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DISCRETISATION OF AN INFINITE DELAY EQUATION

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ABSTRACT. In this paper, a Banach phase space containing $\mathbf{BC}(-\infty, 0]$ and contained in $\mathbf{C}(-\infty, 0]$ is defined with which existence of a solution and convergence of a discrete scheme are proved for an infinite delay differential equation.

1. INTRODUCTION AND PRELIMINARIES

In this paper we prove the existence of a solution and convergence of a discrete scheme for the following functional differential equations with infinite delay.

$$x'(t) = ax(t) + \sum_{k=1}^{\infty} b_k x(t - \tau_k) \ , \quad t \geq 0,$$
$$x(\theta) = \phi(\theta) \ , \quad \theta \in (-\infty, 0]. \tag{1.1}$$

Here a is a non-zero real, $\mathbf{b} = \{b_k\}_{k=1}^{\infty} \in l^1$, with $b_k \neq 0$ for all $k \geq 1$, $\{\tau_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of strictly positive reals such that $\lim_{k\to\infty} \tau_k = \infty$ and $\phi : (-\infty, 0] \longrightarrow \mathbb{R}$ is continuous.

By standard arguments, it can be shown that the solution to this problem gives rise to a semigroup on $\mathbf{BUC}(-\infty, 0]$ whose generator is the derivative operator. If this derivative is replaced by a finite difference, unlike in the case of a finite delay equation, we get an *infinite system* of first order differential equations. To get a finite system of discretised equations, we can confine ourselves to a finite interval at each stage of computation. But, consider $u \in \mathbf{BUC}(-\infty, 0]$ and the sequence $\{u_n\}$ of functions defined as

$$\begin{aligned} u_n(x) &= u(x), \ \ x \in [-n, 0], \\ &= u(-n) \ \ x \in (-\infty, -n]. \end{aligned}$$

Now, we can easily see that, in general, $\lim_{n\to\infty} \|u_n - u\|_\infty \neq 0$. So, approximation schemes which use the initial data restricted to finite intervals at every stage may not converge. In this paper, we define a phase space which enables us to prove a convergence result for a finite difference scheme. This scheme uses the initial data confined to a finite interval at every stage of computation.

It is well known that unlike in the case of finite delays, the choice of a phase space for the infinite delay equation is a difficult one. [1], [3], [4], [5] and [2] are some of the references in this direction. For a Frechét space approach refer to [7].

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If x is the solution of a delay differential equation, then the function φ_t defined as $\varphi_t(\theta) = x(t+\theta)$ is considered as the state of the system at t. In general, φ_t is an element of an infinite dimensional Banach space and its finite dimensional approximations are studied. These finite dimensional approximations are solutions of a system of differential equations. For example, refer to [9]. In [10], the finite dimensional approximations belong to spline functions.

Approximation of infinite delay equations are examined in [11]. For detailed references, see [12].

Our results on discretisation are based on Theorem 1.7 which is a version of the Trotter-Kato theorem proved in [6]. The following definitions and results in the theory of semigroups will be needed. [8] is a standard reference in this context.

Definition 1.1. Let $(X, \|\|)$ be a Banach space. A one parameter family of bounded linear maps $T_t : X \to X$, $t \in [0, \infty)$ is said to be a strongly continuous semigroup of bounded linear operators on X if

- **(i)** $T_0 = I$,
- **(ii)** $T_{t+s} = T_t T_s$ for every $t, s \geq 0$,
- **(iii)** $\lim_{t\to 0} \|T_t x - x\| = 0, \quad x \in X$.

Definition 1.2. A semigroup of bounded linear operators on X is said to be of class $G(M, \omega, X)$ if there exist constants $M \geq 1$ and $\omega \in R$ such that $\|T_t\| \leq Me^{\omega t}$. A strongly continuous semigroup of class $G(1, 0, X)$ is called a semigroup of contractions.

Definition 1.3. Let the linear operator A be defined as follows:

$$D(A) = \left\{ x \in X : \lim_{h\to 0^+} \frac{T_h x - x}{h} \text{ exists } \right\}.$$

For $x \in D(A)$, Ax is defined as

$$Ax = \lim_{h\to 0^+} \frac{T_h x - x}{h}.$$

We say that A generates the semigroup T_t or that A is the infinitesimal generator of T_t.

Theorem 1.4. *If T_t is a semigroup of bounded linear operators, then there are constants $M \geq 1$ and $w \in R$ such that T is of the class $G(M, \omega, X)$. Further, its infinitesimal generator A is a closed and densely defined linear operator which is unique.*

Besides, if $B : X \longrightarrow X$ is a bounded linear operator, then it generates a semigroup which is given by $T_t = e^{tB}$.

Definition 1.5. Let X be a real Banach space and X^* its dual. For $x \in X$, $F(x) \subseteq X^*$ is defined as

$$F(x) = \left\{ \eta \in X^* : \langle x, \eta \rangle = \|x\|^2 = \|\eta\|^2 \right\}.$$

A closed operator A with a dense domain $D(A)$ is said to be dissipative if for every $x \in D(A)$, there is $\eta \in F(x)$ such that $\langle Ax, \eta \rangle \leq 0$.

Theorem 1.6 (Lumer-Philips)**.** *If A is dissipative and there is $\lambda_0 > 0$ such that the range $R(\lambda_0 I - A)$ of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a strongly continuous semigroup of contractions.*

The following version of Trotter-Kato approximation is proved in [6].

Let X and X_n be a Banach spaces with norms $\|.\|$ and $\|\|_n$. For every $n = 1, 2, \dots,$ there exist bounded linear operators $P_n : X \longrightarrow X_n$ and $E_n : X_n \longrightarrow X$ satisfying the following:

(1) $\|P_n\| \leq M_1$, $\|E_n\|_n \leq M_2$ where M_1 and M_2 are independent of n;
(2) $\|E_nP_nx - x\| \longrightarrow 0$ as $n \to \infty$ for all $x \in X$;
(3) $P_nE_n = I_n$ where I_n is the identity operator on X_n.

Theorem 1.7. *Let $A : D(A) \longrightarrow X$ be a closed and densely defined operator in the class $G(M, \omega, X)$ and $A_n : X_n \longrightarrow X_n$ be bounded linear maps in $G(M, \omega, X_n)$. Let the semi-groups generated by A and A_n be denoted by T_t and T_t^n. Then the following are equivalent:*

(a) *For all $u \in D(A)$ there exists a sequence $\overline{u}_n \in X_n$ such that*

$$\lim_{n\to\infty} E_n\overline{u}_n = u \quad \textit{and} \quad \lim_{n\to\infty} E_nA_n\overline{u}_n = Au.$$

(b) $\lim_{n\to\infty} \|E_nT_t^nP_nx - T_tx\| \to 0$ *as $n \to \infty$ for all $x \in X$.*

We require the following definitions and results in the subsequent sections:

Let $J \subset \mathbb{R}$ be a non-compact interval and let J_k be an increasing sequence of compact intervals such that $\bigcup_{n=1}^{\infty} J_k = J$. Let $\mathbf{b} = \{b_k\} \in l^1$. For $u \in \mathbf{C}(J)$, define $p_k(u) = \sup_{x\in J_k} |u(x)|$. $C(J)$ is a Frechét space whose topology is given by the family of semi-norms $\{p_k : k \in \mathbb{N}\}$. We shall define a Banach space whose norm is given by a "linear combination" of these semi-norms. As a consequence of our assumption that $\{b_k\} \in l^1$, $\mathbf{BUC}(J)$ is contained in the new Banach space. Its topology is weaker than that of $\mathbf{BUC}(J)$ and stronger than that of $\mathbf{C}(J)$.

Definition 1.8. Let J_k and the sequence $\{b_k\}$ be as above. Define $\mathbf{C}_\sigma(J)$ as

$$\mathbf{C}_\sigma(J) = \left\{ u \in \mathbf{C}(J) : \sum_{k=1}^{\infty} |b_k| p_k(u) < \infty \right\}.$$

Further, $\|\|_\sigma : \mathbf{C}(J) \to \mathbb{R}^+$ is defined as $\|u\|_\sigma = \sum_{k=1}^{\infty} |b_k| p_k(u)$.

Proposition 1.9. *$\|\|_\sigma$ is a norm on the space $\mathbf{C}_\sigma(J)$ and with respect to this norm, $\mathbf{C}_\sigma(J)$ is a Banach space. The space $\mathbf{BUC}(J)$ is continuously embedded in $\mathbf{C}_\sigma(J)$. Further, we have the following criterion for convergence in this Banach space:*

Let u_n be a sequence of functions in $\mathbf{C}_\sigma(J)$ and $u \in \mathbf{C}_\sigma(J)$. Assume the following:

(i) For a fixed k,

$$\lim_{n\to\infty} p_k(u_n - u) = 0.$$

(ii) There exists a sequence $\{\alpha_k\}$ of non-negative reals with $\sum_{k=1}^{\infty} |b_k|\alpha_k < \infty$ with

$$p_k(u_n - u) \leq \alpha_k$$

for all n and k.

Then,

$$\lim_{n\to\infty} u_n = u.$$

Proof. We only prove the convergence criterion. The other assertions are easy to see:

$$\sum_{k=1}^{\infty}|b_k|p_k(u_n-u)=\sum_{k=1}^{N}|b_k|p_k(u_n-u)+\sum_{k=N+1}^{\infty}|b_k|p_k(u_n-u).$$

For a fixed N, the limit of the finite sum as $k\to\infty$ is zero by (i). The limit of the infinite sum is also zero because of (ii) and $\sum_{k=1}^{\infty}|b_k|\alpha_k<\infty$.

Let $n\in\mathbb{N}$. Consider the finite dimensional vector space $\mathbb{R}^{n^2+1}$. We define the norm $\|.\|_n$ on $\mathbb{R}^{n^2+1}$ as follows:

$$\|v\|_n=\sum_{k=1}^{n}|b_k|\max\{|v_0|,|v_1|,\ldots,|v_{kn}|\}+\sum_{k=n+1}^{\infty}|b_k|\max\{|v_0|,|v_1|,\ldots,|v_{n^2}|\}$$

Here $v=(v_0,v_1,\ldots,v_n,v_{n+1},v_{n+2},\ldots,v_{2n},\ldots,v_{n^2})$.

2. A phase space for the infinite delay equation

Let a, $\{b_k\}$, and τ_k be as in the previous section. Let $[x]$ be the largest integer less than or equal to x. Define $m_1=[-\tau_1]$ and m_k as $[-\tau_i]$ where i is the smallest positive integer such that $-\tau_i<m_{k-1}$. It is clear that m_k is a strictly decreasing sequence of negative integers and $-\tau_k\in[-m_k,0]$.

The space $C_\sigma(-\infty,0]$ is defined as

$$C_\sigma(-\infty,0]=\left\{\phi\in\mathbf{C}(-\infty,0]:\sum_{k=1}^{\infty}|b_k|\sup_{\theta\in[m_k,0]}|\phi(\theta)|<\infty\right\}.$$

The proof of the next proposition follows from Proposition 1.9.

Proposition 2.1. *For $\phi\in\mathbf{C}_\sigma(-\infty,0]$ define*

$$\|\phi\|_\sigma\, as\|\phi\|_\sigma=\sum_{k=1}^{\infty}|b_k|\sup_{\theta\in[m_k,0]}|\phi(\theta)|.$$

Then $\|\|_\sigma$ is a norm on $\mathbf{C}_\sigma(-\infty,0]$ and $\mathbf{C}_\sigma(-\infty,0]$ is a Banach space.

Next we define a linear operator A as follows: Let

$$D(A)=\left\{\phi\in\mathbf{C^1}(-\infty,0]:\phi,\phi'\in\mathbf{C}_\sigma(-\infty,0]\quad\text{and}\quad\phi'(0)=L\phi\right\}$$

where $L:\mathbf{C}_\sigma(-\infty,0]\to\mathbb{R}$ as $L\phi=a\phi(0)+\sum_{k=1}^{\infty}b_k\phi(-\tau_k)$ for $\phi\in D(A)$, $A\phi=\phi'$.

Theorem 2.2. *The operator A defined above generates a strongly continuous semigroup $\{T_t:t\geq 0\}$ of bounded linear operators on $\mathbf{C}_\sigma(-\infty,0]$. Further, for a given $\phi\in\mathbf{C}_\sigma(-\infty,0]$, the map $x:\mathbb{R}\longrightarrow\mathbb{R}$ defined as*

$$\begin{aligned}x(t)&=\phi(t),\ \ t\in(-\infty,0],\\x(t)&=[T_t\phi](0)\ \ t\in(0,\infty),\end{aligned}$$

is a unique solution to (1.1).

Remark. If we can show directly that A generates a semigroup T_t, then defining $x(t)=(T_t\phi)(0)$, we get a solution to (1.1). The standard procedure to show that an unbounded linear operator generates a semigroup, is to use the Hille-Yosida theorem ([8]). But the estimations involved are difficult to obtain. It can be shown that for $\phi\in\mathbf{C}_\sigma(-\infty,0]$, (1.1) has a unique solution and then *we define the semigroup via the solution to* (1.1).

Lemma 2.3. *Let $\phi \in \mathbf{C}_\sigma(-\infty, 0]$. The problem (1.1) has a unique solution $x : \mathbb{R} \longrightarrow \mathbb{R}$. Further, for any $t \in [0, \infty)$, there is a constant $c(t) > 0$ such that*

$$\sup_{s\in[0,t]} |x(s)| \leq c(t)\|\phi\|_\sigma. \tag{2.1}$$

Sketch of the Proof. Consider $t \in [0, \tau_1]$. $\sum_{k=1}^{\infty} b_k\phi(t-\tau_k)$ converges uniformly on $[0, \tau_1]$. Further,

$$\sup_{t\in[0,\tau_1]} |\sum_{k=1}^{\infty} b_k\phi(t-\tau_k)| \leq \|\phi\|_\sigma.$$

Define $y_1 : [0, \tau_1] \to \mathbb{R}$ as the unique solution to the initial value problem

$$\begin{aligned} x'(t) &= ax(t) + \sum_{k=1}^{\infty} b_k\phi(t-\tau_k), \\ x(0) &= \phi(0). \end{aligned} \tag{2.2}$$

We have

$$y_1(t) = \phi(0)e^{at} + e^{at}\int_0^t e^{-as}\left(\sum_{k=1}^{\infty} b_k\phi(s-\tau_k)\right) ds.$$

Define $x_1 : (-\infty, \tau_1] \longrightarrow \mathbb{R}$ as

$$\begin{aligned} x_1(s) &= \phi(s), s \in (-\infty, 0] \\ &= y_1(s), s \in [0, \tau_1]. \end{aligned}$$

Clearly,

$$\sup\{|x_1(t)| : t \in [0,\tau_1]\} \leq \left(\sup_{t\in[0,\tau_1]} e^{at}\right)|\phi(0)| + \left(\sup_{t\in[0,\tau_1]} \frac{e^{at}-1}{a}\right)\|\phi\|_\sigma. \tag{2.3}$$

Here, note that for $r > 0$, $\frac{e^{ar}-1}{a} > 0$ for all $a \neq 0$. From the estimate

$$|b_1|(|\phi(0)|) \leq |b_1| \sup_{\theta\in[-\tau_1,0]} |\phi(\theta)| \leq \|\phi\|_\sigma,$$

we get

$$\sup\{|x_1(t)| : t \in [0,\tau_1]\} \leq \left(\sup_{t\in[0,\tau_1]} e^{at}\right)\frac{1}{|b_1|}\|\phi\|_\sigma + \left(\sup_{t\in[0,\tau_1]} \frac{e^{at}-1}{a}\right)\|\phi\|_\sigma. \tag{2.4}$$

Now, we claim that for each $k \in \mathbb{N}$, there exists a function $x_k : (-\infty, k\tau_1]$ with the following properties:

(i) For each $t \in [0, k\tau_1]$,

$$\sum_{i=1}^{\infty} b_i x_k(t-\tau_i)$$

converges and this summation defines a continuous function on $[0, k\tau_1]$.

(ii) x_k is the unique solution to

$$\begin{aligned} x'(t) &= ax(t) + \sum_{k=1}^{\infty} b_i x_k(t-\tau_i), \quad \text{for} \quad t \in [0, k\tau_1], \\ x(\theta) &= \phi(\theta), \quad \text{for} \quad t \in (-\infty, 0]. \end{aligned} \tag{2.5}$$

(iii) There exist constants $c_k \geq 0$ such that

$$\sup\{|x_k(t)| : t \in [(k-1)\tau_1, k\tau_1]\} \leq c_k(\|\phi\|_\sigma).$$

The case $k = 1$ is already proved and the above claim can be proved by induction.

The solution to (1.1) is obtained by patching up the functions x_k. Note that for every $t \geq 0$, there is k such that $t \in [k\tau_1, (k+1)\tau_1]$ With this k, define $c(t) = \max\{c_1, c_2, \ldots, c_k\}$ and (2.1) is proved.

Proof of Theorem 2.2. Let $\phi \in \mathbf{C}_\sigma(-\infty, 0]$ and x_ϕ be the unique solution to (1.1). Define $T_t(\phi)$ as

$$\begin{aligned}[T_t\phi](\theta) &= \phi(t+\theta), \quad \text{if } \ t+\theta \leq 0 \\ &= x_\phi(t+\theta) \quad \text{if } \ \ t+\theta > 0.\end{aligned}$$

Fix $t \geq 0$. For $\theta \in [m_k, 0]$, $t + \theta \leq [m_k, t]$. We can get the estimate

$$\|T_t\phi\|_\sigma = (c(t)\|\mathbf{b}\|_1 + 1)\, \|\phi\|_\sigma.$$

From the above estimate, it follows that $T_t\phi \in \mathbf{C}_\sigma(-\infty, 0]$ and that T_t is a bounded linear map.

Next, we prove that

$$\lim_{t\to 0}(T_t\phi - \phi) = 0. \tag{2.6}$$

Define $p_k(\phi) = \sup_{\theta\in[m_k,0]}|\phi(\theta)|$. Fix $k \in \mathbb{N}$ and consider

$$\begin{aligned}p_k(T_t\phi - \phi) &= \sup_{\theta\in[m_k,0]} |(T_t\phi)(\theta) - \phi(\theta)| \\ &= \sup_{\theta\in[m_k,0]} |x_\phi(t+\theta) - x_\phi(\theta)|.\end{aligned}$$

Clearly, x is uniformly continuous in $[m_k, 1]$. Thus, given ϵ, there is $\delta^* > 0$ such that $|x(s_1) - x(s_2)| \leq \epsilon$ whenever $s_1, s_2 \in [m_k, 1]$ and $|s_1 - s_2| < \delta^*$. Now, let $\theta \in [m_k, 0]$ and $|t| \leq \min(1, \delta^*)$. We have $m_k \leq t+\theta \leq 1$. Thus, $|x_\phi(t+\theta) - x_\phi(\theta)| \leq \epsilon$.

Thus, by taking $\delta = \min(1, \delta^*)$, the following holds: Given $\epsilon > 0$, there is δ such that for all $|t| \leq \delta \Rightarrow p_k(T_t\phi - \phi) \leq \epsilon$. That is, for a fixed k,

$$\lim_{t\to 0} p_k(T_t\phi - \phi) = 0. \tag{2.7}$$

Now, for any $t \in [0, 1]$,

$$p_k(T_t\phi - \phi) \leq \max\left(p_k(\phi), \sup_{s\in[0,1]} |x(s)|\right). \tag{2.8}$$

Now, (2.7) follows from (2.8), (2.9), and Proposition 1.9.

It is easy to see that $T_{t+s} = T_tT_s$ and hence T_t is a semi-group on $\mathbf{C}_\sigma(-\infty, 0]$. Further, we can show that $\lim_{h\to 0^+} \frac{T_h\phi - \phi}{h}$ exists if and only if $\phi \in D(A)$ and for such a ϕ this limit is equal to ϕ'. The proof is complete. □

3. The space $\mathbf{C}_\sigma(-\infty, 0]$ and a piecewise linear interpolation

In this section, we construct a sequence of piecewise linear approximation $S_n\phi$ for elements of $\mathbf{C}_\sigma(-\infty, 0]$.

Let m_k be as in Section 2.

For a given $n \in \mathbb{N}$, consider the interval $[m_n, 0]$ and its n sub-intervals $[m_k, m_{k-1}]$, $k = 1, 2, \ldots, n$. Divide each of these sub-intervals into n intervals of length $= h_{k,n} =$

$\frac{m_{k-1}-m_k}{n}$. Thus, we get n^2+1 points θ_i, $i=0,1,2,\dots,n^2$ of the interval $[m_n,0]$ which are given by

$$\theta_{kn+j} = m_k + j\left(\frac{m_{k-1}-m_k}{n}\right),$$

where $k=1,2,\dots,n$,$j=0,1,\dots,(n-1)$ and $m_0=0$.

The quantity $\omega(\phi,k,h)$, known as the modulus of continuity, is defined as

$$\omega(\phi,k,h) = \sup\left\{\phi(\theta_1)-\phi(\theta_2)| : \theta_1,\theta_2 \in [m_k,0], |\theta_1-\theta_2| \le h\right\}.$$

The following result is well known:

For fixed $k \in \mathbb{N}$ and ϕ continuous on $[m_k,0]$, $\lim_{h\to 0}\omega(\phi,k,h)=0$.

Define $S_n\phi : (-\infty,0] \longrightarrow \mathbb{R}$ as follows: On $[m_n,0]$, $S_n\phi$ is the piecewise linear interpolation to ϕ at the above n^2+1 points and on $(-\infty,m_n]$ it is identically equal to $\phi(m_n)$. Explicitly, for $\theta \in [\theta_{kn+j+1},\theta_{kn+j}]$,

$$S_n\phi(\theta) = \phi(\theta_{kn+j})\left[1+\frac{\theta-\theta_{kn+j}}{h_{k,n}}\right] - \phi(\theta_{kn+j+1})\left[\frac{\theta-\theta_{kn+j}}{h_{k,n}}\right].$$

Theorem 3.1. *$S_n\phi$ converges to ϕ in the Banach space $\mathbf{C}_\sigma(-\infty,0]$; that is,*

$$\lim_{n\to\infty}\|S_n\phi-\phi\|_\sigma = 0.$$

Proof. We can also express $S_n\phi$ as

$$S_n\phi = \sum_{i=0}^{n^2}\phi(\theta_i)B_i$$

where $B_i : (-\infty,0] \to \mathbb{R}$'s are defined as follows:

$$\begin{aligned}
B_0(\theta) &= \frac{-\theta}{\theta_1}+1, \quad \theta\in[\theta_1,0]\\
&= 0 \quad \text{elsewhere},\\
B_i(\theta) &= \frac{\theta-\theta_{i+1}}{\theta_i-\theta_{i+1}}, \quad \theta\in[\theta_{i+1},\theta_i]\\
&= \frac{\theta-\theta_{i-1}}{\theta_i-\theta_{i-1}}, \quad \theta\in[\theta_i,\theta_{i-1}]\\
&= 0 \quad \text{elsewhere},\\
B_n^2 &= \frac{\theta-\theta_{n^2-1}}{\theta_n^2-\theta_{n^2-1}}, \quad \theta\in[\theta_{n^2},\theta_{n^2-1}]\\
&= 1 \quad \theta\in(-\infty,m_n]\\
&= 0 \quad \text{elsewhere}.
\end{aligned}$$

Since each B_i is bounded and uniformly continuous, $B_i \in \mathbf{C}_\sigma(-\infty,0]$ and hence $S_n\phi \in \mathbf{C}_\sigma(-\infty,0]$. Let $h^*_{n,k} = \max\{h_{n,j} : j=1,2,\dots,k\}$. Let k be fixed and $n>k$. Note that $\theta_n = m_1$, $\theta_{2n}=m_2,\dots,$ and $\theta_{n^2}=m_n$. Thus, for a fixed k, $\lim_{n\to\infty}h^*_{n,k}=0$. The estimate

$$p_k(\phi-S_n\phi) \le 3\omega\left(\phi,m_k,h^*_{n,k}\right)$$

is easily obtained. Now, since k is fixed and $\lim_{n\to\infty}h^*_{n,k}=0$, we have

$$\lim_{n\to\infty}p_k(S_n\phi-\phi)=0. \tag{3.1}$$

For all k, we have $p_k(S_n\phi) \le p_k(\phi)$. Finally, for all k, we obtain

$$p_k(S_n\phi - \phi) \le 2p_k(\phi) \tag{3.2}$$

Now the convergence of $S_n\phi$ to ϕ follows from (3.1), (3.2), and Proposition 1.9.

4. A finite difference scheme for the infinite delay equation

In this section we show the convergence of a finite difference scheme to the solution of the infinite delay equation.

Let θ_i, $i = 0, 1, 2, \ldots, n^2$ be as in the previous section. Define $P_n : \mathbf{C}_\sigma(-\infty, 0] \to \mathbb{R}^{n^2+1}$ as $[P_n(\phi)]_i = \phi(\theta_i)$. Further, define $E_n : \mathbb{R}^{n^2+1} \longrightarrow \mathbf{C}_\sigma(-\infty, 0]$ as follows:

For $v \in \mathbb{R}^{n^2+1}$, $E_n v$ is the piecewise linear function taking values $(E_n v)(\theta_i) = v_i$ and $(E_n v)(\theta) = v_{n^2}$ for all $\theta \in (-\infty, m_n]$. Clearly, $E_n v = \sum_{i=0}^{n^2} v_i B_i$.

Now, consider $\|E_n v\|_\sigma$. From the definitions of $E_n v$ and the norm $\|.\|_\sigma$, it is clear that

$$\|E_n v\|_\sigma = \sum_{k=1}^{n} |b_k| \max\{|v_0|, |v_1|, \ldots, |v_{kn}|\} + \sum_{k=n+1}^{\infty} |b_k| \max\{|v_0|, |v_1|, \ldots, |v_{n^2}|\}.$$

The following result holds:

Proposition 4.1.

(1) $\|P_n\| \le M$, $\|E_n\|_n \le 1$ *where M is independent of n,*
(2) $\|E_n P_n \phi - \phi\|_\sigma \longrightarrow 0$ *as $n \to \infty$ for all $\phi \in \mathbf{C}_\sigma$.*
(3) $P_n E_n = I_n$ *where I_n is the identity operator on X_n.*

Proof. We find that if $\|\|_n$ is the norm in $\mathbb{R}^{n^2+1}$ as defined in Section 1, then $\|E_n v\|_\sigma = \|v\|_n$. Thus $\|E_n\| = 1$. Now, $E_n P_n \phi$ is nothing but $S_n\phi$ where S_n is the piecewise linear approximation of ϕ defined in Section 3. So, $\lim_{n\to\infty} E_n P_n \phi = \lim_{n\to\infty} S_n \phi = \phi$ and hence (2) holds. By the uniform boundedness principle, we have the existence of $M \ge 0$ such that $\|S_n\| \le M$ for all $n \in \mathbb{N}$. Now $\|P_n\phi\|_n = \|E_n P_n \phi\|_\sigma = \|S_n\phi\|_\sigma \le M\|\phi\|_\sigma$. Thus, (1) holds. (3) is obvious from the definition of P_n and E_n.

Next, we discretise A by using finite differences as follows: Define $A_n : \mathbb{R}^{n^2+1} \to \mathbb{R}^{n^2+1}$ as

$$\begin{aligned} (A_n(v))_i &= \frac{(v_{i-1} - v_i)}{\theta_{i-1} - \theta_i}, i = 1, 2, 3, \ldots, n^2, \\ (A_n(v))_0 &= L[(E_n v)]. \end{aligned}$$

Theorem 4.2. *Let A and T_t be as in Proposition 2.1 and let A_n be as above. Let $T_t^{(n)}$ be the semigroup generated by A_n. We have*

$$\lim_{n\to\infty} \|E_n T_t^{(n)} P_n \phi - T_t(\phi)\|_\sigma = 0.$$

We shall apply Theorem 1.7 to prove this result. First, we need the following lemma.

Lemma 4.3. *Fix $v \in \mathbb{R}^{n^2+1}$. Define the function $l : \{1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, n^2\}$ as*

$$l(k) = \max\{0 \le i \le kn : |v_i| \ge |v_j| \ \textit{ for all } \ j \in \{0, 1, ..kn\}\}.$$

Define $\xi \in (\mathbb{R}^{n^2+1})^*$ *as follows:*

$$\langle w, \xi\rangle = \sum_{k=1}^{n} |b_k| \operatorname{sign} v_{l(k)} w_{l(k)} + \sum_{k=n+1}^{\infty} |b_k| \operatorname{sign} v_{l(n)} w_{l(n)}.$$

Then, the continuous linear functional η *is defined as* $\eta = \|v\|\xi \in F(v)$.

Proof.

$$\begin{aligned}|\langle w, \xi\rangle| &\le \sum_{k=1}^{n} |b_k||w_{l(k)}| + \sum_{k=n+1}^{\infty} |b_k||w_{l(n)}| \\ &\le \sum_{k=1}^{n} |b_k| \max\{|w_i| : 0 \le i \le kn\} + \sum_{k=n+1}^{\infty} |b_k| \max\left\{|w_i| : 0 \le i \le n^2\right\} \\ &= \|w\|_n.\end{aligned}$$

Furthermore,

$$\begin{aligned}\langle v, \xi\rangle &= \sum_{k=1}^{n} |b_k||v_{l(k)}| + \sum_{k=n+1}^{\infty} |b_k||v_{l(n)}| \\ &= \sum_{k=1}^{n} |b_k| \max\{|v_i| : 0 \le i \le kn\} + \sum_{k=n+1}^{\infty} |b_k| \max\left\{|v_i| : 0 \le i \le n^2\right\} \\ &= \|v\|_n.\end{aligned}$$

So, it is clear that $\|\xi\| = 1$. Now, it is easy to see that $\eta = \|v\|\xi \in F(v)$.

Proof of Theorem 4.2. Since A_n is a bounded linear operator, it generates the semigroup $T_t^{(n)} = e^{tA_n}$. We prove that each A_n is in the class $G(1, \omega, X_n)$ where $\omega > \|L\|\|\mathbf{b}\|$. Consider $B_n = A_n - \omega I_n$ where $\omega > \|L\|\|\mathbf{b}\|$ and I_n is the identity operator on $\mathbb{R}^{n^2+1}$. We prove that there exists $\eta \in F(v)$ such that $\langle B_n v, \eta\rangle \le 0$. For $v \in D(A_n) = \mathbb{R}^{n^2+1}$, let $\xi \in (\mathbb{R}^{n^2+1})^*$ and $\eta \in F(v)$ be as in the lemma above. Let us observe that for all $k = 1, \ldots, n$ with $l(k) \ne 0$, $|v_{l(k)}| \ge |v_{l(k)-1}|$. Thus,

$$\begin{aligned}\left|\sum_{k\in l^{-1}(0)} |b_k| \operatorname{sign}(v_{l(k)}) L(E_n v)\right| + \left|\sum_{k=n+1}^{\infty} |b_k| \operatorname{sign}(v_{l(n)}) L(E_n v)\right| &\le \|\mathbf{b}\|\|L\|\|E_n\|\|v\|_n \\ &\le (\|\mathbf{b}\|\|L\|)\|v\|_n.\end{aligned}$$

Now,

$$\begin{aligned}\langle B_n v, \xi\rangle &= \langle A_n v, \xi\rangle - \langle \omega v, \xi\rangle \\ &= \sum_{k=1}^{n} |b_k| \operatorname{sign}(v_{l(k)})(A_n v)_{l(k)} + \sum_{k=n+1}^{\infty} |b_k| \operatorname{sign}(v_{l(n)})(A_n v)_{l(n)} - \omega\|v\|_n \\ &= \sum_{k\notin l^{-1}(0)} |b_k| \operatorname{sign}(v_{l(k)}) \frac{v_{l(k)-1} - v_{l(k}}{\theta_{l(k)-1} - \theta_{l(k)}} + \sum_{k\in l^{-1}(0)} |b_k| \operatorname{sign}(v_{l(k)}) L(E_n v) \\ &+ \sum_{k=n+1}^{\infty} |b_k| \operatorname{sign}(v_{l(n)})(A_n v)_{l(n)} - \omega\|v\|_n.\end{aligned}$$

Now we consider two cases, namely $l(n) = 0$ and $l(n) \neq 0$. If $l(n) = 0$, then $(A_n v)_{l(n)} = L(E_n v)$ and we have the estimate

$$\begin{aligned}
\langle B_n v, \xi\rangle &= \sum_{k \notin l^{-1}(0)} |b_k| \operatorname{sign}(v_{l(k)}) \frac{v_{l(k)-1} - v_{l(k}}{\theta_{l(k)-1} - \theta_{l(k)}} \\
&\quad + \sum_{k \in l^{-1}(0)} |b_k| \operatorname{sign}(v_{l(k)}) L(E_n v) \\
&\quad + \sum_{k=n+1}^{\infty} |b_k| \operatorname{sign}(v_{l(n)}) L(E_n v) - \omega \|v\|_n \\
&\leq \sum_{k \in l^{-1}(0)} |b_k| \operatorname{sign}(v_{l(k)}) L(E_n v) \\
&\quad + \sum_{k=n+1}^{\infty} |b_k| \operatorname{sign}(v_{l(n)}) L(E_n v) - \omega \|v\|_n \\
&\leq 0.
\end{aligned}$$

If $l(n) \neq 0$, then $(A_n v)_{l(n)} = \frac{v_{l(n)-1} - v_{l(n)}}{\theta_{l(n)-1} - \theta_{l(n)}}$ and we have the estimate

$$\begin{aligned}
\langle B_n v, \xi\rangle &= \sum_{k \notin l^{-1}(0)} |b_k| \operatorname{sign}(v_{l(k)}) \frac{v_{l(k)-1} - v_{l(k}}{\theta_{l(k)-1} - \theta_{l(k)}} \\
&\quad + \sum_{k \in l^{-1}(0)} |b_k| \operatorname{sign}(v_{l(k)}) L(E_n v) \\
&\quad + \sum_{k=n+1}^{\infty} |b_k| \operatorname{sign}(v_{l(n)}) \frac{v_{l(n)-1} - v_{l(n)}}{\theta_{l(n)-1} - \theta_{l(n)}} - \omega \|v\|_n \\
&\leq \sum_{k \in l^{-1}(0)} |b_k| \operatorname{sign}(v_{l(k)}) L(E_n v) - \omega \|v\|_n \\
&\leq 0.
\end{aligned}$$

Thus, $\langle B_n v, \eta\rangle = \langle B_n v, \|v\|_n \xi\rangle \leq 0$. It is elementary to check that if λ is a real eigenvalue of B_n, then $\lambda \leq 0$. Thus, since $\mathbb{R}^{n^2+1}$ is finite dimensional, if $\lambda > 0$, then $\lambda I_n - B_n$ is invertible and in particular onto. Thus, by Theorem 1.6, it follows that $B_n = A_n v - \lambda I_n$ generates a contraction semigroup. But $e^{-\omega t} T_t^n$ is the semigroup generated by B_n. Thus, $\|e^{-\omega t} T_t^n\| \leq 1$ for all n. We get $\|T_t^n\| \leq e^{\omega t}$. Let T_t be of class $G(M_A, \omega_A, X)$. Now, let $\omega_0 = \max\{\omega_A, \omega\}$. We have now proved that $A \in G(M_A, \omega_0, X)$ and $A_n \in G(M_A, \omega_0, X_n)$.

To complete the proof, we need to check (a) of Theorem 1.7. Choose $\phi \in D(A)$ and $v_n \in \mathbb{R}^{n^2+1}$ as $v_n = P_n(\phi)$. Thus, $(v_n)_i = \phi(\theta_i)$ and $E_n v_n = E_n P_n \phi = S_n \phi$. So, $\lim_{n\to\infty} E_n v_n = \phi$. Next, we have to prove that $E_n A_n v_n \to A\phi$.

By the mean value theorem, for every $i = 1, 2, \dots, n^2$ there exists $\zeta_i \in [\theta_i, \theta_{i-1}]$ such that

$$\phi'(\zeta_i) = \frac{\phi(\theta_{i-1}) - \phi(\theta_i)}{\theta_{i-1} - \theta_i}.$$

Therefore,

$$
\begin{aligned}
&\|E_nA_nv_n(x)-A\phi)\|_\sigma\\
&\quad=\|\sum_{i=1}^{n^2}\frac{(\phi(\theta_{i-1})-\phi(\theta_i)}{\theta_{i-1}-\theta_i}B_i+L(S_n\phi)B_0-\phi'\|_\sigma\\
&\quad=\|\sum_{1=1}^{n^2}\phi'(\zeta_i)B_i+L(S_n\phi)B_0-\phi'\|_\sigma\\
&\quad\le\|\sum_{i=1}^{n^2}\phi'(\zeta_i)B_i+L(S_n\phi)B_0-\sum_{i=0}^{n^2}\phi'(x_i)B_i\|_\sigma+\|\sum_{i=0}^{n^2}\phi'(\theta_i)B_i-\phi'\|_\sigma\\
&\quad\le\|\sum_{i=1}^{n^2}(\phi'(\zeta_i)-\phi'(\theta_i))B_i+L(S_n\phi)B_0-\phi'(0)B_0\|_\sigma+\|\sum_{i=0}^{n^2}\phi'(\theta_i)B_i-\phi'\|_\sigma.
\end{aligned}
$$

Now, $\|\sum_{k=0}^{n^2}\phi'(x_i)B_i-\phi'\|_\sigma=\|S_n(\phi')-\phi'\|_\sigma$ and hence by Theorem 3.1, $\lim_{n\to\infty}\|\sum_{k=0}^{n^2}\phi'(\theta_i)B_i-\phi'\|_\sigma=0$. We need to prove that

$$
\lim_{n\to\infty}\|\sum_{i=1}^{n^2}(\phi'(\zeta_i)-\phi'(\theta_i))B_i+L(S_n\phi)B_0-\phi'(0)B_0\|_\sigma=0. \tag{4.1}
$$

Since $\lim_{n\to\infty}L(S_n\phi)=L(\phi)$, there is $c>0$ such that $|L(S_n\phi)|\le c$ for all n. For $k<n$,

$$
p_k\left(\sum_{i=1}^{n^2}(\phi'(\zeta_i)-\phi'(\theta_i))B_i+L(S_n\phi)B_0-\phi'(0)B_0\right)\le 3p_k(\phi')+c.
$$

For $k\ge n$,

$$
p_k\left(\sum_{i=1}^{n^2}(\phi'(\zeta_i)-\phi'(\theta_i))B_i+L(S_n\phi)B_0-\phi'(0)B_0\right)\le 3p_n(\phi')+c\le 3p_k(\phi')+c.
$$

So, we have proved that for all k,

$$
p_k\left(\sum_{i=1}^{n^2}(\phi'(\zeta_i)-\phi'(\theta_i))B_i+L(S_n\phi)B_0-\phi'(0)B_0\right)\le 3p_k(\phi')+c. \tag{4.2}
$$

Now, fix $k<n$. $B_i\equiv 0$ on $[m_k,0]$ for $i>kn$. Thus,

$$
\begin{aligned}
p_k\left(\sum_{i=1}^{n^2}\phi'(\zeta_i)B_i-\sum_{i=1}^{n^2}\phi'(x_i)B_i\right)&=\sup_{x\in[m_k,0]}|\sum_{i=1}^{n^2}\phi'(\zeta_i)B_i-\sum_{i=1}^{n^2}\phi'(x_i)B_i|\\
&=\sup_{x\in[m_k,0]}|\sum_{i=1}^{kn}(\phi'(\zeta_i)-\phi'(x_i))B_i|\\
&\le\omega\left(\phi',k,h^*_{n,k}\right).
\end{aligned}
$$

Thus, $\lim_{n\to\infty}p_k(\sum_{i=1}^{n^2}\phi'(\zeta_i)B_i-\sum_{k=1}^{n^2}\phi'(x_i)B_i)=0$. Now,

$$
p_k[(L(S_n\phi)-\phi'(0))B_0]\le|L(S_n\phi)-\phi'(0)|.
$$

Hence $\lim_{n\to\infty} p_k((L(S_n\phi) - \phi'(0))B_0) = 0$ Finally, we have

$$\lim_{n\to\infty} p_k(\sum_{i=1}^{n^2} \phi'(\zeta_i)B_i - \sum_{k=1}^{n^2} \phi'(x_i)B_i + (L(S_n(\phi)) - \phi'(0))B_0) = 0. \tag{4.3}$$

From (4.2), (4.3) and Proposition 1.9, (4.1) follows.

5. Numerical examples

In this section we give examples where the initial functions ϕ is *neither bounded nor integrable*

Example 1.

$$\begin{aligned} x'(t) &= x(t) + x(t-0.5) + \sum_{k=2}^{\infty} \frac{1}{k^2} x(t-k), \quad t > 0, \\ x(\theta) &= \sqrt{-\theta}, \quad -\infty < \theta \le 0. \end{aligned} \tag{5.1}$$

In this example, $\tau_1 = 0.5$ and for $k \ge 2$, $\tau_k = k$. For $k \ge 1$, $b_k = \frac{1}{k^2}$, $m_k = -k$.

Thus, $\theta_i = -\frac{i}{n}$, $i = 0, 1, 2, \ldots, n^2$. *Let n be even.* So, $-0.5 = -\tau_1 = \theta_{n/2}$ and for $k \ge 2$, $\tau_k = k = -\theta_{kn}$. Since

$$L(\phi) = a\phi(0) + \phi(-0.5) + \sum_{k=2}^{\infty} \phi(-k),$$

we obtain that

$$L(E_n v) = v_0 + v_{n/2} + \sum_{k=2}^{n} \frac{1}{k^2} v_{kn} + \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right) v_{kn}.$$

With reference to Theorem 4.2, $(T_t^{(n)})v$ is the solution to the system

$$\begin{aligned} v_0'(t) &= v_0(t) + v_{n/2}(t) + \sum_{k=2}^{n} \frac{1}{k^2} v_{kn}(t) + \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right) v_{kn}(t), \\ v_i'(t) &= n(v_{i-1}(t) - v_i(t)), \quad i = 1, 2, \ldots, n^2 \\ v(0) &= v. \end{aligned} \tag{5.2}$$

Taking $v = P_n\phi$ in (5.2), $(E_n T_t^{(n)} P_n \phi)(\theta_i)$ is nothing but $v_i(t)$.

Now, let x be the unique solution to (5.1). Then $(T_t\phi)(\theta_i) = x(t + \theta_i)$.

As per Theorem 4.2, $v_i(t)$ is an approximation to $x(t + \theta_i)$.

x satisfies the equation

$$\begin{aligned} x'(t) &= x(t) + \sqrt{0.5 - t} + \sum_{k=2}^{\infty} \frac{1}{k^2} \sqrt{k - t}, \\ x(0) &= 0, \end{aligned}$$

for $t \in [0, 0.5]$. It is clear that the above equation has no closed form solution. Using the Matlab ODE45 function, we have calculated the values of the solution at $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ to the equation

$$\begin{aligned} x'(t) &= x(t) + \sqrt{0.5 - t} + \sum_{k=2}^{1,00,000} \frac{1}{k^2} \sqrt{k - t}, \\ x(0) &= 0. \end{aligned}$$

The above solution is denoted by x^*.

We solve the system (5.2) using the Matlab ODE45 function. In the table below, we take $i = 0$ and compare the values of $v_0(t)$ and $x^*(t)$ which is an approximation of $x(t)$ in the interval $[0, 0.5]$. We consider $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ and $n = 10, 20, 60, 70$.

t	$n = 10$ $v_0(t)$	$n = 20$ $v_0(t)$	$n = 60$ $v_0(t)$	$n = 70$ $v_0(t)$	$x^*(t)$
0	0	0	0	0	0
0.1	0.2090	0.2168	0.2256	0.2268	0.2386
0.2	0.4288	0.4457	0.4647	0.4671	0.4921
0.3	0.6593	0.6862	0.7170	0.7209	0.7608
0.4	0.9036	0.9386	0.9810	0.9866	1.0439
0.5	1.1685	1.2083	1.2573	1.2640	1.3364
0.6	1.4631	1.5062	1.5589	1.5664	-
0.7	1.7972	1.8449	1.9051	1.9141	-
0.8	2.1807	2.2359	2.3094	2.3206	-
0.9	2.6236	2.6896	2.7810	2.7948	-
1.0	3.1364	3.2161	3.3285	3.3454	-

Remark. To approximately evaluate x in $[0.5, 2]$, we may use a linear interpolation of x in $[0, 0.5]$. Then, to approximately evaluate x in $[2, 3]$, we may use a linear interpolation of x in $[0.5, 2]$ and so on. But the advantage in our procedure is that we use only the initial data ϕ to evaluate $x(t)$ for any t.

Let $n = 70$. For a fixed t, $T_t(\phi) \in C_\sigma(-\infty, 0]$. Now, $E_n T_t^n P_n(\phi)(\theta_i) = v_i(t)$. For each t, an approximation to $T_t(\phi)$ is obtained by interpolating the 4901 values $v_i(t)$ for $i = 0, 1, \ldots, 4901$. In the following tables, we tabulate values of $v_i(t)$ for some values of i. Whenever $t + \theta_i \leq 0$, we compare these values with $T_t\phi(\theta_i) = x(t + \theta_i) = \sqrt{|t + \theta_i|}$:

	$\theta = 0$		$\theta = -1/70$		$\theta = -1/7$	
	Num.	Act.	Num	Act.	Num.	Act
t= 0	0	0	0.1195	0.1195	0.3780	0.3780
t= 0.1	0.2268	-	0.1938	-	0.1945	0.2070
t=0.2	0.4671	-	0.4321	-	0.1543	-
t=0.3	0.7209	-	0.6839	-	0.3644	-
t=0.4	0.9866	-	0.9481	-	0.6118	-
t=0.5	1.2640	-	1.2236	-	0.8725	-
t=0.6	1.5664	-	1.5212	-	1.1448	-
t=0.7	1.9141	-	1.8616	-	1.4359	-
t=0.8	2.3206	-	2.2592	-	1.7638	-
t=0.9	2.7948	-	2.7234	-	2.1450	-
t=1	3.3454	-	3.2626	-	2.5901	-

	$\theta=-4/7$		$\theta=-1$		$\theta=-8/7$	
	Num	Act.	Num.	Act.	Num.	Act.
t= 0	0.7559	0.7559	1.0000	1.0000	1.0690	1.0690
t= 0.1	0.6860	0.6866	0.9485	0.9487	1.0210	1.0212
t=0.2	0.6078	0.6094	0.8939	0.8944	0.9706	0.9710
t=0.3	0.5168	0.5300	0.8357	0.8367	0.9174	0.9181
t=0.4	0.4016	0.4140	0.7730	0.7746	0.8608	0.8619
t=0.5	0.2619	0.2672	0.7045	0.7071	0.8000	0.8018
t=0.6	0.2073	-	0.6278	0.6325	0.7340	0.7368
t=0.7	0.3258	-	0.5388	0.5477	0.6610	0.6655
t=0.8	0.5465	-	0.4297	0.4472	0.5776	0.5855
t=0.9	0.8007	-	0.3118	0.3162	0.4771	0.4928
t=1	1.0704	-	0.2517	0.0000	0.3618	0.3780

	$\theta=-20$		$\theta=-70$	
	Num.	Act	Num	Act.
t= 0	4.4721	4.4721	8.3666	8.3666
t= 0.1	4.4609	4.4609	8.3606	8.3606
t=0.2	4.4497	4.4497	8.3546	8.3546
t=0.3	4.4385	4.4385	8.3487	8.3487
t=0.4	4.4272	4.4272	8.3427	8.3427
t=0.5	4.4159	4.4159	8.3367	8.3367
t=0.6	4.4045	4.4045	8.3307	8.3307
t=0.7	4.3932	4.3932	8.3247	8.3247
t=0.8	4.3818	4.3818	8.3187	8.3126
t=0.9	4.3703	4.3703	8.3126	8.3216
t=1	4.3589	4.3589	8.3066	8.3066

Example 2. Consider

$$\begin{aligned} x'(t) &= x(t)+x(t-0.5)+\sum_{k=2}^{\infty}\frac{1}{k^3}x(t-k), t>0, \\ x(\theta) &= \theta, -\infty<\theta\le 0. \end{aligned} \tag{5.3}$$

It is elementary to show that the solution is

$$\begin{aligned} x(t) &= -\alpha t+(\alpha-\beta)(e^t-1), \ \ 0\le t\le 0.5 \\ &= -pt+(p-q)(e^{t-0.5}-1)+r(t-(0.5))e^t+e^{t-0.5}, \ \ 0.5<t\le 1 \end{aligned}$$

where $\alpha=\sum_{i=1}^{\infty}\frac{1}{i^3}$, $\beta=-0.5-\sum_{i=2}^{\infty}\frac{1}{i^2}$ $p=-1$ $q=-\sum_{i=2}^{\infty}\frac{1}{i^2}+0.5\alpha$ and $r=(\alpha+\beta)e^{-0.5}$.

In this example,

$$L(E_n\phi)=\phi(0)+\phi(-0.5)+\sum_{k=2}^{n}\frac{1}{k^3}\phi(-k)+\sum_{k=n+1}^{\infty}\frac{1}{k^3}\phi(-n).$$

We solve the system

$$\begin{aligned} v_0'(t) &= v_0(t) + v_{n/2}(t) + \sum_{k=2}^{n} \frac{1}{k^3} v_{kn}(t) + \left(\sum_{k=n+1}^{\infty} \frac{1}{k^3} \right) v_{kn}, \\ v_i'(t) &= n(v_{i-1} - v_i), \quad i = 1, 2, \ldots, n^2, \\ v(0) &= v, \end{aligned} \tag{5.4}$$

with $v = P_n \phi$.

Unlike in Example 1, we know the exact solution for $t \in [0, 1]$ and we compare $v_0(t)$ and $x(t)$ in the following table for various values of n:

t	$n = 10$ $v_0(t)$	$n = 20$ $v_0(t)$	$n = 50$ $v_0(t)$	$n = 60$ $v_0(t)$	$x(t)$
0	0	0	0	0	0
0.1	-0.1089	-0.1116	-0.1131	-0.1133	-0.1142
0.2	-0.2168	-0.2223	-0.2256	-0.2259	-0.2278
0.3	-0.3247	-0.3323	-0.3371	-0.3377	-0.3406
0.4	-0.4355	-0.4433	-0.4483	-0.4490	-0.4527
0.5	-0.5533	-0.5597	-0.5625	-0.5628	-0.5639
0.6	-0.6831	-0.6876	-0.6876	-0.6874	-0.6853
0.7	-0.8295	-0.8329	-0.8317	-0.8313	-0.8293
0.8	-0.9970	-1.0005	-0.9996	-0.9994	-0.9982
0.9	-1.1899	-1.1946	-1.1947	-1.1947	-1.1946
1	-1.4126	-1.4189	-1.4203	-1.4205	-1.4212

In the tables below, for $n = 60$, we compare $v_i(t)$ with $x(t + \theta_i)$ for some values of i.

	$\theta = 0$		$\theta = -1/60$		$\theta = -20/60$	
	Num.	Act.	Num	Act.	Num.	Act
t= 0	0	0	-0.0167	-0.0167	-0.3000	-0.3000
t= 0.1	-0.1133	-0.1142	-0.0946	-0.0952	-0.2000	-0.2000
t=0.2	-0.2259	-0.2278	-0.2072	-0.2089	-0.1029	-0.1000
t=0.3	-0.3377	-0.3406	-0.3191	-0.3219	-0.0599	-0.0000
t=0.4	-0.4490	-0.4527	-0.4304	-0.4341	-0.1209	-0.1142
t=0.5	-0.5628	-0.5639	-0.5434	-0.5455	-0.2262	-0.2278
t=0.6	-0.6874	-0.6853	-0.6657	-0.6636	-0.3376	-0.3406
t=0.7	-0.8313	-0.8293	-0.8061	-0.8036	-0.4495	-0. 4527
t=0.8	-0.9994	-0.9982	-0.9700	-0.9682	-0.5652	-0.5639
t=0.9	-1.1947	-1.1946	-1.1606	-1.1598	-0.6920	-0.6853
t=1	-1.4205	-1.4212	-1.3811	-1.3812	-0.8372	-0.8293

	$\theta = -40/60$		$\theta = -1$		$\theta = -80/60$	
	Num	Act.	Num.	Act.	Num.	Act.
t= 0	-0.6667	-0.6667	-1.000	-1.0000	-1.3333	-1.3333
t= 0.1	-0.5667	-0.5667	-0.9000	-0.9000	-1.2333	-1.2333
t=0.2	-0.4667	-0.4667	-0.8000	-0.8000	-1.1333	-1.1333
t=0.3	-0.3667	-0.3667	-0.7000	-0.7000	-1.0333	-1.0333
t=0.4	-0.2667	-0.2667	-0.6000	-0.6000	-0.9333	-0.9333
t=0.5	-0.1701	-0.1667	-0.5000	-0.5000	-0.8333	-0.8333
t=0.6	-0.1000	-0.0667	-0.3999	-0.4000	-0.7333	-0.7333
t=0.7	-0.0975	-0.0381	-0.3004	-0.3000	-0.6333	-0.6333
t=0.8	-0.1642	-0.1521	-0.2050	-0.2000	-0.5331	-0.5333
t=0.9	-0.2650	-0.2655	-0.1317	-0.1000	-0.4334	-0.4333
t=1	-0.3753	-0.3781	-0.1096	-0.0000	-1.3340	-0.3333

	$\theta = -20$		$\theta = -60$	
	Num.	Act	Num	Act.
t= 0	-20.0000	-20.0000	-60.000	-60.0000
t= 0.1	-19.9000	-19.9000	-59.9000	-59.9000
t=0.2	-19.8617	-19.8000	-59.8000	-59.8000
t=0.3	-19.7167	-19.7000	-59.7000	-59.7000
t=0.4	-19.6167	-19.6000	-59.6000	-59.6000
t=0.5	-19.5167	-19.5000	-59.5000	-59.5000
t=0.6	-19.4167	-19.4000	-59.4000	-59.4000
t=0.7	-19.3167	-19.3000	-59.3000	-59.3000
t=0.8	-19.2167	-19.2000	-59.2000	-59.2000
t=0.9	-19.1167	-19.1000	-59.1000	-59.1000
t=1	-19.0167	-19.0000	-59.0000	-59.0000

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