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GOOD LATTICE RULES BASED ON THE GENERAL WEIGHTED STAR DISCREPANCY

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Abstract. We study the problem of constructing rank-1 lattice rules which have good bounds on the "weighted star discrepancy". Here the non-negative weights are general weights rather than the product weights considered in most earlier works. In order to show the existence of such good lattice rules, we use an averaging argument, and a similar argument is used later to prove that these lattice rules may be obtained using a component-by-component (CBC) construction of the generating vector. Under appropriate conditions on the weights, these lattice rules satisfy strong tractability bounds on the weighted star discrepancy. Particular classes of weights known as "order-dependent" and "finite-order" weights are then considered and we show that the cost of the construction can be very much reduced for these two classes of weights.

1. Introduction

We consider rank-1 lattice rules for the approximation of integrals over the d-dimensional unit cube given by

$$I_d(f) = \int_{[0,1]^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

These rank-1 lattice rules are quadrature rules of the form

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\left\{\frac{k\boldsymbol{z}}{n}\right\}\right),$$

where $\boldsymbol{z} \in \mathbb{Z}^d$ is the generating vector whose components are conveniently assumed to be relatively prime with n and the braces around a vector indicate that we take the fractional part of each component of the vector.

Many research papers have been concerned with finding "good" lattice rules. In order to compare the quality of different lattice rules, some criterion needs to be chosen. A number of criteria are based on the idea of "discrepancy". In general terms, the discrepancy may be viewed as a measure of the deviation from the uniform distribution of the quadrature points. In some settings, it may also be considered to be a worst-case error in certain function spaces. Such discrepancy measures have been considered in [3], [4], [8], and [12], or in a more general work such as [13]. A classic example is the star discrepancy which appears in the well-known Koksma-Hlawka inequality (for example, see [13] or [18]). In [12] it was

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proved that there exist d-dimensional rank-1 lattice rules whose star discrepancy is $O\left(n^{-1}(\ln n)^d\right)$ with the implied constant depending only on d. A component-by-component (CBC) construction of the generating vectors for such rules was given in [8].

In this paper we are interested in constructing rank-1 lattice rules by using a weighted star discrepancy as a criterion of goodness. In [9] it was shown that lattice rules with good bounds on the weighted star discrepancy exist and can be obtained by using a CBC construction of $\boldsymbol{z}$ in the situation when n is a prime number and the weights are of a "product" form (see below). In Sections 3 and 4 we extend these results to the general situation where the weights do not necessarily have this product form. Such general weights have been considered in [2], where it was shown that good lattice rules can be obtained for integrands belonging to weighted Korobov spaces. In these spaces the integrands were assumed to be periodic. For the general weighted star discrepancy considered here, the functions belonging to the associated function spaces have no such periodicity assumption.

In [5] it is shown that weighted integrals over possibly unbounded domains may be approximated by suitably transforming points in $[0,1]^d$. As we shall explain later in Section 2, the CBC construction presented here will lead to lattice rules that are appropriate for such weighted integrals.

There are some applications in which it is the low dimensional projections that are the most important. In such cases, it is useful to introduce general weights that allow us to model the relative importance of each group of variables. For example, in some financial applications (see [17] for further details), such a model may be considered. As indicated in [2], weights which are "order-dependent" and/or "finite-order" often provide reasonable assumptions which also present the advantage that computations are very much simplified. The definition of these particular classes of weights and the analysis of their computational costs for the CBC construction are given in Sections 5 and 6.

2. General weighted star discrepancy

Let us consider first the concept of the local discrepancy of a point set in $[0,1]^d$. This can be described as the difference between the proportion of the points that lie in a subset of $[0,1]^d$ and the measure of that subset. If P_n is a set of n points in $[0,1]^d$, then the local star discrepancy of the point set P_n at $\boldsymbol{x} \in [0,1]^d$ is defined by

$$\operatorname{discr}(\boldsymbol{x}, P_n) := \frac{A([\mathbf{0}, \boldsymbol{x}), P_n)}{n} - \prod_{j=1}^{d} x_j. \tag{1}$$

Here $A([\mathbf{0}, \boldsymbol{x}), P_n)$ represents the counting function, namely the number of points in P_n which lie in $[\mathbf{0}, \boldsymbol{x})$ with $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$. The unweighted star discrepancy of the point set P_n is then defined as

$$D^*(P_n) := \sup_{\boldsymbol{x} \in [0,1]^d} \left|\operatorname{discr}(\boldsymbol{x}, P_n)\right|. \tag{2}$$

This is the star discrepancy that arises in the Koksma-Hlawka inequality mentioned above.

In order to introduce the general weighted star discrepancy, now let $\mathfrak{u}$ be an arbitrary non-empty subset of $\mathcal{D} := \{1, 2, \ldots, d-1, d\}$ and denote its cardinality by

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$|\mathfrak{u}|$. For the vector $\boldsymbol{x} \in [0,1]^d$, let $\boldsymbol{x}_\mathfrak{u}$ denote the vector from $[0,1]^{|\mathfrak{u}|}$ containing the components of $\boldsymbol{x}$ whose indices belong to $\mathfrak{u}$. By $(\boldsymbol{x}_\mathfrak{u}, \mathbf{1})$ we mean the vector from $[0,1]^d$ whose j-th component is x_j if $j \in \mathfrak{u}$ and 1 if $j \notin \mathfrak{u}$.

Suppose $Q_{n,d}$ is the quadrature rule given by

$$Q_{n,d}(f) = \frac{1}{n} \sum_{\boldsymbol{x} \in P_n} f(\boldsymbol{x}).$$

Then from Zaremba's identity (see for instance [16] or [18]), we obtain

$$Q_{n,d}(f) - I_d(f) = \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} (-1)^{|\mathfrak{u}|} \int_{[0,1]^{|\mathfrak{u}|}} \operatorname{discr}((\boldsymbol{x}_\mathfrak{u}, \mathbf{1}), P_n) \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{x}_\mathfrak{u}} f((\boldsymbol{x}_\mathfrak{u}, \mathbf{1}))\, \mathrm{d}\boldsymbol{x}_\mathfrak{u}. \tag{3}$$

Now let us introduce a set of non-negative weights $\{\boldsymbol{\gamma}_\mathfrak{u}\}_{\mathfrak{u} \subseteq \mathcal{D}}$ and consider $\boldsymbol{\gamma}_\mathfrak{u}$ as the weight associated with the set $\mathfrak{u}$. We also assume that the weights are independent of the dimension d. Previous research papers such as [9] have assumed that the weights are of a product form, that is, $\boldsymbol{\gamma}_\mathfrak{u} = \prod_{j \in \mathfrak{u}} \gamma_j$ for each subset $\mathfrak{u} \subseteq \mathcal{D}$, where γ_j is the weight associated with the variable x_j. Using (3) we see that we can write

$$Q_{n,d}(f) - I_d(f) = \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} (-1)^{|\mathfrak{u}|} \boldsymbol{\gamma}_\mathfrak{u} \int_{[0,1]^{|\mathfrak{u}|}} \operatorname{discr}((\boldsymbol{x}_\mathfrak{u}, \mathbf{1}), P_n) \boldsymbol{\gamma}_\mathfrak{u}^{-1} \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{x}_\mathfrak{u}} f((\boldsymbol{x}_\mathfrak{u}, \mathbf{1}))\, \mathrm{d}\boldsymbol{x}_\mathfrak{u}.$$

Applying Hölder's inequality for integrals and sums, we obtain

$$\begin{aligned} |Q_{n,d}(f) - I_d(f)| \quad &\leq \quad \left(\max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \sup_{\boldsymbol{x}_\mathfrak{u} \in [0,1]^{|\mathfrak{u}|}} \boldsymbol{\gamma}_\mathfrak{u} \left|\operatorname{discr}((\boldsymbol{x}_\mathfrak{u}, \mathbf{1}), P_n)\right| \right) \\ &\quad \times \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_\mathfrak{u}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{x}_\mathfrak{u}} f((\boldsymbol{x}_\mathfrak{u}, \mathbf{1})) \right| \mathrm{d}\boldsymbol{x}_\mathfrak{u} \right). \end{aligned}$$

Thus the weighted star discrepancy $D^*_{n,\boldsymbol{\gamma}}$ of the point set P_n may be defined by

$$D^*_{n,\boldsymbol{\gamma}} := \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_\mathfrak{u} \sup_{\boldsymbol{x}_\mathfrak{u} \in [0,1]^{|\mathfrak{u}|}} \left|\operatorname{discr}((\boldsymbol{x}_\mathfrak{u}, \mathbf{1}), P_n)\right|. \tag{4}$$

We observe that some of these formulae make sense only for strictly positive weights. If there are some sets $\mathfrak{u} \subseteq \mathcal{D}$ for which $\boldsymbol{\gamma}_\mathfrak{u} = 0$, then we adopt the convention that $0 \cdot \infty = 0$ (the same convention has been used in [2]). As our interest is in rank-1 lattice rules, from now on we shall assume that P_n is the point set $\{\{k\boldsymbol{z}/n\},\ 0 \leq k \leq n-1\}$. The corresponding weighted star discrepancy is then denoted by $D^*_{n,\boldsymbol{\gamma}}(\boldsymbol{z})$.

As mentioned earlier, there are applications for which the lower dimensional projections are the most important. This suggests that the weight associated with a set should not be bigger than the weights associated with any of its subsets. So we shall make the reasonable assumption that for any non-empty subset $\mathfrak{u} \subseteq \mathcal{D}$, we have

$$\boldsymbol{\gamma}_\mathfrak{u} \leq \boldsymbol{\gamma}_\mathfrak{g} \quad \text{for any} \quad \emptyset \neq \mathfrak{g} \subseteq \mathfrak{u}. \tag{5}$$

The next section presents bounds for the general weighted star discrepancy, which allows us to prove the existence of good lattice rules, while in Section 4 we present a CBC construction of $\boldsymbol{z}$.

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3. Bounds on the general weighted star discrepancy

Let us first define

$$E_{n,m}^* := \{\boldsymbol{h} \in \mathbb{Z}^m,\ \boldsymbol{h} \neq \boldsymbol{0} : -n/2 < h_j \leq n/2,\ 1 \leq j \leq m\},$$

for any positive integer m.

From [13, Theorem 5.6], we obtain the following inequality:

$$\sup_{\boldsymbol{x}_{\mathfrak{u}} \in [0,1]^{|\mathfrak{u}|}} |\mathrm{discr}((\boldsymbol{x}_{\mathfrak{u}}, \boldsymbol{1}), P_n)| \leq 1 - (1 - 1/n)^{|\mathfrak{u}|} + \frac{R_n(\boldsymbol{z}, \mathfrak{u})}{2}, \tag{6}$$

where

$$R_n(\boldsymbol{z}, \mathfrak{u}) := \sum_{\substack{\boldsymbol{h} \in E_{n,|\mathfrak{u}|}^* \\ \boldsymbol{h} \cdot \boldsymbol{z}_{\mathfrak{u}} \equiv 0 \pmod{n}}} \prod_{j \in \mathfrak{u}} \frac{1}{\max(1, |h_j|)}.$$

Note that under the assumption that $\gcd(z_j, n) = 1$ for $1 \leq j \leq d$, then $\boldsymbol{z}_{\mathfrak{u}}$ is the generating vector for a $|\mathfrak{u}|$-dimensional lattice rule having n points. This result, together with (4), shows that the general weighted star discrepancy satisfies the inequality

$$D_{n,\boldsymbol{\gamma}}^*(\boldsymbol{z}) \leq \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \left(1 - (1 - 1/n)^{|\mathfrak{u}|} + \frac{R_n(\boldsymbol{z}, \mathfrak{u})}{2}\right). \tag{7}$$

As an aside, let us remark that the bound in (6) also holds for the extreme discrepancy of [13]. This extreme discrepancy is based on the local discrepancy

$$\mathrm{discr}(\boldsymbol{w}, \boldsymbol{x}, P_n) := \frac{A([\boldsymbol{w}, \boldsymbol{x}), P_n)}{n} - \prod_{j=1}^{d} (x_j - w_j),$$

where $0 \leq w_j \leq x_j \leq 1$, $1 \leq j \leq d$. The local star discrepancy of (1) is the special case when $w_j = 0$. In [5] and [6] it is shown that it is appropriate to approximate weighted integrals over possibly unbounded domains by suitably transforming points in $[0,1]^d$ that have what is termed a low weighted L_∞ unanchored discrepancy. Since this latter quantity is a weighted version of the extreme discrepancy of [13], it follows that the CBC construction presented here will produce lattice rules that also have a low weighted L_∞ unanchored discrepancy. So such lattice rules are appropriate for these weighted integrals.

Bernoulli's inequality or a simple direct calculation yields

$$(1 - 1/n)^{|\mathfrak{u}|} \geq 1 - \frac{|\mathfrak{u}|}{n} \quad \text{and so} \quad 1 - (1 - 1/n)^{|\mathfrak{u}|} \leq \frac{|\mathfrak{u}|}{n}.$$

This then leads to

$$\max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \left(1 - (1 - 1/n)^{|\mathfrak{u}|}\right) \leq \frac{1}{n} \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} |\mathfrak{u}| \boldsymbol{\gamma}_{\mathfrak{u}}. \tag{8}$$

It follows from the error theory of lattice rules (for example, see [13, Chapter 5] or [14, Chapter 4]) that we may write $R_n(\boldsymbol{z}, \mathfrak{u})$ as

$$R_n(\boldsymbol{z}, \mathfrak{u}) = \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j \in \mathfrak{u}} \left(1 + {\sum_{-n/2 < h \leq n/2}}' \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|}\right) - 1, \tag{9}$$

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where the $'$ on the sum indicates we omit the $h = 0$ term. Now by defining

$$C_k(z) := \sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hkz/n}}{|h|}, \quad 0 \le k \le n-1,$$

and using the expansion

$$\prod_{j\in\mathfrak{u}}(1+a_j) = 1 + \sum_{\emptyset\ne\mathfrak{g}\subseteq\mathfrak{u}} \prod_{j\in\mathfrak{g}} a_j,$$

we have from (9) that

$$\begin{aligned} R_n(\boldsymbol{z},\mathfrak{u}) &= \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j\in\mathfrak{u}}[1+C_k(z_j)] - 1 = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{\emptyset\ne\mathfrak{g}\subseteq\mathfrak{u}}\prod_{j\in\mathfrak{g}} C_k(z_j) \\ &= \sum_{\emptyset\ne\mathfrak{g}\subseteq\mathfrak{u}} \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j\in\mathfrak{g}} C_k(z_j) = \sum_{\emptyset\ne\mathfrak{g}\subseteq\mathfrak{u}} \widetilde{R}_n(\boldsymbol{z},\mathfrak{g}), \end{aligned}$$

where

$$\widetilde{R}_n(\boldsymbol{z},\mathfrak{g}) := \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j\in\mathfrak{g}}\left(\sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|}\right). \tag{10}$$

For later use, we note that the theory of lattice rules shows that with

$$\widetilde{E}^*_{n,m} := \{\boldsymbol{h}\in\mathbb{Z}^m : -n/2 < h_j \le n/2,\ h_j \ne 0,\ 1 \le j \le m\},$$

we may write $\widetilde{R}_n(\boldsymbol{z},\mathfrak{g})$ as

$$\widetilde{R}_n(\boldsymbol{z},\mathfrak{g}) = \sum_{\substack{\boldsymbol{h}\in\widetilde{E}^*_{n,|\mathfrak{g}|} \\ \boldsymbol{h}\cdot\boldsymbol{z}_{\mathfrak{g}}\equiv 0 \pmod n}} \prod_{j\in\mathfrak{g}} \frac{1}{|h_j|} \ge 0. \tag{11}$$

Hence we have for any $\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}$ that

$$\boldsymbol{\gamma}_{\mathfrak{u}} R_n(\boldsymbol{z},\mathfrak{u}) = \boldsymbol{\gamma}_{\mathfrak{u}} \sum_{\emptyset\ne\mathfrak{g}\subseteq\mathfrak{u}} \widetilde{R}_n(\boldsymbol{z},\mathfrak{g}).$$

Under the assumption given by (5), we obtain

$$\boldsymbol{\gamma}_{\mathfrak{u}} R_n(\boldsymbol{z},\mathfrak{u}) \le \sum_{\emptyset\ne\mathfrak{g}\subseteq\mathfrak{u}} \boldsymbol{\gamma}_{\mathfrak{g}} \widetilde{R}_n(\boldsymbol{z},\mathfrak{g}) \le \sum_{\emptyset\ne\mathfrak{g}\subseteq\mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{g}} \widetilde{R}_n(\boldsymbol{z},\mathfrak{g}).$$

As a consequence, we then conclude that

$$\max_{\emptyset\ne\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} R_n(\boldsymbol{z},\mathfrak{u}) \le \sum_{\emptyset\ne\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \widetilde{R}_n(\boldsymbol{z},\mathfrak{u}).$$

This inequality combined with (7) and (8) then yield the following result:

Lemma 1. *If the weights $\boldsymbol{\gamma}_{\mathfrak{u}}$ satisfy* (5) *for any $\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}$, then*

$$D^*_{n,\boldsymbol{\gamma}}(\boldsymbol{z}) \le \frac{1}{n}\max_{\emptyset\ne\mathfrak{u}\subseteq\mathcal{D}} |\mathfrak{u}|\boldsymbol{\gamma}_{\mathfrak{u}} + \frac{1}{2}e^2_{n,d}(\boldsymbol{z}),$$

where

$$e^2_{n,d}(\boldsymbol{z}) := \sum_{\emptyset\ne\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \widetilde{R}_n(\boldsymbol{z},\mathfrak{u}). \tag{12}$$

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This lemma shows we can then analyse the weighted star discrepancy by considering the quantity $e^2_{n,d}(\boldsymbol{z})$.

From now on, we shall assume that n is a prime number. Since we only consider the fractional part of each component of $k\boldsymbol{z}/n$, we see that we may take each component of the generating vector $\boldsymbol{z}$ as belonging to the set $\mathcal{Z}_n = \{1, 2, \ldots, n-1\}$. We can obtain bounds on $e^2_{n,d}(\boldsymbol{z})$ for the case in which n is prime by obtaining an expression for a certain mean value of $e^2_{n,d}(\boldsymbol{z})$. The mean is taken over all integer vectors $\boldsymbol{z} \in \mathcal{Z}_n^d$. Thus the mean $M_{n,d,\boldsymbol{\gamma}}$ is defined by

$$M_{n,d,\boldsymbol{\gamma}} := \frac{1}{(n-1)^d} \sum_{\boldsymbol{z} \in \mathcal{Z}_n^d} e^2_{n,d}(\boldsymbol{z}).$$

An expression for the mean is given in the next theorem:

Theorem 2. *Let n be a prime number. Then*

$$M_{n,d,\boldsymbol{\gamma}} = \frac{1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} + \frac{n-1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(-\frac{S_n}{n-1} \right)^{|\mathfrak{u}|},$$

where

$$S_n := \sideset{}{'}\sum_{-n/2 < h \leq n/2} \frac{1}{|h|}.$$

Proof. From the definition of the mean and (10) and (12), we have

$$M_{n,d,\boldsymbol{\gamma}} = \frac{1}{(n-1)^d} \sum_{\boldsymbol{z} \in \mathcal{Z}_n^d} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(\frac{1}{n} \sum_{k=0}^{n-1} \prod_{j \in \mathfrak{u}} \left(\sideset{}{'}\sum_{-n/2 < h \leq n/2} \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|} \right) \right).$$

By separating out the $k = 0$ term, we obtain

$$M_{n,d,\boldsymbol{\gamma}} = \frac{1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} + T_{n,d,\boldsymbol{\gamma}}, \tag{13}$$

where

$$\begin{aligned} T_{n,d,\boldsymbol{\gamma}} &= \frac{1}{(n-1)^d} \sum_{\boldsymbol{z} \in \mathcal{Z}_n^d} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(\frac{1}{n} \sum_{k=1}^{n-1} \prod_{j \in \mathfrak{u}} \left(\sideset{}{'}\sum_{-n/2 < h \leq n/2} \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|} \right) \right) \\ &= \frac{1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(\sum_{k=1}^{n-1} \prod_{j \in \mathfrak{u}} \left(\frac{1}{n-1} \sum_{z_j=1}^{n-1} \sideset{}{'}\sum_{-n/2 < h \leq n/2} \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|} \right) \right). \end{aligned}$$

Because n is prime, we have

$$\frac{1}{n-1} \sum_{z_j=1}^{n-1} \sideset{}{'}\sum_{-n/2 < h \leq n/2} \frac{e^{2\pi \mathrm{i} h k z_j / n}}{|h|} = -\frac{S_n}{n-1} \tag{14}$$

for $1 \leq k \leq n-1$ (a complete proof might be found, for instance, in [9]). This leads to

$$T_{n,d,\boldsymbol{\gamma}} = \frac{1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \sum_{k=1}^{n-1} \left(-\frac{S_n}{n-1} \right)^{|\mathfrak{u}|} = \frac{n-1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(-\frac{S_n}{n-1} \right)^{|\mathfrak{u}|}.$$

Now replacing the last term in (13) with this expression, we obtain the desired result. □

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In the case $d = 1$, it is easy to verify that $M_{n,1,\{\gamma_{\{1\}}\}} = 0$. This is to be expected since it is also easy to verify (by using (10)) that $\widetilde{R}_n(\boldsymbol{z}, \mathfrak{g}) = 0$ whenever $|\mathfrak{g}| = 1$.

Corollary 3. *Let n be a prime number. Then there exists a generating vector $\boldsymbol{z}$ such that*

$$e_{n,d}^2(\boldsymbol{z}) \leq M_{n,d,\boldsymbol{\gamma}} \leq \frac{1}{n-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|}.$$

Proof. The first inequality is obvious. The proof of the second inequality is based on the proof of the second assertion in Theorem 1 of [2]. We can write the expression for $M_{n,d,\boldsymbol{\gamma}}$ as

$$M_{n,d,\boldsymbol{\gamma}} = \frac{1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|}(1 + W_n(\mathfrak{u})),$$

where

$$W_n(\mathfrak{u}) = (-1)^{|\mathfrak{u}|}(n-1)\left(\frac{1}{n-1}\right)^{|\mathfrak{u}|}.$$

If $|\mathfrak{u}|$ is odd, then $W_n(\mathfrak{u}) \leq 0$. On the other hand, if $|\mathfrak{u}|$ is even, then $|\mathfrak{u}| \geq 2$ and

$$W_n(\mathfrak{u}) \leq (n-1)\left(\frac{1}{n-1}\right)^2 = \frac{1}{n-1}.$$

So for $|\mathfrak{u}|$ either odd or even, we have

$$M_{n,d,\boldsymbol{\gamma}} \leq \frac{1}{n} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|}\left(1 + \frac{1}{n-1}\right) = \frac{1}{n-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|},$$

which completes the proof. □

This corollary and Lemma 1 then lead to the following result:

Corollary 4. *Suppose the weights satisfy* (5) *and suppose that n is a prime number. Then there exists a vector $\boldsymbol{z} \in \mathcal{Z}_n^d$ such that the general weighted star discrepancy satisfies the bound*

$$D_{n,\boldsymbol{\gamma}}^*(\boldsymbol{z}) \leq \frac{1}{n} \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} |\mathfrak{u}|\gamma_{\mathfrak{u}} + \frac{1}{2(n-1)} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|}.$$

From [12, Lemmas 1 and 2] we have

$$S_n \leq 2\ln n + 2\omega - \ln 4 + \frac{1}{n^2},$$

where ω is the Euler-Mascheroni constant defined by $\omega = \lim_{m \to \infty}\left(\sum_{k=1}^{m} \frac{1}{k} - \ln m\right)$. An approximate value for $2\omega - \ln 4$ is -0.2319. So for any $n \geq 3$, we have $S_n \leq 2\ln n$. In fact, a direct calculation shows that this inequality also holds for $n = 2$. Hence, we conclude that for any prime n, there exists a vector $\boldsymbol{z} \in \mathcal{Z}_n^d$ such that the general weighted star discrepancy satisfies the following bound:

$$D_{n,\boldsymbol{\gamma}}^*(\boldsymbol{z}) \leq \frac{1}{n} \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} |\mathfrak{u}|\gamma_{\mathfrak{u}} + \frac{1}{2(n-1)} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} (2\ln n)^{|\mathfrak{u}|}. \tag{15}$$

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Let $\Gamma = \max_{1 \le j \le d} \gamma_{\{j\}}$. Since $\gamma_{\mathfrak{u}} \le \Gamma$ for all $\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}$ (because of (5)), then in general, we have $\max_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}} |\mathfrak{u}|\gamma_{\mathfrak{u}} \le \Gamma d$. Hence from (15) we have

$$D^*_{n,\boldsymbol{\gamma}}(\boldsymbol{z}) \le \frac{\Gamma d}{n} + \frac{1}{2(n-1)} \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} (2 \ln n)^{|\mathfrak{u}|}.$$

Moreover, we have

$$\sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} (2 \ln n)^{|\mathfrak{u}|} \le \Gamma \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}} (2 \ln n)^{|\mathfrak{u}|} = \Gamma \sum_{j=1}^{d} \binom{d}{j} (2 \ln n)^j \le \Gamma (1 + 2 \ln n)^d.$$

This yields

$$D^*_{n,\boldsymbol{\gamma}}(\boldsymbol{z}) = O\left(n^{-1} (\ln n)^d\right), \tag{16}$$

with the implied constant depending only on d and Γ.

In the situation when all the weights are equal to 1, then

$$D^*_{n,\boldsymbol{\gamma}}(\boldsymbol{z}) = \max_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}} \sup_{\boldsymbol{x}_{\mathfrak{u}} \in [0,1]^{|\mathfrak{u}|}} |\mathrm{discr}((\boldsymbol{x}_{\mathfrak{u}}, \boldsymbol{1}), P_n)| = \sup_{\boldsymbol{x} \in [0,1]^d} |\mathrm{discr}(\boldsymbol{x}, P_n)|$$

is the unweighted star discrepancy defined in (2). For this quantity, the rate of $O\left(n^{-1} (\ln n)^d\right)$ is essentially the best possible (see [10] or [13]). Hence the bound for the weighted star discrepancy given in Corollary 3 is essentially the best possible and so, we consider such a bound to be "good".

4. Component-by-component construction of the generating vector

Since the total number of vectors $\boldsymbol{z} \in \mathcal{Z}_n^d$ is $(n-1)^d$, it is practically impossible to search over all these vectors to find a good one when d and n are large. In this section we propose a cheaper construction of the generating vector, namely the CBC construction, which means that the generating vector is found one component at a time. When we add a new component to the generating vector, the existing components will stay unchanged. Such a CBC construction has been successfully used, for instance, in [2], [8], [9] and the algorithm is given below:

Component-by-component algorithm:

1. Set the value for the first component of the vector, say $z_1 = 1$.
2. For $m = 2, 3, \ldots, d$, find $z_m \in \mathcal{Z}_n$ such that $e^2_{n,m}(z_1, \ldots, z_m)$ is minimized. Here

$$e^2_{n,m}(z_1, \ldots, z_m) = \sum_{\emptyset \ne \mathfrak{u} \subseteq \{1,2,\ldots,m\}} \gamma_{\mathfrak{u}} \widetilde{R}_n((z_1, \ldots, z_m), \mathfrak{u}).$$

Now we are looking to prove that this algorithm does indeed yield good lattice rules. By good, we mean that the $\boldsymbol{z}$ found this way satisfies the bound for $e^2_{n,d}(\boldsymbol{z})$ given in Corollary 3. The following theorem and corollary justify the use of the CBC algorithm.

Theorem 5. *Let n be a prime number. Suppose there exists a $\boldsymbol{z} \in \mathcal{Z}_n^d$ such that*

$$e^2_{n,d}(\boldsymbol{z}) \le \frac{1}{n-1} \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|}. \tag{17}$$

Then there exists $z_{d+1} \in \mathcal{Z}_n$ such that

$$e^2_{n,d+1}(\boldsymbol{z}, z_{d+1}) \le \frac{1}{n-1} \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{D}_1} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|},$$

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where $\mathcal{D}_1 := \mathcal{D} \cup \{d+1\}$. Such a z_{d+1} can be found by minimizing $e^2_{n,d+1}(\boldsymbol{z}, z_{d+1})$ over the set $\mathcal{Z}_n$.

Proof. We have

$$\begin{aligned} e^2_{n,d+1}(\boldsymbol{z}, z_{d+1}) &= \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}_1} \gamma_{\mathfrak{u}} \widetilde{R}_n((\boldsymbol{z}, z_{d+1}), \mathfrak{u}) \\ &= \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \widetilde{R}_n(\boldsymbol{z}, \mathfrak{u}) + \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} \widetilde{R}_n((\boldsymbol{z}, z_{d+1}), \mathfrak{u}). \end{aligned} \tag{18}$$

We recall that if $|\mathfrak{u}| = 1$, then $\widetilde{R}_n(\boldsymbol{z}, \mathfrak{u}) = 0$, so we may assume that $|\mathfrak{u}| \geq 2$ without loss of generality. Also recall that we defined

$$C_k(z) = \sideset{}{'}\sum_{-n/2 < h \leq n/2} \frac{e^{2\pi \mathrm{i} hkz/n}}{|h|}, \quad 0 \leq k \leq n-1.$$

Then clearly $C_0(z) = S_n$. For $\mathfrak{u} \subseteq \mathcal{D}_1$ with $d+1 \in \mathfrak{u}$, we then have

$$\begin{aligned} \widetilde{R}_n((\boldsymbol{z}, z_{d+1}), \mathfrak{u}) &= \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j \in \mathfrak{u}} C_k(z_j) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \left(\prod_{j \in \mathfrak{u} - \{d+1\}} C_k(z_j) \right) C_k(z_{d+1}) \\ &= \frac{S_n^{|\mathfrak{u}|}}{n} + \frac{1}{n} \sum_{k=1}^{n-1} \left(\prod_{j \in \mathfrak{u} - \{d+1\}} C_k(z_j) \right) C_k(z_{d+1}), \end{aligned}$$

where the $k = 0$ term was separated out. Substituting this in (18), we obtain

$$\begin{aligned} e^2_{n,d+1}(\boldsymbol{z}, z_{d+1}) &= e^2_{n,d}(\boldsymbol{z}) + \frac{1}{n} \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1, |\mathfrak{u}| \geq 2 \\ d+1 \in \mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} \\ &\quad + \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D}_1, |\mathfrak{u}| \geq 2 \\ d+1 \in \mathfrak{u}}} \frac{\gamma_{\mathfrak{u}}}{n} \sum_{k=1}^{n-1} \left(\prod_{j \in \mathfrak{u} - \{d+1\}} C_k(z_j) \right) C_k(z_{d+1}). \end{aligned}$$

Next we average $e^2_{n,d+1}(\boldsymbol{z}, z_{d+1})$ over all possible values of $z_{d+1} \in \mathcal{Z}_n$ and consider

$$\mathrm{Avg}(e^2_{n,d+1}(\boldsymbol{z}, z_{d+1})) = \frac{1}{n-1} \sum_{z_{d+1}=1}^{n-1} e^2_{n,d+1}(\boldsymbol{z}, z_{d+1}).$$

As the dependency of $e^2_{n,d+1}(\boldsymbol{z}, z_{d+1})$ on z_{d+1} is only through the $C_k(z_{d+1})$ factor, we next focus on the quantity

$$T_n(k) = \frac{1}{n-1} \sum_{z_{d+1}=1}^{n-1} C_k(z_{d+1}).$$

From (14), we have

$$T_n(k) = -\frac{S_n}{n-1}, \quad 1 \leq k \leq n-1.$$

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It follows that

$$\begin{aligned} &\mathrm{Avg}(e_{n,d+1}^2(\boldsymbol{z}, z_{d+1})) \\ = \quad & e_{n,d}^2(\boldsymbol{z}) + \frac{1}{n} \sum_{\substack{\mathfrak{u}\subseteq\mathcal{D}_1, |\mathfrak{u}|\geq 2 \\ d+1\in\mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} - \frac{S_n}{n(n-1)} \sum_{\substack{\mathfrak{u}\subseteq\mathcal{D}_1, |\mathfrak{u}|\geq 2 \\ d+1\in\mathfrak{u}}} \gamma_{\mathfrak{u}} \sum_{k=1}^{n-1} \prod_{j\in\mathfrak{u}-\{d+1\}} C_k(z_j). \end{aligned}$$

For any $\mathfrak{u} \subseteq \mathcal{D}_1$ with $d+1 \in \mathfrak{u}$ and $|\mathfrak{u}| \geq 2$, we have

$$-\frac{1}{n} \sum_{k=1}^{n-1} \prod_{j\in\mathfrak{u}-\{d+1\}} C_k(z_j) = -\widetilde{R}_n(\boldsymbol{z}, \mathfrak{u}-\{d+1\}) + \frac{S_n^{|\mathfrak{u}|-1}}{n} \leq \frac{S_n^{|\mathfrak{u}|-1}}{n},$$

where we have subtracted and added the $k = 0$ term and used the fact that the quantities $\widetilde{R}_n(\boldsymbol{z}, \mathfrak{g})$ are positive (see (11)) for any subset $\mathfrak{g} \subseteq \mathcal{D}$. Consequently, we have

$$\begin{aligned} &\mathrm{Avg}(e_{n,d+1}^2(\boldsymbol{z}, z_{d+1})) \\ \leq \quad & e_{n,d}^2(\boldsymbol{z}) + \frac{1}{n} \sum_{\substack{\mathfrak{u}\subseteq\mathcal{D}_1, |\mathfrak{u}|\geq 2 \\ d+1\in\mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} + \frac{1}{n(n-1)} \sum_{\substack{\mathfrak{u}\subseteq\mathcal{D}_1, |\mathfrak{u}|\geq 2 \\ d+1\in\mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} \\ = \quad & e_{n,d}^2(\boldsymbol{z}) + \frac{1}{n-1} \sum_{\substack{\mathfrak{u}\subseteq\mathcal{D}_1, |\mathfrak{u}|\geq 2 \\ d+1\in\mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} \\ \leq \quad & e_{n,d}^2(\boldsymbol{z}) + \frac{1}{n-1} \sum_{\substack{\mathfrak{u}\subseteq\mathcal{D}_1 \\ d+1\in\mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|}. \end{aligned}$$

Using the hypothesis, we next obtain

$$\begin{aligned} \mathrm{Avg}(e_{n,d+1}^2(\boldsymbol{z}, z_{d+1})) \quad & \leq \quad \frac{1}{n-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{D}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} + \frac{1}{n-1} \sum_{\substack{\mathfrak{u}\subseteq\mathcal{D}_1 \\ d+1\in\mathfrak{u}}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} \\ & = \quad \frac{1}{n-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{D}_1} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|}. \end{aligned} \tag{19}$$

There exists at least one $z_{d+1} \in \mathcal{Z}_n$ such that $e_{n,d+1}^2(\boldsymbol{z}, z_{d+1}) \leq \mathrm{Avg}(e_{n,d+1}^2(\boldsymbol{z}, z_{d+1}))$ and this z_{d+1} may be chosen by minimizing $e_{n,d+1}^2(\boldsymbol{z}, z_{d+1})$ over the set $\mathcal{Z}_n$. From (19), it is clear now that for the chosen z_{d+1}, we have

$$e_{n,d+1}^2(\boldsymbol{z}, z_{d+1}) \leq \frac{1}{n-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{D}_1} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|},$$

which is the desired result. □

From this result we can deduce the following:

Corollary 6. *Let n be a prime number. Then for $1 \leq m \leq d$ we can construct a vector $\boldsymbol{z} \in \mathcal{Z}_n^d$ such that*

$$e_{n,m}^2(z_1, \ldots, z_m) \leq \frac{1}{n-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq\{1,2,\ldots,m\}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|}.$$

We can set $z_1 = 1$ and for $2 \leq m \leq d$, every z_m can be found by minimizing $e_{n,m}^2(z_1, \ldots, z_m)$ over the set $\mathcal{Z}_n$.

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Proof. Recall that $\widetilde{R}_n(\boldsymbol{z},\mathfrak{u}) = 0$ for all subsets $\mathfrak{u} \subseteq \mathcal{D}$ with $|\mathfrak{u}| = 1$. It follows that $e_{n,1}^2(z) = 0$ for any $z \in \mathcal{Z}_n$, so the inequality (17) holds for $d = 1$. The result then follows immediately from Theorem 5. □

Since $S_2 = 1$ and $S_3 = 2$, observe that if $n \geq 3$, then $|\mathfrak{u}| \leq S_n^{|\mathfrak{u}|}$. Now suppose that the weights are such that (5) is satisfied and

$$\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} S_n^{|\mathfrak{u}|} \leq C(\boldsymbol{\gamma},\delta) n^{\delta},$$

for some $\delta > 0$, where $C(\boldsymbol{\gamma},\delta)$ is independent of d and n. Then Lemma 1 shows that for any odd prime n, the CBC algorithm yields a $\boldsymbol{z}$ for which the weighted star discrepancy satisfies the strong tractability error bound

$$D_{n,\boldsymbol{\gamma}}^*(\boldsymbol{z}) \leq 2C(\boldsymbol{\gamma},\delta) n^{-1+\delta}.$$

An example of weights $\boldsymbol{\gamma}_{\mathfrak{u}}$ having this property is when the $\boldsymbol{\gamma}_{\mathfrak{u}}$ are product weights, that is, $\boldsymbol{\gamma}_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$, and the γ_j are summable. Further details may be found in [9].

We remark that the approach to the general weight case used here is slightly different to the approach used in [9] for the product weight case. If we apply the results obtained here to that case, then the bounds on the weighted star discrepancy are better than those in [9]. However, the approach in [9] has the advantage that it yields bounds on the weighted L_p discrepancy, whereas here we are essentially restricted to the L_∞ case.

5. CBC construction for special classes of weights

In practical situations the weights may satisfy further assumptions. Special classes of weights are the so-called "order-dependent" and "finite-order" weights, which were mentioned in the first section and defined in [2]. The tractability of multivariate integration for the latter class of weights has been studied in [15].

Assume first that the weights are order-dependent. This means that their dependence on $\mathfrak{u}$ is only through the cardinality of $\mathfrak{u}$. It might be reasonable to assume that sets having the same cardinality have equal values of the associated weights. So we assume that instead of using $2^d - 1$ weights, we can use just d weights, say $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$, where Γ_ℓ denotes the weight associated with any set containing ℓ elements for $1 \leq \ell \leq d$. For the bound on the weighted star discrepancy given in Lemma 1 to hold, we require these weights to be in non-increasing order, that is, $\Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_d$.

The next result follows directly from Theorem 5 and Corollary 6 by taking $\boldsymbol{\gamma}_{\mathfrak{u}} = \Gamma_\ell$ whenever $|\mathfrak{u}| = \ell$ and noting that the number of subsets of $\mathcal{D}$ with cardinality ℓ is $\binom{d}{\ell}$.

Corollary 7. *Let n be a prime number and suppose the weights are order-dependent. Then a generating vector $\boldsymbol{z} \in \mathcal{Z}_n^d$ may be constructed component-by-component such that*

$$e_{n,d}^2(\boldsymbol{z}) \leq \frac{1}{n-1} \sum_{\ell=1}^{d} \Gamma_\ell \binom{d}{\ell} S_n^\ell \leq \frac{1}{n-1} \sum_{\ell=1}^{d} \Gamma_\ell \binom{d}{\ell} (2 \ln n)^\ell.$$

Let us assume now that the weights are finite-order. This means that there exists a positive integer q such that $\boldsymbol{\gamma}_{\mathfrak{u}} = 0$ for all $\mathfrak{u}$ with $|\mathfrak{u}| > q$. We shall take q^* to be the smallest integer satisfying this condition. Of course, it makes sense to assume

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that $q^* < d$, otherwise it will be no different from the situation already discussed. We then obtain the following result:

Corollary 8. *Let n be a prime number and suppose the weights are finite-order. Then a generating vector $\boldsymbol{z} \in \mathcal{Z}_n^d$ may be constructed component-by-component such that*

$$e_{n,d}^2(\boldsymbol{z}) \le \frac{1}{n-1} \sum_{\substack{\mathfrak{u}\subseteq\mathcal{D} \\ 1\le|\mathfrak{u}|\le q^*}} \gamma_{\mathfrak{u}} S_n^{|\mathfrak{u}|} \le \frac{1}{n-1} \sum_{\substack{\mathfrak{u}\subseteq\mathcal{D} \\ 1\le|\mathfrak{u}|\le q^*}} \gamma_{\mathfrak{u}} (2\ln n)^{|\mathfrak{u}|}.$$

We can combine these two classes of weights to consider the situation when the weights are both order-dependent and finite-order.

Corollary 9. *Let n be a prime number and suppose the weights are both order-dependent and finite-order. Then a generating vector $\boldsymbol{z} \in \mathcal{Z}_n^d$ may be constructed component-by-component such that*

$$e_{n,d}^2(\boldsymbol{z}) \le \frac{1}{n-1} \sum_{\ell=1}^{q^*} \Gamma_\ell \binom{d}{\ell} S_n^\ell \le \frac{1}{n-1} \sum_{\ell=1}^{q^*} \Gamma_\ell \binom{d}{\ell} (2\ln n)^\ell.$$

Lattice rules with order-dependent and/or finite-order weights present the advantage that the costs of the CBC construction are significantly reduced. The computational costs of the CBC construction are analysed in the next section.

6. Computational costs of the CBC algorithm

6.1. **The cost of the CBC algorithm in the general case.** In this subsection we analyse the complexity of the CBC algorithm, which was presented at the beginning of Section 4.

In order to analyse the cost of the construction, first recall from (10) that $\widetilde{R}_n(\boldsymbol{z}, \mathfrak{u})$ is given by

$$\widetilde{R}_n(\boldsymbol{z}, \mathfrak{u}) = \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j\in\mathfrak{u}} C_k(z_j), \quad \text{where} \quad C_k(z) = \sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi i hkz/n}}{|h|}.$$

It is easy to see that the cost of calculating each $\widetilde{R}_n(\boldsymbol{z}, \mathfrak{u})$ by using this formula is $O\left(n^2|\mathfrak{u}|\right)$ operations. However, it is shown in [7] (see also [9, Appendix A]) that this cost can be reduced at the expense of extra storage. The idea is based on the fact that because n is prime, then $\{kz_j/n\} = \ell/n$ for some ℓ satisfying $0 \le \ell \le n-1$. So to calculate $\widetilde{R}_n(\boldsymbol{z}, \mathfrak{u})$, we need the values of

$$\sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi i h\ell/n}}{|h|},$$

for $0 \le \ell \le n-1$. As shown in [7], these n values may be calculated at a total cost of $O(n)$ operations and then stored. It follows that the number of operations required to calculate each $\widetilde{R}_n(\boldsymbol{z}, \mathfrak{u})$ is of $O(n|\mathfrak{u}|)$ operations at the expense of $O(n)$ extra storage.

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Recall that

$$\begin{aligned} e_{n,m}^2(z_1,\dots,z_m) &= \sum_{\emptyset\neq\mathfrak{u}\subseteq\{1,2,\dots,m\}} \gamma_{\mathfrak{u}} \widetilde{R}_n((z_1,\dots,z_m),\mathfrak{u}) \\ &= e_{n,m-1}^2(z_1,\dots,z_{m-1}) + \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\dots,m\}\\ m\in\mathfrak{u}}} \gamma_{\mathfrak{u}} \widetilde{R}_n((z_1,\dots,z_m),\mathfrak{u}) \\ &= e_{n,m-1}^2(z_1,\dots,z_{m-1}) \\ &\quad + \frac{1}{n} \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\dots,m\}\\ m\in\mathfrak{u}}} \gamma_{\mathfrak{u}} \sum_{k=0}^{n-1} C_k(z_m) \prod_{j\in\mathfrak{u}-\{m\}} C_k(z_j). \end{aligned} \tag{20}$$

Now it may be the case that some of the 2^d-1 weights are zero. To take into account the computational savings that arise, let τ_m be the number of non-zero weights $\gamma_{\mathfrak{u}}$ for which $\mathfrak{u}\subseteq\{1,2,\dots,m\}$ with $m\in\mathfrak{u}$. Then $0\le\tau_m\le 2^{m-1}$. Also, let τ be the total number of non-zero weights, that is,

$$\tau = \sum_{m=1}^{d} \tau_m \le 2^d - 1.$$

Then to find z_m which minimizes $e_{n,m}^2(z_1,\dots,z_m)$, we need to calculate the last term in (20) for all $z_m\in\mathcal{Z}_n$. This requires $O(nm\tau_m)$ operations. Since there are $n-1$ choices for z_m, this means that the cost of adding a new component z_m to the already existing components is $O(n^2m\tau_m)$ operations for each m. Taking m from 2 to d, we conclude that the total operation count of the CBC algorithm to obtain a d-dimensional $\boldsymbol{z}$ is $O(n^2d\tau)$.

We observe that if all the weights are non-zero, we have a total of $\tau=2^d-1$ weights and so the total cost of the construction will be $O(n^2d2^d)$. In practice such a cost is unacceptable as 2^d grows very quickly when d increases, but it can be considerably reduced under further assumptions on the weights.

6.2. **The cost of the construction for finite-order weights.** Let q^* be the smallest integer for which $\gamma_{\mathfrak{u}}=0$ whenever $|\mathfrak{u}|>q^*$. In this case the total number of weights is $\tau=\sum_{\ell=1}^{q^*}\binom{d}{\ell}$. For $d\ge 2$ and $q^*<d$, it may be proved by induction that

$$\sum_{\ell=1}^{q^*}\binom{d}{\ell} \le d^{q^*}.$$

From the previous subsection, it will follow that the total operation count of the CBC algorithm with finite-order weights is then $O(n^2d^{q^*+1})$. As pointed out in [2], the cost of the construction is exponential in q^*, but this is not dangerous as long as q^* is not large.

6.3. **The cost of the construction for order-dependent weights.** In this case, because there are at most d distinct weights, the cost of the construction can be significantly reduced by using a similar technique as in [2]. First, we observe that

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the quantity $e^2_{n,m}(z_1, z_2, \ldots, z_m)$ can be expanded as

$$\begin{aligned} e^2_{n,m}(z_1, z_2, \ldots, z_m) &= \sum_{\ell=1}^{m} \Gamma_\ell \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\ldots,m\}\\ |\mathfrak{u}|=\ell}} \widetilde{R}_n(\boldsymbol{z}, \mathfrak{u}) \\ &= \sum_{\ell=1}^{m} \Gamma_\ell \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\ldots,m\}\\ |\mathfrak{u}|=\ell}} \left(\frac{1}{n} \sum_{k=0}^{n-1} \prod_{j\in\mathfrak{u}} C_k(z_j) \right) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\ell=1}^{m} \Gamma_\ell \sigma_k(m, \ell), \end{aligned}$$

where

$$\sigma_k(m, \ell) = \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\ldots,m\}\\ |\mathfrak{u}|=\ell}} \prod_{j\in\mathfrak{u}} C_k(z_j) \quad \text{for} \quad 1 \le \ell \le m.$$

Then we can obtain a recursive formula to compute the quantities $\sigma_k(m, \ell)$. Indeed, we have

$$\begin{aligned} \sigma_k(m, \ell) &= \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\ldots,m-1\}\\ |\mathfrak{u}|=\ell}} \prod_{j\in\mathfrak{u}} C_k(z_j) + C_k(z_m) \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\ldots,m-1\}\\ |\mathfrak{u}|=\ell-1}} \prod_{j\in\mathfrak{u}} C_k(z_j) \\ &= \sigma_k(m-1, \ell) + C_k(z_m)\sigma_k(m-1, \ell-1), \end{aligned}$$

for $m \ge 2$ and $\ell \ge 2$. It is easy to see that $\sigma_k(1, 1) = C_k(z_1)$. We also have

$$\sigma_k(m, 1) = \sum_{j=1}^{m} C_k(z_j) \quad \text{and} \quad \sigma_k(m, m) = \prod_{j=1}^{m} C_k(z_j).$$

For each k, the quantities $\sigma_k(m, \ell)$ may be viewed as being the elements of a lower triangular matrix. Then to compute the quantities $\sigma_k(m, \ell)$ required for $e^2_{n,m}(z_1, z_2, \ldots, z_m)$, we can use the following algorithm (with $\sigma_k(1, 1) = C_k(z_1)$):

Set $\sigma_k(m, 1) = \sum_{j=1}^{m} C_k(z_j)$.

Set $\sigma_k(m, m) = \prod_{j=1}^{m} C_k(z_j)$.

For $\ell = 2, 3, \ldots, m-1$ do:

$$\sigma_k(m, \ell) = \sigma_k(m-1, \ell) + C_k(z_m)\sigma_k(m-1, \ell-1).$$

Now it is clear that if the quantities $\sigma_k(m-1, \ell)$ for $\ell = 1, 2, \ldots, m-1$ have been computed and stored using $O(m)$ memory, then the computation of all $\sigma_k(m, \ell)$ as well as of $\sum_{\ell=1}^{m} \Gamma_\ell \sigma_k(m, \ell)$ will require only $O(m)$ operations for each k, assuming that the values of $C_k(z_m)$ have also been stored as indicated in Section 6.1. Since there are n possible values for k, the amount of storage required is $O(nd)$ for a complete run of the algorithm. In conclusion, the computation of $e^2_{n,m}(z_1, z_2, \ldots, z_m)$ for each z_m requires $O(nm)$ operations, and the total cost of the CBC algorithm will be $O(n^2d^2)$. This shows that the complexity of the CBC construction is smaller for order-dependent weights than for finite-order weights.

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6.4. **The cost of the construction for weights which are both order-dependent and finite-order.** If we assume that the order-dependent weights are also finite-order, then

$$e_{n,d}^2(\boldsymbol{z}) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\ell=1}^{q^*} \Gamma_\ell \sigma_k(d, \ell).$$

With the assumption that $q^* < d$, the total cost of the construction will be reduced to $O(n^2 d q^*)$, with additional $O(nq^*)$ memory required for storage.

6.5. **Speeding up the CBC construction.** A fast CBC construction has recently been proposed by Nuyens and Cools in [11] for shift-invariant reproducing kernel Hilbert spaces. Their technique is based on writing the CBC algorithm appropriate for these function spaces in terms of matrix-vector multiplications and then applying a fast algorithm to do these multiplications. For multiplication of an $n \times n$ matrix with an n-vector, the operation count is reduced to $O(n \ln n)$ from the normal $O(n^2)$.

Their technique can be modified so that it applies to the CBC algorithm given in Section 4. Thus for the case of general weights, the $O(n^2 d 2^d)$ operation count may be reduced to $O(n \ln(n) d 2^d)$, while for finite-order weights the operation count may be reduced to $O(n \ln(n) d^{q^*+1})$. In the case of order-dependent weights, by first doing a summation over all weights and then applying the fast matrix-vector multiplication, the total operation count may actually be reduced to $O(nd \ln(n) + nd^2)$ with $O(nd)$ additional storage as mentioned in Section 6.3. Further details of such a fast algorithm may by found in [1, Section 4]. In that work, a function of the form

$$\frac{1}{n} \sum_{k=0}^{n-1} \sum_{\ell=1}^{d} \Gamma_\ell \sum_{\substack{\mathfrak{u} \subseteq \mathcal{D} \\ |\mathfrak{u}| = \ell}} \prod_{j \in \mathfrak{u}} \omega\left(\left\{\frac{k z_j}{n}\right\}\right)$$

is minimized. For the weighted star discrepancy considered here, we see from Section 6.3 that we can apply their fast algorithm by taking

$$\omega(x) = {\sum_{-\frac{n}{2} < h \leq \frac{n}{2}}}' \frac{e^{2\pi \mathrm{i} h x}}{|h|}, \quad x \in [0, 1].$$

Finally, if the weights are both order-dependent and finite-order, then the cost of the construction will be $O(nd \ln(n) + ndq^*)$ with $O(nq^*)$ additional storage.

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