MATHEMATICS OF COMPUTATION
Volume 76, Number 258, April 2007, Pages 745–776
S 0025-5718(06)01927-2
Article electronically published on November 30, 2006

TRACTABILITY OF QUASILINEAR PROBLEMS II: SECOND-ORDER ELLIPTIC PROBLEMS

A. G. WERSCHULZ AND H. WOŹNIAKOWSKI

Abstract. In a previous paper, we developed a general framework for establishing tractability and strong tractability for quasilinear multivariate problems in the worst case setting. One important example of such a problem is the solution of the Helmholtz equation $-\Delta u + qu = f$ in the d-dimensional unit cube, in which u depends linearly on f, but nonlinearly on q. Here, both f and q are d-variate functions from a reproducing kernel Hilbert space with finite-order weights of order ω. This means that, although d can be arbitrarily large, f and q can be decomposed as sums of functions of at most ω variables, with ω independent of d.

In this paper, we apply our previous general results to the Helmholtz equation, subject to either Dirichlet or Neumann homogeneous boundary conditions. We study both the absolute and normalized error criteria. For all four possible combinations of boundary conditions and error criteria, we show that the problem is *tractable*. That is, the number of evaluations of f and q needed to obtain an ε-approximation is polynomial in ε^{-1} and d, with the degree of the polynomial depending linearly on ω. In addition, we want to know when the problem is *strongly tractable*, meaning that the dependence is polynomial only in ε^{-1}, independently of d. We show that if the sum of the weights defining the weighted reproducing kernel Hilbert space is uniformly bounded in d and the integral of the univariate kernel is positive, then the Helmholtz equation is strongly tractable for three of the four possible combinations of boundary conditions and error criteria, the only exception being the Dirichlet boundary condition under the normalized error criterion.

1. Introduction

The worst case complexity of solving many important d-dimensional problems, such as integration, approximation, and elliptic partial differential equations, is known to be exponential in d when the input functions belong to standard Sobolev spaces; see, e.g., [11, Chapter 3] and [7] for discussion and references. This *curse of dimensionality* means that such problems are intractable. One major goal of information-based complexity research has been to vanquish the curse of dimensionality by shrinking the class of input functions, so that such problems can be made tractable in the worst case setting.

Received by the editor November 8, 2005 and, in revised form, January 17, 2006.

2000 *Mathematics Subject Classification.* Primary 65N15; Secondary 41A65.

Key words and phrases. Complexity, tractability, high-dimensional problems, elliptic partial differential equations, reproducing kernel hilbert spaces, quasi-linear problems, finite-order weights.

This research was supported in part by the National Science Foundation.

Much attention has been lavished on the tractability of *linear* multivariate problems; see, e.g., [12] and the references contained therein. However, many important problems are *nonlinear*. Perhaps the simplest kinds of nonlinear problems to analyze are problems that appear to be linear, but have "hidden" nonlinearities. For example, consider the solution of the Helmholtz equation $-\Delta u + qu = f$ on the d-dimensional unit cube, with Dirichlet or Neumann boundary conditions. If we treat q as a fixed known function, then we are only interested in the dependence of u on f; this is a linear problem. However, if we treat both f and q as unknown functions, the nonlinear dependence of u on q means that we now have a nonlinear problem.

The Helmholtz equation is an example of a *quasilinear* problem. A quasilinear multivariate problem is determined by giving, for each positive integer d, an operator $S_d\colon F_d \times Q_d \to G_d$, where

(1) F_d and Q_d are sets of d-variate functions,
(2) F_d and G_d are normed spaces,
(3) $S_d(\cdot, q)$ is a linear operator for each $q \in Q_d$, and
(4) S_d satisfies a Lipschitz condition with respect to its two variables.

Note that the presence of Q_d distinguishes quasilinear problems from well-posed linear problems, as defined in [10]. For example, a linear partial differential equation $Lu = f$ yields a linear problem if we are only interested in how u depends on f; however, if we also want to study how u depends on the coefficients of L, we will have a quasilinear problem.

In this paper, we consider algorithms that use the values of linear functionals of f and q. We will be interested in algorithms that allow the evaluation of any linear functionals of f and q, as well as those that only allow the evaluation of f and q at points of the unit cube. Let $\mathrm{card}(\varepsilon, S_d)$ denote the minimal number of such evaluations needed to compute an ε-approximation in the worst case setting.[1] A family $S = \{S_d\}_{d=1}^{\infty}$ of problems is said to be *tractable* if $\mathrm{card}(\varepsilon, S_d)$ is bounded by a polynomial in ε^{-1} and d. If this bound is independent of d, then S is said to be *strongly tractable*.

Of course, tractability results depend on how we choose F_d and Q_d. One idea that has worked well for linear problems has been to choose *weighted* spaces. These are spaces for which the dependence on successive variables or groups of variables is moderated by corresponding weights; see [9] where this idea was probably studied for the first time, and [7] for a survey. Recently, spaces with *finite-order* weights have been thoroughly analyzed. These spaces were introduced in [4] for the integration problem; they were first studied for general linear problems in [12], and for quasilinear problems in [14].

The main idea behind finite-order weights is as follows. We want to solve problems S_d, where d may be arbitrarily large. This means that we want to approximate $S_d(f, q)$, where the functions f and q depend on d variables. However, we restrict our attention to spaces for which f and q can be decomposed as sums of functions that depend on at most ω variables, where ω is independent of d. We stress that algorithms using function values of f and q do *not* use the values of the terms appearing in the decomposition of f and q. These decompositions only serve as a theoretical tool to prove error bounds and tractability.

[1] These concepts, among others, will be precisely defined in Section 2.

By considering only input functions belonging to spaces of finite-order weights, we find that the number of evaluations needed to obtain an ε-approximation is at most $C_\omega(1/\varepsilon)^{a_\omega} d^{b_\omega}$, which is polynomial in $1/\varepsilon$ and d. The degrees a_ω and b_ω depend at most linearly on ω; however, the leading coefficient C_ω may depend exponentially on ω. Thus, we would hope that ω is relatively small. As an example, in quantum mechanics, one commonly encounters sums

$$q(\mathbf{x}_1, \ldots, \mathbf{x}_{d/3}) = \sum_{1 \le i < j \le d/3} \frac{1}{(\|\mathbf{x}_i - \mathbf{x}_j\|^2_{\ell_2(\mathbb{R}^3)} + \alpha^2)^{1/2}}$$

of modified[2] Coulomb pair potentials; see, e.g., [6, pg. 71]. Here, each $\mathbf{x}_i$ belongs to $\mathbb{R}^3$, so that q depends on d scalar variables; however, each term of q only depends on 6 variables. Hence, $\omega = 6$ for this example.

The paper [12] developed a general framework for studying the tractability of *linear* multivariate problems over reproducing kernel Hilbert spaces with finite-order weights. One of the main results of [12] is that such problems are always tractable, and they are sometimes even strongly tractable. In a recent paper [14], we showed how the framework of [12] can be extended to cover *quasilinear* problems. Using this framework, we presented general conditions for determining when quasilinear multivariate problems are tractable or strongly tractable.

In this paper, we verify these general conditions for specific important multivariate problems. Namely, for a nonnegative function q on I^d, where $I = (0, 1)$, we study the variational formulation of the *Helmholtz equation*

$$-\Delta u + qu = f \qquad \text{in } I^d, \tag{1.1}$$

subject to one of two kinds of homogeneous boundary conditions:

(1) *Dirichlet* boundary conditions

$$u = 0 \qquad \text{on } \partial I^d.$$

In this case, we will take $G_d = H_0^1(I^d)$.

(2) *Neumann* boundary conditions

$$\partial_\nu u = 0 \qquad \text{on } \partial I^d,$$

where ∂_ν is the outer-directed normal derivative. In this case, we will take $G_d = H^1(I^d)$.

Hence we will be measuring the error in the H^1-sense.

As already mentioned, we assume that we can compute function values of f and q or, more generally, arbitrary linear functionals of f and q. The set F_d of right-hand-side functions f will be a reproducing kernel Hilbert space $H(K_d)$, and Q_d will be chosen so that the variational form of the solution $u = S_d(f, q)$ exists for all $f \in H(K_d)$ and $q \in Q_d$. We consider the worst case setting, in which we want to compute an ε-approximation to the solution u for all $f \in H(K_d)$ and $q \in Q_d \cap H(K_d)$, assuming additionally that the norms of f and q are bounded by given numbers.

We study two error criteria:

(1) The *absolute error criterion*: Here, we want to guarantee that the worst case error of an algorithm is at most ε.

[2]The modification is the inclusion of the positive term α. Physicists often include a small α as a regularization parameter to make q smooth.

(2) The *normalized error criterion*: Here, we want to guarantee that the worst case error is at most ε times the initial error. (By the *initial error*, we mean the minimal error we can attain without sampling the functions f and q.)

Combining the two kinds of boundary conditions with the two error criteria, we see that there are four different combinations to consider. Furthermore, each of these four combinations is considered, both for algorithms using function values and for algorithms using continuous linear functionals.

We consider reproducing kernel Hilbert spaces spaces $H(K_d)$ with finite-order weights of order ω and prove tractability results for both the Dirichlet and Neumann problems. Moreover, we find that the problem is strongly tractable in three of the four possible combinations mentioned above, provided that the sum of the finite-order weights is uniformly bounded in d and the integral of the univariate kernel is positive; the only exception is the Dirichlet boundary condition under the normalized error criterion, which is open.

We now present the main results of this paper in more precise terms. Let $\Lambda \in \{\Lambda^{\text{all}}, \Lambda^{\text{std}}\}$, where Λ^{all} denotes the case where we use arbitrary linear functionals and Λ^{std} denotes the case where we only use function evaluations. As before, $\text{card}(\varepsilon, S_d) = \text{card}(\varepsilon, S_d, \Lambda)$ denotes the minimal number of evaluations needed to compute an ε-approximation in the worst case setting under the absolute or normalized error criterion.

To prove our tractability results, we use a maximum principle. For the Dirichlet problem, we use the result found in [5], which bounds the L_∞-norm of the solution by the L_∞-norm of the right-hand-side function. For the Neumann problem, we could not find such a result in the literature, and so a proof (based on suggestions of T. I. Seidman) is provided in this paper.

Let p_{err} and p_{dim} denote ε- and d-exponents of tractability, so that

$$\text{card}(\varepsilon, S_d, \Lambda) \le C \left(\frac{1}{\varepsilon}\right)^{p_{\text{err}}} d^{p_{\text{dim}}} \qquad \forall\, \varepsilon \in (0,1), d \in \mathbb{Z}^{++},$$

and let p_{strong} denote the exponent of strong tractability, so that

$$\text{card}(\varepsilon, S_d, \Lambda) \le C \left(\frac{1}{\varepsilon}\right)^{p_{\text{strong}}}.$$

Here, C is an absolute constant, independent of both ε and d.

We assume that the reproducing kernel K_d of the weighted RKHS $H(K_d)$ has the form

$$K_d(\mathbf{x}, \mathbf{y}) = \sum_{\mathfrak{u} \in \{1,\dots,d\},\, |\mathfrak{u}| \le \omega} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} K(x_j, y_j),$$

where K is the reproducing kernel of a Hilbert space $H(K)$ of univariate functions, and $\gamma_{d,\mathfrak{u}}$ are nonnegative numbers (weights). Let

$$\kappa_2 = \int_0^1 \int_0^1 K(x,y)\, dx\, dy < \infty.$$

Since K is a reproducing kernel we know that $\kappa_2 \ge 0$. Our results depend on whether κ_2 is positive or zero, and whether we are dealing with the general case for finite-order weights of order ω or whether we are dealing with finite-order weights of order ω with a uniformly bounded sum, i.e., for which

$$\sup_{1\le d<\infty} \sum_{\substack{\mathfrak{u}\in\{1,\dots,d\}\\ |\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}} < \infty.$$

Then we have the following results:

(1) For the Dirichlet and Neumann problems under the absolute error criterion, we have

	General case		Bounded sum
	$\kappa_2 > 0$	$\kappa_2 = 0$	$\kappa_2 > 0$
Λ^{all}	$p_{\mathrm{err}} \le 2, p_{\mathrm{dim}} \le 2\omega$	$p_{\mathrm{err}} \le 2, p_{\mathrm{dim}} \le 3\omega$	$p_{\mathrm{strong}} \le 2$
Λ^{std}	$p_{\mathrm{err}} \le 4, p_{\mathrm{dim}} \le 4\omega$	$p_{\mathrm{err}} \le 2, p_{\mathrm{dim}} \le 6\omega$	$p_{\mathrm{strong}} \le 4$

We see that both these problems are tractable. Moreover, if the sum of weights is uniformly bounded and $\kappa_2 > 0$, then these problems are strongly tractable.

(2) For the Dirichlet problem under the normalized error criterion, we have

	$\kappa_2 > 0$
Λ^{all}	$p_{\mathrm{err}} \le 2, p_{\mathrm{dim}} \le 2+\omega$
Λ^{std}	$p_{\mathrm{err}} \le 4, p_{\mathrm{dim}} \le 4+2\omega$

Hence, this problem is tractable. However, we do not know conditions that guarantee strong tractability for this problem. The case $\kappa_2 = 0$ is also open.

(3) For the Neumann problem under the normalized error criterion, we have

	General case		Bounded sum
	$\kappa_2 > 0$	$\kappa_2 = 0$	$\kappa_2 > 0$
Λ^{all}	$p_{\mathrm{err}} \le 2, p_{\mathrm{dim}} \le \omega$	$p_{\mathrm{err}} \le 2, p_{\mathrm{dim}} \le 2\omega$	$p_{\mathrm{strong}} \le 2$
Λ^{std}	$p_{\mathrm{err}} \le 4, p_{\mathrm{dim}} \le 2\omega$	$p_{\mathrm{err}} \le 2, p_{\mathrm{dim}} \le 4\omega$	$p_{\mathrm{strong}} \le 4$

Thus, this problem is tractable. Moreover, if the sum of weights is uniformly bounded and $\kappa_2 > 0$, then the problem is strongly tractable.

We stress that these results hold for the kernels K_d with *any* finite-order weights of order ω and *any* univariate kernel K. Of course, the smoothness of functions from $H(K_d)$ will depend on the kernel K, which may be chosen arbitrarily. Therefore, it may be possible to improve the exponents of tractability and strong tractability for a given choice of the kernel and weights by using an algorithm specially tailored to the particular situation.

For the class Λ^{all}, the results are constructive; that is, we know which linear functionals we should use to obtain the bounds on $\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{all}})$. For the class Λ^{std}, the results are *not* constructive, since they are based on probabilistic arguments. Making these results constructive has been an open problem for a long time.

Finally, as in [14], we emphasize that our results for the Dirichlet and Neumann problems give bounds only on the *information cost*, i.e., on the number of evaluations of f and q needed to obtain an ε-approximation. We have not considered the problem of how many arithmetic operations are needed to implement the algorithms that use these evaluations. These algorithms have the following form:

(1) Obtain approximations $\tilde{f}$ of f and $\tilde{q}$ of q.
(2) Calculate $S_d(\tilde{f}, \tilde{q})$ as an appropriate ε-approximation.

Note that the first stage uses linear algorithms to compute the needed approximations. The coefficients used by these linear algorithms may be precomputed independently of f and q. If the cost of precomputation is not counted, the arithmetic cost of the first stage is proportional to the information cost. However, the second stage introduces some difficulty. Since the operator S_d is not linear, it is not a priori clear how hard it will be to compute $S_d(\tilde{f}, \tilde{q})$ or an approximation thereof. Hence, our positive tractability results on the number of evaluations must be augmented with positive results on the approximate computation of $S_d(\tilde{f}, \tilde{q})$ if we wish to claim that the quasilinear Dirichlet and Neumann problems are computationally feasible for large d.

We have already mentioned some open problems. Let us close this Introduction by posing two more.

(1) For simplicity's sake, we have restricted our attention to *homogeneous* Dirichlet and Neumann boundary conditions. To what extent do the results of this paper still hold when the boundary conditions are *nonhomogeneous*? To maintain the spirit of this paper, the functions describing the boundary conditions should also belong to a space of finite-order weights on each face of the unit cube. If such is the case, we expect that similar tractability results will hold for both the homogeneous and nonhomogeneous cases.

(2) We have not discussed lower bounds for elliptic problems over spaces of finite-order weights. It is easy to see that a lower bound is given by the problem of approximating the embedding operator from $H(K_d)$ to $H^{-1}(I^d)$. Note that the target space for this approximation problem is $H^{-1}(I^d)$, rather than the more familiar space $L_2(I^d)$. Moreover, in the sequel, we show that the Dirichlet problem is at least as hard as computing the most difficult weighted average of $H(K_d)$ functions, the weights coming from $H_0^1(I^d)$; furthermore, the Neumann problem is at least as hard as computing the integral of $H(K_d)$ functions. The problem of finding lower bounds for all these subsidiary problems has not yet been studied and remains open.

2. Notation and assumptions

In this section, we first recall some notation and concepts from [14, Sect. 2], which the reader should consult for motivation and more detailed explanation. In addition, we precisely define the Dirichlet and Neumann problems that we study.

Let us first establish a few notational conventions. If R is an ordered ring, then R^+ and R^{++} respectively denote the nonnegative and positive elements of R. If X and Y are normed linear spaces, then $\mathrm{Lin}[X, Y]$ denotes the space of bounded linear transformations of X into Y. We write $\mathrm{Lin}[X]$ for $\mathrm{Lin}[X, X]$, and X^* for $\mathrm{Lin}[X, \mathbb{R}]$. Finally, we use the standard notation for Sobolev inner products, seminorms, norms, and spaces, found in, e.g., [8, 13].

Let K be a measurable reproducing kernel defined on $\bar{I} \times \bar{I}$ with $I = (0, 1)$. We will require that

$$\kappa_0 := \operatorname*{ess\,sup}_{x \in I} K(x, x) < \infty, \tag{2.1}$$

from which it follows that

$$0 \le \kappa_2 \le \kappa_1 \le \kappa_0,$$

where

$$\kappa_1 = \int_0^1 K(x,x)\,dx \tag{2.2}$$

and

$$\kappa_2 = \int_0^1 \int_0^1 K(x,y)\,dx\,dy. \tag{2.3}$$

Without loss of generality, we assume that κ_1 is positive, since the problem will be trivial otherwise. It then follows that κ_0 is also positive. However, κ_2 may be either positive or zero. It turns out that $\kappa_2 = 0$ can occur for kernels that arise in practice; see Remark 2.2. The distinction between the cases $\kappa_2 > 0$ and $\kappa_2 = 0$ will affect the error bounds for our problem.

Let $\mathscr{P}_d$ be the power set of $\{1,\dots,d\}$, and let

$$\boldsymbol{\gamma} = \{\, \gamma_{d,\mathfrak{u}} : \mathfrak{u} \in \mathscr{P}_d, d \in \mathbb{Z}^{++} \,\}$$

be a set of nonnegative numbers $\gamma_{d,\mathfrak{u}}$ (which we call *weights*), with

$$\boldsymbol{\gamma}_{\max} := \sup_{d\in\mathbb{Z}^{++}} \max_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} < \infty.$$

We shall assume that $\boldsymbol{\gamma}$ is a set of *finite-order* weights of *order* $\omega \in \mathbb{Z}^{++}$, see [4], i.e., that

$$\gamma_{d,\mathfrak{u}} \neq 0 \quad \text{only if} \quad |\mathfrak{u}| \le \omega \qquad \forall\, \mathfrak{u} \in \mathscr{P}_d, d \in \mathbb{Z}^{++}, \tag{2.4}$$

where ω is the smallest positive integer such that (2.4) holds and $|\mathfrak{u}|$ is the cardinality of $\mathfrak{u}$.

For each $d \in \mathbb{Z}^{++}$, the space $H(K_d)$ is the reproducing kernel Hilbert space (RKHS) whose reproducing kernel is

$$K_d = \sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} K_{d,\mathfrak{u}},$$

with

$$K_{d,\mathfrak{u}}(\mathbf{x},\mathbf{y}) = \prod_{j\in\mathfrak{u}} K(x_j,y_j) \qquad \forall\, \mathbf{x},\mathbf{y} \in \bar{I}^d, \mathfrak{u} \in \mathscr{P}_d.$$

For $f \in H(K_d)$ we know (see, e.g., [12]) that

$$\|f\|_{L_2(I^d)} \le \sigma_d(\kappa_1)\|f\|_{H(K_d)}, \tag{2.5}$$

where, here and elsewhere, we will often use the function

$$\sigma_d(\theta) = \left(\sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}}\, \theta^{|\mathfrak{u}|} \right)^{1/2} \qquad \forall\, \theta \in \mathbb{R}^+. \tag{2.6}$$

Hence, $H(K_d)$ is embedded in $L_2(I^d)$ for arbitrary weights $\boldsymbol{\gamma}$. For finite-order weights of order ω, we can estimate $\sigma_d(\theta)$ by

$$\sigma_d(\theta) \le \sqrt{2\,\max\{\theta^\omega, 1\}\,\boldsymbol{\gamma}_{\max}}\, d^{\,\omega/2}; \tag{2.7}$$

see [14, Lemma 6].

Example 2.1. We illustrate our approach by the *min-kernel*

$$K(x,y) = K_{\min}(x,y) := \min\{x,y\} \qquad \forall\, x,y \in [0,1], \tag{2.8}$$

which has been studied in many papers and is related to the Wiener measure and the Sobolev space of univariate functions. More precisely, the space $H(K)$ consists of absolutely continuous functions vanishing at zero and whose first derivatives belong to $L_2(I)$, with the inner product

$$\langle f,g\rangle_{H(K)} \;=\; \int_I f'(x)g'(x)\,dx.$$

In this case, we have $\kappa_0 = 1$, $\kappa_1 = \frac{1}{2}$, $\kappa_2 = \frac{1}{3}$.

For the d-variate case, the space $H(K_d)$ with finite-order weights of order ω consists of functions $f : I^d \to \mathbb{R}$ that can be uniquely decomposed as

$$f(\mathbf{x}) \;=\; \sum_{\mathfrak{u}\in\mathscr{P}_d,\,|\mathfrak{u}|\le\omega} f_{\mathfrak{u}}(\mathbf{x}),$$

with $\mathbf{x} = [x_1, x_2, \dots, x_d]$, where $f_{\mathfrak{u}}(\mathbf{x}) = f(\mathbf{x}_{\mathfrak{u}})$ depends only on x_j for $j \in \mathfrak{u}$, and $f_{\mathfrak{u}} \in H(K_{d,\mathfrak{u}})$. Furthermore,

$$\|f\|^2_{H(K_d)} \;=\; \sum_{\mathfrak{u}\in\mathscr{P}_d,\,|\mathfrak{u}|\le\omega} \gamma_{d,\mathfrak{u}}^{-1}\, \|f_{\mathfrak{u}}\|^2_{H(K_{d,\mathfrak{u}})},$$

where

$$\|f_{\mathfrak{u}}\|^2_{H(K_{d,\mathfrak{u}})} \;=\; \int_{I^{|\mathfrak{u}|}} \left(\frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x}_{\mathfrak{u}})\right)^2 d\mathbf{x}_{\mathfrak{u}}.$$

Here, by convention, we have $0/0 = 0$. That is, if $\gamma_{d,\mathfrak{u}} = 0$, then the corresponding component $f_{\mathfrak{u}} = 0$.

Observe that the constant function $f(\mathbf{x}) = c$ for all $\mathbf{x} \in I^d$ belongs to $H(K_d)$ iff $\gamma_{d,\emptyset} > 0$, in which case we have $\|f\|_{H(K_d)} = |c|/\gamma_{d,\emptyset}^{1/2}$. □

Remark 2.2. As we shall see, tractability results will be different for the cases $\kappa_2 > 0$ and $\kappa_2 = 0$. For the min-kernel we have $\kappa_2 > 0$. For some other kernels, we may have $\kappa_2 = 0$. For instance, consider the *Korobov kernel* $K(x,y) = B_2(|x-y|)$, where $B_2(t) = t^2 - t + \frac{1}{6}$ is the Bernoulli polynomial of degree 2. Then the space $H(K_d)$ differs from the Sobolev space with the min-kernel by replacing the condition $f(0) = 0$ by $\int_0^1 f(x)\,dx = 0$; more properties of these and similar spaces may be found in, e.g., [9]. For the Korobov kernel, we have $\kappa_2 = 0$. □

We now recall the standard variational forms of the Dirichlet and Neumann problems for the Helmholtz equation (1.1); see (e.g.) [3, pp. 35–40]. In what follows, we write

$$B_d(v,w;q) = \int_{I^d} [\nabla v\cdot\nabla w + qvw] \qquad \forall\, v,w \in H^1(I^d),\; q \in L_\infty(I^d).$$

(1) For the *Dirichlet problem*, let

$$Q_d^* = \{\, q \in L_\infty(I^d) : q \ge 0 \,\}.$$

For $f \in H(K_d)$ and $q \in Q_d^*$, a solution element $u = S_d^{\mathrm{DIR}}(f,q) \in H_0^1(I^d)$ is defined such that

$$B_d(u,w;q) = \langle f,w\rangle_{L_2(I^d)} \qquad \forall\, w \in H_0^1(I^d). \tag{2.9}$$

(2) For the *Neumann problem*, let q_0 be a positive number, independent of d. Define

$$Q_d^{**} = \{ q \in L_\infty(I^d) : q \geq q_0 \}.$$

For $f \in H(K_d)$ and $q \in Q_d^{**}$, a solution element $u = S_d^{\mathrm{NEU}}(f,q) \in H^1(I^d)$ is defined such that

$$B_d(u, w; q) = \langle f, w \rangle_{L_2(I^d)} \qquad \forall\, w \in H^1(I^d). \tag{2.10}$$

The well-definedness of S_d^{DIR} and S_d^{NEU} will be addressed in the sequel.

Let

$$(S_d, Q_d, G_d) = \begin{cases} \big(S_d^{\mathrm{DIR}}, Q_d^*, H_0^1(I^d)\big) & \text{for the Dirichlet problem,} \\ \big(S_d^{\mathrm{NEU}}, Q_d^{**}, H^1(I^d)\big) & \text{for the Neumann problem.} \end{cases}$$

We want to efficiently compute approximations of $S_d(f,q)$ for $[f,q] \in H_{d,\rho_1} \times (Q_d \cap H_{d,\rho_2})$, where ρ_1, $\rho_2 \in \mathbb{R}^{++}$ are independent of d, and

$$H_{d,\rho} = \{ f \in H(K_d) : \|f\|_{H(K_d)} \leq \rho \}$$

is the ball of $H(K_d)$ of radius $\rho > 0$.

For the Neumann problem to be well-defined, we must assume that $Q_d^{**} \cap H_{d,\rho_2}$ is nonempty. This holds if $1 \in H(K_d)$, i.e., the constant function 1 belongs to $H(K_d)$, and $\|1\|_{H(K_d)} \leq \rho_2/q_0$. Then the constant function q_0 belongs to $Q_d^{**} \cap H_{d,\rho_2}$. It is known, see [2], that $1 \in H(K_d)$ if $\gamma_{d,\emptyset} > 0$, and then $\|1\|_{H(K_d)} \leq \gamma_{d,\emptyset}^{-1/2}$. Furthermore, if $1 \notin H(K)$, then $\|1\|_{H(K_d)} = \gamma_{d,\emptyset}^{-1/2}$. Hence, if $q_0 \gamma_{d,\emptyset}^{-1/2} \leq \rho_2$, then $Q_d^{**} \cap H_{d,\rho_2}$ is nonempty.

Let $A_{d,n}$ be an algorithm using n information evaluations from a class Λ of linear functionals on $H(K_d)$. Here, Λ is either the class Λ^{all} of all continuous linear functionals on $H(K_d)$, or the class Λ^{std} of *standard information* consisting of function evaluations.

The worst case *error* of $A_{d,n}$ is given by

$$e(A_{d,n}, S_d, \Lambda) = \sup_{[f,q] \in H_{d,\rho_1} \times Q_d \cap H_{d,\rho_2}} \|S_d(f,q) - A_{d,n}(f,q)\|_{G_d},$$

and the nth *minimal error* is defined to be

$$e(n, S_d, \Lambda) = \inf_{A_{d,n}} e(A_{d,n}, S_d, \Lambda),$$

the infimum being over all algorithms using at most n information evaluations from Λ. Note that the operator $S_d(\cdot, q)\colon H(K_d) \to G_d$ is linear for any $q \in Q_d$. Hence the *initial error* $e(0, S_d)$ is

$$e(0, S_d) = \rho_1 \sup_{q \in Q_d \cap H_{d,\rho_2}} \|S_d(\cdot, q)\|_{\mathrm{Lin}[H(K_d), G_d]}. \tag{2.11}$$

We shall prove later that $e(0, S_d)$ is finite.

If $\varepsilon \in (0,1)$, we say that the algorithm $A_{d,n}$ provides an ε-approximation to S_d if

$$e(A_{d,n}, S_d, \Lambda) \leq \varepsilon \cdot \mathrm{ErrCrit}(S_d).$$

Here, ErrCrit will be one of the two error criteria

$$\mathrm{ErrCrit}(S_d) = \begin{cases} 1 & \text{for absolute error,} \\ e(0, S_d) & \text{for normalized error.} \end{cases}$$

Let

$$\mathrm{card}(\varepsilon, S_d, \Lambda) = \min\{\, n \in \mathbb{Z}^+ : e(n, S_d, \Lambda) \le \varepsilon \cdot \mathrm{ErrCrit}(S_d)\,\}$$

denote the minimal number of information evaluations from Λ needed to compute an ε-approximation to S_d. The family $S = \{S_d\}_{d\in\mathbb{Z}^{++}}$ is said to be *tractable* in the class Λ if there exist nonnegative numbers C, p_{err}, and p_{dim} such that

$$\mathrm{card}(\varepsilon, S_d, \Lambda) \le C \left(\frac{1}{\varepsilon}\right)^{p_{\mathrm{err}}} d^{p_{\mathrm{dim}}} \qquad \forall\, \varepsilon \in (0,1), d \in \mathbb{Z}^{++}. \tag{2.12}$$

Numbers $p_{\mathrm{err}} = p_{\mathrm{err}}(S,\Lambda)$ and $p_{\mathrm{dim}} = p_{\mathrm{dim}}(S,\Lambda)$ such that (2.12) holds are called ε- and d-*exponents of tractability*; these need not be uniquely defined. If $p_{\mathrm{dim}} = 0$ in (2.12), then S is *strongly tractable* in Λ, and we define

$$p_{\mathrm{strong}}(\Lambda) = \inf\Bigg\{\, p_{\mathrm{err}} \ge 0 : \ \exists\, C \ge 0 \text{ such that} \\ \mathrm{card}(\varepsilon, S_d, \Lambda) \le C\left(\frac{1}{\varepsilon}\right)^{p_{\mathrm{err}}} \forall\, \varepsilon \in (0,1), d \in \mathbb{Z}^{++} \Bigg\}$$

to be the *exponent of strong tractability*.

Of course, a problem's tractability or strong tractability will depend on the error criterion used. Hence in the sequel, we will write $p^{\mathrm{abs}}_{\mathrm{err}}$, $p^{\mathrm{abs}}_{\mathrm{dim}}$, and $p^{\mathrm{abs}}_{\mathrm{strong}}$ for the ε- and d-exponents of tractability and the exponent of strong tractability under the absolute error criterion; these exponents will be denoted by $p^{\mathrm{nor}}_{\mathrm{err}}$, $p^{\mathrm{nor}}_{\mathrm{dim}}$, and $p^{\mathrm{nor}}_{\mathrm{strong}}$ when we are using the normalized error criterion.

We will establish tractability of the Dirichlet and Neumann problems by using the results of [14]. Suppose that the following conditions hold:

(1) S_d is *quasilinear*. That is, there exists a function $\phi\colon H(K_d) \to Q_d$, as well as a nonnegative number C_d, such that

$$\|S_d(f,q) - S_d(\tilde f, \phi(\tilde q))\|_{G_d} \le C_d \left[\|f - \tilde f\|_{L_2(I^d)} + \|q - \tilde q\|_{L_2(I^d)}\right] \\ \forall\, [f,q] \in H_{d,\rho_1} \times Q_d,\ [\tilde f, \tilde q] \in H(K_d) \times H(K_d). \tag{2.13}$$

(2) There exists $\alpha \ge 0$ such that

$$N_\alpha := \sup_{d\in\mathbb{Z}^{++}} \frac{C_d \|\,\mathrm{App}_d\,\|_{\mathrm{Lin}[H(K_d), L_2(I^d)]}}{d^\alpha\, \mathrm{ErrCrit}(S_d)} < \infty. \tag{2.14}$$

Here, C_d is from (2.13) and App_d is the embedding, $\mathrm{App}_d\, f = f$, of $H(K_d)$ into $L_2(I^d)$.

Under these assumptions, [14, Theorem 3] tells us that the quasilinear problem $S = \{S_d\}_{d\in\mathbb{Z}^{++}}$ is tractable if $\alpha > 0$ and strongly tractable if $\alpha = 0$. More specific estimates with the exponents of tractability or strong tractability will be presented later.

The first assumption (2.13) establishes a Lipschitz condition for S_d. It also implies that for any $q \in Q_d$, the linear operator $S_d(\cdot, q)\colon H(K_d) \to G_d$ is continuous. To see this, note that if we take $\tilde q = q$ and $\tilde f = 0$, then $S_d(\tilde f, \phi(\tilde q)) = 0$, so that (2.5) and (2.13) imply that

$$\|S_d(f,q)\|_{G_d} \le C_d \|f\|_{L_2(I^d)} \le C_d \sigma_d(\kappa_1) \|f\|_{H(K_d)},$$

as claimed.

To verify that the second assumption (2.14) holds, we will need to estimate the norm of App_d. Note that (2.5) implies that the embedding App_d is well-defined, with

$$\|\mathrm{App}_d\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} \le \sigma_d(\kappa_1). \tag{2.15}$$

More precise results for $\|\mathrm{App}_d\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}$ are given in [12]:

(1) There exists $c_d \in [\kappa_2, \kappa_1]$ such that

$$\|\mathrm{App}_d\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} = \sigma_d(c_d).$$

This result holds for any value of $\kappa_2 \ge 0$.

(2) When $\kappa_2 = 0$, we have the explicit formula

$$\|\mathrm{App}_d\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} = \max_{\mathfrak{u}\in\mathscr{P}_d} \left[\gamma_{d,\mathfrak{u}} \|W\|^{|\mathfrak{u}|}_{\mathrm{Lin}[H(K)]}\right]^{1/2},$$

where the operator $W \in \mathrm{Lin}[H(K)]$ is defined as

$$Wf = \int_0^1 K(x,\cdot) f(x)\,dx \qquad \forall f \in H(K). \tag{2.16}$$

Since K is nonzero, the norm of W is positive.

3. The Dirichlet problem

We now apply the machinery of [14] to the problem of approximating solutions to the variational form of the Dirichlet problem for the Helmholtz equation.

3.1. **Some preliminary bounds.**

We already know that $H(K_d)$ is embedded in $L_2(I^d)$. Using condition (2.1), it is easy to see that $H(K_d)$ is also embedded in $L_\infty(I^d)$.

Lemma 3.1.

$$\|g\|_{L_\infty(I^d)} \le \sigma_d(\kappa_0)\|g\|_{H(K_d)} \qquad \forall\, g \in H(K_d).$$

Proof. For any $g \in H(K_d)$ and $\mathbf{x} \in I^d$, we have

$$g(\mathbf{x}) = \langle g, K_d(\cdot,\mathbf{x})\rangle_{H(K_d)},$$

and thus

$$|g(\mathbf{x})| \le \|g\|_{H(K_d)} \|K_d(\cdot,\mathbf{x})\|_{H(K_d)} = \|g\|_{H(K_d)} \sqrt{K_d(\mathbf{x},\mathbf{x})}.$$

Moreover,

$$K_d(\mathbf{x},\mathbf{x}) = \sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} K(x_j,x_j) \le \sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \kappa_0^{|\mathfrak{u}|} = \sigma_d^2(\kappa_0)$$

for almost every $\mathbf{x} \in I^d$. Thus

$$\|g\|_{L_\infty(I^d)} \le \|g\|_{H(K_d)} \sup_{\mathbf{x}\in I^d} \sqrt{K_d(\mathbf{x},\mathbf{x})} \le \sigma_d(\kappa_0)\|g\|_{H(K_d)},$$

as claimed. □

Although it is known that the bilinear form $B_d(\cdot,\cdot;q)$ is strongly $H_0^1(I^d)$-coercive and bounded for any $q \in Q_d^*$, we include a formal proof of this fact, so that we can establish values of the coercivity and bounding factors.

Lemma 3.2. *For any $q \in Q_d^*$, we have*

$$B_d(v, v; q) \geq \tfrac{2}{3}\|v\|^2_{H_0^1(I^d)} \qquad \forall\, v \in H_0^1(I^d),$$

and

$$|B_d(v, w; q)| \,\leq\, \max\{1, \|q\|_{L_\infty(I^d)}\}\, \|v\|_{H_0^1(I^d)}\|w\|_{H_0^1(I^d)} \qquad \forall\, v, w \in H_0^1(I^d).$$

Proof. Let $v, w \in H_0^1(I^d)$. From the proof of Poincaré's inequality [1, Lemma 6.30], we see that

$$\|\cdot\|_{L_2(I^d)} \leq \frac{1}{\sqrt{2}}\,|\cdot|_{H^1(I^d)} \qquad \text{on } H_0^1(I^d). \tag{3.1}$$

Hence

$$\begin{aligned} B_d(v, v; q) &= \int_{I^d}[|\nabla v|^2 + qv^2] \geq \int_{I^d}|\nabla v|^2 \\ &= \tfrac{1}{3}\int_{I^d}|\nabla v|^2 + \tfrac{2}{3}\int_{I^d}|\nabla v|^2 \geq \tfrac{2}{3}\left[\int_{I^d}|v|^2 + \int_{I^d}|\nabla v|^2\right] \\ &= \tfrac{2}{3}\|v\|^2_{H_0^1(I^d)}. \end{aligned} \tag{3.2}$$

On the other hand,

$$B_d(v, v; q) \leq \max\{1, \|q\|_{L_\infty(I^d)}\}\|v\|^2_{H_0^1(I^d)}. \tag{3.3}$$

Using (3.2) and (3.3), we see that $B_d(\cdot, \cdot; q)$ is an inner product on $H_0^1(I^d)$; its associated norm $B_d^{1/2}(\cdot, \cdot; q)$ is equivalent to the usual norm $\|\cdot\|_{H_0^1(I^d)}$. Hence using the Cauchy-Schwarz inequality, along with (3.3), we find that

$$|B_d(v, w; q)| \leq \sqrt{B_d(v, v; q)}\sqrt{B_d(w, w; q)} \leq \max\{1, \|q\|_{L_\infty(I^d)}\}\|v\|_{H_0^1(I^d)}\|w\|_{H_0^1(I^d)}$$

holds, as required. □

Since $H(K_d)$ is embedded in $L_2(I^d)$, the Lax-Milgram Lemma [3, pg. 29] and Lemma 3.2 tell us that for any $[f, q] \in H(K_d) \times Q_d^*$, the problem (2.9) has a unique solution $u = S_d^{\mathrm{DIR}}(f, q) \in H_0^1(I^d)$. In other words, the solution operator $S_d^{\mathrm{DIR}}\colon H(K_d) \times Q_d^* \to H_0^1(I^d)$ is well-defined.

We now show that S_d^{DIR} satisfies a Lipschitz condition.

Lemma 3.3. *Let*

$$C_d^{\mathrm{DIR}} = \tfrac{3}{2}\max\{1, \rho_1(e-1)\sigma_d(\kappa_0)\}. \tag{3.4}$$

For any $[f, q] \in H_{d,\rho_1} \times Q_d^*$ and $[\tilde f, \tilde q] \in H(K_d) \times Q_d^*$, we have

$$\|S_d^{\mathrm{DIR}}(f, q) - S_d^{\mathrm{DIR}}(\tilde f, \tilde q)\|_{H_0^1(I^d)} \leq C_d^{\mathrm{DIR}}\left[\|f - \tilde f\|_{L_2(I^d)} + \|q - \tilde q\|_{L_2(I^d)}\right].$$

Proof. Let $u = S_d^{\mathrm{DIR}}(f, q)$ and $\tilde u = S_d^{\mathrm{DIR}}(\tilde f, \tilde q)$. For any $w \in H_0^1(I^d)$, we have

$$\begin{aligned} \langle f - \tilde f, w\rangle_{L_2(I^d)} &= B_d(u, w; q) - B_d(\tilde u, w; \tilde q) \\ &= \int_{I^d}[\nabla(u - \tilde u)\cdot\nabla w + \tilde q(u - \tilde u)w] + \langle q - \tilde q, uw\rangle_{L_2(I^d)} \\ &= B_d(u - \tilde u, w; \tilde q) + \langle q - \tilde q, uw\rangle_{L_2(I^d)}. \end{aligned}$$

Taking $w = u - \tilde u$, we have

$$B_d(w, w; \tilde q) = \langle f - \tilde f, w\rangle_{L_2(I^d)} - \langle q - \tilde q, uw\rangle_{L_2(I^d)}.$$

From Lemma 3.2, we have

$$B_d(w, w; \tilde{q}) \geq \tfrac{2}{3}\|w\|^2_{H_0^1(I^d)},$$

and thus

$$\|w\|^2_{H_0^1(I^d)} \leq \tfrac{3}{2}\left[\left|\langle f - \tilde{f}, w\rangle_{L_2(I^d)}\right| + \left|\langle q - \tilde{q}, uw\rangle_{L_2(I^d)}\right|\right]. \tag{3.5}$$

Now

$$\left|\langle f - \tilde{f}, w\rangle_{L_2(I^d)}\right| \leq \|f - \tilde{f}\|_{L_2(I^d)}\|w\|_{H_0^1(I^d)}. \tag{3.6}$$

Theorem 3.7 of [5] allows us to estimate the L_∞-norm of the solution u in terms of the same norm of the right-hand-side function f. More precisely, we have

$$\|u\|_{L_\infty(I^d)} \leq (e-1)\|f\|_{L_\infty(I^d)}.$$

Applying Lemma 3.1, we obtain

$$\|u\|_{L_\infty(I^d)} \leq (e-1)\sigma_d(\kappa_0)\|f\|_{H(K_d)} \leq \rho_1(e-1)\sigma_d(\kappa_0),$$

and thus

$$\begin{aligned} \left|\langle q - \tilde{q}, uw\rangle_{L_2(I^d)}\right| &\leq \|q - \tilde{q}\|_{L_2(I^d)}\|u\|_{L_\infty(I^d)}\|w\|_{L_2(I^d)} \\ &\leq \rho_1(e-1)\sigma_d(\kappa_0)\|w\|_{H_0^1(I^d)}\|q - \tilde{q}\|_{L_2(I^d)}. \end{aligned} \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5) and remembering that $w = u - \tilde{u}$, we immediately get

$$\begin{aligned} \|u - \tilde{u}\|_{H_0^1(I^d)} &\leq \tfrac{3}{2}\left[\|f - \tilde{f}\|_{L_2(I^d)} + \rho_1(e-1)\sigma_d(\kappa_0)\|q - \tilde{q}\|_{L_2(I^d)}\right] \\ &\leq \tfrac{3}{2}\max\{1, \rho_1(e-1)\sigma_d(\kappa_0)\}\left[\|f - \tilde{f}\|_{L_2(I^d)} + \|q - \tilde{q}\|_{L_2(I^d)}\right], \end{aligned}$$

as claimed. □

Since $H(K_d)$ is embedded in $L_\infty(I^d)$, we can define a mapping $\phi\colon H(K_d) \to Q_d^*$ by

$$\phi(v)(\mathbf{x}) = v_+(\mathbf{x}) := \max\{v(\mathbf{x}), 0\} \qquad \forall\, \mathbf{x} \in I^d, v \in H(K_d).$$

We are now ready to show that S_d^{DIR} for our elliptic Dirichlet problem is quasilinear, i.e., (2.13) holds.

Lemma 3.4. *Let C_d^{DIR} be defined as in Lemma 3.3. Then*

$$\begin{aligned} \|S_d^{\mathrm{DIR}}(f,q) - S_d^{\mathrm{DIR}}(\tilde{f}, \phi(\tilde{q}))\|_{H_0^1(I^d)} &\leq C_d^{\mathrm{DIR}}\left[\|f - \tilde{f}\|_{L_2(I^d)} + \|q - \tilde{q}\|_{L_2(I^d)}\right] \\ &\forall\, [f,q] \in H_{d,\rho_1} \times Q_d^*,\ [\tilde{f}, \tilde{q}] \in H(K_d) \times H(K_d). \end{aligned}$$

Hence, S_d^{DIR} is quasilinear.

Proof. We first claim that

$$\|q - \phi(\tilde{q})\|_{L_2(I^d)} \leq \|q - \tilde{q}\|_{L_2(I^d)}.$$

Indeed, let

$$A = \{\, \mathbf{x} \in I^d : \tilde{q}(\mathbf{x}) \geq 0 \,\} \qquad \text{and} \qquad B = \{\, \mathbf{x} \in I^d : \tilde{q}(\mathbf{x}) < 0 \,\},$$

so that

$$\phi(\tilde{q})(\mathbf{x}) = \begin{cases} \tilde{q}(\mathbf{x}) & \text{if } \mathbf{x} \in A, \\ 0 & \text{if } \mathbf{x} \in B. \end{cases}$$

Now for any $\mathbf{x} \in B$, we have $\tilde{q}(\mathbf{x}) < 0$ and $q(\mathbf{x}) \geq 0$, and thus $0 \leq q(\mathbf{x}) < q(\mathbf{x}) - \tilde{q}(\mathbf{x})$. Hence $\|q\|^2_{L_2(B)} \leq \|q - \tilde{q}\|^2_{L_2(B)}$, and so

$$\begin{aligned} \|q - \phi(\tilde{q})\|^2_{L_2(I^d)} &= \|q - \tilde{q}\|^2_{L_2(A)} + \|q\|^2_{L_2(B)} \leq \|q - \tilde{q}\|^2_{L_2(A)} + \|q - \tilde{q}\|^2_{L_2(B)} \\ &= \|q - \tilde{q}\|^2_{L_2(I^d)}, \end{aligned}$$

as claimed. Using this inequality along with Lemma 3.3, we have

$$\begin{aligned} \|S_d^{\mathrm{DIR}}(f, q) - S_d^{\mathrm{DIR}}(\tilde{f}, \phi(\tilde{q}))\|_{H_0^1(I^d)} &\leq C_d^{\mathrm{DIR}} \left[\|f - \tilde{f}\|_{L_2(I^d)} + \|q - \phi(\tilde{q})\|_{L_2(I^d)} \right] \\ &\leq C_d^{\mathrm{DIR}} \left[\|f - \tilde{f}\|_{L_2(I^d)} + \|q - \tilde{q}\|_{L_2(I^d)} \right], \end{aligned}$$

as required. This proves that S_d^{DIR} is quasilinear, as claimed. □

3.2. **The absolute error criterion.** We are now ready to begin establishing tractability results for the elliptic Dirichlet problem. Our first result establishes tractability under the absolute error criterion. Since $\mathrm{ErrCrit}(S_d) = 1$, finding α for which (2.14) is satisfied means that we need to determine α such that

$$C_d^{\mathrm{DIR}} \| \mathrm{App}_d \|_{\mathrm{Lin}[H(K_d), L_2(I^d)]} = O(d^\alpha).$$

Theorem 3.5. *The elliptic Dirichlet problem, defined for the spaces $H(K_d)$ with finite-order weights of order ω, is tractable for the absolute error. More precisely, for N_ω defined by (2.14), we have*

$$N_\omega \leq \tfrac{3}{2} \max\left\{1, \rho_1(e-1)\sqrt{2\max\{1, \kappa_0^\omega\}\gamma_{\max}}\right\} \sqrt{2\max\{1, \kappa_1^\omega\}\gamma_{\max}}\,, \tag{3.8}$$

and the following bounds hold:

(1) *Suppose that $\kappa_2 > 0$.*
  (a) *For the class Λ^{all}, we have*

$$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \leq 2(\rho_1 + \rho_2)^2 N_\omega^2 \left(\frac{\kappa_1}{\kappa_2}\right)^\omega \left(\frac{1}{\varepsilon}\right)^2 d^{2\omega}.$$

  Hence

$$p_{\mathrm{err}}^{\mathrm{abs}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \leq 2 \qquad \textit{and} \qquad p_{\mathrm{dim}}^{\mathrm{abs}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \leq 2\omega.$$

  (b) *For the class Λ^{std}, we have*

$$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d^{\mathrm{DIR}}, \Lambda^{\mathrm{std}}) \leq \left\lceil 8(\rho_1 + \rho_2)^4 N_\omega^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{4\omega} \right\rceil + 1,$$

  and so

$$p_{\mathrm{err}}^{\mathrm{abs}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{std}}) \leq 4 \qquad \textit{and} \qquad p_{\mathrm{dim}}^{\mathrm{abs}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{std}}) \leq 4\omega.$$

(2) *Suppose that $\kappa_2 = 0$, and let*

$$\Gamma = \frac{\max\{1, \kappa_1\}}{\min\{1, \|W\|_{\mathrm{Lin}[H(K)]}\}}. \tag{3.9}$$

  Then we have the following results:

(a) *For the class* Λ^{all}*, we have*

$$\operatorname{card}^{\text{abs}}(\varepsilon, S_d^{\text{DIR}}, \Lambda^{\text{all}}) \le 4(\rho_1+\rho_2)^2 N_\omega^2 \Gamma^\omega \left(\frac{1}{\varepsilon}\right)^2 d^{3\omega},$$

and so

$$p_{\text{err}}^{\text{abs}}(S^{\text{DIR}}, \Lambda^{\text{all}}) \le 2 \qquad \textit{and} \qquad p_{\text{dim}}^{\text{abs}}(S^{\text{DIR}}, \Lambda^{\text{all}}) \le 3\omega.$$

(b) *For the class* Λ^{std}*, we have*

$$\operatorname{card}^{\text{abs}}(\varepsilon, S_d^{\text{DIR}}, \Lambda^{\text{std}}) \le \left\lceil 32(\rho_1+\rho_2)^4 N_\omega^4 \Gamma^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{6\omega} \right\rceil + 1,$$

and so

$$p_{\text{err}}^{\text{abs}}(S^{\text{DIR}}, \Lambda^{\text{std}}) \le 4 \qquad \textit{and} \qquad p_{\text{dim}}^{\text{abs}}(S^{\text{DIR}}, \Lambda^{\text{std}}) \le 6\omega.$$

Proof. Using (2.7), (2.15), and (3.4), we find that

$$C_d^{\text{DIR}} \| \operatorname{App}_d \|_{\operatorname{Lin}[H(K_d), L_2(I^d)]} \le \tfrac{3}{2} \max\left\{1, \rho_1(e-1)\sqrt{2\max\{1,\kappa_0^\omega\}\boldsymbol{\gamma}_{\max}}\right\} \sqrt{2\max\{1,\kappa_1^\omega\}\boldsymbol{\gamma}_{\max}} \cdot d^\omega.$$

Hence setting $\alpha = \omega$ in (2.14), we obtain (3.8). The remaining results of this theorem now follow from [14, Theorem 7], with $\alpha = \omega$. ☐

Example 3.6. Suppose that K is the min-kernel $K_{\min}$. Since $\kappa_0 = 1$ and $\kappa_1 = \frac{1}{2}$, we have

$$N_\omega \le \tfrac{3}{2} \max\left\{1, \rho_1(e-1)\sqrt{2\boldsymbol{\gamma}_{\max}}\right\} \sqrt{2\boldsymbol{\gamma}_{\max}}$$

from (3.8). Furthermore, since $\kappa_2 = \frac{1}{3} \neq 0$, we see that case 1 holds in Theorem 3.5. Hence we find that the elliptic Dirichlet problem is now tractable under the absolute error criterion, with

$$p_{\text{err}}^{\text{abs}}(S^{\text{DIR}}, \Lambda^{\text{all}}) \le 2 \qquad \text{and} \qquad p_{\text{dim}}^{\text{abs}}(S^{\text{DIR}}, \Lambda^{\text{all}}) \le 2\omega$$

for continuous linear information, and

$$p_{\text{err}}^{\text{abs}}(S^{\text{DIR}}, \Lambda^{\text{std}}) \le 4 \qquad \text{and} \qquad p_{\text{dim}}^{\text{abs}}(S^{\text{DIR}}, \Lambda^{\text{std}}) \le 4\omega$$

for standard information. ☐

Theorem 3.5 tells us that the elliptic Dirichlet problem for the absolute error criterion is tractable for any finite-order weighted RKHS, no matter what set of weights is used. The reason we are unable to establish strong tractability in this case is that the Lipschitz constant C_d^{DIR} and $\| \operatorname{App}_d \|_{\operatorname{Lin}[H(K_d), L_2(I^d)]}$ are expressed in terms of $\sigma_d(\kappa_0)$ and $\sigma(\kappa_1)$, whose product is bounded by a polynomial of degree ω in d. Hence we can only guarantee that N_ω is finite. It is proved in [14, Theorem 7] that strong tractability holds if $\kappa_2 > 0$ and if N_0 is finite. We can guarantee that N_0 is finite if we follow the approach taken in [14, Theorem 8].

Theorem 3.7. *Suppose that* $\kappa_2 > 0$ *and*

$$\rho_3 := \sup_{d \in \mathbb{Z}^{++}} \sum_{u \in \mathscr{P}_d} \gamma_{d,u} < \infty. \tag{3.10}$$

The elliptic Dirichlet problem defined for the spaces $H(K_d)$ with finite-order weights of order ω satisfying (3.10) *is strongly tractable for the absolute error. More precisely, for N_0 defined by* (2.14)*, we have*

$$N_0 \le \tfrac{3}{2}\rho_3^{1/2}\max\{1,\kappa_1^{\omega/2}\}\max\left\{1,\rho_1\rho_3^{1/2}(e-1)\max\{1,\kappa_0^{\omega/2}\}\right\}, \tag{3.11}$$

and the following bounds hold:

(1) *For the class Λ^{all}, we have*

$$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \le 2(\rho_1+\rho_2)^2 N_0^2 \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega}\left(\frac{1}{\varepsilon}\right)^2.$$

Hence

$$p_{\mathrm{strong}}^{\mathrm{abs}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \le 2.$$

(2) *For the class Λ^{std}, we have*

$$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d^{\mathrm{DIR}}, \Lambda^{\mathrm{std}}) \le \left\lceil 8(\rho_1+\rho_2)^4 N_0^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega}\left(\frac{1}{\varepsilon}\right)^4\right\rceil + 1.$$

Hence

$$p_{\mathrm{strong}}^{\mathrm{abs}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{std}}) \le 4.$$

Proof. Using (3.10), it follows that

$$\sigma_d(\theta) \;\le\; \rho_3^{1/2}\max\{1,\theta^{\omega/2}\} \qquad \forall\,\theta\in\mathbb{R}^+. \tag{3.12}$$

From (2.14), (2.15), and (3.12), we have

$$N_0 \le C^{\mathrm{DIR}}\rho_3^{1/2}\max\{1,\kappa_1^{\omega/2}\},$$

where

$$\begin{aligned} C^{\mathrm{DIR}} = \sup_{d\in\mathbb{Z}^{++}} C_d^{\mathrm{DIR}} &= \tfrac{3}{2}\max\left\{1,\rho_1(e-1)\sup_{d\in\mathbb{Z}^{++}}\sigma_d(\kappa_0)\right\} \\ &\le \tfrac{3}{2}\max\left\{1,\rho_1(e-1)\rho_3^{1/2}\max\{1,\kappa_0^{\omega/2}\}\right\} \end{aligned}$$

by (3.4) and (3.12). Combining these results, we obtain (3.11). The desired result now follows from [14, Theorem 8]. □

Example 3.8. Suppose once again that $K = K_{\min}$. Assume that (3.10) holds. Then the conditions of Theorem 3.7 are satisfied with

$$N_0 \le \tfrac{3}{2}\rho_3^{1/2}\max\{1,\rho_1\rho_3^{1/2}(e-1)\} \qquad \text{and} \qquad \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} = \left(\frac{3}{2}\right)^{\omega}.$$

Hence, the elliptic Dirichlet problem is now strongly tractable under the absolute error criterion, with

$$p_{\mathrm{strong}}^{\mathrm{abs}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \le 2 \qquad \text{and} \qquad p_{\mathrm{strong}}^{\mathrm{abs}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \le 4. \quad \square$$

3.3. **The normalized error criterion.** We now consider the elliptic Dirichlet problem for finite-order weights under the normalized error criterion. For this error criterion, we need a lower bound estimate on the initial error.

Lemma 3.9. *Define the set*

$$H^1_{0,*}(I) = \left\{ \theta \in H^1_0(I) : \int_0^1 \theta(x)\,dx = 1 \right\}.$$

Then for any $d \in \mathbb{Z}^{++}$, we have

$$e(0, S_d^{\mathrm{DIR}}) \geq \rho_1 \sqrt{\frac{2}{3d}} \sup_{\theta \in H^1_{0,*}(I)} \frac{\sigma_d\big(\tau(\theta)\big)}{\|\theta\|^{d-1}_{L_2(I)} \|\theta'\|_{L_2(I)}},$$

where

$$\tau(\theta) = \int_0^1 \int_0^1 \theta(x)\theta(y)K(x,y)\,dx\,dy \qquad \forall\, \theta \in L_\infty(I). \tag{3.13}$$

Proof. Since our problem is quasilinear, we may use (2.11) to see that

$$\begin{aligned} e(0, S_d^{\mathrm{DIR}}) &= \rho_1 \sup_{q \in Q_d^* \cap H_{d,\rho_2}} \|S_d^{\mathrm{DIR}}(\cdot, q)\|_{\mathrm{Lin}[H(K_d), H^1_0(I^d)]} \\ &\geq \rho_1 \|S_d^{\mathrm{DIR}}(\cdot, 0)\|_{\mathrm{Lin}[H(K_d), H^1_0(I^d)]}. \end{aligned} \tag{3.14}$$

Now let $f \in H(K_d)$ and $w \in H^1_0(I^d)$. Let $u = S_d^{\mathrm{DIR}}(f, 0)$. Then

$$\|u\|_{H^1_0(I^d)} \|w\|_{H^1_0(I^d)} \geq |u|_{H^1_0(I^d)} |w|_{H^1_0(I^d)} \geq \left| \int_{I^d} \nabla u \cdot \nabla w \right| = \left| \int_{I^d} f w \right|.$$

It is easy to see that

$$\mathrm{Int}_{d,w}(g) = \int_{I^d} g(\mathbf{x}) w(\mathbf{x})\,d\mathbf{x} \qquad \forall\, g \in H(K_d)$$

is a continuous linear functional. From [12, Lemma 2], we know that

$$\|\,\mathrm{Int}_{d,w}\,\|^2_{[H(K_d)]^*} = \int_{I^d} \int_{I^d} w(\mathbf{x}) w(\mathbf{y}) K_d(\mathbf{x}, \mathbf{y})\,d\mathbf{x}\,d\mathbf{y}. \tag{3.15}$$

The previous inequality may be rewritten as

$$\frac{\|S_d^{\mathrm{DIR}}(f,0)\|_{H^1_0(I^d)}}{\|f\|_{H(K_d)}} \geq \frac{1}{\|w\|_{H^1_0(I^d)}} \frac{|\,\mathrm{Int}_{d,w}(f)|}{\|f\|_{H(K_d)}}.$$

Since $f \in H(K_d)$ and $w \in H^1_0(I^d)$ are arbitrary, this implies that

$$\|S_d^{\mathrm{DIR}}(\cdot, 0)\|_{\mathrm{Lin}[H(K_d), H^1_0(I^d)]} \geq \sup_{w \in H^1_0(I^d)} \frac{\|\,\mathrm{Int}_{d,w}\,\|_{[H(K_d)]^*}}{\|w\|_{H^1_0(I^d)}}. \tag{3.16}$$

Now let $\theta \in H^1_{0,*}(I)$, and define

$$w_{d,\theta}(\mathbf{x}) = \theta(x_1) \dots \theta(x_d) \qquad \forall\, \mathbf{x} = (x_1, \dots, x_d) \in \bar{I}^d. \tag{3.17}$$

Since $w_{d,\theta}$ vanishes on ∂I^d, we have $w_{d,\theta} \in H^1_0(I^d)$. Let us calculate an upper bound on $\|w_{d,\theta}\|_{H^1_0(I^d)}$. Using (3.2), we have

$$\|w_{d,\theta}\|^2_{H^1_0(I^d)} \leq \tfrac{3}{2} \int_{I^d} |\nabla w_{d,\theta}|^2 = \tfrac{3}{2} \sum_{j=1}^d \|\partial_j w_{d,\theta}\|^2_{L_2(I^d)}. \tag{3.18}$$

Now for any $j \in \{1, \dots, d\}$, we have

$$\partial_j w_{d,\theta}(\mathbf{x}) = \Bigg[\prod_{\substack{1\le i\le d\\ i\ne j}} \theta(x_i)\Bigg]\theta'(x_j), \qquad \forall\, \mathbf{x} = (x_1, \dots, x_d) \in \bar{I}^d,$$

and so

$$\|\partial_j w_{d,\theta}\|^2_{L_2(I^d)} = \|\theta\|^{2d-2}_{L_2(I)}\, \|\theta'\|^2_{L_2(I)}.$$

Substituting this equality into (3.18), we find

$$\|w_{d,\theta}\|_{H^1(I^d)} \le \sqrt{\frac{3d}{2}}\, \|\theta\|^{d-1}_{L_2(I)}\, \|\theta'\|_{L_2(I)}. \tag{3.19}$$

Using (3.15), we find that

$$\begin{aligned}
\|\operatorname{Int}_{d,w_{d,\theta}}\|_{[H(K_d)]^*} &= \left(\int_{I^d}\int_{I^d} w_{d,\theta}(\mathbf{x}) w_{d,\theta}(\mathbf{y}) K_d(\mathbf{x},\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}\right)^{1/2}\\
&= \left(\sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \int_{I^d}\int_{I^d} \prod_{j=1}^d \theta(x_j)\theta(y_j) \prod_{j\in\mathfrak{u}} K(x_j, y_j)\, d\mathbf{x}\, d\mathbf{y}\right)^{1/2}\\
&= \left(\sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \left(\int_0^1\int_0^1 \theta(x)\theta(y)K(x,y)\, dx\, dy\right)^{|\mathfrak{u}|}\right)^{1/2}\\
&= \left(\sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \tau(\theta)^{|\mathfrak{u}|}\right)^{1/2} = \sigma\big(\tau(\theta)\big).
\end{aligned}$$

Using this result, (3.16), and (3.19), we get our desired lower bound on the initial error. □

To use Lemma 3.9, we need to choose a function $\theta \in H^1_{0,*}(I)$ for each $d \in \mathbb{Z}^{++}$ and to estimate $\sigma_d\big(\tau(\theta)\big)/\Big(\|\theta\|^{d-1}_{L_2(I)}\|\theta'\|_{L_2(I)}\Big)$ from below. One possibility is as follows.

For $\delta \in (0, \frac{1}{2}]$, let

$$\theta_\delta(x) = \begin{cases} \dfrac{x}{\delta(1-\delta)} & \text{if } 0 \le x \le \delta,\\[2ex] \dfrac{1}{1-\delta} & \text{if } \delta \le x \le 1-\delta,\\[2ex] \dfrac{1-x}{\delta(1-\delta)} & \text{if } 1-\delta \le x \le 1. \end{cases} \tag{3.20}$$

Clearly, $\theta_\delta \in H^1_{0,*}(I)$. A straightforward calculation yields

$$\int_0^1 \theta_\delta^2(x)\, dx = \frac{3-4\delta}{3(1-\delta)^2}$$

and

$$\int_0^1 [\theta'_\delta(x)]^2\, dx = \frac{2}{\delta(1-\delta)^2}.$$

Hence

$$\|\theta_\delta\|_{L_2(I)}^{d-1}\|\theta'_\delta\|_{L_2(I)} = \sqrt{\left(\frac{3-4\delta}{3(1-\delta)^2}\right)^{d-1}\frac{2}{\delta(1-\delta)^2}}\,. \tag{3.21}$$

We now choose $\delta = \delta_d$ such that (3.21) is of order $\sqrt{d}$. It is easy to see that this can be achieved by taking $\delta_d = \Theta(d^{-1})$. Since we want to control the constants, we need to see the details, which are as follows.

For $d = 1$ we choose $\delta = \delta_1 = \frac{1}{3}$ and obtain

$$\|\theta_\delta\|_{L_2(I)}^{d-1}\|\theta'_\delta\|_{L_2(I)} = \tfrac{3}{2}\sqrt{6} = \tfrac{3}{2}\sqrt{6d}.$$

For $d \geq 2$, let

$$\alpha_d = \left(\frac{4}{3}\right)^{1/(d-1)}, \tag{3.22}$$

so that

$$\alpha_2 = \tfrac{4}{3} > \alpha_3 > \cdots > 1 \qquad \text{with} \qquad \lim_{d\to\infty} \alpha_d = 1.$$

Let

$$\delta_d = 1 - \frac{1}{3\alpha_d}(2 + \sqrt{4 - 3\alpha_d}), \tag{3.23}$$

which is a solution to

$$\frac{3-4\delta_d}{3(1-\delta_d)^2} = \alpha_d. \tag{3.24}$$

Since $\alpha_d \in (1, \frac{4}{3}]$, we see that $\delta_d \in (0, \frac{1}{2}]$. Clearly, for large d we have

$$\alpha_d \approx 1 + \frac{\ln 4/3}{d-1} \qquad \text{and} \qquad \delta_d \approx \frac{3/2 \ln 4/3}{d-1}.$$

Now

$$\frac{2}{\delta_d(1-\delta_d)^2} = \zeta(\alpha_d) := \frac{54\alpha_d^3}{(2+\sqrt{4-3\alpha_d})^2(3\alpha_d - 2 - \sqrt{4-3\alpha_d})}.$$

Moreover, we have

$$\frac{2}{d \cdot \delta_d(1-\delta_d)^2} = \eta(\alpha_d) := \frac{\zeta(\alpha_d)}{1 + \dfrac{\ln\frac{4}{3}}{\ln \alpha_d}}.$$

Plotting the function η, we see that η is increasing over the interval $[1, \frac{4}{3}]$, with $\eta(\frac{4}{3}) = 8$. Hence

$$\frac{2}{\delta_d(1-\delta_d)^2} \leq 8d. \tag{3.25}$$

Using (3.21)–(3.25), we find that for $d \geq 2$ we have

$$\|\theta_\delta\|_{L_2(I)}^{d-1}\|\theta'_\delta\|_{L_2(I)} \leq \tfrac{4}{3}\sqrt{6d}.$$

Combining the two cases for $d = 1$ and $d \geq 2$ we write

$$\|\theta_\delta\|_{L_2(I)}^{d-1}\|\theta'_\delta\|_{L_2(I)} \leq \left(\tfrac{3}{2}\delta_{d,1} + \tfrac{4}{3}(1-\delta_{d,1})\right)\sqrt{6d},$$

where $\delta_{d,1}$ denotes the Kronecker delta.

Applying Lemma 3.9 with $\theta = \theta_{\delta_d}$, we have proved the following lemma.

Lemma 3.10. *Let*

$$\tau_{0,d} = \tau\left(\theta_{\delta_d}\right),$$

where

- $\tau(\cdot)$ *is given by* (3.13), *and*
- θ_{δ_d} *is given by* (3.20), *with*

$$\delta = \begin{cases} \frac{1}{3} & \text{for } d = 1, \\ \delta_d \text{ as defined in (3.22)–(3.23)} & \text{for } d \geq 2. \end{cases}$$

Then for any $d \in \mathbb{Z}^{++}$, *we have*

$$e(0, S_d^{\mathrm{DIR}}) \geq \frac{2\rho_1 \sigma_d(\tau_{0,d})}{9\delta_{d,1} + 8(1-\delta_{d,1})} \cdot \frac{1}{d}. \quad \square$$

We now find that the elliptic Dirichlet problem is always tractable for finite-order weights, modulo one technical assumption. Recall the definitions (3.13) and (3.20) of the functions τ and θ_δ, respectively. We will require that

$$\exists\, \tau_0 > 0 \text{ such that } \tau(\theta_\delta) \geq \tau_0 \qquad \forall\, \delta \in (0, \tfrac{1}{2}]. \tag{3.26}$$

Note the following:

(1) Condition (3.26) can only hold for $\tau_0 \leq \kappa_2$. To see that this is true, note that $\lim_{\delta\to 0} \theta_\delta = 1$ in $(0,1)$. Using the Lebesgue dominated convergence theorem, we find that

$$\lim_{\delta\to 0} \tau(\theta_\delta) = \tau(1) = \kappa_2. \tag{3.27}$$

In particular, this means that (3.26) cannot hold if $\kappa_2 = 0$.

(2) We claim that condition (3.26) automatically holds whenever $\kappa_2 > 0$ and the kernel K is strictly positive definite. Indeed, under these conditions, we have $\tau(\theta_\delta) > 0$ for all $\delta \in (0, \frac{1}{2}]$ and $\tau(1) = \kappa_2 > 0$. Using (3.27), we see that $\delta \mapsto \tau(\theta_\delta)$ is a continuous function from $[0, \frac{1}{2}] \to \mathbb{R}^{++}$. Hence (3.26) holds, as claimed.

We are now ready to prove the following tractability result.

Theorem 3.11. *Suppose that* (3.26) *holds, so that* $\kappa_2 > 0$. *Then the elliptic Dirichlet problem, defined for the spaces* $H(K_d)$ *with finite-order weights of order* ω, *is tractable for the normalized error. More precisely, for* $N_{1+\omega/2}$ *defined by* (2.14), *we have*

$$N_{1+\omega/2} \leq \frac{27 \max\left\{1, \rho_1(e-1)\sqrt{2\gamma_{\max}}\max\{1, \kappa_0^{\omega/2}\}\right\}}{\rho_1} \left(\frac{\kappa_1}{\tau_0}\right)^{\omega/2}, \tag{3.28}$$

and the following bounds hold:

(1) *For the class* Λ^{all}, *we have*

$$\mathrm{card}^{\mathrm{nor}}(\varepsilon, S_d^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \leq 2(\rho_1+\rho_2)^2 N_{1+\omega/2}^2 \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} \left(\frac{1}{\varepsilon}\right)^2 d^{2+\omega}.$$

Hence

$$p_{\mathrm{err}}^{\mathrm{nor}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \leq 2 \qquad \text{and} \qquad p_{\mathrm{dim}}^{\mathrm{nor}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \leq 2 + \omega.$$

(2) *For the class* Λ^{std}, *we have*

$$\mathrm{card}^{\mathrm{nor}}(\varepsilon, S_d^{\mathrm{DIR}}, \Lambda^{\mathrm{std}}) \le \left\lceil 8(\rho_1+\rho_2)^4 N_{1+\omega/2}^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{4+2\omega} \right\rceil + 1,$$

and so

$$p_{\mathrm{err}}^{\mathrm{nor}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{std}}) \le 4 \qquad \text{and} \qquad p_{\mathrm{dim}}^{\mathrm{nor}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{std}}) \le 4 + 2\omega.$$

Proof. We first prove (3.28). Using Lemmas 3.3 and 3.10, along with condition (3.26), we have

$$C_d^{\mathrm{DIR}} = \tfrac{3}{2}\max\{1, \rho_1(e-1)\sigma_d(\kappa_0)\},$$
$$e(0, S_d^{\mathrm{DIR}}) \ge \frac{2\rho_1\sigma_d(\tau_0)}{(9\delta_{d,1} + 8(1-\delta_{d,1}))\ d}.$$

Hence we find that

$$\frac{C_d^{\mathrm{DIR}}\|\operatorname{App}_d\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}}{d^{1+\omega/2}e(0,S_d^{\mathrm{DIR}})} \le \frac{3\max\{1,\rho_1(e-1)\sigma_d(\kappa_0)\}}{\rho_1\, d^{\omega/2}}\ \frac{\sigma_d(\kappa_1)}{\sigma_d(\tau_0)}\,(9\delta_{d,1} + 8(1-\delta_{d,1}))\,.$$

From (2.7) we have

$$\sigma_d(\kappa_0) \le \sqrt{2\gamma_{\max}}\ \max\{1, \kappa_0^{\omega/2}\}\, d^{\omega/2},$$

and since $\tau_0 \le \kappa_1$, we have

$$\frac{\sigma_d(\kappa_1)}{\sigma_d(\tau_0)} = \left(\frac{\sum_{\mathfrak{u}\in\mathscr{P}_d,\,|\mathfrak{u}|\le\omega}\gamma_{d,\mathfrak{u}}\,\kappa_1^{|\mathfrak{u}|}}{\sum_{\mathfrak{u}\in\mathscr{P}_d,\,|\mathfrak{u}|\le\omega}\gamma_{d,\mathfrak{u}}\,\tau_0^{|\mathfrak{u}|}}\right)^{1/2} \le \left(\frac{\kappa_1}{\tau_0}\right)^{\omega/2}. \tag{3.29}$$

Hence

$$N_{1+\omega/2} = \sup_{d\in\mathbb{Z}^{++}} \frac{C_d^{\mathrm{DIR}}\|\operatorname{App}_d\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}}{d^{1+\omega/2}e(0,S_d^{\mathrm{DIR}})} \le \frac{27\max\left\{1, \rho_1(e-1)\sqrt{2\gamma_{\max}}\max\{1,\kappa_0^{\omega/2}\}\right\}}{\rho_1}\left(\frac{\kappa_1}{\tau_0}\right)^{\omega/2},$$

establishing (3.28). The theorem now follows immediately from [14, Theorem 7], with $\alpha = 1 + \omega/2$. □

Example 3.12. Let us once again consider the min-kernel $K = K_{\min}$. A straightforward (but tedious) calculation reveals that

$$\tau(\theta_\delta) = \tfrac{1}{3}(1 + \delta - \delta^2),$$

and thus (3.26) holds with $\tau_0 = \frac{1}{3}$. Since $\kappa_2 > 0$, we may use Theorem 3.11 to see that for Λ^{all}, we have

$$p_{\mathrm{err}}^{\mathrm{nor}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \le 2 \qquad \text{and} \qquad p_{\mathrm{dim}}^{\mathrm{nor}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \le 2 + \omega,$$

whereas for Λ^{std}, we have

$$p_{\mathrm{err}}^{\mathrm{nor}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{std}}) \le 4 \qquad \text{and} \qquad p_{\mathrm{dim}}^{\mathrm{nor}}(S^{\mathrm{DIR}}, \Lambda^{\mathrm{all}}) \le 4 + 2\omega. \quad \square$$

Unfortunately, we are not able to provide a strong tractability result for the elliptic Dirichlet problem under the normalized error criterion. The reason for this is that the best lower bound we know for the initial error goes linearly with d^{-1} to zero. Hence, we are unable to show that N_0 is finite, which is needed for strong tractability.

4. The Neumann problem

We now apply the machinery of [14] to the problem of approximating solutions to the variational form of the Neumann problem for the Helmholtz equation. Recall that for the Neumann problem to be well-defined, we must assume that $Q_d^{**} \cap H_{d,\rho_2}$ is nonempty. This holds, in particular, if $\gamma_{d,\emptyset} > 0$ and $q_0 \gamma_{d,\emptyset}^{-1/2} \le \rho_2$, as explained before.

4.1. **Some preliminary bounds.** It is known that for any $q \in Q_d^{**}$, the bilinear form $B_d(\cdot,\cdot;q)$ is strongly $H^1(I^d)$-coercive and bounded. However, we provide a proof of this fact, so that we can establish values for the coercivity and bounding constants, just as we did in Section 3.1.

Lemma 4.1. *For any $q \in Q_d^{**}$, we have*

$$B_d(v,v;q) \ge \min\{1,q_0\} \|v\|_{H^1(I^d)}^2 \qquad \forall\, v \in H^1(I^d),$$

and

$$|B_d(v,w;q)| \le \max\{1, \|q\|_{L_\infty(I^d)}\} \|v\|_{H^1(I^d)} \|w\|_{H^1(I^d)} \qquad \forall\, v,w \in H^1(I^d).$$

Proof. For $q \in Q_d^{**}$, we have $q \ge q_0$ and therefore

$$B_d(v,v;q) = \int_{I^d} [|\nabla v|^2 + qv^2] \ge \min\{1,q_0\} \int_{I^d} [|\nabla v|^2 + v^2] = \min\{1,q_0\} \|v\|_{H^1(I^d)}^2.$$

The rest is as in Lemma 3.2. □

Note that $q \in Q^{**}$ implies that $\|q\|_{L_\infty(I^d)} \ge q_0$. Therefore $\min\{1,q_0\} \le \max\{1, \|q\|_{L_\infty(I^d)}\}$ and the bounds in Lemma 4.1 make sense.

As in Section 3.1, the Lax-Milgram Lemma [3, pg. 29] and Lemma 4.1 tell us that for any $[f,q] \in H(K_d) \times Q_d^{**}$, the problem (2.10) has a unique solution $u = S_d^{\mathrm{NEU}}(f,q) \in H^1(I^d)$. Hence the solution operator $S_d^{\mathrm{NEU}} \colon H(K_d) \times Q_d^{**} \to H^1(I^d)$ is well-defined.

We now show that S_d^{NEU} satisfies a Lipschitz condition. This requires two preliminary steps. First, we establish a maximum principle for our problem.

Lemma 4.2. *Let $f \in H(K_d)$ and $q \in Q_d^{**}$. Then*

$$S_d^{\mathrm{NEU}}(f,q) \le \frac{M(f)}{q_0} \qquad \text{a.e. in } I^d,$$

where

$$M(f) = \operatorname*{ess\,sup}_{\mathbf{x} \in I^d} f(\mathbf{x}) \le \sigma_d(\kappa_0) \|f\|_{H(K_d)}.$$

Proof. Since the bound on $M = M(f)$ follows immediately from Lemma 3.1, we need only prove the inequality for $u = S_d^{\mathrm{NEU}}(f,q)$. Let

$$A = \left\{ \mathbf{x} \in I^d : u(\mathbf{x}) > \frac{M}{q_0} \right\}.$$

We claim that the Lebesgue measure of A is zero. Indeed, suppose otherwise, i.e., that A has positive measure. Define

$$u^*(\mathbf{x}) = \max\left\{u(\mathbf{x}) - \frac{M}{q_0}, 0\right\} \qquad \forall\, \mathbf{x} \in I^d.$$

By [15, Cor. 2.1.8], we have $u^* \in H^1(I^d)$, with

$$\nabla u^* = \begin{cases} \nabla u & \text{in } A, \\ 0 & \text{in } I^d \setminus A, \end{cases}$$

noting that $u^* > 0$ almost everywhere in A. Now in A, we have $\nabla u^* = \nabla u$, and so $|\nabla u^*|^2 = \nabla u^* \cdot \nabla u^* = \nabla u \cdot \nabla u^*$. In the complement of A, we have $\nabla u^* = 0$, so that $|\nabla u^*|^2 = 0 = \nabla u \cdot \nabla u^*$. Hence, $|\nabla u^*|^2 = \nabla u \cdot \nabla u^*$ everywhere in I^d. Moreover,

$$u(\mathbf{x}) > \frac{M}{q_0} \geq \frac{f(\mathbf{x})}{q(\mathbf{x})} \qquad \mathbf{x} \in A,$$

and so

$$f - qu < 0 \qquad \text{in } A.$$

Note that the function u^* is an admissible test function for the Neumann problem, i.e., we can take $w = u^*$ in (2.10). We thus have

$$\begin{aligned} 0 \leq \int_A |\nabla u^*|^2 &= \int_{I^d} |\nabla u^*|^2 = \int_{I^d} \nabla u \cdot \nabla u^* = B_d(u, u^*; q) - \int_{I^d} quu^* \\ &= \langle f, u^* \rangle_{L_2(I^d)} - \int_{I^d} quu^* = \int_{I^d} (f - qu)u^* = \int_A (f - qu)u^* < 0, \end{aligned}$$

which is a contradiction. Thus, A has measure zero, which implies that $u \leq M/q_0$ a.e. in I^d, establishing the lemma. □

Using this maximum principle, we can obtain an L_∞-bound for the Neumann problem:

Lemma 4.3. *Let $f \in H(K_d)$ and $q \in Q_d^{**}$. Then*

$$\|S_d^{\mathrm{NEU}}(f, q)\|_{L_\infty(I^d)} \leq \frac{1}{q_0} \|f\|_{L_\infty(I^d)} \leq \frac{\sigma_d(\kappa_0)}{q_0} \|f\|_{H(K_d)}.$$

Proof. Since the second equality follows immediately from Lemma 3.1, we need only prove the first inequality. Let $u = S_d^{\mathrm{NEU}}(f, q)$. For a.e. $\mathbf{x} \in I^d$, we may use Lemma 4.2 (once with f and once with $-f$) to find that

$$u(\mathbf{x}) \leq \frac{1}{q_0} \operatorname*{ess\,sup}_{\mathbf{y} \in I^d} f(\mathbf{y})$$

and

$$-u(\mathbf{x}) \leq \frac{1}{q_0} \operatorname*{ess\,sup}_{\mathbf{y} \in I^d} -f(\mathbf{y}).$$

Hence

$$\begin{aligned} |u(\mathbf{x})| = \max\{u(\mathbf{x}), -u(\mathbf{x})\} &\leq \frac{1}{q_0} \operatorname*{ess\,sup}_{\mathbf{y} \in I^d} \max\{f(\mathbf{y}), -f(\mathbf{y})\} \\ &= \frac{1}{q_0} \operatorname*{ess\,sup}_{\mathbf{y} \in I^d} |f(\mathbf{y})| = \frac{1}{q_0} \|f\|_{L_\infty(I^d)}, \end{aligned}$$

as required. □

Following the same ideas as in Lemma 3.3, we now show that S_d^{NEU} satisfies a Lipschitz condition.

Lemma 4.4. *Let*

$$C_d^{\mathrm{NEU}} = \frac{\max\left\{1, \dfrac{\rho_1 \sigma_d(\kappa_0)}{q_0}\right\}}{\min\{1, q_0\}}.$$

For any $[f, q] \in H_{d,\rho_1} \times Q_d^{**}$ *and* $[\tilde f, \tilde q] \in H(K_d) \times Q_d^{**}$*, we have*

$$\|S_d^{\mathrm{NEU}}(f, q) - S_d^{\mathrm{NEU}}(\tilde f, \tilde q)\|_{H^1(I^d)} \le C_d^{\mathrm{NEU}} \left[\|f - \tilde f\|_{L_2(I^d)} + \|q - \tilde q\|_{L_2(I^d)}\right].$$

Proof. Let $w = u - \tilde u$, where $u = S_d^{\mathrm{NEU}}(f, q)$ and $\tilde u = S_d^{\mathrm{NEU}}(\tilde f, \tilde q)$. As in the proof of Lemma 3.3, we have

$$B_d(w, w; \tilde q) = \langle f - \tilde f, w\rangle_{L_2(I^d)} - \langle q - \tilde q, uw\rangle_{L_2(I^d)}.$$

From Lemma 4.1, we have

$$B_d(w, w; \tilde q) \ge \min\{1, q_0\}\|w\|^2_{H^1(I^d)},$$

and thus

$$\min\{1, q_0\}\|w\|^2_{H^1(I^d)} \le \left|\langle f - \tilde f, w\rangle_{L_2(I^d)}\right| + \left|\langle q - \tilde q, uw\rangle_{L_2(I^d)}\right|. \tag{4.1}$$

Now

$$\left|\langle f - \tilde f, w\rangle_{L_2(I^d)}\right| \le \|f - \tilde f\|_{L_2(I^d)}\|w\|_{H^1(I^d)}. \tag{4.2}$$

Using Lemma 4.3, we have

$$\|u\|_{L_\infty(I^d)} \le \frac{\rho_1 \sigma_d(\kappa_0)}{q_0},$$

and thus

$$\begin{aligned} \left|\langle q - \tilde q, uw\rangle_{L_2(I^d)}\right| &\le \|q - \tilde q\|_{L_2(I^d)}\|u\|_{L_\infty(I^d)}\|w\|_{L_2(I^d)} \\ &\le \frac{\rho_1 \sigma_d(\kappa_0)}{q_0}\|w\|_{H^1(I^d)}\|q - \tilde q\|_{L_2(I^d)}. \end{aligned} \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1) and remembering that $w = u - \tilde u$, we immediately get

$$\begin{aligned} \|u - \tilde u\|_{H^1(I^d)} &\le \frac{1}{\min\{1, q_0\}}\left[\|f - \tilde f\|_{L_2(I^d)} + \frac{\rho_1 \sigma_d(\kappa_0)}{q_0}\|q - \tilde q\|_{L_2(I^d)}\right] \\ &\le \frac{\max\left\{1, \dfrac{\rho_1 \sigma_d(\kappa_0)}{q_0}\right\}}{\min\{1, q_0\}}\left[\|f - \tilde f\|_{L_2(I^d)} + \|q - \tilde q\|_{L_2(I^d)}\right], \end{aligned}$$

as claimed. □

Let us define $\phi\colon H(K_d) \to Q_d^{**}$ as

$$\phi(q)(\mathbf{x}) = \max\{q(\mathbf{x}), q_0\} = \big(q(\mathbf{x}) - q_0\big)_+ + q_0 \qquad \forall\, \mathbf{x} \in I^d, q \in H(K_d).$$

As in the previous section, we conclude that $\phi(q)$ belongs to Q_d^{**}. We are now ready to show that (2.13) holds for our elliptic Neumann problem.

Lemma 4.5. *Let C_d^{NEU} be as in Lemma 4.4. Then*

$$\|S_d^{\mathrm{NEU}}(f,q) - S_d^{\mathrm{NEU}}(\tilde{f},\phi(\tilde{q}))\|_{H^1(I^d)} \le C_d^{\mathrm{NEU}} \left[\|f-\tilde{f}\|_{L_2(I^d)} + \|q-\tilde{q}\|_{L_2(I^d)} \right]$$
$$\forall\, [f,q] \in H_{d,\rho_1} \times Q_d^{**}, [\tilde{f},\tilde{q}] \in H(K_d) \times H(K_d).$$

Hence, S_d^{NEU} is quasilinear.

Proof. We use a slight variation of the proof of Lemma 3.4. We claim that

$$\|q - \tilde{\phi(q)}\|_{L_2(I^d)} \le \|q-\tilde{q}\|_{L_2(I^d)}.$$

Indeed, let

$$A = \{\, \mathbf{x} \in I^d : \tilde{q}(\mathbf{x}) \ge q_0 \,\} \qquad \text{and} \qquad B = \{\, \mathbf{x} \in I^d : \tilde{q}(\mathbf{x}) < q_0 \,\},$$

so that

$$\tilde{\phi(q)}(\mathbf{x}) = \begin{cases} \tilde{q}(\mathbf{x}) & \text{if } \mathbf{x} \in A, \\ q_0 & \text{if } \mathbf{x} \in B. \end{cases}$$

Now for any $\mathbf{x} \in B$, we have $\tilde{q}(\mathbf{x}) < q_0$ and $q(\mathbf{x}) \ge q_0$, and thus $0 \le q(\mathbf{x}) - q_0 < q(\mathbf{x}) - \tilde{q}(\mathbf{x})$. Hence $\|q-q_0\|^2_{L_2(B)} \le \|q-\tilde{q}\|^2_{L_2(B)}$, and so

$$\begin{aligned} \|q - \tilde{\phi(q)}\|^2_{L_2(I^d)} &= \|q-\tilde{q}\|^2_{L_2(A)} + \|q-q_0\|^2_{L_2(B)} \le \|q-\tilde{q}\|^2_{L_2(A)} + \|q-\tilde{q}\|^2_{L_2(B)} \\ &= \|q-\tilde{q}\|^2_{L_2(I^d)}, \end{aligned}$$

as claimed. Using this inequality along with Lemma 4.4, we have

$$\begin{aligned} \|S_d^{\mathrm{NEU}}(f,q) - S_d^{\mathrm{NEU}}(\tilde{f},\tilde{\phi(q)})\|_{H^1(I^d)} &\le C_d^{\mathrm{NEU}} \left[\|f-\tilde{f}\|_{L_2(I^d)} + \|q-\tilde{\phi(q)}\|_{L_2(I^d)} \right] \\ &\le C_d^{\mathrm{NEU}} \left[\|f-\tilde{f}\|_{L_2(I^d)} + \|q-\tilde{q}\|_{L_2(I^d)} \right], \end{aligned}$$

as claimed. □

4.2. **The absolute error criterion.** We are now ready to begin establishing tractability results for the elliptic Neumann problem. Our first result establishes tractability under the absolute error criterion.

Theorem 4.6. *The elliptic Neumann problem, defined for the spaces $H(K_d)$ with finite-order weights of order ω, is tractable for the absolute error. More precisely, for N_ω defined by (2.14), we have*

$$N_\omega \le \frac{\max\left\{1, \dfrac{\rho_1}{q_0}\sqrt{2\max\{1,\kappa_0^\omega\}\gamma_{\max}}\right\}\sqrt{2\max\{1,\kappa_1^\omega\}\gamma_{\max}}}{\min\{1,q_0\}}, \tag{4.4}$$

and the following bounds hold:

(1) *Suppose that $\kappa_2 > 0$.*

(a) *For the class Λ^{all}, we have*

$$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d^{\mathrm{NEU}}, \Lambda^{\mathrm{all}}) \le 2(\rho_1+\rho_2)^2 N_\omega^2 \left(\frac{\kappa_1}{\kappa_2}\right)^\omega \left(\frac{1}{\varepsilon}\right)^2 d^{2\omega}.$$

Hence

$$p_{\mathrm{err}}^{\mathrm{abs}}(S^{\mathrm{NEU}}, \Lambda^{\mathrm{all}}) \le 2 \qquad \textit{and} \qquad p_{\mathrm{dim}}^{\mathrm{abs}}(S^{\mathrm{NEU}}, \Lambda^{\mathrm{all}}) \le 2\omega.$$

(b) *For the class $\Lambda^{\rm std}$, we have*

$$\operatorname{card}^{\rm abs}(\varepsilon, S_d^{\rm NEU}, \Lambda^{\rm std}) \le \left\lceil 8(\rho_1+\rho_2)^4 N_\omega^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{4\omega} \right\rceil + 1,$$

and so

$$p_{\rm err}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm std}) \le 4 \qquad \textit{and} \qquad p_{\rm dim}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm std}) \le 4\omega.$$

(2) *Suppose that $\kappa_2 = 0$. Let Γ be as in* (3.9).

(a) *For the class $\Lambda^{\rm all}$, we have*

$$\operatorname{card}^{\rm abs}(\varepsilon, S_d^{\rm NEU}, \Lambda^{\rm all}) \le 4(\rho_1+\rho_2)^2 N_\omega^2 \Gamma^\omega \left(\frac{1}{\varepsilon}\right)^2 d^{3\omega},$$

and so

$$p_{\rm err}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm all}) \le 2 \qquad \textit{and} \qquad p_{\rm dim}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm all}) \le 3\omega.$$

(b) *For the class $\Lambda^{\rm std}$, we have*

$$\operatorname{card}^{\rm abs}(\varepsilon, S_d^{\rm NEU}, \Lambda^{\rm std}) \le \left\lceil 32(\rho_1+\rho_2)^4 N_\omega^4 \Gamma^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{6\omega} \right\rceil + 1,$$

and so

$$p_{\rm err}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm std}) \le 4 \qquad \textit{and} \qquad p_{\rm dim}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm std}) \le 6\omega.$$

Proof. Using (2.7), (2.15), and Lemma 4.4, we find that

$$C_d^{\rm NEU} \| \operatorname{App}_d \|_{\operatorname{Lin}[H(K_d), L_2(I^d)]} \le \frac{\max\left\{1, \dfrac{\rho_1}{q_0}\sqrt{2\max\{1,\kappa_0^\omega\}\gamma_{\max}}\right\}}{\min\{1,q_0\}} \sqrt{2\max\{1,\kappa_1^\omega\}\gamma_{\max}}\; d^\omega.$$

Hence setting $\alpha = \omega$ in (2.14), we obtain (4.4). The remaining results of this theorem now follow from [14, Theorem 7], with $\alpha = \omega$. □

Example 4.7. Suppose that K is the min-kernel $K_{\min}$. Since $\kappa_0 = 1$ and $\kappa_1 = \frac{1}{2}$, we can use (4.4) to see that

$$N_\omega \le \frac{\max\left\{1, \dfrac{\rho_1\sqrt{2\gamma_{\max}}}{q_0}\right\}\sqrt{2\gamma_{\max}}}{\min\{1,q_0\}}\ .$$

Furthermore, since $\kappa_2 > 0$, we see that case 1 holds in Theorem 4.6. Hence we find that the elliptic Neumann problem is tractable under the absolute error criterion, with

$$p_{\rm err}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm all}) \le 2 \qquad \text{and} \qquad p_{\rm dim}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm all}) \le 2\omega$$

for continuous linear information, and

$$p_{\rm err}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm std}) \le 4 \qquad \text{and} \qquad p_{\rm dim}^{\rm abs}(S^{\rm NEU}, \Lambda^{\rm std}) \le 4\omega$$

for standard information. □

Hence, the elliptic Neumann problem for the absolute error criterion is tractable for *any* set of finite-order weights and arbitrary spaces $H(K_d)$. The reason we are unable to establish strong tractability in this case is the same as for the Dirichlet problem, since the Lipschitz constant C_d^{NEU} and $\|\operatorname{App}_d\|_{\operatorname{Lin}[H(K_d),L_2(I^d)]}$ are expressed in terms of $\sigma_d(\kappa_0)$ and $\sigma_d(\kappa_1)$, whose product is bounded by a polynomial of degree ω in d. Hence we can only guarantee that N_ω is finite. If we want to establish strong tractability, we need to prove that N_0 is finite. Just as in the Dirichlet problem, we can do this if we assume that $\kappa_2 > 0$ and the sum of the weights is uniformly bounded.

Theorem 4.8. *Suppose that $\kappa_2 > 0$ and that condition (3.10) holds. Then the elliptic Neumann problem, defined for the spaces $H(K_d)$ with finite-order weights of order ω satisfying (3.10), is strongly tractable under the absolute error criterion. More precisely, for N_0 defined by (2.14), we have*

$$N_0 \le \frac{\rho_3^{1/2} \max\left\{1, \dfrac{\rho_1 \rho_3^{1/2}}{q_0} \max\{1, \kappa_0^{1/2}\}\right\} \max\{1, \kappa_1^{\omega/2}\}}{\min\{1, q_0\}}, \tag{4.5}$$

and the following bounds hold:

(1) *For the class Λ^{all}, we have*

$$\operatorname{card}^{\mathrm{abs}}(\varepsilon, S_d^{\mathrm{NEU}}, \Lambda^{\mathrm{all}}) \le 2(\rho_1 + \rho_2)^2 N_0^2 \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} \left(\frac{1}{\varepsilon}\right)^2.$$

Hence

$$p_{\mathrm{strong}}^{\mathrm{abs}}(S^{\mathrm{NEU}}, \Lambda^{\mathrm{all}}) \le 2.$$

(2) *For the class Λ^{std}, we have*

$$\operatorname{card}^{\mathrm{abs}}(\varepsilon, S_d^{\mathrm{NEU}}, \Lambda^{\mathrm{std}}) \le \left\lceil 8(\rho_1 + \rho_2)^4 N_0^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 \right\rceil + 1.$$

Hence

$$p_{\mathrm{strong}}^{\mathrm{abs}}(S^{\mathrm{NEU}}, \Lambda^{\mathrm{std}}) \le 4.$$

Proof. As in the proof of Theorem 3.7, we have

$$N_0 \le C^{\mathrm{NEU}} \rho_3^{1/2} \max\{1, \kappa_1^{\omega/2}\},$$

where

$$C^{\mathrm{NEU}} = \sup_{d \in \mathbb{Z}^{++}} C_d^{\mathrm{NEU}}.$$

Using Lemma 4.4 and (3.12), we have

$$C_d^{\mathrm{NEU}} = \frac{\max\left\{1, \dfrac{\rho_1 \sigma_d(\kappa_0)}{q_0}\right\}}{\min\{1, q_0\}} \le \frac{\max\left\{1, \dfrac{\rho_1 \rho_3^{1/2}}{q_0} \max\{1, \kappa_0^{\omega/2}\}\right\}}{\min\{1, q_0\}}.$$

Combining these results, we obtain obtain (4.5). The desired result now follows from [14, Theorem 8]. ☐

Example 4.9. Suppose once again that $K = K_{\min}$. We find that the conditions of Theorem 4.8 hold, with

$$N_0 \le \frac{\rho_3^{1/2} \max\left\{1, \dfrac{\rho_1 \rho_3^{1/2}}{q_0}\right\}}{\min\{1, q_0\}}.$$

Hence the elliptic Dirichlet problem is strongly tractable under the absolute error criterion, with

$$p^{\mathrm{abs}}_{\mathrm{strong}}(S^{\mathrm{NEU}}, \Lambda^{\mathrm{all}}) \le 2 \qquad \text{and} \qquad p^{\mathrm{abs}}_{\mathrm{strong}}(S^{\mathrm{NEU}}, \Lambda^{\mathrm{all}}) \le 4. \quad \square$$

4.3. **The normalized error criterion.** We now consider the elliptic Neumann problem for finite-order weights under the normalized error criterion. For this case, we will need to make an additional assumption, namely, that $1 \in H(K_d)$ and $\|1\|_{H(K_d)} \le \rho_2/q_0$. As already mentioned in Section 2, this implies that $q_0 \in Q_d^{**} \cap H_{d,\rho_2}$. We need this assumption to establish a lower bound on the initial error of the Neumann problem.

Lemma 4.10.

$$e(0, S_d^{\mathrm{NEU}}) \ge \rho_1 \sigma_d(\kappa_2).$$

Proof. Define $\mathrm{Int}_d \in [H(K_d)]^*$ as

$$\mathrm{Int}_d(g) = \int_{I^d} g(\mathbf{x})\, d\mathbf{x} \qquad \forall\, g \in H(K_d).$$

From [12, Lemma 2], we know that

$$\| \mathrm{Int}_d \|_{[H(K_d)]^*} = \sigma_d(\kappa_2).$$

Hence, it suffices to show that

$$e(0, S_d^{\mathrm{NEU}}) \ge \rho_1 \| \mathrm{Int}_d \|_{[H(K_d)]^*}. \tag{4.6}$$

As mentioned above, the constant function q_0 is an element of $Q_d^{**} \cap H_{d,\rho_2}$. Choose $f \in H(K_d)$, and let $u = S_d^{\mathrm{NEU}}(f, q_0)$. Since $q_0 \in H^1(I^d)$, we have

$$\begin{aligned}\|u\|_{H^1(I^d)} &\ge \frac{|\langle u, q_0\rangle_{H^1(I^d)}|}{\|q_0\|_{H^1(I^d)}} = |\langle u, 1\rangle_{H^1(I^d)}| = |B_d(u,1;1)| = |\langle f, 1\rangle_{L_2(I^d)}| \\ &= \left| \int_{I^d} f(\mathbf{x})\, d\mathbf{x} \right| = |\mathrm{Int}_d(f)|.\end{aligned}$$

Hence

$$\frac{\|S_d^{\mathrm{NEU}}(f, q_0)\|_{H^1(I^d)}}{\|f\|_{H(K_d)}} \ge \frac{|\mathrm{Int}_d(f)|}{\|f\|_{H(K_d)}}.$$

Since $f \in H(K_d)$ is arbitrary, this inequality and (2.11) imply that

$$e(0, S_d^{\mathrm{NEU}}) \ge \rho_1 \|S_d^{\mathrm{NEU}}(\cdot, q_0)\|_{\mathrm{Lin}[H(K_d), H^1(I^d)]} \ge \rho_1 \| \mathrm{Int}_d \|_{[H(K_d)]^*}.$$

This yields (4.6), which establishes the lemma. $\square$

We are now ready to prove the following result.

Theorem 4.11. *The elliptic Neumann problem, defined for the spaces $H(K_d)$ with finite-order weights of order ω, is tractable for the normalized error. More precisely for $N_{\omega/2}$ defined by (2.14), we have*

$$N_{\omega/2} \leq \frac{1}{\rho_1 \min\{1, q_0\}} \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega/2} \max\left\{1, \frac{\rho_1}{q_0}\sqrt{2\max\{\kappa_0^\omega, 1\}\gamma_{\max}}\right\}, \tag{4.7}$$

and the following bounds hold:

(1) *Suppose that $\kappa_2 > 0$.*

(a) *For the $\Lambda^{\rm all}$, we have*

$$\mathrm{card}^{\rm nor}(\varepsilon, S_d^{\rm NEU}, \Lambda^{\rm all}) \leq 2(\rho_1+\rho_2)^2 N_{\omega/2}^2 \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega}\left(\frac{1}{\varepsilon}\right)^2 d^\omega.$$

Hence

$$p_{\rm err}^{\rm nor}(S^{\rm NEU}, \Lambda^{\rm all}) \leq 2 \qquad \textit{and} \qquad p_{\rm dim}^{\rm nor}(S^{\rm NEU}, \Lambda^{\rm all}) \leq \omega.$$

(b) *For the class $\Lambda^{\rm std}$, we have*

$$\mathrm{card}^{\rm nor}(\varepsilon, S_d^{\rm NEU}, \Lambda^{\rm std}) \leq \left\lceil 8(\rho_1+\rho_2)^4 N_{\omega/2}^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega}\left(\frac{1}{\varepsilon}\right)^4 d^{2\omega}\right\rceil + 1,$$

and so

$$p_{\rm err}^{\rm nor}(S^{\rm NEU}, \Lambda^{\rm std}) \leq 4 \qquad \textit{and} \qquad p_{\rm dim}^{\rm nor}(S^{\rm NEU}, \Lambda^{\rm std}) \leq 2\omega.$$

(2) *Suppose that $\kappa_2 = 0$. Let Γ be as in (3.9).*

(a) *For the class $\Lambda^{\rm all}$, we have*

$$\mathrm{card}^{\rm nor}(\varepsilon, S_d^{\rm NEU}, \Lambda^{\rm all}) \leq 4(\rho_1+\rho_2)^2 N_{\omega/2}^2 \Gamma^\omega \left(\frac{1}{\varepsilon}\right)^2 d^{2\omega},$$

and so

$$p_{\rm err}^{\rm nor}(S^{\rm NEU}, \Lambda^{\rm all}) \leq 2 \qquad \textit{and} \qquad p_{\rm dim}^{\rm nor}(S^{\rm NEU}, \Lambda^{\rm all}) \leq 2\omega.$$

(b) *For the $\Lambda^{\rm std}$, we have*

$$\mathrm{card}^{\rm nor}(\varepsilon, S_d^{\rm NEU}, \Lambda^{\rm std}) \leq \left\lceil 32(\rho_1+\rho_2)^4 N_{\omega/2}^4 \Gamma^{2\omega}\left(\frac{1}{\varepsilon}\right)^4 d^{3\omega}\right\rceil + 1,$$

and so

$$p_{\rm err}^{\rm nor}(S^{\rm NEU}, \Lambda^{\rm std}) \leq 4 \qquad \textit{and} \qquad p_{\rm dim}^{\rm nor}(S^{\rm NEU}, \Lambda^{\rm std}) \leq 3\omega.$$

Proof. Using Lemmas 4.4 and 4.10, we find that

$$\frac{C_d^{\rm NEU}\|\,\mathrm{App}_d\,\|_{\mathrm{Lin}[H(K_d), L_2(I^d)]}}{e(0, S_d^{\rm NEU})} \leq \frac{\max\left\{1, \dfrac{\rho_1\sigma_d(\kappa_0)}{q_0}\right\}\sigma_d(\kappa_1)}{\rho_1\min\{1,q_0\}\sigma_d(\kappa_2)}.$$

From (3.29), we have

$$\frac{\sigma_d(\kappa_1)}{\sigma_d(\kappa_2)} \leq \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega/2},$$

and so (2.7) yields

$$\frac{C_d^{\mathrm{NEU}}\|\operatorname{App}_d\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}}{e(0,S_d^{\mathrm{NEU}})} \le \frac{1}{\rho_1\min\{1,q_0\}}\left(\frac{\kappa_1}{\kappa_2}\right)^{\omega/2}\max\left\{1,\frac{\rho_1}{q_0}\sqrt{2\max\{1,\kappa_0^\omega\gamma_{\max}\}}\right\}d^{\omega/2}.$$

Hence setting $\alpha=\omega/2$ in (2.14), we obtain (4.7). The remaining results of this theorem now follow from [14, Theorem 7], with $\alpha=\omega/2$. □

Example 4.12. Suppose that K is the min-kernel $K_{\min}$. Since $\kappa_0=1$, $\kappa_1=\frac{1}{2}$, and $\kappa_2=\frac{1}{3}$, we can use (4.7) to obtain that

$$N_{\omega/2}\le\frac{\max\left\{1,\dfrac{\rho_1\sqrt{2\gamma_{\max}}}{q_0}\right\}}{\rho_1\min\{1,q_0\}}\left(\frac{3}{2}\right)^{\omega/2}.$$

Furthermore, since $\kappa_2\ne0$, we see that case 1 holds in Theorem 4.11. Hence we find that the elliptic Neumann problem is tractable under the normalized error criterion, with

$$p_{\mathrm{err}}^{\mathrm{nor}}(S^{\mathrm{NEU}},\Lambda^{\mathrm{all}})\le2\qquad\text{and}\qquad p_{\dim}^{\mathrm{nor}}(S^{\mathrm{NEU}},\Lambda^{\mathrm{all}})\le\omega$$

for continuous linear information, and

$$p_{\mathrm{err}}^{\mathrm{nor}}(S^{\mathrm{NEU}},\Lambda^{\mathrm{std}})\le4\qquad\text{and}\qquad p_{\dim}^{\mathrm{nor}}(S^{\mathrm{NEU}},\Lambda^{\mathrm{std}})\le2\omega$$

for standard information. □

Hence the elliptic Neumann problem is tractable for *any* set of finite-order weights if we are using the normalized error criterion. The reason we are unable to establish strong tractability in this case is similar to that for the Dirichlet problem, namely, we can only establish that $N_{\omega/2}$ is finite. If we want to establish strong tractability, we need to prove that N_0 is finite. As before, we can do this if $\kappa_2>0$ and the sum of the weights is uniformly bounded.

Theorem 4.13. *Suppose that $\kappa_2>0$ and that condition* (3.10) *holds. Then the elliptic Neumann problem, defined for the spaces $H(K_d)$ with finite-order weights of order ω satisfying* (3.10)*, is strongly tractable under the normalized error criterion. More precisely, for N_0 defined by* (2.14)*, we have*

$$N_0\le\frac{\rho_3^{1/2}\max\left\{1,\dfrac{\rho_1}{q_0}\max\{1,\kappa_0^{\omega/2}\}\right\}\max\{1,\kappa_1^{\omega/2}\}}{\min\{1,q_0\}},\tag{4.8}$$

and the following bounds hold:

(1) *For the class Λ^{all}, we have*

$$\operatorname{card}^{\mathrm{nor}}(\varepsilon,S_d^{\mathrm{NEU}},\Lambda^{\mathrm{all}})\le2(\rho_1+\rho_2)^2N_0^2\left(\frac{\kappa_1}{\kappa_2}\right)^{\omega}\left(\frac{1}{\varepsilon}\right)^2.$$

Hence

$$p_{\mathrm{strong}}^{\mathrm{nor}}(S^{\mathrm{NEU}},\Lambda^{\mathrm{all}})\le2.$$

(2) *For the class Λ^{std}, we have*

$$\operatorname{card}^{\text{nor}}(\varepsilon, S_d^{\text{NEU}}, \Lambda^{\text{std}}) \le \left\lceil 8(\rho_1+\rho_2)^4 N_0^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 \right\rceil + 1.$$

Hence

$$p_{\text{strong}}^{\text{nor}}(S^{\text{NEU}}, \Lambda^{\text{std}}) \le 4.$$

Proof. As in the proof of Theorem 3.7, we have

$$N_0 \le C^{\text{NEU}} \rho_3^{1/2} \max\{1, \kappa_0^{\omega/2}\},$$

where

$$C^{\text{NEU}} = \sup_{d \in \mathbb{Z}^{++}} C_d^{\text{NEU}}.$$

Using Lemma 4.4, we find that

$$C_d^{\text{NEU}} = \frac{\max\left\{1, \dfrac{\rho_1 \sigma_d(\kappa_0)}{q_0}\right\}}{\min\{1, q_0\}} \le \frac{\max\left\{1, \dfrac{\rho_1 \rho_3^{1/2} \max\{1, \kappa_0^{\omega/2}\}}{q_0}\right\}}{\min\{1, q_0\}}.$$

Combining these results, we obtain (4.8). The desired result now follows from [14, Theorem 8]. □

Example 4.14. Suppose once again that $K = K_{\min}$. We find that the conditions of Theorem 4.13 hold, with

$$N_0 \le \frac{\rho_3^{1/2} \max\left\{1, \dfrac{\rho_1 \rho_3^{1/2}}{q_0}\right\}}{\min\{1, q_0\}}.$$

Hence, the elliptic Dirichlet problem is strongly tractable under the normalized error criterion, with

$$p_{\text{strong}}^{\text{abs}}(S^{\text{NEU}}, \Lambda^{\text{all}}) \le 2 \qquad \text{and} \qquad p_{\text{strong}}^{\text{abs}}(S^{\text{NEU}}, \Lambda^{\text{all}}) \le 4. \quad \square$$

In closing, we note that we have found conditions guaranteeing strong tractability for the Neumann problem under the normalized error criterion when $\kappa_2 > 0$. We have only tractability results for this problem when $\kappa_2 = 0$.

Acknowledgments

We are delighted to thank a number of people for their contributions. T. I. Seidman (University of Maryland, Baltimore County) suggested the maximum principle for the Neumann problem. E. Novak and H. Triebel (University of Jena) made us aware of the reference [15]. M. Dryja (University of Warsaw), S. Heinrich (University of Kaiserslautern), and J. F. Traub (Columbia University) provided useful comments on the paper itself.

References

1. R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, second ed., Pure and Applied Mathematics, vol. 140, Academic Press, Boston, 2003.
2. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404. MR0051437 (14:479c)
3. P. G. Ciarlet, *Basic error estimates for elliptic problems*, Handbook of Numerical Analysis, Vol. II, North-Holland, Amsterdam, 1991, pp. 17–351. MR1115237
4. J. Dick, I. H. Sloan, X. Wang, and H. Woźniakowski, *Good lattice rules in weighted Korobov spaces with general weights*, To appear in *Numerische Mathematik*.
5. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Grundlehren der mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin, 1983. MR0737190 (86c:35035)
6. A. Messiah, *Quantum mechanics*, Dover, Mineola, NY, 1999.
7. E. Novak and H. Woźniakowski, *When are integration and discrepancy tractable?*, Foundations of Computational Mathematics, Proceedings of Oxford 1999 (R. A. DeVore, A. Iserles, and E. Süli, eds.), Cambridge University Press, 2001, pp. 211–266. MR1836619 (2002h:68083)
8. J. T. Oden and J. N. Reddy, *An introduction to the mathematical theory of finite elements*, Wiley-Interscience, New York, 1976. MR0461950 (57:1932)
9. I. H. Sloan and H. Woźniakowski, *When are quasi-Monte Carlo algorithms efficient for high-dimensional integrals?*, J. Complexity **14** (1998), no. 1, 1–33.
10. J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski, *Information-based complexity*, Academic Press, Boston, 1988. MR0958691 (90f:68085)
11. J. F. Traub and A. G. Werschulz, *Complexity and information*, Cambridge University Press, Cambridge, 1998. MR1692462 (2000m:65170)
12. G. W. Wasilkowski and H. Woźniakowski, *Finite-order weights imply tractability of linear multivariate problems*, J. Approx. **130** (2004), 57–77. MR2086810 (2005d:41048)
13. A. G. Werschulz, *The computational complexity of differential and integral equations: An information-based approach*, Oxford University Press, New York, 1991. MR1144521 (93a:68061)
14. A. G. Werschulz and H. Woźniakowski, *Tractability of quasilinear problems. I: General results*, Tech. Report CUCS-025-05, Columbia University, Department of Computer Science, New York, 2005, Submitted to *J. Approx. Theory* for publication. Available at `http://mice.cs.columbia.edu/getTechreport.php?techreportID=248&format=pdf&`.
15. William P. Ziemer, *Weakly differentiable functions: Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989. MR1014685 (91e:46046)

Department of Computer and Information Sciences, Fordham University, New York, New York 10023 and Department of Computer Science, Columbia University, New York, New York 10027

E-mail address: `agw@cs.columbia.edu`

Department of Computer Science, Columbia University, New York, New York 10027 and Institute of Applied Mathematics, University of Warsaw, Poland

E-mail address: `henryk@cs.columbia.edu`