with \(\Delta_{(p, l)} \neq 0\). For positive integers \(n, r\) we have
\[
\mathcal{D}^{r+1} \zeta_{p, l, n}(s) \equiv 0 \pmod{p^{nr} \mathbb{Z}_p}, \quad s \in \mathbb{Z}_p,
\]
and
\[
\zeta_{p, l, n}(s) \equiv \zeta_{p, l, n}(t) \pmod{p^r \mathbb{Z}_p}, \quad s, t \in \mathbb{Z}_p,
\]
when \(s \equiv t \pmod{p^r \mathbb{Z}_p}\).

Proof. First assume that \(s \in \mathbb{N}_0\). In analogy to Corollary 2.5, we have to modify Theorem 2.4 in a similar way, since \(\zeta_{p, l, n}(0)\) corresponds to the related irregular pair of order \(n\). Rewriting (2.2) in the case \(k = 1\) gives (4.1). Since \(\mathbb{Z}\) is dense in \(\mathbb{Z}_p\), we can extend (4.1) to values \(s \in \mathbb{Z}_p\) by means of the interpolation property of \(\zeta_{p, l, n}\). By the same arguments, the Kummer congruences, given in Remark 4.3, can be extended to values in \(\mathbb{Z}_p\). Let \(s \equiv t \pmod{p^r \mathbb{Z}_p}\) and write \(s' = \psi_{n-1}(\chi_{(p, l)}) + p^{n-1}s + t' = \psi_{n-1}(\chi_{(p, l)}) + p^{n-1}t\). Then \(s' \equiv t' \pmod{p^{n+1} \mathbb{Z}_p}\) and therefore \(\zeta_{p, l}(s') \equiv \zeta_{p, l}(t') \pmod{p^{n+1} \mathbb{Z}_p}\). By Definition 4.20 this gives (4.2).

Proposition 4.22. Let \((p, l) \in \Psi_1^{irr}\) with \(\Delta_{(p, l)} \neq 0\). For \(n \geq 1\) we have
\[
\zeta_{p, l, n}(s) \equiv \Delta_{(p, l)} (s_{n+1} - s) \pmod{p \mathbb{Z}_p}, \quad s \in \mathbb{Z}_p,
\]
where \(s_{n+1}\) is defined by \(\chi_{(p, l)} = s_2 + s_3 p + \cdots\). There exists the Mahler expansion
\[
\zeta_{p, l, n}(s) = \zeta_{p, l, n}(0) + \sum_{\nu \geq 1} \frac{p^{n(\nu-1)} z_{\nu}}{\nu} \left( \frac{s}{\nu} \right), \quad s \in \mathbb{Z}_p,
\]
with \(z_{\nu} \in \mathbb{Z}_p\) where \(z_{\nu} = (-1)^{\nu} \pi_{-n(\nu-1)} D^{\nu} \zeta_{p, l, n}(0)\) and \(z_1 \equiv -\Delta_{(p, l)} \pmod{p \mathbb{Z}_p}\).

Proof. Proposition 2.7 also works with \(\alpha_j \equiv \zeta_{p, l, n}(j) \pmod{p \mathbb{Z}_p}\) for \(j \in \mathbb{N}_0\). Since \(\zeta_{p, l, n}(0)\) corresponds to the related irregular pair of order \(n\), we then have
\[
\zeta_{p, l, n}(0) \equiv \Delta_{(p, l)} s_{n+1} \pmod{p \mathbb{Z}_p},
\]
where \(s_{n+1}\) is given by Definition 4.4. A further consequence of Proposition 2.7 extended to \(s \in \mathbb{Z}_p\), is that
\[
D \zeta_{p, l, n}(s) \equiv \Delta_{(p, l)} \pmod{p \mathbb{Z}_p},
\]
and furthermore
\[
\zeta_{p, l, n}(s) \equiv \Delta_{(p, l)} (s_{n+1} - s) \pmod{p \mathbb{Z}_p}.
\]
By construction \(\zeta_{p, l, n}\) is continuous; this is also a consequence of (4.2). Theorem 4.19 shows that \(\zeta_{p, l, n}\) has a Mahler series with the coefficients \(a_{\nu} = (-1)^{\nu} D^{\nu} \zeta_{p, l, n}(0)\) for \(\nu \geq 0\). Using (4.1) we see that \(a_{\nu} \in p^{n(\nu-1)} \mathbb{Z}_p\). Thus, we can set \(z_{\nu} = p^{-n(\nu-1)} a_{\nu}\) for \(\nu \geq 1\) to obtain the proposed series expansion above. From (4.3) we deduce that \(z_1 \equiv -\Delta_{(p, l)} \pmod{p \mathbb{Z}_p}\).

\[\square\]
Corollary 4.23. Let \((p, l) \in \Psi^{irr}_1\) with \(\Delta_{(p, l)} \neq 0\). Let

\[
\zeta_{p, l, 1}(s) = \zeta_{p, l, 1}(0) + \sum_{\nu \geq 1} p^{\nu-1} z_{\nu} \left( \frac{s}{\nu} \right), \quad s \in \mathbb{Z}_p,
\]

be the Mahler expansion of \(\zeta_{p, l, 1}\) given by Proposition 4.22. We have for \(r \geq 1\) that

\[
\zeta_{p, l}(s) \equiv \zeta_{p, l}(0) + \sum_{\nu = 1}^{r-1} p^{\nu} z_{\nu} \left( \frac{s}{\nu} \right) \pmod{p^r \mathbb{Z}_p}, \quad s \in \mathbb{Z}_p.
\]

Special cases are given by

\[
\zeta_{p, l}(0) = -\sum_{\nu \geq 1} p^{\nu} z_{\nu} \left( \frac{\chi_{(p, l)}}{\nu} \right)
\]

and for \(r \geq 1\):

\[
\zeta_{p, l}(0) \equiv -\sum_{\nu = 1}^{r-1} p^{\nu} z_{\nu} \left( \frac{\chi_{(p, l)}}{\nu} \right) \pmod{p^r \mathbb{Z}_p}.
\]

Proof. We rewrite the Mahler expansion above using \(\zeta_{p, l, 1}(s) = p^{-1} \zeta_{p, l}(s)\), which yields

\[
\zeta_{p, l}(s) = \zeta_{p, l}(0) + \sum_{\nu \geq 1} p^{\nu} z_{\nu} \left( \frac{s}{\nu} \right), \quad s \in \mathbb{Z}_p.
\]

By the assumption \(\Delta_{(p, l)} \neq 0\) we have the zero \(\chi_{(p, l)}\) of \(\zeta_{p, l}\). The finite sums follow easily \((\pmod{p^r \mathbb{Z}_p})\), since the coefficients \(p^{\nu} z_{\nu}\) vanish for \(\nu \geq r\).

Hence, we can use Corollary 4.23 to verify calculations of the coefficients \(z_{\nu}\) and of the zero \(\chi_{(p, l)}\). Now, we are almost ready to prove Theorem 4.10; we first recall the following lemma from [17] p. 227.

Lemma 4.24. For \(k \geq 1\) and \(p^i \leq k < p^{i+1}\), we have

\[
\left| \left( \frac{s}{k} \right) - \left( \frac{t}{k} \right) \right|_p \leq p^i |s-t|_p, \quad s, t \in \mathbb{Z}_p.
\]

Proof of Theorem 4.10. Let \(s, t \in \mathbb{Z}_p\) with \(s \neq t\) and set \(r = \text{ord}_p(s-t) \geq 0\). Then we have by the Kummer congruences that

\[
(4.4) \quad s \equiv t \pmod{p^r \mathbb{Z}_p} \quad \Rightarrow \quad \zeta_{p, l}(s) \equiv \zeta_{p, l}(t) \pmod{p^{r+1} \mathbb{Z}_p}.
\]

We show that

\[
(4.5) \quad \frac{\zeta_{p, l}(s) - \zeta_{p, l}(t)}{p(s-t)} \equiv -\Delta_{(p, l)} \pmod{p \mathbb{Z}_p} \quad \text{for} \quad s \neq t
\]

which also implies the converse to (4.4). Set \(n = \min \{ \text{ord}_p \psi_{n-1}(\chi_{(p, l)}) + p^{n-1} s', \text{ord}_p \psi_{n-1}(\chi_{(p, l)}) + p^{n-1} t' \}\) with some \(s', t' \in \mathbb{Z}_p\). Set \(u = r + 1 - n \geq 0\). Note that \(\text{ord}_p(s' - t') = u\). By the implication in (4.4) we then have

\[
p^{-n}(\zeta_{p, l}(s) - \zeta_{p, l}(t)) \equiv \zeta_{p, l, n}(s') - \zeta_{p, l, n}(t') \equiv p^{u} c \pmod{p^{u+1} \mathbb{Z}_p}
\]
with some integer \( c \). On the other side, we can use the Mahler expansion of \( \zeta_{p, t, n} \) given by Proposition 4.22. We obtain
\[
\zeta_{p, t, n}(s') - \zeta_{p, t, n}(t') \equiv z_1(s' - t') + p^n z_2 \left( \binom{s'}{2} - \binom{t'}{2} \right) + p^{2n} z_3 \left( \binom{s'}{3} - \binom{t'}{3} \right) + \cdots \pmod{p^{u+1} \mathbb{Z}_p}.
\]
Since \( \text{ord}_p(s' - t') = u \), Lemma \ref{lemma:order} shows that
\[
\text{ord}_p \left( \binom{s'}{k} - \binom{t'}{k} \right) \geq u - \lfloor \log_p k \rfloor,
\]
where \( \log_p \) is the real-valued logarithm with base \( p \). Since we have \( p \geq 5 \) we obtain the estimate
\[
\text{ord}_p \left( p^{n(k-1)} z_k \left( \binom{s'}{k} - \binom{t'}{k} \right) \right) \geq u + n(k - 1) - \lfloor \log_p k \rfloor \geq u + 1
\]
for all \( k \geq 2 \). Hence, all terms vanish \( \pmod{p^{u+1}} \) except for \( k = 1 \):
\[
\zeta_{p, t, n}(s') - \zeta_{p, t, n}(t') \equiv z_1(s' - t') \pmod{p^{u+1} \mathbb{Z}_p}.
\]
By Proposition \ref{prop:order} and \( \text{ord}_p(s' - t') = u \) we finally get
\[
z_1(s' - t') \equiv -\Delta_{(p, t)}(s' - t') \equiv p^u c \pmod{p^{u+1} \mathbb{Z}_p}
\]
which shows that \( c \neq 0 \pmod{p} \). Since \( p^{-u-1}(s - t) = s' - t' \), this deduces \eqref{eq:derivative}.

Now, we have to determine the derivative of \( \zeta_{p, t} \). Taking any sequence \( (t_\nu)_{\nu \geq 1} \) with \( t_\nu \neq s \) for all \( \nu \geq 1 \) and \( \lim_{\nu \to \infty} t_\nu = s \), \eqref{eq:derivative} yields the derivative with
\[
\zeta'_{p, 1}(s) \equiv -p \Delta_{(p, t)} \pmod{p^2 \mathbb{Z}_p}.
\]

\textbf{Remark 4.25.} In general the converse of the Kummer congruences does not hold. The first nontrivial counterexample is given by \( p = 13 \) and \( B_{16/16} - B_{4/4} = -7 \cdot 13^2/2720 \). The only case where the divided Bernoulli numbers are equal is \( B_{14/14} - B_2/2 = 0 \).

At the end of this section we revisit the \( p \)-adic zeta functions \( \zeta_{p, s_1} \) in the case \( s_1 \neq 0 \) as introduced by Definition \ref{def:zeta}. We can transfer some results to these functions. Moreover, we can state a formula equivalent to the Kummer congruences, but it involves values of \( p \)-adic zeta functions at the smallest possible argument values.

\textbf{Proposition 4.26.} Let \( p \) be a prime with \( p \geq 5 \) and \( s_1 \in \{2, 4, \ldots, p - 3\} \) be fixed. Then
\[
\mathcal{D} \zeta_{p, s_1}(s) \equiv 0 \pmod{p^r \mathbb{Z}_p}, \quad s \in \mathbb{Z}_p.
\]
There exists the Mahler expansion
\[
\zeta_{p, s_1}(s) = \zeta_{p, s_1}(0) + \sum_{\nu \geq 1} p^\nu z_\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p,
\]
with \( z_\nu \in \mathbb{Z}_p \) where \( z_\nu = (-1)^{s_1} p^{s_1} \mathcal{D}^\nu \zeta_{p, s_1}(0) \).

\textbf{Proof.} In analogy to Propositions \ref{prop:order} and \ref{prop:kummer} in the case \( n = 1 \), congruence \eqref{eq:derivative} is a consequence of Theorem \ref{thm:order}. Since \( \zeta_{p, s_1} \) is a continuous function on \( \mathbb{Z}_p \), we obtain a Mahler expansion where the proposed coefficients follow by \eqref{eq:derivative}. \qed
Kummer congruences (1.3) for $r$

Remark 4.29

Corollary 4.28. Note that $s$

Proof. This is a reformulation of Theorem 4.27 using Definition 4.2 where $T_{r,k}$ are functions on $\mathbb{Z}$

Theorem 4.27. Let $p$ be a prime with $p \geq 5$ and $s_1 \in \{2, 4, \ldots, p - 3\}$ be fixed. For $r \geq 0$ we have

$$\zeta_{p, s_1}(s) \equiv \sum_{k=0}^{r} \zeta_{p, s_1}(k) T_{r,k}(s) \pmod{p^{r+1} \mathbb{Z}_p}, \quad s \in \mathbb{Z}_p,$$

where the polynomials $T_{r,k} : \mathbb{Z}_p \to \mathbb{Z}_p$ with $T_{r,k} \in \mathbb{Q}[x]$ and $\deg T_{r,k} = r$ are given by

$$T_{r,k}(x) = \sum_{j=k}^{r}(-1)^{j+k} \binom{j}{k} \left(\frac{x}{j}\right).$$

Proof. We rewrite the Mahler expansion of $\zeta_{p, s_1}$ given in (1.7). Let $s \in \mathbb{Z}_p$. For $r \geq 0$ we obtain the finite expansion

$$\zeta_{p, s_1}(s) \equiv \zeta_{p, s_1}(0) + \sum_{p^r \leq s} p^r z_p \frac{s^k}{p^r} \equiv \sum_{j=0}^{r}(-1)^{j} D^j \zeta_{p, s_1}(0) \left(\binom{s}{j}\right) \pmod{p^{r+1} \mathbb{Z}_p}.$$ 

By Definition 4.18 we have

$$D^j \zeta_{p, s_1}(0) = \sum_{k=0}^{j} \binom{j}{k} (-1)^k \zeta_{p, s_1}(k) = \sum_{k=0}^{r} \binom{j}{k} (-1)^k \zeta_{p, s_1}(k).$$

Reordering the finite sums and terms yields the definition of $T_{r,k}$ above. Since $x \mapsto \binom{s}{j}$, which is a polynomial of degree $j$, defines a function on $\mathbb{Z}_p$, the polynomials $T_{r,k}$ are functions on $\mathbb{Z}_p$. The coefficients of $T_{r,k}$ are rational; e.g., rewrite $T_{r,r}(x) = \binom{x}{r}$ as a polynomial in $x$. We deduce that $\deg T_{r,k} = r$, because the term $\binom{x}{r}$ which occurs only once gives the maximal degree.

Corollary 4.28. Let $n$ be an even positive integer and $p$ be a prime where $p - 1 \nmid n$. Define the integer $s_1$ by $s_1 \equiv n \pmod{p - 1}$ with $0 < s_1 < p - 1$. Set $s = (n - s_1)/(p - 1)$. For $r \geq 1$ we have

$$(1 - p^{n-1}) \tilde{B}(n) \equiv \sum_{k=0}^{r-1} (1 - p^{s_1+k(p-1)-1}) \tilde{B}(s_1 + k(p - 1)) T_{r-1,k}(s) \pmod{p^r \mathbb{Z}_p}$$

with $T_{r,k}$ as defined above.

Proof. This is a reformulation of Theorem 4.27 using Definition 4.2 where $s_1 + s(p - 1) = n$.

Remark 4.29. The case $r = 1$ of Corollary 4.28 reduces to a special case of the Kummer congruences (1.3) for $r = 1$. We explicitly give the cases $r = 2, 3, 4$ of Corollary 4.28. Note that $s_1 \geq 2$ and $p \geq 5$. Case $r = 2$:

$$(1 - p^{n-1}) \tilde{B}(n) \equiv -(s - 1) (1 - p^{s_1-1}) \tilde{B}(s_1) + s \tilde{B}(s_1 + p - 1) \pmod{p^2 \mathbb{Z}_p}.$$ 

Case $r = 3$:

$$(1 - p^{n-1}) \tilde{B}(n) \equiv \frac{1}{2} (s^2 - 3s + 2) (1 - p^{s_1-1}) \tilde{B}(s_1) - s(s - 2) \tilde{B}(s_1 + p - 1) + \frac{1}{2} s(s - 1) \tilde{B}(s_1 + 2(p - 1)) \pmod{p^3 \mathbb{Z}_p}.$$
Case $r = 4$:

$$(1 - p^{n-1})\hat{B}(n) \equiv -\frac{1}{6}(s^3 - 6s^2 + 11s - 6)(1 - p^{s_1-1})\hat{B}(s_1) + \frac{1}{2}s(s^2 - 5s + 6)\hat{B}(s_1 + p - 1) - \frac{1}{2}s(s^2 - 4s + 3)\hat{B}(s_1 + 2(p - 1)) + \frac{1}{6}s(s - 1)(s - 2)\hat{B}(s_1 + 3(p - 1)) \pmod{p^l\mathbb{Z}_p}.$$ 

5. Algorithms

Here we will give some algorithms for calculating irregular pairs of higher order assuming we are in the nonsingular case $\Delta_{(p, l)} \neq 0$. As a result of Theorems 3.1 and 4.1, one must first calculate $\Delta_{(p, l)}$, then the irregular pair of order $n$, resp. the corresponding divided Bernoulli number, provides the next related irregular pair of order $n + 1$. This is not practicable for higher orders, say $n \geq 3$. Proposition 2.10 shows a more effective way of determining related irregular pairs of higher order. Starting from an irregular pair $(p, l) \in \Psi^\text{irr}_n$ with $n \geq 1$ and requiring that $l > (r - 1)n$ with some $r \geq 2$ we can obtain a related irregular pair $(p, l') \in \Psi^\text{irr}_{nl}$. If the corresponding sequence $(\alpha_j)_{j \geq 0}$ is equidistant (mod $p^{(r-1)n}$), then one can easily apply this proposition. If not, one has to calculate successively all elements $\alpha_j$ in order to find $\alpha_s \equiv 0 \pmod{p^{(r-1)n}}$, where $0 \leq s < p^{(r-1)n}$, which exists uniquely by the assumption $\Delta_{(p, l)} \neq 0$. To shorten these calculations, this search can be accomplished step by step, moving each time from a sequence $(\alpha_j, k)_{j \geq 0}$ to a sequence $(\alpha_{j, k+1})_{j \geq 0}$ which are assigned to the irregular pair of order $k$, resp. $k + 1$.

**Proposition 5.1.** Let $(p, l) \in \Psi^\text{irr}_n$, $n \geq 1$, with $\Delta_{(p, l)} \neq 0$. Let $r$, $u$ be positive integers with $r > 1$ and $u = (r - 1)n$. Assume that $l > u$. Let the elements

$$\alpha_{j, 0} \equiv p^{-n}\hat{B}(l + j\varphi(p^n)) \pmod{p^u}, \quad j = 0, \ldots, r - 1,$$

be given. For each step $k = 0, \ldots, u - 1$ proceed as follows.

The elements $\alpha_{j, k}$ with $j = 0, \ldots, rp - 1$ have to be calculated successively by

$$\alpha_{j+r, k} \equiv (-1)^{r+1}\sum_{\nu=0}^{r-1} \binom{r}{\nu}(-1)^\nu\alpha_{j+r, k} \pmod{p^{u-k}}.$$ 

Set $s_k \equiv -\alpha_{0, k}\Delta_{(p, l)}^{-1}(\text{mod } p)$ with $0 \leq s_k < p$. The elements $\alpha_{j, k}$, which are divisible by $p$, are given by

$$\alpha_{s_k + \mu p, k} \equiv 0 \pmod{p}, \quad \mu = 0, \ldots, r - 1.$$ 

For $k < u - 1$, set

$$\alpha_{j+k+1} = \alpha_{s_k + \mu p, k}/p, \quad j = 0, \ldots, r - 1,$$

and go to the next step $k+1$; otherwise, stop. Let $(p, t_1, \ldots, t_n) \in \Psi^\text{irr}_n$ be the associated irregular pair with $(p, l)$, then

$$(p, t_1, \ldots, t_n, s_0, \ldots, s_{u-1}) \in \Psi^\text{irr}_{rn}$$ 

is the only related irregular pair of order $rn$. 

Proposition 2.10 shows that

\[ \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu \alpha_{j+\nu,0} \equiv 0 \pmod{p^\nu}. \]

(5.1)

All elements of the sequence \((\alpha_{j,0})_{j \geq 0}\) can be calculated successively; this sequence is induced by the first elements \(\alpha_{j,0}\) with \(j = 0, \ldots, r - 1\). Using Corollary 2.15 with \(\omega = p^k \varphi(p^n)\), \(0 \leq k < u\), (5.1) then becomes

\[ \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu \alpha_{j+\nu+p,0} \equiv 0 \pmod{p^\nu}, \]

(5.2)

whereby the sequence \((\alpha_{j+p^\nu,0})_{\mu \geq 0}\) can also be calculated successively. Note that now \(j\) is fixed and \(\mu\) runs. The sequences \((\alpha_{j,k})_{j \geq 0}\), which we will consider, are subsequences of \((\alpha_{j,0})_{j \geq 0}\) in a suitable manner. Essentially, these sequences are given by (5.4). The existence of these sequences and that they correspond to the related irregular pair of order \(n + k\) will be shown by induction on \(k\) for \(k = 0, \ldots, u - 1\). Set \(l_n = l_1\). By Proposition 2.14 and Theorem 3.1, there exist certain integers \(s_k\) and related irregular pairs of higher order for \(k = 0, \ldots, u - 1\) where

\[ (p, l_{n+k+1}) \in \Psi_{n+k+1}^{\text{ir}}, \quad l_{n+k+1} = l_{n+k} + s_k \varphi(p^{n+k}) \quad \text{with} \quad 0 \leq s_k < p. \]

Basis of induction \(k = 0\): The sequence \((\alpha_{j,0})_{j \geq 0}\) is given by (5.1) and we have

\[ \alpha_{j,0} \equiv p^{-n} \widehat{B}(l_n + j \varphi(p^n)) \pmod{p^u}. \]

Inductive step \(k \mapsto k + 1\): Assume this is true for \(k\) prove for \(k + 1\). The elements \(\alpha_{j,k}\) with \(j = 0, \ldots, r - 1\) are given and the following elements are calculated by

\[ \alpha_{j+r,k} \equiv (-1)^{r+1} \sum_{\nu=0}^{r-1} \binom{r}{\nu} (-1)^\nu \alpha_{j+\nu,k} \pmod{p^{u-k}} \]

(5.3)

up to index \(j = rp - 1\). Proposition 2.14 provides

\[ s_k \equiv -\alpha_{0,k} \Delta_{(p,t)}^{-1} \equiv 0 \pmod{p} \]

(5.4)

In the case \(k < u - 1\), Lemma 2.6 additionally ensures that only \(\alpha_{s_k+jp,k} \equiv 0 \pmod{p}\) for all \(j\). Thus, we can define a new sequence by

\[ \alpha_{j,k+1} \equiv \frac{\alpha_{s_k+jp,k}}{p} \equiv p^{-(n+k+1)} \widehat{B}(l_{n+k} + (s_k + j) \varphi(p^{n+k})) \]

(5.5)

for \(j = 0, \ldots, r - 1\). By definition \((p, \alpha_{j,k+1})_{j=0, \ldots, r-1}\) is a subsequence of \((\alpha_{j,k})_{j \geq 0}\). Inductively \((p^{k+1}, \alpha_{j,k+1})_{j=0, \ldots, r-1}\) is a subsequence of \((\alpha_{j,0})_{j \geq 0}\) and satisfies (5.2) in a suitable manner and therefore also (5.5) considering case \(k + 1\). On the other side, congruence (5.5) shows that the new sequence also corresponds to the related irregular pair of order \(n + k + 1\).

Let \((p, t_1, \ldots, t_n) \in \Psi_n^{\text{ir}}\) be the associated irregular pair with \((p, l)\). Congruence (5.4) provides a unique integer \(s_k\) for each step. Thus, \((p, t_1, \ldots, t_n, s_0, \ldots, s_{u-1}) \in \Psi_n^{\text{ir}}\) is the only related irregular pair of order \(rn\).

Remark 5.2. Unfortunately, Proposition 5.1 has the restriction that for an irregular pair \((p, l) \in \Psi_n^{\text{ir}}\) of order \(n\) and parameter \(r\) it must hold that

\[ l > (r - 1)n. \]
Consider \((691, 12) \in \Psi_1^{irr}\). In this case one only could calculate related irregular pairs up to order 12. However, this restriction can be removed by shifting the index of the starting sequence \((\alpha_{j,0})_{j \geq 0}\). Then shifting \(j \mapsto j + t\) yields
\[
l + t \varphi(p^n) > (r - 1)n
\]
enabling one to choose a greater value of \(r\). In general, one has to proceed as follows. Moving from a sequence \((\alpha_{j,k})_{j \geq 0}\) to \((\alpha_{j,k+1})_{j \geq 0}\), one has to determine elements \(\alpha_{j,k} \equiv 0 \pmod{p}\). If one starts with a shifted sequence \((\alpha'_{j,k})_{j \geq 0} = (\alpha_{j,k})_{j \geq t}\) having elements \(\alpha_{j,k} \equiv 0 \pmod{p}\) with \(0 \leq j < t\), then the calculated sequence \((\alpha'_{j,k+1})_{j \geq 0}\) is also shifted in the index compared to \((\alpha_{j,k+1})_{j \geq 0}\). This can be easily corrected by comparing the sequences, resp. the resulting integers \(s_k\), with unshifted sequences calculated with a lower \(r' < r\). In this case determining the integer \(s_k\) is better done by searching \(\alpha_{j,k} \equiv 0 \pmod{p}\) rather than calculating via Theorem 4.27.

The main result can be stated as follows. Irregular pairs of higher order can be determined with little effort by calculating a small number of divided Bernoulli numbers with small indices. We shall give another algorithm in terms of the \(p\)-adic zeta function \(\zeta_{p,l}\) which produces a truncated \(p\)-adic expansion of \(\chi_{(p,l)}\).

**Proposition 5.3.** Let \((p, l) \in \Psi_1^{irr}\) with \(\Delta_{(p,l)} \neq 0\). Let \(n\) be a positive integer. Initially calculate the values
\[
\zeta_{p,l,1}(k) \equiv p^{-1} \zeta_{p,l}(k) \pmod{p^n \mathbb{Z}_p}
\]
for \(k = 0, \ldots, n\) and
\[
\Delta_{(p,l)} \equiv \zeta_{p,l,1}(0) - \zeta_{p,l,1}(1) \pmod{p \mathbb{Z}_p}
\]
where \(0 < \Delta_{(p,l)} < p\). Set \(t_1 = 0\). For each step \(r = 1, \ldots, n\) proceed as follows.

Calculate
\[
\zeta_r \equiv \sum_{k=0}^{r} \zeta_{p,l,1}(k) T_{r,k}(t_r) \pmod{p^r \mathbb{Z}_p}
\]
with the polynomials \(T_{r,k}\) as defined in Theorem 4.27. Then \(\zeta_r \in p^{r-1} \mathbb{Z}_p\). Set
\[
s_{r+1} \equiv \Delta_{(p,l)} p^{1-r} \zeta_r \pmod{p \mathbb{Z}_p}
\]
where \(0 \leq s_{r+1} < p\). Set \(t_{r+1} = t_r + s_{r+1} p^{r-1}\) and go to the next step while \(r < n\). Finally \(t_{n+1} \equiv \chi_{(p,l)} \pmod{p^n \mathbb{Z}_p}\).

**Proof.** By Definition 4.20 we have \(\zeta_{p,l,1}(k) = p^{-1} \zeta_{p,l}(k)\). The value \(\Delta_{(p,l)}\) is given by its definition. Define \(t'_r = \psi_{r-1}(\chi_{(p,l)})\) for any \(r \geq 1\) where the expansion of the zero of \(\zeta_{p,l}\) is given by
\[
\chi_{(p,l)} = s_2 + 3p + \cdots + s_{r+1} p^{r-1} + \cdots.
\]
For now, let \(r \in \{1, \ldots, n\}\) be fixed. Theorem 4.27 provides
\[
\zeta_{p,l,1}(s) \equiv \sum_{k=0}^{r} \zeta_{p,l,1}(k) T_{r,k}(s) \pmod{p^r \mathbb{Z}_p}
\]
for \(s \in \mathbb{Z}_p\). From Definition 4.20 we have
\[
\zeta_{p,l,1}(0) = p^{-r} \zeta_{p,l}(\psi_{r-1}(\chi_{(p,l)})) = p^{1-r} \zeta_{p,l,1}(t'_r).
\]
Proposition 4.22 shows that
\[
\zeta_{p,l,1}(0) \equiv \Delta_{(p,l)} s_{r+1} \pmod{p \mathbb{Z}_p}.
\]
Proof. Let \( n \) be an even positive integer, then
\[
B_n = (-1)^{\frac{n}{2}} \frac{2}{n} \prod_{p \mid n} p^\nu \prod_{p \mid n} p
\]
where
\[
\nu(n) := \sum_{\nu=1}^{\infty} \#(\Psi^\nu \cap \{(p, n \mod \nu(p))\}).
\]
Here, as in Definition \ref{def:nu}, \( x \mod y \) denotes the least nonnegative residue of \( x \) modulo \( y \).

Proposition \ref{prop:bn} gives an unconditional representation of the Bernoulli numbers by means of the sets \( \Psi^\nu \). Theorem \ref{thm:psi} also gives a representation by zeros \( \chi(p, l) \) assuming the \( \Delta \)-Conjecture. Of course, the problem of determining the occurring irregular prime factors remains open. On the other side, for instance, if one has calculated the first irregular pairs of order 10 for the first irregular primes \( p_1, \ldots, p_r \), as in Table \ref{table:irregular}, then one can specify \textit{ad hoc} all irregular prime powers \( p^\nu \) with \( \nu \leq \nu(p) \leq r \) of \( B_n \), resp. \( \zeta(1-n) \), up to index \( n = 4 \cdot 10^{15} \). Note that the lower bound is determined here by the first irregular prime 37 and order 10.

Define for positive integers \( n \) and \( m \) the summation function of consecutive integer powers by \( S_n(m) = \sum_{n=0}^{m-1} \nu^n \). Many congruences concerning the function \( S_n \) are naturally related to the Bernoulli numbers.

Proposition \ref{prop:delta}. Let \( (p, l) \in \Psi^\nu \). Assume that \( n \equiv l \pmod{p-1} \) where \( n > 0 \), then
\[
S_n(p) = \frac{S_n(p)}{n} \pmod{p^2}.
\]
Moreover
\[
\Delta(p, l) \equiv p^{-2} \left( \frac{S_{n-l-1}(p)}{l-1} - \frac{S_l(p)}{l} \right) \pmod{p}
\]
with \( 0 \leq \Delta(p, l) < p \).

Proof. Let \( n \equiv l \pmod{p-1} \). The well-known formula of \( S_n \) (see \cite[p. 234]{19}) is given by
\[
S_n(p) = \frac{B_n}{n} + \left( \frac{n-1}{1} \right) B_{n-2} \frac{p^2}{2 \cdot 3} + \sum_{k=3}^{n} \left( \frac{n-1}{k-1} \right) B_{n-k} \frac{p^k}{k(k+1)}.
\]
where the equation is divided by \( n \) and \( p \). Note that \( p \geq 37 \) and \( n \geq l \geq 12 \) (see Tables \ref{table:irregular} and \ref{table:table}), because 37 is the first irregular prime and \( B_{32}/12 \) is the first divided Bernoulli number which has a numerator greater than one. Now, the
properties of (1.4) and (1.5) provide that $B_{n}/n$ and $B_{n-2}$ are $p$-integral. For the other terms with $B_{n-k} \neq 0$ it follows that $pB_{n-k}$ is $p$-integral and $\text{ord}_p \left( \frac{p^{k-1}/(k(k+1))}{k} \right) \geq 2$ by a standard counting argument. Therefore, equation (5.8) is $p$-integral and holds (mod $p^2$) whereas all terms of the right-hand side vanish except for $B_{n}/n$. This gives congruence (5.6). From Definition 2.3 we have

$$p \Delta_{(p, l)} \equiv \hat{B}(l + p - 1) - \hat{B}(l) \pmod{p^2}.$$ 

We can apply (5.6) in the congruence above. Reducing a $p$-power and considering that $l + p - 1 \equiv l - 1 \neq 0 \pmod{p}$ finally yields (5.7).

Looking at each line of Table A.3 the product of the first three entries $\Delta_{(p, l)}$, $s_1$, and $s_2$ are connected with the function $S_n$. Thus, one can easily verify these values.

**Proposition 5.6.** Let $(p, l) \in \Psi_1^{\text{irr}}$ with $\Delta_{(p, l)} \neq 0$. Let $(p, s_1, s_2) \in \hat{\Psi}_2^{\text{irr}}$ be the related irregular pair of order two with $l = s_1$. Then

$$\Delta_{(p, l)} s_1 s_2 \equiv -p^{-2} S_l(p) \pmod{p}.$$ 

**Proof.** By Proposition 2.7 we have

$$s_2 \equiv -p^{-1} \frac{B_l}{l} \Delta_{(p, l)}^{-1} \pmod{p}.$$ 

We then obtain by Proposition 5.5 that

$$p s_2 \Delta_{(p, l)} \equiv -\frac{B_l}{l} \equiv -\frac{S_l(p)}{l p} \pmod{p^2},$$

which deduces the result since $s_1 = l < p$. □

Now, we shall give some reasons why a prediction or description of the occurrence of irregular prime factors of Bernoulli numbers seems to be impossible in general. For example, we have with an extremely small index $n = 42$ that

$$B_{42} = \frac{1520097643918070802691}{1806},$$

observing that the numerator is a large irregular prime! As mentioned in Section 3 Bernoulli numbers $B_n$ with an index up to $n = 152$ have large irregular prime factors with 30 up to 100 digits. This is even now the greatest mystery of the Bernoulli numbers!

The connection with the Riemann zeta function $\zeta(s)$ via (1.1) leads to methods of calculating Bernoulli numbers directly in a fast and effective way (see [11, Section 2.7]), noting that the main part of the calculation can be done using integers only. Let $n$ be an even positive integer and $|B_n| = U_n/V_n$ with $(U_n, V_n) = 1$. Then (1.1) reads

$$U_n = \tau_n \zeta(n), \quad \tau_n = 2V_n \frac{n!}{(2\pi)^n}, \quad V_n = \prod_{p \nmid n} p,$$

with $V_n$ given by (1.4). Since $\zeta(n) \rightarrow 1$ for $n \rightarrow \infty$, $\tau_n$ is a first approximation of the numerator $U_n$. Considering the decimal digit representation of $U_n$ and $\tau_n$, about $n/3$ digits of the most significant decimal digits of $U_n$ and $\tau_n$ are equal; see
Satz 2.7.9, p. 75] for a more precise statement and formula. Visiting $B_{42}$ again, we observe 12 identical digits:

\[ U_{42} = 1\ 520\ 697\ 643\ 918\ 070\ 802\ 691, \]

\[ \tau_{42} = 2 \cdot 1806 \cdot 42! / (2\pi)^{32} \approx 1\ 520\ 697\ 643\ 917\ 725\ 172\ 488.7773. \]

How can we interpret this result? A part of the most significant digits of the numerator of the Bernoulli number $B_n$ is determined in a certain way by all primes $p \leq n + 1$ and the reciprocal $n$-th power of $\pi$. That the digits of $\pi$ are involved in the numerators of the Bernoulli numbers is quite remarkable.

6. Connections with Iwasawa theory

The $\Delta$-Conjecture is directly connected with Iwasawa theory of cyclotomic fields over $\mathbb{Q}$. Let $p$ be an odd prime and $\mu_{p^n}$ be the set of $p^n$-th roots of unity where $n$ is a positive integer. For the cyclotomic field $\mathbb{Q}(\mu_{p^n})$ let $\mathbb{Q}(\mu_{p^n})^+$ be its maximal real subfield. The class number $h_p = h(\mathbb{Q}(\mu_p))$ can be factored by $h_p = h_p^+ h_p^-$ where $h_p^+ = h(\mathbb{Q}(\mu_p)^+)$ and $h_p^-$ is the relative class number introduced by Kummer. Define

\[ B_{1,\omega^m} = \frac{1}{p} \sum_{a=1}^{p-1} a \omega^m(a) \]

as the generalized Bernoulli number assigned to the Teichmüller character $\omega$. This character is defined by $\omega: \mathbb{Z}_p^* \to \mathbb{Z}_p^*$ and $\omega(a) \equiv a \pmod{p\mathbb{Z}_p}$ for $a \in \mathbb{Z}_p^*$ giving the $(p-1)$-th roots of unity in $\mathbb{Q}_p$. We have for even $m > 0$ and $p - 1 \nmid m$ the following relation (see [23, Corollary 5.15, p. 61]):

\[ B_{1,\omega^m-1} \equiv \hat{B}(m) \pmod{p\mathbb{Z}_p}. \]

For the detailed theory, especially of Iwasawa invariants and cyclotomic $\mathbb{Z}_p$-extensions, see Washington [23] and Greenberg [8]. The results of Iwasawa, Ferrero and Washington, Vandiver and Kummer provide the following theorem; see [23 Cor. 10.17, p. 202].

**Theorem 6.1.** Let $p$ be an irregular prime. Assume the following conditions for all irregular pairs $(p, l)$:

1. The conjecture of Kummer–Vandiver holds: $p \nmid h_{p}^+$.
2. The Kummer congruence does not hold (mod $p^2$):
   \[ \hat{B}(l + p - 1) \neq \hat{B}(l) \pmod{p^2}. \]
3. The generalized Bernoulli number is not divisible by $p^2$:
   \[ B_{1,\omega^m-1} \neq 0 \pmod{p^2\mathbb{Z}_p}. \]

If these are satisfied, then

\[ \text{ord}_p h(\mathbb{Q}(\mu_{p^n})) = i(p) n \quad \text{for all} \quad n \geq 1. \]

All conditions of Theorem 6.1 hold for all irregular primes $p < 12\ 000\ 000$ as verified in [2]. In the case of a regular prime $p$ the formula of Theorem 6.1 is also valid, because then we have $i(p) = 0$ and

\[ p \nmid h_p = h(\mathbb{Q}(\mu_p)) \iff p \nmid h(\mathbb{Q}(\mu_{p^n})) \quad \text{for all} \quad n \geq 1. \]

This follows, e.g., from Iwasawa theory. To get another point of view we can exchange two conditions of the previous theorem by our results. Conditions equivalent to those of Theorem 6.1 are as follows.
(2') The $\Delta$-Conjecture holds: $\Delta_{(p, l)} \neq 0$.

(3') A special irregular pair of order two does not exist: $(p, l, l - 1) \notin \tilde{\Psi}_{2r}^r$.

Now, the $\Delta$-Conjecture with its consequences gives a significant reason to believe that condition (2), resp. (2'), may hold in general. We still have to show that condition (3') is equivalent to condition (3).

**Proposition 6.2.** Let $(p, l) \in \Psi_1^{irr}$. Then

$$B_1, \omega|^{-1} \equiv B_{1+(p-1)(l-1)} \pmod{p^2 \mathbb{Z}_p}$$

and

$$B_1, \omega|^{-1} \equiv 0 \pmod{p^2 \mathbb{Z}_p} \iff (p, l, l - 1) \in \tilde{\Psi}_{2r}^r.$$  

To prove this result, we first need some properties of the Teichmüller character $\omega$. Since $\omega(a)$ is defined by $\omega(a) = \lim_{n \to \infty} a^{p^n}$ in $\mathbb{Z}_p$, the following lemma is easily derived.

**Lemma 6.3.** Let $a, p$ be integers with $p$ an odd prime and $0 < a < p$. Then

$$\omega(a) \equiv a^p \pmod{p^2 \mathbb{Z}_p} \quad \text{and} \quad \omega(a) \equiv a^p + p(a^p - a) \pmod{p^3 \mathbb{Z}_p}.$$  

**Proof of Proposition 6.2.** Since $\omega^{l-1}(a) = \omega(a^{l-1})$, we have

$$pB_{1, \omega|^{-1}} = \sum_{a=1}^{p-1} a \omega^{l-1}(a) = \sum_{a=1}^{p-1} a \omega(a^{l-1}),$$

Using Lemma 6.3 we obtain

$$a \omega(a^{l-1}) \equiv a^{p(l-1)+1} + p(a^{p(l-1)+1} - a^l) \pmod{p^3 \mathbb{Z}_p}.$$  

From the definition of $S_n$ and Proposition 6.3, we have

$$pB_{1, \omega|^{-1}} \equiv S_{p(l-1)+1}(p) + pS_{p(l-1)+1}(p) - pS_{1}(p) \equiv B_{1+(p-1)(l-1)} \pmod{p^3 \mathbb{Z}_p}.$$  

Since $p(l-1)+1 = l + (p-1)(l-1) \equiv l \pmod{p-1}$ and $(p, l) \in \Psi_1^{irr}$, only the first term $S_{p(l-1)+1}^{irr}$ does not vanish. Note that $p \nmid p(l-1) + 1$. Therefore, we get

$$0 \equiv B_{1+(p-1)(l-1)} \equiv B(l + (p-1)(l-1)) \pmod{p^2}$$

if and only if $(p, l, l - 1) \in \tilde{\Psi}_{2r}^r$. \hfill \Box

**Remark 6.4.** From Propositions 5.5 and 6.6, the conditions $\Delta_{(p, l)} \neq 0$ and $(p, l, l - 1) \notin \tilde{\Psi}_{2r}^r$ are equivalent to the system

$$l S_{l+p-1}(p) - (l - 1) S_l(p) \neq 0 \pmod{p^3},$$

$$l S_{l+p-1}(p) - (l - 2) S_l(p) \neq 0 \pmod{p^3}.$$  

7. The singular case

In Section 4 we derived most of the results assuming the $\Delta$-Conjecture. Theorem 4.8 conjecturally describes a closed formula for $\zeta(1-n)$ by zeros $\chi(p, l)$. The following theorem gives an equivalent formulation for the Bernoulli numbers.

**Theorem 7.1.** Let $n$ be an even positive integer. Under the assumption of the $\Delta$-Conjecture we have

$$B_n = (-1)^{\frac{n}{2}-1} \prod_{p-1|n} |n|^{-1} \prod_{(p, l) \in \Psi_1^{irr}} \prod_{l \equiv n \pmod{p-1}} \prod_{p-1|n} p^{-1}.$$  

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Proof. We have to modify the formula of Theorem 4.8. The product formula gives
\[
1 = \prod_{p \in \mathbb{P} \cup \{\infty\}} |n|_p = |n|_\infty \prod_{p-1|n} |n|_p \prod_{p-1|n} |n|_p.
\]
Since \(-B_n/n = \zeta(1-n)\), the proposed formula follows easily. \qed

To get an unconditional formula for \(B_n\), resp. \(\zeta(1-n)\), we have to include the case of a singular \(\Delta(p, l)\). However, no such singular \(\Delta(p, l)\) has been found yet. Theorem 3.2 describes the more complicated behavior of related irregular pairs of higher order in the singular case which can be described by a rooted \(p\)-ary tree; see Diagram 3.3.

Let \((p, l) \in \Psi_1^{\text{irr}}\) with a singular \(\Delta(p, l)\). We construct the rooted \(p\)-ary tree of related irregular pairs of higher order which is a consequence of Theorem 3.2. Each node contains one related irregular pair of higher order. Note that these pairs are not necessarily distinct. We denote this tree by \(T^0_{(p, l)}\) assigned to the root node \((p, l)\).

The tree \(T^0_{(p, l)}\) has the property that each node of height \(r\) lies in \(\Psi_1^{\text{irr}}\). A tree \(T^0_{(p, l)} = \{(p, l)\}\) is called a trivial tree having height 0. If the tree \(T^0_{(p, l)}\) is of height \(\geq 1\), then it contains the root node \((p, l)\) and its \(p\) child nodes \((p, l + j\varphi(p))\) \(\in \Psi_1^{\text{irr}}\) for \(j = 0, \ldots, p - 1\).

In the nonsingular case, we have a zero of the \(p\)-adic zeta function. In contrast the singular case does not guarantee that related irregular pairs of higher order exist at all. The discovery of a singular \(\Delta(p, l)\) is not incompatible with Theorem 7.1 but the formula becomes more complicated, because we then have to consider the complete tree \(T^0_{(p, l)}\). By combining both cases we obtain an unconditional formula which is given by the following theorem. Recall Definition 2.1.

**Theorem 7.2.** Let \(n\) be an even positive integer. Then
\[
B_n = (-1)^{\frac{n-1}{2}} \prod_{p-1|n} |n|_p^{-1} \prod_{(p, l) \in \Psi_1^{\text{irr}}, \Delta(p, l) \neq 0} |p(\varphi(p) - \frac{n}{p-1})|_p^{-1}
\]
\[
\times \prod_{(p, l) \in \Psi_1^{\text{irr}}, \Delta(p, l) = 0 \atop l \equiv n \pmod{p-1}} p^{1 + h^0_{(p, l)}(n)} \prod_{p-1|n} p^{-1}
\]
with the height \(h^0_{(p, l)}(n)\) of \(n\) defined by
\[
h^0_{(p, l)}(n) = \max \left\{ \text{height}((p, l')) : (p, l') \in T^0_{(p, l)} \cap \{(p, n \mod \varphi(p^r))_{r \geq 1}\} \right\}.
\]
Moreover, \(h^0_{(p, l)}(n) = 0 \iff\) the tree \(T^0_{(p, l)}\) is trivial.

**Proof.** The case \(\Delta(p, l) \neq 0\) is already covered by Theorem 7.1. Next we assume that \(\Delta(p, l) = 0\) with a given tree \(T^0_{(p, l)}\) where \(n \equiv l \pmod{p - 1}\). As a consequence of the construction of \(T^0_{(p, l)}\) and Remark 2.2 we have to determine the maximal height of a node \((p, l, v) \in T^0_{(p, l)} \cap \Psi_1^{\text{irr}}\) where \((p, l, v) = (p, n \mod \varphi(p^r))\). The root node \((p, l)\) has height 0, so the exponent equals \(1 + h^0_{(p, l)}(n)\).

If the tree \(T^0_{(p, l)}\) has the height \(\geq 1\), then \((p, n \mod \varphi(p^2)) \in T^0_{(p, l)}\); this implies that \(h^0_{(p, l)}(n) \geq 1\). A trivial tree \(T^0_{(p, l)}\) implies that \(h^0_{(p, l)}(n) = 0\). Conversely, if \(h^0_{(p, l)}(n) = 0\), then the height of the tree \(T^0_{(p, l)}\) must be zero, otherwise we would get a contradiction. \qed
Corollary 7.3. Let \( n \) be an even positive integer, then
\[
\zeta(1 - n) = (-1)^{\frac{n}{2}} \prod_{\substack{(p, l) \in \Phi^{\text{irr}}, \Delta(p, l) \neq 0 \\ n \equiv 0 \pmod{p-1}}} p \left( \chi(p, l) - \frac{n+1}{p-1} \right)^{-1} \prod_{\substack{(p, l) \in \Phi^{\text{irr}}, \Delta(p, l) = 0 \\ n \equiv 0 \pmod{p-1}}} p^{1 + h_{(p, l)}^0(n)} \prod_{p-1|n} \frac{|n|}{p}
\]
with \( h_{(p, l)}^0 \) as defined above.

Proof. This is a reformulation of Theorem 7.2 by \( \zeta(1 - n) = -B_n/n \) and (7.1). \( \square \)

8. An Extension of Adams’ Theorem

Let \( n \) be an even positive integer. The trivial factor of \( B_n \), given by (1.5), is a consequence of the implication, known as Adams’ theorem, that
\[
p^r \mid n \quad \text{with} \quad p - 1 \nmid n \quad \Longrightarrow \quad p^r \mid B_n,
\]
where \( p \) is a prime and \( r \) is some positive integer. It was, however, never proved by Adams. In 1878 he computed a table of Bernoulli numbers \( B_n^* = |B_{2n}| \) for \( 1 \leq n \leq 62 \). On the basis of this table he conjectured\(^1\) that \( p \mid n \) implies \( p \mid B_n^* \) for primes with \( p - 1 \nmid n \); see [1]. Note that the property that \( B_n/n \) is a \( p \)-integer for \( p - 1 \nmid n \) is needed to formulate the Kummer congruences (1.3). The case \( r = 1 \) of these congruences was proved by Kummer [15] earlier in 1851.

By Theorem 2.2 and the definitions of \( h_{(p, l)}^0 \) and \( \chi(p, l) \), we can state an extended version of Adams’ theorem. We introduce the following notation. We write \( p^r \mid | n \) when \( p^r \mid n \) but \( p^{r+1} \nmid n \); i.e., \( r = \text{ord}_p(n) \).

Theorem 8.1. Let \( n \) be an even positive integer. Let \( p \) be a prime with \( p^r \mid | n \), \( r \geq 1 \), and \( p - 1 \nmid n \). Let \( l \equiv n \pmod{p-1} \) with \( 0 < l < p - 1 \). Then \( p^r \delta \mid B_n \) with the following cases.

1. If \( p \) is regular, then \( \delta = 0 \).
2. If \( p \) is irregular with \( (p, l) \notin \Phi^{\text{irr}} \), then \( \delta = 0 \).
3. If \( p \) is irregular with \( (p, l) \in \Phi^{\text{irr}}, \Delta(p, l) = 0 \), then \( \delta = 1 + \text{ord}_p \left( \chi(p, l) - \frac{n+1}{p-1} \right) \).
4. If \( p \) is irregular with \( (p, l) \in \Phi^{\text{irr}}, \Delta(p, l) \neq 0 \), then \( \delta = 1 + h_{(p, l)}^0(n) \).

Admittedly, in case (3), resp. (4), if \( (p, l) \notin \hat{\Phi}^{\text{irr}} \), then \( \delta = 1 \); otherwise \( \delta \geq 2 \).

Proof. We have to consider the formula of Theorem 7.2. The first product yields \( p^r \mid B_n \). Only the second, resp. third, product can give additional \( p \)-factors. Therefore cases (1) and (2) are given by definition. We can now assume that \( (p, l) \in \Phi^{\text{irr}} \).

Case (3). A nonsingular \( \Delta(p, l) \) provides
\[
\delta = \text{ord}_p \left( p \left( \chi(p, l) - \frac{n+1}{p-1} \right) \right) = 1 + \text{ord}_p \left( \chi(p, l) - \frac{n+1}{p-1} \right).
\]
By assumption \( n = p^n n' \) with some integer \( n' \). We have to evaluate
\[
d = \text{ord}_p \left( \chi(p, l) - \frac{n+1}{p-1} \right) = \text{ord}_p \left( p \chi(p, l) - \chi(p, l) + l - p^r n' \right).
\]
Since \( r \geq 1 \), we \( p \)-adically obtain
\[
(p, l) \in \hat{\Phi}^{\text{irr}} \iff \chi(p, l) = l + s_3 p + \cdots \iff d \geq 1.
\]
Conversely, \( (p, l) \notin \hat{\Phi}^{\text{irr}} \) yields \( d = 0 \).

\(^1\) “. . . I have also observed that if \( p \) be a prime factor of \( n \) which is not likewise a factor of the denominator of the \( n \)th number of Bernoulli, then the numerator of that number will be divisible by \( p \). I have not succeeded, however, in obtaining a general proof of this proposition, though I have no doubt of its truth.”
Case (4). A singular \( \Delta_{(p, l)} \) provides \( \delta = 1 + h^0_{(p, l)}(n) \). The definition of \( T^0_{(p, l)} \) and Theorem \( \PageIndex{2} \) show that

\[
(p, l, l) \notin \hat{\Psi}^\text{irr}_2 \iff \text{the tree } T^0_{(p, l)} \text{ is trivial } \iff h^0_{(p, l)}(n) = 0.
\]

Conversely, \((p, l, l) \in \hat{\Psi}^\text{irr}_2 \) yields \( h^0_{(p, l)}(n) \geq 1 \). \( \Box \)

So far, no \((p, l, l) \in \hat{\Psi}^\text{irr}_2 \) has been found. The following corollary theoretically shows examples where \( \delta \) is arbitrarily large.

**Corollary 8.2.** Assume that \((p, l, \ldots, l) \in \hat{\Psi}^\text{irr}_{r+1} \) exists with some \( r \geq 1 \). Set \( n = lp^r \). Then we have \( p^r \parallel n \) and \( p^{2r+1} \mid B_n \); i.e., \( \delta \geq r + 1 \).

**Proof.** By Definition \( \PageIndex{2} \)(\( \PageIndex{1} \)) \((p, n) \in \hat{\Psi}^\text{irr}_{r+1} \) is associated with \((p, l, \ldots, l) \in \hat{\Psi}^\text{irr}_{r+1} \), since \( n = lp^r = \sum_{\nu=1}^{r+1} l\varphi(p^{\nu-1}) \). Thus, \( p^{r+1} \mid B_n/n \) and finally \( p^{2r+1} \mid B_n \). \( \Box \)

**Remark 8.3.** As mentioned above, Johnson \( \cite{10} \) calculated the now-called irregular pairs \((p, s_1, s_2) \in \hat{\Psi}^\text{irr}_2 \) of order two for \( p < 8000 \). He also proved that \((p, l, l) \notin \hat{\Psi}^\text{irr}_2 \), resp. \( s_1 \neq s_2 \), in that range. In a similar manner, the nonexistence of irregular pairs \((p, l, l - 1) \) of order two plays an important role in Iwasawa theory as seen in Section \( \PageIndex{6} \). One may conjecture that no such special irregular pairs \((p, l, l) \) and \((p, l, l - 1) \) of order two exist. But there is still a long way to prove such results, even to understand properly which role the zeros \( \chi_{(p, l)} \) play. Now, we have the relation

\[
(p, l, l) \notin \hat{\Psi}^\text{irr}_2 \iff p^2 \nmid B(lp) \iff p^3 \nmid B_{lp}.
\]

Yamaguchi \( \cite{24} \) also verified by calculation that \( p^3 \nmid B_{lp} \) for all irregular pairs \((p, l) \) with \( p < 5500 \), noting that this was conjectured earlier by Morishima in general. The condition \( p^3 \nmid B_{lp} \) is related to the second case of FLT; see \( \cite{23} \) Thm. 9.4, p. 174] and also \( \cite{23} \) Cor. 8.23, p. 162] for a different context. Under the assumption of the conjecture of Kummer–Vandiver and that no \((p, l, l) \in \hat{\Psi}^\text{irr}_2 \) exists, the second case of FLT is true for the exponent \( p \). For details we refer to the references cited above.

The converse of Adams’ theorem does not hold, but one can state a somewhat different result which deals with the common prime factors of numerators and denominators of Bernoulli numbers with indices close to each other.

**Proposition 8.4.** Let \( S = \{2, 4, 6, 8, 10, 14\} \) be the set of all even indices \( m \) where the numerator of \( |B_m/n| \) equals 1. Write \( B_n = A_n/V_n \) with \( (A_n, V_n) = 1 \). Let \( k, n \) be even positive integers with \( k \in S \) and \( n - k \geq 2 \). Then

\[
D = (A_n, V_{n-k}) \iff D \mid n.
\]

Moreover, if \( D > 1 \), then \( D = p_1 \cdots p_r \) with some \( r \geq 1 \). The primes \( p_1, \ldots, p_r \) are pairwise different and \( p_\nu \mid V_k, p_\nu \nmid B_n/n \) for \( \nu = 1, \ldots, r \).
The critical cases to consider are \( m > \inf \) infinitely many examples of \( p \). Then we have

\[ p \equiv B_{n} \equiv 0 \pmod{p_{\nu}}, \]

since \( n \equiv k \pmod{p_{\nu} - 1} \). By the definition of the set \( S \) we have

\[ (8.1) \]

\[ \frac{B_{k}}{k} \neq 0 \pmod{p_{\nu}}, \]

which yields a contradiction. This shows that \( p_{\nu} | n \) and \( p_{\nu} \nmid B_{n}/n \). Finally it follows that \( D | n \).

Now, the set \( S \) cannot be enlarged, because \( 8.1 \) does not hold in general for numerators having prime factors. For example, let \( p = 691 \) and \( n = 12 + (p - 1) = 702 \). Then we have \( p | B_{12}/12 \) and \( D = (A_{n}, V_{n}) = pc \nmid n \) with some \( c \geq 1 \). On the other hand, one trivially obtains for \( k \in S \), \( p \) prime with \( p - 1 \nmid k \), and \( n = kp \) infinitely many examples of \( D > 1 \). In the following proposition, Proposition 8.4 plays a crucial role. Recall the definition of \( S_{n}(m) \).

**Proposition 8.5.** Let \( n, m \) be positive integers with even \( n \). For \( r = 1, 2 \) we have

\[ m^{r+1} | S_{n}(m) \iff m^{r} | B_{n}. \]

**Proof.** We can assume that \( m > 1 \), since \( m = 1 \) is trivial. The case \( n = 2 \) follows by \( B_{2} = \frac{1}{2} \) and that \( m^{2} \nmid \frac{1}{2}m(m - 1)(2m - 1) = S_{2}(m) \) for \( m > 1 \). For now we assume that \( n \geq 4 \). We have (see [5.8]) that

\[ (8.2) \]

\[ S_{n}(m) = B_{n} m + \binom{n}{2} B_{n-2} m^{3} + \sum_{k=3}^{n} \binom{n}{k} B_{n-k} \frac{m^{k+1}}{k+1}. \]

By [1.4] and the cases \( B_{0} = 1 \) and \( B_{1} = -\frac{1}{2} \), the denominator of all nonzero Bernoulli numbers is squarefree. For each prime power factor \( p^{s} \mid m \) and \( k \) where \( B_{n-k} \neq 0 \) \((2 \leq k \leq n) \), we have

\[ (8.3) \]

\[ \ord_{p} \left( \binom{n}{k} B_{n-k} \frac{m^{k+1}}{k+1} \right) \geq s(k + 1) - 1 - \ord_{p}(k + 1) \geq \lambda s \]

with the following cases:

1. \( \lambda = 1 \) for \( k \geq 2, p \geq 2 \);
2. \( \lambda = 2 \) for \( k \geq 2, p \geq 5 \);
3. \( \lambda = 3 \) for \( k \geq 4, p \geq 5 \).

The critical cases to consider are \( p = 2, 3, 5 \) and \( s = 1 \). Now, we are ready to evaluate \( 8.2 \pmod{m^{r}} \) for \( r = 1, 2 \). Write \( B_{n} = A_{n}/V_{n} \) with \((A_{n}, V_{n}) = 1\).
Case $r = 1$: Assume that $(m, V_n) > 1$. By (8.3) (case $k \geq 2$, $p \geq 2$) we obtain

$$S_n(m) \equiv B_n m \equiv \frac{A_n}{V_n} m \not\equiv 0 \pmod{m}.$$ 

Therefore, $(m, V_n) = 1$ must hold which implies $2 \nmid m$, $3 \nmid m$, and $p \geq 5$. Hence, by (8.4) (case $k \geq 2$, $p \geq 5$), we can write $S_n(m) \equiv B_n m \pmod{m^2}$. This yields

$$m^2 \mid S_n(m) \iff m \mid B_n. \tag{8.4}$$

Case $r = 2$: We have $m \mid B_n$ and $(m, 6) = 1$, because either $m^2 \mid B_n$ or $m^3 \mid S_n(m)$ is assumed. The latter case implies $m^2 \mid S_n(m)$ and therefore $m \mid B_n$ by (8.4). Since $|A_4| = 1$, we can assume that $n \geq 6$. We then have $B_{n-3} = 0$ and we can apply (8.3) (case $k \geq 4$, $p \geq 5$) to obtain

$$S_n(m) \equiv B_n m + \frac{n(n-1)A_{n-2}}{6V_{n-2}} m^3 \pmod{m^3}. \tag{8.5}$$

Our goal is to show that the second term of the right side of (8.5) vanishes, but the denominator $V_{n-2}$ could possibly remove prime factors from $m$. Proposition 8.4 asserts that $(A_n, V_{n-2}) \mid n$. We also have $(m, V_{n-2}) \mid n$ since $m \mid B_n$. This means that the factor $n$ contains those primes that $V_{n-2}$ possibly removes from $m$. Therefore the second term of (8.5) vanishes (mod $m^3$). The rest follows by $S_n(m) \equiv B_n m \equiv 0 \pmod{m^3}$. \hfill $\square$

One cannot improve the value $r$ in general. Choose $p = 37$ and $l = 37580$. Since $(p, l) \in \Psi_{37}^m$ we have $p^3 \nmid B_l$, but $p^4 \nmid S_l(p)$ which was checked with Mathematica.

Example 8.6.

1. We have $B_{42} = 1520097643918070802691/1806$. Since the numerator $A_{42}$ is a large irregular prime, we obtain for $m > 1$ that

$$m^2 \mid S_{42}(m) \iff m = 1520097643918070802691.$$  

2. We have $A_{50} = 5^2 \cdot 417202699 \cdot 4746449777438199$ and $V_{48} = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$. Hence, for $m > 1$ we have

$$m^3 \mid S_{50}(m) \iff m = 5.$$  

Appendix A. Calculations

Table A.1. $B_n$ and $B_n/n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>4</th>
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<th>8</th>
<th>10</th>
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<td>1/42</td>
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<td>1/66</td>
<td>-691/2730</td>
<td>7/6</td>
<td>-3617/510</td>
<td>174611/330</td>
<td></td>
</tr>
<tr>
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<td>1/30</td>
<td>1/42</td>
<td>-1/36</td>
<td>1/66</td>
<td>-691/2730</td>
<td>7/6</td>
<td>-3617/510</td>
<td>174611/330</td>
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<td></td>
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</tbody>
</table>

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Table A.2. Calculated irregular pairs of order 100 of primes 37, 59, and 67.

<table>
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<tr>
<th>Case</th>
<th>p</th>
<th>Zeros of the sequence ((s_\nu)) occur at index 19 and 81.</th>
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</tr>
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<td>11</td>
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Table A.3. Calculated irregular pairs of order 10 of primes below 1000.

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<th>(s_3)</th>
<th>(s_4)</th>
<th>(s_5)</th>
<th>(s_6)</th>
<th>(s_7)</th>
<th>(s_8)</th>
<th>(s_9)</th>
<th>(s_{10})</th>
</tr>
</thead>
<tbody>
<tr>
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<td>21</td>
<td>3</td>
<td>7</td>
<td>28</td>
<td>21</td>
<td>30</td>
<td>4</td>
<td>17</td>
<td>26</td>
<td>13</td>
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Table A.4. Calculation: $n = 1$, $p = 37$, $l = 32$, $(p, l) \in \Psi^{irr}_1$.

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Using Proposition 2.10 with $r = 3$ and $(r - 1)n = 2$ yields $s \equiv 1043 \pmod{p^2}$ and $l_3 = 32 + s\varphi(p) = 37580$. We obtain $(37, 284) \in \Psi^{irr}_2$, $(37, 37580) \in \Psi^{irr}_3$, and $(37, 32, 7, 28) \in \hat{\Psi}^{irr}_3$.

Table A.5. Calculation: $n = 3$, $p = 37$, $l = 37580$, $(p, l) \in \Psi^{irr}_3$.

<table>
<thead>
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Using Proposition 2.9 yields $s \equiv 6607 \pmod{p^3}$ and $l_6 = 37580 + s\varphi(p^3) = 32556968$. We obtain $(37, 32556968) \in \Psi^{irr}_6$, $(37, 55777784) \in \Psi^{irr}_5$, $(37, 1072544) \in \Psi^{irr}_4$, and $(37, 32, 7, 28, 21, 30, 4) \in \hat{\Psi}^{irr}_6$.

Table A.6. Calculation: $n = 3$, $p = 37$, $l = 37580$, $(p, l) \in \Psi^{irr}_3$.

<table>
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Using Proposition 5.1 with $r = 4$ and $(r - 1)n = 9$ yields the sequence $21, 30, 4, \ldots, 27$ which provides $(37, 32, 7, 28, 21, 30, 4, 17, 26, 13, 32, 35, 27) \in \Psi^{irr}_{12}$.

Note that Tables A.2 and A.3 were calculated with smallest possible indices of the Bernoulli numbers using Proposition 5.1. They agree with the results above. Additionally, the results were checked by Corollary 4.23 and Proposition 5.3. The program calcbn [11, Section 2.7] was used to calculate these large Bernoulli numbers extremely quickly.

Acknowledgments

The author wishes to thank Prof. Samuel J. Patterson for his numerous helpful comments and suggestions to improve this paper. The author is thankful to Prof. L. C. Washington for some comments on Iwasawa theory and reference [16]. The author is also grateful to the referee for useful suggestions and detailed remarks. This paper is based on the author’s diploma thesis [11].

References


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