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CONVERGENT DIFFERENCE SCHEMES FOR THE HUNTER–SAXTON EQUATION

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Abstract. We propose and analyze several finite difference schemes for the Hunter–Saxton equation

$$u_t + uu_x = \frac{1}{2}\int_0^x (u_x)^2\,dx, \qquad x > 0,\ t > 0. \tag{HS}$$

This equation has been suggested as a simple model for nematic liquid crystals. We prove that the numerical approximations converge to the unique dissipative solution of (HS), as identified by Zhang and Zheng. A main aspect of the analysis, in addition to the derivation of several a priori estimates that yield some basic convergence results, is to prove strong convergence of the discrete spatial derivative of the numerical approximations of u, which is achieved by analyzing various renormalizations (in the sense of DiPerna and Lions) of the numerical schemes. Finally, we demonstrate through several numerical examples the proposed schemes as well as some other schemes for which we have no rigorous convergence results.

1. Introduction

Liquid crystals are mesophases, i.e., intermediate states of matter between the liquid and the crystal phase [15]. They exhibit characteristics of fluid flow and have optical properties typically associated with crystals. Liquid crystals consist of strongly elongated molecules (typical sizes are $5 - 10$Å) that can be considered invariant under rotation by an angle of π. Nematic liquid crystals are commonly described by two linearly independent vector fields; one describing the fluid flow and one describing the orientation of the so-called director field that gives the orientation of the rod-like molecule. In this paper we will specialize to stationary flow, and hence focus exclusively on the dynamics of the director field, a map $\mathbf{n}\colon \mathbb{R}^3 \to \mathbb{S}^3$ from the Euclidean space to the unit ball; see Saxton [14]. It is common to consider the Oseen–Franck expression for the internal energy (see [14], [15], [6])

$$W(\mathbf{n}, \nabla\mathbf{n}) = \alpha\,|\mathbf{n}\times(\nabla\times\mathbf{n})|^2 + \beta(\nabla\cdot\mathbf{n})^2 + \gamma(\mathbf{n}\cdot\nabla\times\mathbf{n})^2,$$

where α, β, and γ are constants. Physically, α correlates with "splay"; β correlates with "twist"; and γ with "bend" (see, e.g., [15] for an extensive discussion). The

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dynamics of n is governed by the action principle

$$\frac{\delta}{\delta \mathbf{n}} \iint (\mathbf{n}_t^2 - W(\mathbf{n}, \nabla \mathbf{n}))\, dx\, dt. \tag{1.1}$$

Here we further specialize to consider planar director fields given by

$$\mathbf{n} = \mathbf{n}(x,t) = \cos(\psi(x,t))\mathbf{e}_x + \sin(\psi(x,t))\mathbf{e}_y,$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are orthonormal vectors in the x and y direction, respectively. Inserting this into (1.1) we find the Lagrangian

$$\mathcal{L} = \iint \left(\psi_t^2 - c^2(\psi)\psi_x^2\right) dx\, dt$$

with

$$c(\psi)^2 = \alpha \cos^2 \psi + \beta \sin^2 \psi,$$

which yields the Euler–Lagrange equation

$$\psi_{tt} = c(\psi)(c(\psi)\psi_x)_x.$$

We now consider the equation satisfied by expansions around the constant state. More precisely, assume [6]

$$\psi(x,t,\varepsilon) = \psi_0 + \varepsilon\psi_1(\theta,\tau) + \mathcal{O}(\varepsilon^2)$$

with $\theta = x - c(\psi_0)t$ (assuming $c'(\psi_0) \neq 0$) and $\tau = \varepsilon t$. Introduce $u = c'(\psi_0)\psi_1$ and redefine x by $x = \operatorname{sign}(c'(\psi_0))\theta$. Then

$$(u_t + uu_x)_x = \frac{1}{2}(u_x)^2, \quad u|_{t=0} = u_0,$$

or

$$u_t + uu_x = \frac{1}{2}\int_0^x (u_x)^2\, dx, \quad u|_{t=0} = u_0, \tag{1.2}$$

which is the Hunter–Saxton equation [6]. By introducing

$$v = u_x,$$

we may write this as

$$v_t + uv_x = -\frac{1}{2}v^2, \quad v = u_x \tag{1.3}$$

or

$$v_t + (uv)_x = \frac{1}{2}v^2, \quad v = u_x.$$

The equation possesses many intriguing properties: it is completely integrable [7]; indeed, let

$$L = \partial_x \frac{1}{u_{xx}}\partial_x, \quad A = \frac{1}{2}(u\partial_x + \partial_x u).$$

Then

$$L_t = [L, A] \qquad \text{is formally equivalent to} \qquad (u_{xt} + uu_{xx} + \frac{1}{2}u_x^2)_x = 0.$$

Equation (1.3) also has infinitely many conservation laws (see [7]); the first few reading

$$\begin{aligned}(|v_x|^{1/2})_t + (u\,|v_x|^{1/2})_x &= 0,\\ (v^2)_t + (uv^2)_x &= 0,\\ (uv^2)_t - (2uvu_t + u_t^2)_x &= 0.\end{aligned}$$

Furthermore, it is bivariational and bi-Hamiltonian (see [7]). Characteristics are given by

$$\frac{d}{dt}\Phi(x,t) = u(\Phi(x,t),t), \quad \Phi(x,0) = x.$$

We consider the half-line problem and assume $u(0,t) = 0$ and $v(x,0) = v_0$. If $v_0 \geq 0$, then

$$\begin{aligned} \Phi(x,t) &= \int_0^x (1 + \tfrac{1}{2}v_0(y)t)^2 \, dy, \\ u(\Phi(x,t),t) &= \int_0^x (1 + \tfrac{1}{2}v_0(y)t)v_0(y) \, dy, \\ v(\Phi(x,t),t) &= \frac{2v_0(x)}{2 + v_0(x)t}. \end{aligned}$$

In contrast to hyperbolic conservation laws where characteristics in general will collide, the characteristics for the Hunter–Saxton equation will only focus, that is, approach the same tangent.

Smooth solutions of (1.3) can be expressed as the solution of a system (see [8])

$$\begin{aligned} u &= u_0(\xi) + tg(\xi) + h'(\xi), \\ x &= \xi + tu_0(\xi) + \frac{1}{2}t^2 g(\xi) + h(\xi), \end{aligned}$$

where h is any function with $h(0) = h'(0) = 0$, and $g'(\xi) = \frac{1}{2}u_0'(\xi)^2$. However, the Hunter–Saxton equation will not in general enjoy classical solutions. More precisely, if u_0 is *not* monotone increasing, then

$$\inf(u_x) \to -\infty \text{ as } t \uparrow t^* = 2/\sup(-u_0'). \tag{1.4}$$

The concept of a weak solution is more complicated. Two different solution concepts can be found in the literature, namely that of a *conservative solution* and that of a *dissipative solution* (see Hunter and Zheng [8, 9] and Zhang and Zheng [18]). Before we recall these definitions, and for later reference, let us state the problem that we intend to study in this paper, i.e., the Hunter–Saxton equation augmented with appropriate initial and boundary conditions:

$$\begin{aligned} & v_t + uv_x = -\frac{1}{2}v^2, \quad u_x = v, \quad (x,t) \in Q_T, \\ & v(x,0) = v_0(x), \quad x \in \mathbb{R}^+, \\ & u(0,t) = 0, \quad t \in [0,T], \end{aligned} \tag{1.5}$$

where $T > 0$ is a fixed final time ($T = \infty$ is possible) and Q_T denotes the space-time cylinder $\mathbb{R}^+ \times (0,T)$, where $\mathbb{R}^+ = (0,\infty)$. Sometimes we also use the notation $\overline{Q_T}$ for the set $\mathbb{R}_0^+ \times [0,T]$, where $\mathbb{R}_0^+$ is shorthand for the half-closed interval $[0,\infty)$.

Conservative solutions of (1.5) are defined as triplets (v, u, Φ) satisfying

$$
\begin{aligned}
&v \in L^\infty((0,T); L^2(\mathbb{R}^+)), \quad u \in C(\overline{Q_T}), \quad \Phi, \Phi_t \in C(\overline{Q_T}),\\
&\left.\begin{aligned} &v_t + (uv)_x = \tfrac{1}{2}v^2, \quad u_x = v\\ &(v^2)_t + (uv^2)_x = 0 \end{aligned}\right\} \text{ in the sense of distributions on } Q_T,\\
&\partial_t \Phi(x,t) = u(\Phi(x,t),t), \quad \Phi_0(x) = x,\\
&\int_{\Phi(x_1,t)}^{\Phi(x_2,t)} v(y,t)^2\,dy = \int_{\Phi(x_1,0)}^{\Phi(x_2,0)} v(y,0)^2\,dy, \quad x_1 < x_2.
\end{aligned}
$$

Moreover, the function $u(x,t)$ is zero at $x = 0$ as a continuous function in x for each $t \in [0,T]$, while the function $v(x,t)$ takes on the initial data $v_0(x)$ at $t = 0$ in the sense of $C(\mathbb{R}^+, L^1(\mathbb{R}^+))$. Since we are not interested in conservative solutions in this paper, we refer to the papers [8, 9, 16, 17, 18, 2, 1] by Bressan, Constantin, Hunter, Zhang, and Zheng for more information about them and their properties.

However, in this paper we are going to work with dissipative solutions, so we choose to state this notion of solution explicitly in a definition. It is convenient to first define a weak solution.

Definition 1.1. A pair of functions (v, u) is a weak solution of (1.5) provided

$$
\begin{aligned}
&v \in L^\infty((0,T); L^2(\mathbb{R}^+)), \quad u \in C(\overline{Q_T}),\\
&v_t + (uv)_x = \frac{1}{2}v^2 \text{ and } u_x = v \text{ in the sense of distributions on } Q_T,\\
&\int_0^\infty v(x,t)^2\,dx \le \int_0^\infty v_0(x)^2\,dx \quad \text{for almost all } t \in (0,T),\\
&u(x,t) \to 0 \text{ as } x \to 0 \text{ for each } t \in [0,T],\\
&v(x,t) \to v_0(x) \text{ in } C([0,T]; L^1(\mathbb{R}^+)) \text{ as } t \to 0+.
\end{aligned}
$$

Definition 1.2. A pair of functions (v, u) is a dissipative solution of (1.5) provided the pair (v, u) is a weak solution of (1.5) and

$$
v \le \frac{2}{t} \quad \text{a.e. on } Q_T.
$$

Dissipative solutions of the Hunter–Saxton equation are well understood, and we refer to a long series of papers by Hunter, Zhang, and Zheng [8, 9, 16, 17, 18] for various types of results. This series culminated with the paper [18] by Zhang and Zheng, which established the existence and uniqueness of dissipative solutions for the (natural) case of pure L^2 initial data v_0.

Thus the Hunter–Saxton equation is well studied from a mathematical point of view. However, there has been no rigorous analysis of numerical schemes for (1.5). For general initial data, there are no closed-form solutions to the Hunter–Saxton equation, and therefore the study of numerical schemes is important. It is the chief purpose of this paper to propose and analyze some numerical schemes of finite difference type for the Hunter–Saxton equation.

The numerical schemes that we propose are deliberately based on discretizing the nonconservative form (1.3) and not the conservative form (1.2). One might expect the latter form to be natural since it can be viewed as a perturbation of Burgers' equation, where the perturbation takes the form of a nonlocal integro operator.

For Burgers' equation and other nonlinear conservation laws there exist a rich literature on several types of numerical schemes. Many of the schemes developed for conservation laws are also accompanied by a sound theoretical foundation, sometimes using rather sophisticated analytical tools like, for example, compensated compactness. However, we have not been able to prove that any "reasonable" finite difference scheme based on the conservation law form (1.2) converges to a dissipative solution. For this reason we will focus exclusively on the form (1.3), which is a linear transport equation with a quadratic right-hand side plus an additional side constraint relating the derivative of the "velocity" u to the unknown v.

Let us be a bit more precise about our achievements in this paper. In the case where the initial data v_0 is a nonnegative function in $L^1 \cap L^q$ with $q > 2$, we describe semi-discrete, implicit, and explicit upwind finite difference schemes, and for all these schemes we show that the corresponding approximate solutions converge to the unique dissipative solution of the Hunter–Saxton equation (1.5). Then we consider the more complicated case where v_0 does not have a definite sign and merely belongs to $L^1 \cap L^2$. Here we define a semi-discrete upwind scheme and again we show that the suggested scheme converges to the unique dissipative solution of the Hunter–Saxton equation.

The fact that our numerical schemes are of *upwind* type means that the finite differencing of the transport part uv_x is biased in the direction of incoming waves, making it possible to resolve discontinuities without excessive smearing. We stress that the use of upwind schemes is quite natural, as one would expect them to dissipate the energy and as such generate dissipative solutions in the limit as the discretization parameters tend to zero. Our analysis confirms this intuition. However, we stress that the results are not obvious. One can, for instance, show that certain "natural" schemes for the related Camassa–Holm equation either diverge or converge to a wrong solution; see [5]. Furthermore, some of the schemes tested in Section 7 for the Hunter–Saxton equation indicate nonconvergent behavior.

Regarding the convergence analysis, we derive several a priori estimates in Lebesgue and Sobolev spaces, which yield some basic convergence results. An interesting mathematical issue is that we need to prove that the spatial derivative of the numerical solutions, i.e., $v_{\Delta x} = (u_{\Delta x})_x$, which only is weakly compact a priori, in fact converges strongly. Strong convergence of $v_{\Delta x}$ is necessary if we want to recover the Hunter–Saxton equation when we take the limit in the finite difference schemes. Strong convergence of $v_{\Delta x}$ is obtained by analyzing various renormalizations (in the sense of DiPerna and Lions [3, 12, 13]) of the numerical schemes and corresponding defect measures. In addition, to prevent $v_{\Delta x}^2$ from exhibiting concentrations as $\Delta x \to 0$, we need to derive higher (than L^2) integrability estimates for $v_{\Delta x}$. Our arguments can be viewed as discrete counterparts of those employed by Zhang and Zheng [16, 17, 18] to prove existence of a dissipative solution.

The organization of this paper goes as follows: in Section 2 we introduce some (finite difference) notation and recall a few well-known mathematical results useful for the subsequent analysis. In Section 3 we present and analyze the semi-discrete scheme. The particular form of the scheme and the analysis rely on the assumption that the initial data are nonnegative and belong to $L^1 \cap L^q$ for some $q > 2$. Sections 4 and 5 are devoted to similar analyses for implicit and explicit upwind schemes. In Section 6 we extend our analysis to the case of initial data in $L^1 \cap L^2$. Finally, in Section 7 we present several numerical examples, which demonstrate the proposed numerical schemes as well as some other schemes which do not have a theoretical foundation.

2. Some preliminaries

We start by introducing some notation needed to define the finite difference schemes. Throughout this paper we reserve Δx and Δt to denote two small positive numbers that represent the spatial and temporal discretization parameters, respectively, of the numerical schemes. Given $\Delta x, \Delta t > 0$, let $D_\pm$ denote the discrete forward and backward differences, respectively, in the spatial direction, i.e.,

$$D_\pm g(x) = \pm \frac{1}{\Delta x}\big(g(x \pm \Delta x) - g(x)\big)$$

for any function $g\colon \mathbb{R} \to \mathbb{R}$ admitting pointvalues. Similarly, we let $D_\pm^t$ denote the forward and backward differences, respectively, in the time direction, i.e.,

$$D_\pm^t h(x,t) = \pm \frac{1}{\Delta t}\big(h(x, t \pm \Delta t) - h(x,t)\big)$$

for any function $h\colon Q_T \to \mathbb{R}$ admitting pointvalues.

For $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we set $x_j = j\Delta x$, and for $n = 0, 1, \dots, N$, where $N\Delta t = T$ for some fixed time horizon $T > 0$, we set $t_n = n\Delta t$.

For any function $g = g(x)$ admitting pointvalues we write $g_j = g(x_j)$, and similarly for any function $h = h(x,t)$ admitting pointvalues we write $h_j^n = h(x_j, t_n)$. Furthermore, let us introduce the spatial and temporal grid cells

$$I_j = [x_{j-1/2}, x_{j+1/2}), \qquad I_j^n = I_j \times [t_n, t_{n+1}),$$

where $x_{j\pm 1/2} = x_j \pm \Delta x/2$. Thus in this notation, $D_\pm g_j = \pm(g_{j\pm 1} - g_j)/\Delta x$. Also, a discrete Leibniz rule holds:

$$D_\pm (g_j h_j) = g_j D_\pm h_j + h_{j\pm 1} D_\pm g_j. \tag{2.1}$$

If we extend a sequence $\{g_j\}_{j\in\mathbb{N}_0}$ to a piecewise constant function defined on $\mathbb{R}_0^+$ (actually on $[-\Delta x/2, \infty)$) by

$$g_{\Delta x}(x) = \sum_{j\in\mathbb{N}_0} g_j \mathbf{1}_{I_j}(x), \tag{2.2}$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A, viz.

$$\mathbf{1}_A(x) = \begin{cases} 1, & \text{for } x \in A, \\ 0, & \text{for } x \notin A, \end{cases}$$

then clearly

$$\|g_{\Delta x}\|_{L^p(\mathbb{R}^+)} = \Big(\Delta x \sum_{j\in\mathbb{N}_0} |g_j|^p\Big)^{1/p}.$$

Let f be a C^2 function. By using a Taylor expansion we find

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(\xi)(b-a)^2 \tag{2.3}$$

for some ξ between a and b. Let $\{v_j\}_{j\in\mathbb{N}_0}$ be a given sequence. Using the Taylor expansion (2.3) on the sequence $\{f(v_j)\}_{j\in\mathbb{N}_0}$ we obtain

$$D_\pm f(v_j) = f'(v_j) D_\pm v_j \pm \frac{\Delta x}{2} f''(\xi_j^\pm)(D_\pm v_j)^2 \tag{2.4}$$

for some $\xi_j^{\pm}$ between $v_{j\pm 1}$ and v_j. We will make frequent use of (2.4), which states that a discrete chain rule holds up to an error term of order $\Delta x (D_\pm v_j)^2$.

In this paper we will exploit some well-known results related to weak convergence and convex functions. For the convenience of the reader we collect these results in a lemma (for proofs, see, for example, [4]).

Lemma 2.1. *Let O be a bounded open subset of $\mathbb{R}^M$, with $M \geq 1$.*

Let $\{v_n\}_{n\geq 1}$ be a sequence of measurable functions on O for which

$$\sup_{n\geq 1} \int_O \Phi(|v_n(y)|)\, dy < \infty,$$

for some continuous function $\Phi : [0,\infty) \to [0,\infty)$. Then there exists a subsequence (which is not relabeled) such that

$$g(v_n) \rightharpoonup \overline{g(v)} \text{ in } L^1(O)$$

for all continuous functions $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|v|\to\infty} \frac{|g(v)|}{\Phi(|v|)} = 0.$$

Let $g \colon \mathbb{R} \to (-\infty, \infty]$ be a lower semicontinuous convex function and $\{v_n\}_{n\geq 1}$ a sequence of measurable functions on O, for which

$$v_n \rightharpoonup v \text{ in } L^1(O),\ g(v_n) \in L^1(O) \text{ for each } n,\ g(v_n) \rightharpoonup \overline{g(v)} \text{ in } L^1(O).$$

Then

$$g(v) \leq \overline{g(v)} \text{ a.e. on } O.$$

Moreover, $g(v) \in L^1(O)$ and

$$\int_O g(v)(y)\, dy \leq \liminf_{n\to\infty} \int_O g(v_n)(y)\, dy.$$

If, in addition, g is strictly convex on an open interval $(a,b) \subset \mathbb{R}$ and

$$g(v) = \overline{g(v)} \text{ a.e. on } O,$$

then, passing to a subsequence if necessary,

$$v_n(y) \to v(y) \text{ for a.e. } y \in \{y \in O \mid v(y) \in (a,b)\}.$$

Occasionally we will use the following standard interpolation inequality.

Lemma 2.2. *Let O be an open subset of $\mathbb{R}^M$, with $M \geq 1$. Let $1 \leq p_0 < p_\theta < p_1 \leq \infty$, $\theta \in (0,1)$, and*

$$\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

Then, for any $v \in L^{p_0}(O) \cap L^{p_1}(O)$,

$$\|v\|_{L^{p_\theta}(O)} \leq \|v\|_{L^{p_0}(O)}^{\theta} \|v\|_{L^{p_1}(O)}^{1-\theta} \leq \|v\|_{L^{p_0}(O)} + \|v\|_{L^{p_1}(O)}.$$

Finally, let us recall the definition of a standard mollifier, which will be used several times in this paper. Let $\omega(x)$ be a smooth non-negative function with support inside $[-1,1]$, $\omega(-x) = \omega(x)$, and $\int \omega\, dx = 1$. Then a standard mollifier $\omega_\varepsilon = \omega_\varepsilon(x)$, $\varepsilon > 0$, is defined by

$$\omega_\varepsilon(x) = \frac{1}{\varepsilon} \omega\left(\frac{x}{\varepsilon}\right), \qquad x \in \mathbb{R}.$$

3. The semi-discrete upwind scheme

In this section we present and analyze the semi-discrete scheme, relying on the notation introduced in Section 2.

For the analysis in this section we assume that the initial function satisfies

$$v_0 \geq 0 \text{ and } v_0 \in L^1(\mathbb{R}^+) \cap L^q(\mathbb{R}^+) \text{ for some } q > 2. \tag{3.1}$$

By interpolation the function v_0 belongs to $L^p(\mathbb{R}^+)$ for any $1 \leq p \leq q$. The general case where v_0 belongs merely to L^2 and may change sign is more involved and will be treated in Section 6. The L^1 requirement is a natural replacement of the compact support condition on v_0 used by Zhang and Zheng [16, 17, 18].

Let $\{v_j^0\}_{j\in\mathbb{N}_0}$ be sequence of discrete initial data chosen such that

$$v_{\Delta x}^0(x) = \sum_{j\in\mathbb{N}_0} v_j^0 \mathbf{1}_{I_j}(x) \tag{3.2}$$

converges to the initial data v_0 in $L^2(\mathbb{R}^+)$ as $\Delta x \to 0$. We make the approximation such that $v_j^0 \geq 0$ and $v_j^0 = 0$ for $j > J_{\Delta x} := 1/(\Delta x^2)$. It is not hard to construct such a sequence. For example, we may take

$$v_j^0 = \frac{1}{\Delta x}\int_{I_j} v_0(x)\,dx, \qquad j = 1, 2, \dots, J_{\Delta x},$$

and set $v_0^0 = v_1^0$ and $v_j^0 = 0$ for all $j \geq J_{\Delta x}$. For $t \geq 0$, let $\{(v_j(t), u_j(t))\}_{j\in\mathbb{N}_0}$ satisfy the finite system of ordinary differential equations

$$\begin{aligned} &\dot v_j + u_j D_- v_j = -\frac{1}{2}(v_j)^2, \quad j \in [0, J_{\Delta x}], \qquad v_j = 0, \quad j > J_{\Delta x}, \\ &D_+ u_j = v_j, \quad j \in \mathbb{N}_0, \qquad u_0(t) = 0, \\ &v_j|_{t=0} = v_j^0, \quad j \in [0, J_{\Delta x}], \end{aligned} \tag{3.3}$$

where $\dot v_j$ denotes differentiation of v_j with respect to t. Whenever it is convenient we also extend v_j and u_j to be zero for $j < 0$. Observe that it follows from (3.3) that

$$u_j(t) = \Delta x \sum_{i=0}^{j-1} v_i(t) \quad \text{for } j \in \mathbb{N}.$$

Using the discrete Leibniz rule (2.1), we have

$$D_-(u_j v_j) = u_j D_- v_j + v_{j-1} D_- u_j = u_j D_- v_j + (v_{j-1})^2,$$

and hence we may write the scheme (3.3) in conservative form:

$$\dot v_j + D_-(u_j v_j) = \frac{1}{2}(v_j)^2 + (v_{j-1})^2 - (v_j)^2 = \frac{1}{2}(v_j)^2 - \Delta x\, D_-(v_j)^2. \tag{3.4}$$

For positive Δx, equation (3.3) is a finite-dimensional system of ordinary differential equations, which has a C^1 solution at least until some blowup time. Below (see Lemma 3.2) we shall show that blowup does not happen. For the convergence analysis, we need to introduce the two pointwise defined functions

$$v_{\Delta x}(x,t) = \sum_{j\in\mathbb{N}_0} v_j(t)\mathbf{1}_{I_j}(x) \quad \text{and} \quad u_{\Delta x}(x,t) = \int_0^x v_{\Delta x}(y,t)\,dy, \tag{3.5}$$

which are piecewise constant and piecewise linear and continuous, respectively.

Before we continue we need to establish that the numerical solution $(u_{\Delta x}, v_{\Delta x})$ remains nonnegative if it initially started nonnegative. We also prove that $v_{\Delta x}$ is bounded from above, independently of Δx, as soon as $t > 0$. This latter estimate is a consequence of an Oleĭnik-type (one-sided Lipschitz) estimate for $u_{\Delta x}$. Besides ensuring uniqueness of the dissipative solution, the Oleĭnik-type estimate is not used directly in the convergence proof in this and the next two sections. It will, however, play an important role in the convergence proof in Section 6, where we allow v_0 (and thus the solution) to change sign. We emphasize that for the arguments in this and the next two sections it is important that the functions $u_{\Delta x}, v_{\Delta x}$ are nonnegative.

Lemma 3.1. *For $t > 0$ and $j \in \mathbb{N}_0$ we have*

$$0 \leq v_j(t) \leq \frac{2}{t}. \tag{3.6}$$

Proof. We have that $v_0(0) = v_0^0 \geq 0$. Since

$$\dot{v}_0 = -\frac{1}{2}(v_0)^2,$$

it trivially follows that $v_0(t) \geq 0$ for all t. Let $t_0 \geq 0$ and $k > 0$ be such that $v_k(t_0) = 0$, and $v_j(t_0) \geq 0$ for all $j < k$. Then $D_- v_k(t_0) \leq 0$ and $u_k(t_0) \geq 0$, and hence

$$\dot{v}_k(t_0) = -u_k D_- v_k(t_0) \geq 0.$$

Hence $v_j(t) \geq 0$ and $u_j(t) \geq 0$ for all j and t.

Set

$$\bar{k}(t) = \sup\{k \mid v_k(t) \geq v_j(t) \text{ for all } j\}$$

and $\bar{v}_{\Delta x}(t) = v_{\bar{k}(t)}(t)$. Since $\bar{v}_{\Delta x}(t)$ is the maximum of a finite number of continuously differentiable functions, it is continuous and differentiable almost everywhere. At every differentiable point, we know that

$$\frac{d}{dt}\bar{v}_{\Delta x}(t) \leq -\frac{1}{2}\bar{v}_{\Delta x}(t)^2,$$

since if $\bar{k} > 0$, then $D_- v_{\bar{k}}(t) \geq 0$, while if $\bar{k} = 0$ the above inequality is an equality. Now the comparison principle for ordinary differential equations yields the last inequality of the lemma. $\square$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. Multiplying the scheme (3.3) by $f'(v_j)$ and using the discrete chain rule (2.4), we find that

$$\frac{d}{dt}f(v_j) + u_j D_- f(v_j) + \frac{\Delta x}{2} u_j f''(\xi_j)(D_- v_j)^2 = -\frac{1}{2}f'(v_j)(v_j)^2. \tag{3.7}$$

This is our main tool for proving the next lemma, which collects some uniform a priori estimates satisfied by the numerical approximations.

Lemma 3.2. *Suppose* (3.1) *holds. Then for any $t > 0$ we have*

$$\|v_{\Delta x}(\,\cdot\,,t)\|_{L^p(\mathbb{R}^+)} \leq \|v_{\Delta x}(\,\cdot\,,0)\|_{L^p(\mathbb{R}^+)} \leq C, \quad p \in [2,q]. \tag{3.8}$$

Furthermore, there holds

$$\|v_{\Delta x}\|_{L^{q+1}(Q_T)} \leq \frac{2}{q-2}\,\|v_{\Delta x}(\,\cdot\,,0)\|_{L^q(\mathbb{R}^+)} \leq C.$$

For any $t > 0$ there holds

$$\|v_{\Delta x}(\,\cdot\,,t)\|_{L^1(\mathbb{R}^+)} \leq \|v_{\Delta x}(\,\cdot\,,0)\|_{L^1(\mathbb{R}^+)} + \frac{t}{2}\,\|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)} \leq C(t).$$

For any $t > 0$ there holds

$$\|u_{\Delta x}(\cdot\,,t)\|_{L^\infty(\mathbb{R}^+)} \le \|v_{\Delta x}(\cdot\,,0)\|_{L^1(\mathbb{R}^+)} + \frac{t}{2}\|v_{\Delta x}(\cdot\,,0)\|_{L^2(\mathbb{R}^+)} \le C(t).$$

Remark 3.3. The first estimate (3.8) also states that the approximate solutions remain inside some ball in "$\mathbb{R}^{J_{\Delta x}}$", and thus do not blow up. Therefore the solution of the system of ordinary differential equations (3.3) exists for all $t > 0$.

Proof. Choosing $f(v) = v^p$ in (3.7), we obtain

$$\frac{d}{dt}(v_j)^p + u_j D_-(v_j)^p + \frac{p(p-1)}{2}u_j \xi_j^{p-2}\left(D_- v_j\right)^2 \Delta x = -\frac{p}{2}(v_j)^{p+1}, \tag{3.9}$$

with $\xi = \{\xi_j\}_{j\in\mathbb{N}_0}$ being a sequence of nonnegative numbers. Multiplying (3.9) with Δx and summing over j yields (using that u_j and v_j are nonnegative) the fundamental identity

$$\begin{aligned}\frac{d}{dt}\|v^p_{\Delta x}(\cdot\,,t)\|_{L^1(\mathbb{R}^+)} &+ \frac{p(p-1)}{2}(\Delta x)^2\sum_j u_j\xi_j^{p-2}\left(D_- v_j\right)^2 \\ &= -\Delta x\sum_j u_j D_-(v_j)^p + \frac{p}{2}\left\|v^{p+1}_{\Delta x}(\cdot\,,t)\right\|_{L^1(\mathbb{R}^+)} \\ &= \Delta x \sum_j (D_+u_j)(v_j)^p - \frac{p}{2}\left\|v^{p+1}_{\Delta x}(\cdot\,,t)\right\|_{L^1(\mathbb{R}^+)} \\ &= \left(1-\frac{p}{2}\right)\left\|v^{p+1}_{\Delta x}(\cdot\,,t)\right\|_{L^1(\mathbb{R}^+)}.\end{aligned} \tag{3.10}$$

Integrating (3.10) from 0 to t we end up with

$$\begin{aligned}\|v^p_{\Delta x}(\cdot\,,t)\|_{L^1(\mathbb{R}^+)} &+ (\Delta x)^2\frac{p(p-1)}{2}\int_0^t\sum_j u_j\xi_j^{p-2}\left(D_-v_j\right)^2\,ds \\ &= \left(1-\frac{p}{2}\right)\left\|v^{p+1}_{\Delta x}\right\|_{L^1(Q_T)} + \|v^p_{\Delta x}(\cdot\,,0)\|_{L^1(\mathbb{R}^+)} \le C,\end{aligned} \tag{3.11}$$

for some constant C independent of Δx. As the second term of the left-hand side is nonnegative and the first term on the right-hand side is nonpositive, (3.11) implies that the first and second claims of the lemma hold.

Next, we set $p = 1$ in (3.10) and (3.8) with $p = 2$ to obtain

$$\frac{d}{dt}\|v_{\Delta x}(\cdot\,,t)\|_{L^1(\mathbb{R}^+)} = \frac{1}{2}\|v_{\Delta x}(\cdot\,,t)\|^2_{L^2(\mathbb{R}^+)} \le \frac{1}{2}\|v_{\Delta x}(\cdot\,,0)\|^2_{L^2(\mathbb{R}^+)},$$

which proves the third claim. The fourth claim follows from the third one, since

$$\begin{aligned}|u_{\Delta x}(x,t)| &\le \|v_{\Delta x}(\cdot\,,t)\|_{L^1(\mathbb{R}^+)} \\ &\le \|v_{\Delta x}(\cdot\,,0)\|_{L^1(\mathbb{R}^+)} + \frac{t}{2}\|v_{\Delta x}(\cdot\,,0)\|^2_{L^2(\mathbb{R}^+)}.\end{aligned}$$

□

Using the estimates above we can prove some useful convergence results.

Lemma 3.4. *Suppose v_0 satisfies the conditions in* (3.1)*. Extracting subsequences if necessary, we have the following basic convergence results as $\Delta x \to 0$:*

$$\begin{aligned}&u_{\Delta x} \to u \textit{ uniformly in } [0,R]\times[0,T] \textit{ for each } R>0 \textit{ and pointwise in } \overline{Q_T},\\ &\textit{and the limit } u \textit{ belongs to } W^{1,q+1}(\overline{Q_T});\end{aligned} \tag{3.12}$$

$$\begin{aligned}&v_{\Delta x} = \partial_x u_{\Delta x} \rightharpoonup \partial_x u =: v \textit{ in } L^{q+1}(Q_T),\\ &\textit{and } v_{\Delta x} = \partial_x u_{\Delta x} \overset{\star}{\rightharpoonup} \partial_x u =: v \textit{ in } L^\infty((0,T);L^1(\mathbb{R}^+)\cap L^q(\mathbb{R}^+));\end{aligned} \tag{3.13}$$

$$\begin{aligned}&(v_{\Delta x})^2 \rightharpoonup w \textit{ in } L^{\frac{q+1}{2}}(Q_T),\\ &\textit{and } (v_{\Delta x})^2 \overset{\star}{\rightharpoonup} w \textit{ in } L^\infty((0,T);L^1(\mathbb{R}^+)\cap L^{\frac{q}{2}}(\mathbb{R}^+));\end{aligned} \tag{3.14}$$

$$\begin{aligned}&u_{\Delta x} v_{\Delta x} \rightharpoonup uv \textit{ in } L^{q+1}(Q_T),\\ &\textit{and } u_{\Delta x} v_{\Delta x} \overset{\star}{\rightharpoonup} uv \textit{ in } L^\infty((0,T);L^1(\mathbb{R}^+)\cap L^q(\mathbb{R}^+)).\end{aligned} \tag{3.15}$$

Proof. The second part of Lemma 3.2 shows that $\partial_x u_{\Delta x} = v_{\Delta x}$ is bounded in $L^{q+1}(Q_T)$ independently of Δx. Next, we bound $\partial_t u_{\Delta x}$. Recalling that $u_{-1} = v_{-1} = 0$, we find that

$$\begin{aligned}\frac{d}{dt}u_j &= \Delta x \sum_{i=0}^{j-1} \dot{v}_i\\ &= \Delta x \sum_{i=0}^{j-1}\left[-D_-\left(u_i v_i\right) + \frac{1}{2}(v_i)^2 - \Delta x D_- v_i^2\right]\\ &= -u_{j-1}v_{j-1} - \Delta x v_{j-1}^2 + \frac{\Delta x}{2}\sum_{i=0}^{j-1} v_i^2.\end{aligned}$$

Thus, using (3.8), we find

$$\left|\frac{d}{dt}u_j\right| \le \|u_{\Delta x}\|_{L^\infty(Q_T)}\left|v_{j-1}\right| + \|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}.$$

Fix any $R>0$ and let J be an integer such that $J\Delta x \le R$. Then it follows that

$$\Delta x \sum_{j=0}^{J}\left|\frac{d}{dt}u_j\right|^{q+1} \le C_1 + \|v_{\Delta x}\|_{L^{q+1}(Q_T)}^{q+1} \le C_2,$$

where C_1 and C_2 depend on R but are independent of Δx. Consequently, $u_{\Delta x}$ is uniformly bounded in $W^{1,q+1}([0,R]\times[0,T])$, a space which is compactly embedded into the Hölder space $C^{0,\ell}([0,R]\times[0,T])$, where $\ell = 1-2/(q+1)$. In other words, there exists a continuous function $u\colon \overline{Q_T}\to\mathbb{R}$ such that the following convergence holds, extracting a subsequence if necessary:

$$u_{\Delta x} \to u \text{ uniformly on } [0,R]\times[0,T] \text{ and pointwise in } \overline{Q_T} \text{ as } \Delta x \to 0.$$

Now the claim (3.12) follows from this and a standard diagonal argument on a sequence $R_\ell \to \infty$.

The claims (3.13) and (3.14) are consequences of the uniform L^{q+1} bound on $v_{\Delta x}$, while (3.15) holds thanks to (3.12) and (3.13). □

Remark 3.5. By the weak lower semicontinuity property of norms, the limits u, v inherit the a priori bounds in Lemma 3.2, that is, Lemma 3.2 holds with $u_{\Delta x}, v_{\Delta x}$ replaced by u, v, respectively.

We are going to prove strong convergence of $\{v_{\Delta x}\}_{\Delta x>0}$ by analyzing a particular renormalization (in the sense of DiPerna–Lions) of the numerical scheme and its limit. As mentioned before, strong convergence is needed if we want to prove that the weak limit v solves the Hunter–Saxton equation.

Lemma 3.6. *The limit triplet* (v, u, w) *from Lemma* 3.4 *satisfies*

$$v_t + (uv)_x = \frac{1}{2}w, \qquad u_x = v, \tag{3.16}$$

in the sense of distributions on Q_T, *and*

$$v \in C([0,T]; L^p(\mathbb{R}^+)), \quad \lim_{t\to 0} \|v(\,\cdot\,,t) - v_0\|_{L^p(\mathbb{R}^+)} = 0, \tag{3.17}$$

for any $p \in [1, q]$. *Moreover,*

$$w_t + (uw)_x \leq 0 \tag{3.18}$$

in the sense of distributions on Q_T *and*

$$\lim_{t\to 0} \int_0^\infty \left(w(x,t) - v_0(x)^2\right) dx = 0. \tag{3.19}$$

Proof. Set

$$\varphi_j(t) = \frac{1}{\Delta x}\int_{I_j} \varphi(x,t)\,dx,$$

where φ is a nonnegative test function, that is, $0 \leq \varphi \in C_c^\infty(Q_T)$. We multiply (3.4) with $\Delta x \varphi_j$, integrate from 0 to T, and sum over j, obtaining

$$\begin{aligned}
&-\int_0^T \Delta x \sum_{j\in\mathbb{N}_0} v_j \varphi_j'\,dt - \int_0^T \Delta x \sum_{j\in\mathbb{N}_0} u_j v_j D_+\varphi_j\,dt \\
&\quad = \int_0^T \Delta x \sum_{j\in\mathbb{N}_0} \frac{1}{2}(v_j)^2 \varphi_j\,dt + \underbrace{\int_0^T \Delta x \sum_{j\in\mathbb{N}_0} (v_j)^2 \Delta x D_+\varphi_j\,dt}_{E_1},
\end{aligned} \tag{3.20}$$

after a partial integration in t and a partial summation in j. Due to the choice of φ_j, we can rewrite this as

$$\begin{aligned}
&-\iint_{Q_T} \left[v_{\Delta x}\varphi_t + u_{\Delta x} v_{\Delta x}\varphi_x + \frac{1}{2}v_{\Delta x}^2\varphi\right] dtdx \\
&\quad = E_1 + \underbrace{\int_0^T \sum_{j\in\mathbb{N}_0} \left[\Delta x u_j v_j D_+\varphi_j - \int_{I_j} v_{\Delta x} u_{\Delta x}\varphi_x\,dx\right] dt}_{E_2}.
\end{aligned}$$

To establish (3.16), we must show that $\lim_{\Delta x\to 0}(E_1 + E_2) = 0$. Observe

$$\begin{aligned}
E_1 &\leq \Delta x \,\|\varphi_x\|_{L^\infty(Q_T)} \int_0^T \int_0^\infty v_{\Delta x}^2(x,t)\,dx\,dt \\
&\leq \Delta x \,\|\varphi_x\|_{L^\infty(Q_T)} \int_0^T \int_0^\infty v_{\Delta x}^2(x,0)\,dx\,dt \\
&= \Delta x \,\|\varphi_x\|_{L^\infty(Q_T)}\, T \,\|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}\,,
\end{aligned}$$

and thus E_1 vanishes with Δx. Regarding E_2 we have that the integrand equals

$$\sum_{j\in\mathbb{N}_0} v_j \int_{I_j} \big[\underbrace{u_j D_+\varphi_j - u_{\Delta x}\varphi_x}_{A}\big]\,dx.$$

We split the integrand A above by writing

$$A = (u_j - u_{\Delta x})\,D_+\varphi_j + u_{\Delta x}\,(D_+\varphi_j - \varphi_x)\,.$$

For $x \in I_j$ we have

$$u_j(t) - u_{\Delta x}(x,t) = (x_{j-1/2} - x)v_j$$

and

$$\begin{aligned} D_+\varphi_j(t) - \varphi_x(x,t) &= \frac{1}{\Delta x}\int_{I_j}\left[\frac{\varphi(y+\Delta x,t)-\varphi(y,t)}{\Delta x} - \varphi_x(x,t)\right]dy \\ &= \frac{1}{(\Delta x)^2}\int_{I_j}\int_y^{y+\Delta x}\big[\varphi_x(z,t) - \varphi_x(x,t)\big]\,dzdy \\ &= \frac{1}{(\Delta x)^2}\int_{I_j}\int_y^{y+\Delta x}\int_x^z \varphi_{xx}(w,t)\,dw\,dz\,dy. \end{aligned}$$

Therefore

$$|D_+\varphi_j(t) - \varphi_x(x,t)| \le \|\varphi_{xx}\|_{L^\infty(Q_T)}\,\Delta x.$$

Collecting this we find that

$$\begin{aligned} |E_2| &\le \int_0^T \sum_{j\in\mathbb{N}_0} v_j \int_{I_j}\Big[\big(x_{j-1/2}-x\big)\,v_j\,\|\varphi_x\|_{L^\infty(Q_T)} + \Delta x\,\|\varphi_{xx}\|_{L^\infty(Q_T)}\Big]\,dx\,dt \\ &\le \left(\frac{1}{2}\|\varphi_x\|_{L^\infty(Q_T)} + \|\varphi_{xx}\|_{L^\infty(Q_T)}\right)\Delta x\int_0^T\sum_{j\in\mathbb{N}_0} v_j\Delta x\,dt \\ &\le \Delta x\left(\frac{1}{2}\|\varphi_x\|_{L^\infty(Q_T)} + \|\varphi_{xx}\|_{L^\infty(Q_T)}\right)T\,\|u_{\Delta x}\|_{L^\infty(Q_T)}\,. \end{aligned}$$

From this we also see that E_2 vanishes when Δx becomes small, and the first part of (3.16) holds. The second part of (3.16) follows from (3.13).

To prove the time continuity/initial data statements (3.17) we can apply standard renormalization arguments; see for example [16, 17].

To prove that (3.18) holds, we recall that we have a scheme for $(v_j)^2$ using (3.9) with $p = 2$:

$$\frac{d}{dt}(v_j)^2 + u_jD_-(v_j)^2 + u_j\,(D_-v_j)^2\,\Delta x = -(v_j)^3.$$

Using the Leibniz identity (2.1), we can rewrite this as

$$\frac{d}{dt}(v_j)^2 + D_-\left(u_j(v_j)^2\right) + u_j\,(D_-v_j)^2\,\Delta x = -\Delta x D_-(v_j)^3. \tag{3.21}$$

The third term above is certainly nonnegative, so after multiplying with $\Delta x\,\varphi_j$, summing over j and integrating over t, we find that

$$
\begin{aligned}
-\iint_{Q_T} &\left[v_{\Delta x}^2\varphi_t + u_{\Delta x}v_{\Delta x}^2\varphi_x\right]\,dx\,dt \\
&\le \Delta x\,\|\varphi_x\|_{L^\infty(Q_T)}\iint_{Q_T} v_{\Delta x}^3\,dx \\
&\quad + \underbrace{\int_0^T\left[\sum_{j\in\mathbb{N}_0}\Delta x u_j(v_j)^2 D_+\varphi_j - \int_{I_j} u_{\Delta x}v_{\Delta x}^2\varphi_x\,dx\right]dt}_{E_3}.
\end{aligned}
$$

Since $v_{\Delta x}\in L^3(Q_T)$ with an L^3 norm that is independent of Δx, the first term on the right-hand side vanishes with Δx. The second term E_3 is similar to E_2, the only difference being that we have $v_{\Delta x}^2$ instead of $v_{\Delta x}$. Hence we can bound E_3 as

$$
\begin{aligned}
|E_3| &\le \left(\frac{1}{2}\|\varphi_x\|_{L^\infty(Q_T)} + \|\varphi_{xx}\|_{L^\infty(Q_T)}\right)\Delta x\int_0^T \|v_{\Delta x}(\,\cdot\,,t)\|_{L^2(\mathbb{R}^+)}^2\,dt \\
&\le \Delta x\left(\frac{1}{2}\|\varphi_x\|_{L^\infty(Q_T)} + \|\varphi_{xx}\|_{L^\infty(Q_T)}\right)T\,\|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}^2.
\end{aligned}
$$

Consequently, $\lim_{\Delta x\to 0}E_3 = 0$ and (3.18) holds.

Finally, let us prove (3.19), which also follows from standard arguments. Thanks to (3.13), we have that

$$
v^2(x,t)\le w(x,t)\ \text{for a.e. in}\ (x,t)\in Q_T, \tag{3.22}
$$

so that by the energy estimate (the first part of Lemma 3.2 with $p=2$, cf. also Remark 3.5) we obtain

$$
\begin{aligned}
\lim_{t\to 0}\int_0^\infty v(x,t)^2\,dx &\le \liminf_{t\to 0}\int_0^\infty w(x,t)\,dx \\
&\le \limsup_{t\to 0}\int_0^\infty w(x,t)\,dx \le \int_0^\infty v_0(x)^2\,dx.
\end{aligned}
$$

On the other hand, (3.17) yields

$$
\lim_{t\to 0}\int_0^\infty v(x,t)^2\,dx = \int_0^\infty v_0(x)^2\,dx,
$$

which finishes the proof of (3.19). $\square$

We state and prove the next lemma in a form that is slightly more general than what we actually need in this section to conclude that the sequence $\{v_{\Delta x}\}_{\Delta x>0}$ is strongly convergent.

Lemma 3.7. *Suppose u is bounded and continuous in $\overline{Q_T}$ with $u(0,t)=0$ for $t\in[0,T]$, $v\in L^\infty((0,T);L^2(\mathbb{R}^+))\cap L^3(Q_T)$, $v\ge 0$ a.e. in Q_T, $w\in L^\infty((0,T);L^1(\mathbb{R}^+))\cap L^{\frac{3}{2}}(Q_T)$, and $w\ge v^2$ a.e. in Q_T. Assume that*

$$
\lim_{t\to 0}\int_0^\infty \left(w-v^2\right)(\,\cdot\,,t)\,dx = 0 \tag{3.23}
$$

and that the triplet (v, u, w) *satisfies the system*

$$v_t + (uv)_x = \frac{1}{2}w, \tag{3.24}$$

$$w_t + (uw)_x \leq 0, \tag{3.25}$$

$$u_x = v \tag{3.26}$$

in the sense of distributions on Q_T. *Then*

$$w = v^2 \text{ a.e. in } Q_T.$$

Proof. The proof is a standard exercise in the theory of renormalized solutions, so we include it only for the sake of completeness. Set $v^\varepsilon = v \star \omega_\varepsilon$, $w^\varepsilon = w \star \omega_\varepsilon$, where ω_ε is a standard mollifier acting on the spatial variable. Then according to the DiPerna–Lions folklore lemma [3], as well as (3.24) and (3.26), v^ε solves

$$v_t^\varepsilon + uv_x^\varepsilon = \frac{1}{2}w^\varepsilon - (v^\varepsilon)^2 + r^\varepsilon, \tag{3.27}$$

where $r^\varepsilon = uv_x^\varepsilon - (uv_x) \star \omega_\varepsilon + (v^\varepsilon)^2 - v^2 \star \omega_\varepsilon$ and

$$r^\varepsilon \to 0 \text{ in } L^p(Q_T) \text{ for any } p \in [1, 3/2].$$

Multiplying this equation by v^ε we get

$$\left(\frac{(v^\varepsilon)^2}{2}\right)_t + u\left(\frac{(v^\varepsilon)^2}{2}\right)_x = \frac{1}{2}w^\varepsilon v^\varepsilon - (v^\varepsilon)^3 + r^\varepsilon v^\varepsilon,$$

or, thanks to (3.26),

$$\left(\frac{(v^\varepsilon)^2}{2}\right)_t + \left(u\frac{(v^\varepsilon)^2}{2}\right)_x = \frac{1}{2}w^\varepsilon v^\varepsilon - (v^\varepsilon)^3 + v\frac{(v^\varepsilon)^2}{2} + r^\varepsilon v^\varepsilon.$$

Sending $\varepsilon \downarrow 0$, we obtain

$$\left(v^2\right)_t + \left(uv^2\right)_x = w\,v - v^3 = \left(w - v^2\right) v \geq 0,$$

where we have used (3.22) to derive the last inequality. Comparing this inequality with (3.25), keeping in mind that $w \geq v^2$ a.e. in Q_T, we find

$$\left(w - v^2\right)_t + \left(u\left(w - v^2\right)\right)_x \leq 0$$

in the sense of distributions on Q_T. In particular, this implies that

$$\int_0^T \int_0^\infty \left(w - v^2\right)(x,t)\partial_t\psi\,dx\,dt \geq 0$$

for any nonnegative $\psi \in C_c^\infty((0,T))$. Hence, for any two Lebesgue points $t_1, t_2 \in (0,T)$, $t_1 < t_2$, of the L^1 function

$$(0,T) \ni t \mapsto \int_0^\infty \left(w - v^2\right)(x,t)\,dx,$$

we obtain

$$\int_0^\infty \left(w - v^2\right)(x,t_2)\,dx \leq \int_0^\infty \left(w - v^2\right)(x,t_1)\,dx,$$

and combining this with (3.23) we have proved the lemma. □

We summarize our findings in the following main theorem.

Theorem 3.8. *Let v_0 be a function satisfying* (3.1). *Define the semi-discrete finite difference approximation $(v_{\Delta x}, u_{\Delta x})$ for Δx positive using* (3.5), (3.2), *and* (3.3). *Then $\{(v_{\Delta x}, u_{\Delta x})\}_{\Delta x>0}$ converges to a dissipative solution (v, u) of* (1.5) *in the sense of Definition* 1.2. *More precisely, as $\Delta x \to 0$*

$$\|u_{\Delta x} - u\|_{L^\infty(Q_T)} \to 0, \qquad \|v_{\Delta x} - v\|_{L^p(Q_T)} \to 0 \quad \text{for any } p \in [1, q+1). \tag{3.28}$$

Proof. In view of Lemmas 3.6, 3.7, and 2.1 we conclude that $w = v^2$ a.e. in Q_T and that there exists a subsequence of $\{v_{\Delta x}\}_{\Delta x>0}$ that converges to v a.e. in Q_T, where v is the (weak) limit from Lemma 3.4. Moreover, Lemma 3.4 implies that

$$v \in L^\infty((0,T); L^p(\mathbb{R}^+)) \cap L^{q+1}(Q_T) \cap C([0,T]; L^p(\mathbb{R}^+)), \quad p \in [1, q],$$

which clearly proves the second part of (3.28). The first part follows from (3.12).

The fact that the limit (v, u) solves the Hunter–Saxton equation (1.5) in the sense of distributions (i.e., the second requirement in Definition 1.2) follows from Lemma 3.6 and the identification $w = v^2$ a.e. in Q_T.

The remaining requirements in Definition 1.2 are straightforward consequences of the subsequent strong convergence of $\{v_{\Delta x}\}_{\Delta x>0}$ and Lemmas 3.1, 3.2, 3.4, 3.6.

For any given sequence we have proved that we can find a subsequence $\Delta x_j \to 0$ for which all statements hold. However, Zhang and Zheng [18] have proved that the Hunter–Saxton equation has a *unique* global dissipative solution. Hence the limit exists for all subsequences, which concludes the proof of the theorem. □

Remark 3.9. In addition to the properties stated in Theorem 3.8, the proof also shows that the limits u, v possess the following properties:

$$\begin{aligned} &u \in W^{1.q+1}(\overline{Q_T}), \\ &v \in L^\infty((0,T); L^p(\mathbb{R}^+)) \cap L^{q+1}(Q_T), \quad p \in [1, q], \\ &v \in C([0,T]; L^p(\mathbb{R}^+)), \quad p \in [1, q]. \end{aligned}$$

Moreover, since $v_0 \in L^q(\mathbb{R}^+)$ with $q > 2$, which implies $v \in L^{q+1}(Q_T)$ and in particular $v \in L^3([0,R] \times [0,T])$ for any $R > 0$, the dissipative solution constructed in Theorem 3.8 is energy conservative, that is, for any $t > 0$

$$\int_0^\infty v(x,t)^2\, dx = \int_0^\infty v_0(x)^2\, dx.$$

Formally this is obtained by multiplying the equation for v by v, which gives

$$\partial_t \left(\frac{v^2}{2}\right) + \partial_x \left(u \frac{v^2}{2}\right) = 0,$$

from which the claim follows. To make this argument rigorous one appeals to the DiPerna–Lions folklore lemma [3] and the local L^3 estimate on v.

4. The implicit upwind scheme

In this section we show how to extend the convergence analysis from the previous section to an implicit upwind difference scheme, where we still work under the initial data assumption (3.1). Since many of the arguments are very similar, we have attempted to make this section brief.

Referring to Section 2 for the notation, the implicit finite difference solution

$$\left\{ \left(v_j^n, u_j^n\right) \mid j \in \mathbb{N}_0,\ n = 0, 1, 2, \dots, N \right\}$$

is defined by

$$D_+^t v_j^n + u_j^{n+1} D_- v_j^{n+1} = -\frac{1}{2}\left(v_j^{n+1}\right)^2, \qquad D_+ u_j^{n+1} = v_j^{n+1}, \tag{4.1}$$

for $0 \le j \le J_{\Delta x}$ and $n = 0, \ldots, N-1$, where we have set $v_j^n = 0$ for $j > J_{\Delta x}$ and set $v_{-1} = 0$. The final step N is chosen such that $N\Delta t = T$. The initial values $\left\{v_j^0\right\}_{j\in\mathbb{N}_0}$ are defined as in Section 3 and boundary values as specified as $u_0^n = 0$ for $n = 0, 1, \ldots, N$. Based on $\left\{(v_j^n, u_j^n)\right\}$ we define the functions $v_{\Delta x}$ and $u_{\Delta x}$ as in Section 3 by

$$v_{\Delta x}(x,t) = \sum_{\substack{j\in\mathbb{N}_0\\ n=0,\ldots,N}} v_j^n \mathbf{1}_{I_j^n} \quad \text{and} \quad u_{\Delta x}(x,t) = \int_0^x v_{\Delta x}(y,t)\,dy. \tag{4.2}$$

As for the semi-implicit scheme, we can derive a conservative form of (4.1):

$$D_+^t v_j^n + D_-\left(u_j^{n+1} v_j^{n+1}\right) = \frac{1}{2}(v_j^{n+1})^2 - \Delta x D_-(v_j^{n+1})^2. \tag{4.3}$$

We can solve (4.1) "upwards from left to right", by rewriting it as

$$u_0^{n+1} = 0, \tag{4.4}$$

$$\begin{aligned} u_j^{n+1} &= u_{j-1}^{n+1} + \Delta x v_{j-1}^{n+1}, \qquad 0 < j \le J_{\Delta x},\ 0 \le n, \\ v_j^{n+1} &= \frac{1}{\Delta t}\left[\sqrt{\left(1+\lambda u_j^{n+1}\right)^2 + 2\Delta t\left(v_j^n + \lambda u_j^{n+1} v_{j-1}^{n+1}\right)} - \left(1+\lambda u_j^{n+1}\right)\right], \end{aligned} \tag{4.5}$$

where (the constant) $\lambda = \Delta t/\Delta x$. We have chosen the plus sign in front of the square root, since otherwise v_j^{n+1} would be negative.

Lemma 4.1. *Assume that the initial approximations are chosen so that*

$$\lim_{\Delta t\to 0} \Delta t \max_j \left\{v_j^0\right\} = 0.$$

Then for $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$ we have

$$0 \le v_j^n \le \frac{2}{t_n K_{\Delta t}}, \tag{4.6}$$

where $t_n = n\Delta t$ and $\{K_{\Delta t}\}$ is a bounded sequence such that $\lim_{\Delta t\to 0} K_{\Delta t} = 1$.

Proof. From (4.4) and (4.5) it is straightforward to see that if $v_j^0 \ge 0$, then also $v_j^n \ge 0$ and $u_j^n \ge 0$. In order to show the upper bound, note first that if $v_j^{n+1} \ge v_{j-1}^{n+1}$, then $u_j^{n+1} D_- v_j^{n+1} \ge 0$, and hence, using (4.1),

$$v_j^{n+1} \le v_j^n - \frac{\Delta t}{2}\left(v_j^{n+1}\right)^2 \quad \text{or} \quad v_j^{n+1} \le \frac{1}{\Delta t}\left[\sqrt{1+2\Delta t v_j^n} - 1\right].$$

Set $\bar{v}_n = \max_j \left\{v_j^n\right\}$. Since $v_j^{n+1} \ge v_{j-1}^{n+1}$ if $v_j^{n+1} = \bar{v}_{n+1}$, we deduce that

$$\bar{v}_{n+1} = v_j^{n+1} \le \frac{1}{\Delta t}\left[\sqrt{1+2\Delta t\bar{v}_n} - 1\right] = \frac{2\bar{v}_n}{\sqrt{1+2\Delta t\bar{v}_n}+1}.$$

Thus in particular we see that $\bar v_n \le \bar v_0$, and we can use this to deduce that

$$\begin{aligned}
\frac{1}{\Delta t}(\bar v_{n+1}-\bar v_n) &\le \frac{1}{\Delta t}\left(\frac{2\bar v_n}{\sqrt{1+2\Delta t\bar v_n}+1}-\bar v_n\right)\\
&= \frac{\bar v_n}{\Delta t}\left(\frac{1-\sqrt{1+2\Delta t\bar v_n}}{1+\sqrt{1+2\Delta t\bar v_n}}\right)\\
&= -\frac{2\bar v_n^2}{(1+\sqrt{1+2\Delta t\bar v_n})^2}\\
&\le -\frac{1}{2}\bar v_n^2\left(\frac{2}{1+\sqrt{1+2\Delta t\bar v_0}}\right)^2\\
&= -\frac{1}{2}\bar v_n^2 K_{\Delta t}. \qquad (4.7)
\end{aligned}$$

Applying (2.4) with $f(v)=1/v$ we find

$$D_+^t\frac{1}{\bar v_n}-\frac{\Delta t}{\xi_n^3}\left(D_+^t\bar v_n\right)^2 \ge \frac{1}{2}K_{\Delta t}.$$

Multiplying by Δt and summing over the time variable yields

$$\frac{1}{\bar v_{n+1}} \ge \frac{1}{\bar v_0}+\frac{t_nK_{\Delta t}}{2}+P,$$

where

$$P=\Delta t^2\sum_{j=0}^{n}\frac{1}{\xi_j^3}\left(D_+^t\bar v_j\right)^2\ge 0.$$

Rearranging we finally get

$$\bar v_{n+1}\le \frac{2\bar v_0}{\bar v_0 t_nK_{\Delta t}+2+2\bar v_0P}\le \frac{2}{K_{\Delta t}t_n}.$$

Using that

$$\Delta t\bar v_0\to 0 \quad \text{as } \Delta t\to 0,$$

i.e., $K_{\Delta t}\to 1$, we conclude the proof. □

Similar to (3.7), if we multiply the scheme (4.1) by $f'(v_j^{n+1})$ we get

$$\begin{aligned}
&D_+^t f\left(v_j^n\right)+u_j^{n+1}D_-f\left(v_j^{n+1}\right)\\
&\quad+\frac{1}{2}\left[\Delta t f''(\eta_j^n)\left(D_+^tv_j^n\right)^2+\Delta x f''(\xi_j^{n+1})\left(D_-v_j^{n+1}\right)^2\right]=-\frac{1}{2}f'\left(v_j^{n+1}\right)\left(v_j^{n+1}\right)^2,
\end{aligned} \qquad (4.8)$$

where η_j^n is between v_j^{n+1} and v_j^n, and ξ_j^n between v_j^n and v_{j-1}^n. With this we can show the following result.

Lemma 4.2. *Lemma* 3.2 *holds also for* $v_{\Delta x}$ *and* $u_{\Delta x}$ *defined by* (4.2) *and* (4.1).

Proof. Choosing $f(v)=v^p$ in (4.8) yields

$$\begin{aligned}
&\left(D_+^t\left(v_j^n\right)^p+u_j^{n+1}D_-\left(v_j^{n+1}\right)^p\right)\\
&\quad+\frac{p(p-1)}{2}\left(u_j^{n+1}\left(\xi_j^{n+1}\right)^{p-2}\left(D_-v_j^{n+1}\right)^2\Delta x+\left(\eta_j^{n+1}\right)^{p-2}\left(D_+^tv_j^n\right)^2\Delta t\right)\\
&\quad=-\frac{p}{2}\left(v_j^{n+1}\right)^{p+1}.
\end{aligned} \qquad (4.9)$$

Therefore, we can proceed as in the semi-discrete case (cf. (3.9)–(3.10)) to find

$$
\begin{aligned}
(4.10)\quad & D_+^t \left\| v_{\Delta x}^p(\,\cdot\,,t_n)\right\|_{L^1(\mathbb{R}^+)} \\
&\quad + \frac{p(p-1)}{2}\Delta x \sum_{j\in\mathbb{N}_0} \left(u_j^{n+1} \left(\xi_j^{n+1}\right)^{p-2} \left(D_- v_j^{n+1}\right)^2 \Delta x + \left(\eta_j^{n+1}\right)^{p-2} \left(D_+^t v_j^n\right)^2 \Delta t \right) \\
&\qquad = \left(1-\frac{p}{2}\right) \left\| v_{\Delta x}^{p+1}(\,\cdot\,,t_{n+1})\right\|_{L^1(\mathbb{R}^+)} .
\end{aligned}
$$

Multiplying by Δt and summing over the time variable yields

$$
(4.11)\quad \left\| v_{\Delta x}^p(\,\cdot\,,t_{n+1})\right\|_{L^1(\mathbb{R}^+)} + P = \left(1-\frac{p}{2}\right) \left\| v_{\Delta x}^{p+1}\right\|_{L^1(Q_T)} + \left\| v_{\Delta x}^p(\,\cdot\,,0)\right\|_{L^1(\mathbb{R}^+)} ,
$$

where

$$
\begin{aligned}
P = \Delta t \Delta x \frac{p(p-1)}{2} \sum_{\substack{j\in\mathbb{N}_0\\ n=0,\dots,N}} \Big\{ & u_j^{n+1} \left(\xi_j^{n+1}\right)^{p-2} \left(D_- v_j^{n+1}\right)^2 \Delta x \\
& + \left(\eta_j^{n+1}\right)^{p-2} \left(D_-^t v_j^{n+1}\right)^2 \Delta t \Big\}.
\end{aligned}
$$

Recall that $p \in [2,q]$, hence the first term on the right-hand side of (4.11) is nonpositive; similarly P is nonnegative, hence (3.8) holds. The proof of the rest of the lemma is identical to the proof of Lemma 3.2. □

We continue as in the previous section to prove the following result.

Lemma 4.3. *The conclusions of Lemma* 3.4 *hold for the sequences* $\{v_{\Delta x}\}$ *and* $\{u_{\Delta x}\}$ *defined by* (4.2) *and* (4.1).

Proof. The proof is almost identical to the proof of Lemma 3.4. We estimate $D_+^t u_j^n$:

$$
\begin{aligned}
D_+^t u_j^n &= \Delta x \sum_{i=0}^{j-1} D_+^t v_i^n \\
&= \Delta x \sum_{i=0}^{j-1} \Big[-D_- \left(u_i^{n+1} v_i^{n+1}\right) + \frac{1}{2}\left(v_i^{n+1}\right)^2 - \Delta x D_- \left(v_i^{n+1}\right)^2 \Big] \\
&= -u_{j-1}^{n+1} v_{j-1}^{n+1} - \Delta x \left(v_{j-1}^{n+1}\right)^2 + \frac{\Delta x}{2} \sum_{i=0}^{j-1} \left(v_i^{n+1}\right)^2 .
\end{aligned}
$$

Next define $\tilde{u}_{\Delta x}(x,t)$ as

$$
\tilde{u}_{\Delta x}(x,t) = \frac{1}{\Delta t}\left((t_{n+1}-t) u_{\Delta x}(x,t_n) + (t-t_n) u_{\Delta x}(x,t_{n+1})\right)
$$

for $t \in [t_n, t_{n+1})$. Then $\partial_t \tilde{u}_{\Delta x} = D_+^t u_j^n$ for $(x,t) \in I_j^n$. Furthermore, $\partial_x \tilde{u}_{\Delta x}$ is a convex combination of v_j^n and v_j^{n+1}. Therefore $\tilde{u}_{\Delta x}$ is uniformly bounded in $W^{1,q+1}([0,R]\times[0,T])$, and this space is compactly embedded in $C^{0,\ell}([0,R]\times[0,T])$ with $\ell = 1 - 2/(q+1)$. Thus there is a continuous function $u : \overline{Q_T} \to \mathbb{R}$ such that (if necessary for a subsequence)

$$
\tilde{u}_{\Delta x} \to u \quad \text{uniformly on } [0,R]\times[0,T] \text{ and pointwise in } \overline{Q_T} \text{ as } \Delta x \to 0.
$$

Furthermore, by the definition of $\tilde{u}_{\Delta x}$, we also have

$$|\tilde{u}_{\Delta x}(x,t) - u_{\Delta x}(x,t)| \leq \Delta t \, |\partial_t u_{\Delta x}(x,t)| \leq \Delta t C,$$

for some constant not depending on Δx. Hence $u_{\Delta x}$ also converges uniformly to u. This concludes the proof of Lemma 4.3. □

Lemma 4.4. *Lemma* 3.6 *holds for the triplet* (v,u,w) *from Lemma* 4.3*.*

Proof. First we claim that (3.16) holds, i.e.,

$$-\iint_{Q_T} \left[v\varphi_t + uv\varphi_x + \frac{1}{2} w\varphi \right] dx\, dt = 0. \tag{4.12}$$

To show this, we choose a test function $\varphi \in C_c^\infty(Q_T)$ and set

$$\varphi_j^n = \frac{1}{\Delta x \Delta t} \iint_{I_j^n} \varphi(x,t)\, dxdt.$$

Next we multiply the scheme with $\Delta x \Delta t \varphi_j^{n+1}$, sum over $n = 0, \dots, N-1$, where $N\Delta t = T$ and $j \in \mathbb{N}_0$, to find that

$$-\Delta x \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{N}_0} v_j^n D_+^t \varphi_j^n \tag{4.13}$$

$$- \Delta x \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{N}_0} u_j^n v_j^n D_+ \varphi_j^n \tag{4.14}$$

$$- \Delta x \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{N}_0} \frac{1}{2} \left(v_j^n\right)^2 \varphi_j^n \tag{4.15}$$

$$\begin{aligned} & - \Delta t \Delta x \sum_{j\in\mathbb{N}_0} u_j^N v_j^N D_+ \varphi_j^N + \Delta x \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{N}_0} \left(v_j^{n+1}\right)^2 D_+ \varphi_j^n \Delta x \\ &= 0. \end{aligned} \tag{4.16}$$

The first term in (4.16) can be bounded as

$$(4.16)_1 \leq \Delta t \, \|u_{\Delta x}\|_{L^\infty(Q_T)} \, \|\varphi_x\|_{L^\infty(Q_T)} \, \|v_{\Delta x}(\,\cdot\,, T-\Delta t)\|_{L^1(\mathbb{R}^+)} \to 0$$

as $\Delta t \to 0$. Similarly, the second term in (4.16) can be bounded as

$$(4.16)_2 \leq \Delta x T \, \|v_{\Delta x}(\,\cdot\,, 0)\|_{L^2(\mathbb{R}^+)} \, \|\varphi_x\|_{L^\infty(Q_T)},$$

which also vanishes when Δx becomes small. Hence the whole line (4.16) will vanish in the limit. We compare the remaining expressions (4.13)–(4.16) with their expected limits. To this end first note that for $(x,t) \in I_j^n$ we have that

$$\left|\varphi_x(x,t) - D_+\varphi_j^n\right| \leq C(\Delta x + \Delta t) \quad \text{and} \quad \left|\varphi_t(x,t) - D_+^t\varphi_j^n\right| \leq C(\Delta x + \Delta t),$$

for some constant C depending on φ but not on Δx or Δt. Now

$$\begin{aligned}\Big|\iint_{Q_T} v_{\Delta x}\varphi_t\,dxdt - \Delta x\Delta t\sum_{n,j} v_j^n D_+^t\varphi_j^n\Big| &\le C(\Delta x+\Delta t)\int_0^T \left\|v_{\Delta x}(\,\cdot\,,t)\right\|_{L^1(\mathbb{R}^+)}\,dt\\ &\le C(\Delta x+\Delta t)\left(T\left\|v_{\Delta x}(\,\cdot\,,0)\right\|_{L^1(\mathbb{R}^+)}+\frac{T^2}{4}\left\|v_{\Delta x}(\,\cdot\,,0)\right\|_{L^2(\mathbb{R}^+)}^2\right).\end{aligned}$$

We also find that

$$\begin{aligned}&\sum_{n,j}\Big|\Delta x\Delta t\, u_j^n v_j^n D_+\varphi_j^n - \iint_{I_j^n} u_{\Delta x}v_{\Delta x}\varphi_x\,dxdt\Big|\\ &\quad\le \sum_{n,j}\iint_{I_j^n}\left|u_j^n-u_{\Delta x}(x,t)\right| v_{\Delta x}\left|D_+\varphi_j^n\right|\,dxdt\\ &\qquad+\sum_{n,j}\iint_{I_j^n} u_{\Delta x}v_{\Delta x}\left|D_+\varphi_j^n-\varphi_x\right|\,dxdt\\ &\quad\le \sum_{n,j}\iint_{I_j^n} v_{\Delta x}^2\left|x-x_{j-1/2}\right|\left|D_+\varphi_j^n\right|\,dxdt\\ &\qquad+C(\Delta x+\Delta t)\left\|u_{\Delta x}\right\|_{L^\infty(Q_T)}\int_0^T\left\|v_{\Delta x}(\,\cdot\,,t)\right\|_{L^1(\mathbb{R}^+)}\,dt\\ &\quad\le \Delta xT\left\|v_{\Delta x}(\,\cdot\,,0)\right\|_{L^2(\mathbb{R}^+)}^2\left\|\varphi_x\right\|_{L^\infty(Q_T)}\\ &\qquad+C(\Delta x+\Delta t)\left\|u_{\Delta x}\right\|_{L^\infty(Q_T)}\big(T\left\|v_{\Delta x}(\,\cdot\,,0)\right\|_{L^1(\mathbb{R}^+)}\\ &\qquad\qquad+\frac{T^2}{4}\left\|v_{\Delta x}(\,\cdot\,,0)\right\|_{L^2(\mathbb{R}^+)}^2\big).\end{aligned}$$

Collecting all these results, and noting that

$$(4.15) = -\iint_{Q_T}\frac{1}{2}v_{\Delta x}^2\varphi\,dxdt,$$

we end up with

$$\begin{aligned}&-\iint_{Q_T}\left[v_{\Delta x}\varphi_t+u_{\Delta x}v_{\Delta x}\varphi_x+\frac{1}{2}v_{\Delta x}^2\varphi\right]dx\,dt\\ &\qquad+\int_0^\infty v_{\Delta x}\varphi\,\Big|_0^T\,dx=\mathcal{O}\left(\Delta x+\Delta t\right).\end{aligned}\tag{4.17}$$

Hence (4.12) is proved.

Next, we claim that (3.18) also holds, i.e.,

$$w_t+(uw)_x\le 0,\tag{4.18}$$

weakly in Q_T. To demonstrate this we consider (4.9) with $p=2$, giving

$$\begin{aligned}D_-^t\left(v_j^{n+1}\right)^2+D_-\left(u_j^{n+1}\left(v_j^{n+1}\right)^2\right)+\left[u_j^{n+1}\left(D_-v_j^{n+1}\right)^2\Delta x+\left(D_-^t v_j^{n+1}\right)^2\Delta t\right]&\\ =-\Delta xD_-(v_j^{n+1})^3.&\end{aligned}$$

The terms in the square brackets above are non-negative, hence

(4.19)
$$
\begin{aligned}
-\Delta t\Delta x\sum_{n=1}^{N-1}\sum_{j\in\mathbb{N}_0}\left[\left(v_j^n\right)^2 D_+^t\varphi_j^n+u_j^n\left(v_j^n\right)^2 D_+\varphi_j^n\right]+\Delta t\Delta x\sum_{j\in\mathbb{N}_0}u_j^N\left(v_j^N\right)^2 D_+\varphi_j^n \\
\le \Delta t\Delta t\sum_{n=1}^{N-1}\sum_{j\in\mathbb{N}_0}D_+\varphi_j^n\left(v_j^{n+1}\right)^3\Delta x.
\end{aligned}
$$

In the same manner as proving (4.17) this can be used to verify (4.18); the details are left to the reader. The only additional ingredient is that we use that $v_{\Delta x}$ is uniformly bounded in $L^3(Q_T)$ in order to prove that the right-hand side of the above inequality vanishes with Δx. □

Theorem 4.5. *Theorem* 3.8 *remains valid if* $v_{\Delta x}$ *and* $u_{\Delta x}$ *are defined by the implicit difference scheme* (4.2) *and* (4.1).

Proof. The proof is identical to the proof of Theorem 3.8. In order to conclude that $w=v^2$ a.e. in Q_T, we appeal to Lemma 3.7 and note that the limit triplet (v,u,w) satisfies all the assumptions of that lemma. □

5. The explicit upwind scheme

In this section we analyze an explicit version of the scheme from the previous section. This presents some additional technical difficulties, but the analysis has many similarities with what we have already done.

We assume that the initial data (3.1) is nonnegative, bounded with compact support, specifically,

$$
0\le v_0\le M,\quad\text{and}\quad \operatorname{supp}(v_0)\subset[0,X], \tag{5.1}
$$

for positive constants M and X. The explicit scheme we shall study is similar to the implicit scheme. It is defined by

$$
\left.\begin{aligned}
D_+^t v_j^n+u_j^n D_- v_j^n&=-\frac{1}{2}\left(v_j^n\right)^2\\
D_+u_j^n=v_j^n,\quad u_0^n&=0
\end{aligned}\right\}\quad n\in\mathbb{N}_0,\quad 0\le j\le J_{\Delta x}, \tag{5.2}
$$

with the initial data $\left\{v_j^0\right\}$ given by (3.2), and $J_{\Delta x}=X/\Delta x$. For convenience we define $u_{-1}^n=v_{-1}^n=0$. We define the functions $v_{\Delta x}$ and $u_{\Delta x}$ as

$$
v_{\Delta x}(x,t)=\sum_{\substack{j\in\mathbb{N}_0\\ n=0,\dots,N}}v_j^n\mathbf{1}_{I_j^n}, \tag{5.3}
$$

$$
u_{\Delta x}(x,t)=\int_0^x v_{\Delta x}(y,t)\,dy. \tag{5.4}
$$

On conservative form the scheme reads

$$
D_+^t v_j^n+D_-\left(u_j^n v_j^n\right)=\frac{1}{2}(v_j^n)^2-\Delta x\,D_-(v_j^n)^2. \tag{5.5}
$$

Fix $T>0$ and let $N=T/\Delta t$. The scheme has finite speed of propagation, and if

$$
\operatorname{supp}(v_{\Delta x}(\,\cdot\,,T))\subseteq[0,X_T], \tag{5.6}
$$

then

$$X_T \leq X + N\Delta x = X + \frac{T}{\lambda}.$$

Choosing

$$\Delta x = 4CMX\Delta t, \tag{5.7}$$

where M is given by (5.1), or

$$\lambda = \frac{1}{4CMX},$$

where $C \geq 1$ is a constant to be decided later (cf. (5.23)), we find that

$$X_T \leq X + 4MCXT. \tag{5.8}$$

Thus we can without loss of generality assume that X is so large that $v_j^n = 0$ for all $n \leq N$ and all $j \leq J_{\Delta x}$.

For convenience we will use the notation

$$a = v_{j-1}^n, \quad b = v_j^n, \quad c = v_j^{n+1}, \quad \text{and} \quad \alpha = \frac{\Delta t}{\Delta x} u_j^n. \tag{5.9}$$

In this notation, the difference scheme (5.2) reads

$$c = \left(1 - \alpha - \frac{\Delta t}{2} b\right) b + \alpha a. \tag{5.10}$$

Lemma 5.1. *Let $\Delta t < 1/(2M)$ and assume that*

$$0 \leq v_j^0 \leq M \tag{5.11}$$

for all $j \in \mathbb{N}_0$. Then

$$0 \leq v_j^n \leq M$$

for all $n \leq N$ and all $j \in \mathbb{N}_0$.

Proof. Assume that the lemma holds for some n. Then we get

$$0 \leq u_j^n = \Delta x \sum_{i=0}^{j-1} v_i^n \leq \Delta x J_{\Delta x} M \leq XM.$$

Hence $0 \leq \alpha \leq \lambda MX \leq 1/(4C) \leq 1/4$. Observe next that we have

$$X\lambda + \frac{\Delta t}{2} \leq \frac{1}{2M}, \tag{5.12}$$

from our assumptions $\Delta t < 1/(2M)$ and (5.7). The condition (5.12) implies

$$\alpha + \frac{\Delta t}{2} b \leq \frac{1}{2},$$

in the notation (5.9), and thus

$$c \geq \frac{b}{2} + \alpha a.$$

From this it follows that if $v_j^n \geq 0$, then also $v_j^{n+1} \geq 0$. Hence $u_j^{n+1} \geq 0$ for all n and j. Therefore we also get the bound

$$c \leq (1 - \alpha) b + \alpha a,$$

which trivially yields, since $0 \leq \alpha \leq 1/4$, that

$$\max_j v_j^{n+1} \leq \max_j v_j^n \leq M. \tag{5.13}$$

□

We will from now on tacitly assume that the initial approximation satisfies (5.11).

Now we proceed as before, similar to (4.8), that by multiplying the scheme (5.2) with $f'(v_j^n)$, we get

$$
\begin{aligned}
(5.14)\quad & D_+^t f\left(v_j^n\right) + u_j^n D_- f\left(v_j^n\right) \\
&+ \frac{1}{2}\left[-\Delta t f''(\eta_j^n)\left(D_+^t v_j^n\right)^2 + \Delta x f''(\xi_j^n)\left(D_- v_j^n\right)^2\right] = -\frac{1}{2} f'\left(v_j^n\right)\left(v_j^n\right)^2,
\end{aligned}
$$

where η_j^n is between v_j^{n+1} and v_j^n, and ξ_j^n is between v_j^n and v_{j-1}^n.

Lemma 5.2. *Suppose* (5.1) *and* (5.7) *hold, and that* Δt *satisfies*

$$
\Delta t \leq \frac{1}{4M}. \tag{5.15}
$$

Then

$$
\|v_{\Delta x}\|_{L^4(Q_T)}^4 \leq MT \, \|v_{\Delta x}(\,\cdot\,,0)\|_{L^3}^3 . \tag{5.16}
$$

For any $0 < t \leq T$ *we have*

$$
\|v_{\Delta x}(\,\cdot\,,t)\|_{L^2(\mathbb{R}^+)}^2 \leq \|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}^2 + \frac{\Delta t}{2}\|v_{\Delta x}\|_{L^4(Q_T)}^4 , \tag{5.17}
$$

$$
\begin{aligned}
\|v_{\Delta x}(\,\cdot\,,t)\|_{L^1(\mathbb{R}^+)} \leq \|v_{\Delta x}(\,\cdot\,,0)\|_{L^1(\mathbb{R}^+)} &+ \frac{t}{2}\|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}^2 \\
&+ \frac{t\Delta t}{4}\|v_{\Delta x}\|_{L^4(Q_T)}^4 ,
\end{aligned} \tag{5.18}
$$

and

$$
\|u_{\Delta x}(\,\cdot\,,t)\|_{L^\infty(\mathbb{R}^+)} \leq \|v_{\Delta x}(\,\cdot\,,t)\|_{L^1(\mathbb{R}^+)} . \tag{5.19}
$$

Proof. Observe first that (5.19) follows if we can establish the other bounds. Choosing $f(v) = v^p$ in (5.14) yields

(5.20)

$$
\begin{aligned}
D_+^t \Delta x \sum_j \left(v_j^n\right)^p &+ \Delta x \sum_j \frac{p(p-1)}{2}\left(\xi_j^n\right)^{p-2} u_j^n \left(D_- v_j^n\right)^2 \Delta x \\
&= \left(1 - \frac{p}{2}\right)\Delta x \sum_j \left(v_j^n\right)^{p+1} + \frac{p(p-1)}{2}\Delta x \sum_j \left(\eta_j^n\right)^{p-2}\left(D_+^t v_j^n\right)^2 \Delta t.
\end{aligned}
$$

The reason for the inconvenient extra term on the right-hand side is that we had to expand the Taylor series about v_j^n rather than v_j^{n+1}. By the definition of the scheme

$$
\left(D_+^t v_j^n\right)^2 \leq 2\left(u_j^n\right)^2\left(D_- v_j^n\right)^2 + \frac{1}{2}\left(v_j^n\right)^4 .
$$

We group the first part of this with the second term on the left-hand side of (5.20). In order to make this approach work we must then ensure that

$$
\left(\xi_j^n\right)^{p-2} u_j^n \left(D_- v_j^n\right)^2 \Delta x - 2\left(u_j^n\right)^2\left(\eta_j^n\right)^{p-2}\left(u_j^n\right)^2\left(D_- v_j^n\right)^2 \Delta t \geq 0. \tag{5.21}
$$

This will hold if we can choose Δt so small that

$$
\Delta x - 2X_j^n u_j^n \Delta t \geq 0, \tag{5.22}
$$

where

$$
X_j^n = \left(\frac{\eta_j^n}{\xi_j^n}\right)^{p-2}.
$$

Since $u_j^n \leq MX$, this can easily be achieved for $p = 2$. We need to be able to do this also for $p > 2$, so we investigate X_j^n further. In terms of a, b and c from (5.9), we have that

$$\begin{aligned} c^p &= b^p + pb^{p-1}(c-b) + \frac{p(p-1)}{2}(c-b)^2 \left(\eta_j^n\right)^{p-2}, \\ a^p &= b^p + pb^{p-1}(a-b) + \frac{p(p-1)}{2}(a-b)^2 \left(\xi_j^n\right)^{p-2}. \end{aligned}$$

This gives

$$\begin{aligned} X_j^n &= \frac{\left(c^p + b^p(p-1) - pb^{p-1}c\right)(a-b)^2}{\left(a^p + b^p(p-1) - pb^{p-1}a\right)(c-b)^2} \\ &= \frac{w^p - pw + (p-1)}{(w-1)^2}\,\frac{(y-1)^2}{y^p - py + (p-1)}, \end{aligned}$$

where $w = c/a$ and $y = a/b$. Now we have that

$$z^p - pz + (p-1) = (z-1)^2 \sum_{k=2}^{p} (k-1) z^{p-k}.$$

Thus we arrive at

$$X_j^n = \frac{q(w)}{q(y)},$$

where q is the polynomial

$$q(z) = \sum_{k=2}^{p} (k-1) z^{p-k}.$$

For $z \geq 0$, the function q is clearly increasing, $q'(z) > 0$, and satisfies $q(z) \geq p-1$. By the bounds on c, we have that

$$\frac{1}{2} + \alpha y \leq w \leq (1-\alpha) + \alpha y,$$

where α is defined in (5.9), and therefore

$$\frac{q\left(\frac{1}{2} + \alpha y\right)}{q(y)} \leq X_j^n \leq \frac{q\left((1-\alpha) + \alpha y\right)}{q(y)} \leq \sup_{\alpha \in [0,1], y \in \mathbb{R}} \frac{q\left((1-\alpha) + \alpha y\right)}{q(y)} =: C. \tag{5.23}$$

The constant C, which we may assume is greater than one, is the one appearing in (5.7). Now Δt is so small that

$$\frac{\Delta t}{\Delta x} CMX \leq \frac{1}{4}, \tag{5.24}$$

and therefore (5.21) holds for any $p \geq 1$. Consequently

$$\begin{aligned} &D_+^t \Big[\Delta x \sum_j \left(v_j^n\right)^p\Big] \\ &\qquad \leq \left(1 - \frac{p}{2}\right) \Delta x \sum_j \left(v_j^n\right)^{p+1} + \frac{p(p-1)}{4} \Delta x \sum_j \left(\eta_j^n\right)^{p-2} \left(v_j^n\right)^4 \Delta t. \end{aligned} \tag{5.25}$$

Next, (5.15) also implies that $\eta_j^n\Delta t \le 1/4$. Using this we can derive a number of useful estimates from (5.25). First we set $p=3$ to find that

$$\begin{aligned} D_+^t\Big[\Delta x\sum_j \left(v_j^n\right)^3\Big] &\le -\frac{1}{2}\Delta x\sum_j \left(v_j^n\right)^4 + \frac{3}{2}\Delta x\sum_j \left(v_j^n\right)^4\left(\Delta t\eta_j^n\right) \\ &\le \frac{1}{2}\left(\frac{3}{4}-1\right)\Delta x\sum_j \left(v_j^n\right)^4 \le 0. \end{aligned}$$

Hence

$$\|v_{\Delta x}(\,\cdot\,,t_n)\|^3_{L^3(\mathbb{R}^+)} \le \|v_{\Delta x}(\,\cdot\,,0)\|^3_{L^3(\mathbb{R}^+)}\,. \tag{5.26}$$

This also implies that (5.16) holds by using $v_j^n \le M$. Now we are ready to tackle $p=2$, which yields in (5.25)

$$D_+^t\Big[\Delta x\sum_j \left(v_j^n\right)^2\Big] \le \frac{\Delta t}{2}\Delta x\sum_j \left(v_j^n\right)^4. \tag{5.27}$$

Summing (5.27) over n after multiplying with Δt gives

$$\Delta x\sum_j \left(v_j^n\right)^2 \le \Delta x\sum_j \left(v_j^0\right)^2 + \frac{(\Delta t)^2}{2}\Delta x\sum_j \left(v_j^n\right)^4, \tag{5.28}$$

which implies (5.17). Finally, setting $p=1$ in (5.25) we find, using (5.28), that

$$\begin{aligned} D_+^t\Big[\Delta x\sum_j v_j^n\Big] &\le \frac{1}{2}\Delta x\sum_j \left(v_j^n\right)^2 \\ &\le \frac{1}{2}\Delta x\sum_j \left(v_j^0\right)^2 + \frac{\Delta t}{4}\,\|v_{\Delta x}\|^4_{L^4(Q_T)}\,, \end{aligned}$$

which gives the L^1 bound (5.18). □

Lemma 5.3. *Suppose v_0 satisfies the condition* (5.1), *and that Δx, Δt satisfy* (5.7) *and* (5.15). *Then, extracting subsequences if necessary, we have the following convergence results:*

$$u_{\Delta x}\to u \text{ uniformly in } [0,X]\times[0,T] \text{ for each } X>0 \text{ and pointwise in } \overline{Q_T},$$
$$\text{and the limit } u \text{ belongs to } W^{1,4}(Q_T); \tag{5.29}$$

$$v_{\Delta x}=\partial_x u_{\Delta x}\rightharpoonup \partial_x u = v \text{ in } L^3(Q_T),$$
$$\text{and } v_{\Delta x}=\partial_x u_{\Delta x}\overset{\star}{\rightharpoonup}\partial_x u = v \text{ in } L^\infty((0,T);L^1(\mathbb{R}^+)\cap L^3(\mathbb{R}^+)); \tag{5.30}$$

$$(v_{\Delta x})^2\rightharpoonup w \text{ in } L^2(Q_T),$$
$$\text{and } (v_{\Delta x})^2\overset{\star}{\rightharpoonup} w \text{ in } L^\infty((0,T);L^1(\mathbb{R}^+)\cap L^{3/2}(\mathbb{R}^+)); \tag{5.31}$$

$$u_{\Delta x}v_{\Delta x}\rightharpoonup uv \text{ in } L^3(Q_T),$$
$$\text{and } u_{\Delta x}v_{\Delta x}\overset{\star}{\rightharpoonup} uv \text{ in } L^\infty((0,T);L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)). \tag{5.32}$$

Proof. We can bound $D_+^t u_j^n$ as for the implicit scheme, the only difference is that the terms on the right-hand side are evaluated at $t=t_n$ instead of t_{n+1}. We end up with

$$D_+^t u_j^n = -u_{j-1}^n v_{j-1}^n - \Delta x\left(v_{j-1}^n\right)^2 + \frac{\Delta x}{2}\sum_{i=0}^{j-1}\left(v_i^n\right)^2.$$

Via the bilinear interpolant $\tilde{u}_{\Delta x}$ we conclude that $u_{\Delta x}$ is uniformly bounded in $W^{1,4}([0,X]\times[0,T])$, and this space is compactly embedded in $C^{0,1/2}([0,X]\times[0,T])$. The rest of the proof is identical to the proof of Lemma 3.4. □

Lemma 5.4. *Lemma* 3.6 *holds for the triplet* (v,u,w) *from Lemma* 5.3.

Proof. Repeating the arguments from the implicit case, it is straightforward to show that

$$v_t + (uv)_x = \frac{1}{2}w, \tag{5.33}$$

in the sense of distributions in Q_T. To show (3.18) we consider the explicit scheme for $(v_j^n)^2$,

$$\begin{aligned} D_+^t\left(v_j^n\right)^2 + D_-\left(u_j^n\left(v_j^n\right)^2\right) + \Delta x u_j^n\left(D_-v_j^n\right)^2 \\ = -\Delta x D_-\left(v_j^n\right)^3 + \Delta t\left(D_+^t v_j^n\right)^2. \end{aligned}$$

After the same type of manipulations that we have carried out so far we find that

$$\begin{aligned} D_+^t\left(v_j^n\right)^2 + D_-\left(u_j^n\left(v_j^n\right)^2\right) + u_j^n\left(D_-v_j^n\right)^2\left(\Delta x - 2u_j^n\Delta t\right) \\ \le -\Delta x D_-\left(v_j^n\right)^3 + \frac{\Delta t}{2}\left(v_j^n\right)^4. \end{aligned}$$

Since $u_j^n \le MX$ and we have (5.7), the last term on the left is positive. Therefore this term can be dropped, and, with the L^p bounds that $v_{\Delta x}$ satisfies, it is not difficult establish that

$$w_t + (uw)_x \le 0.$$

□

Theorem 5.5. *Let* v_0 *be a function satisfying* (5.1). *Define the explicit difference approximations* $(v_{\Delta x}, u_{\Delta x})$ *by* (5.2)–(5.4). *Assume that* Δx, Δt *satisfy* (5.7) *and* (5.15). *Then* $\{(v_{\Delta x}, u_{\Delta x})\}$ *converges to a weak dissipative solution* (v,u) *of* (1.5) *in the sense of Definition* 1.1. *Precisely, we have that*

$$\left\|u_{\Delta x} - u\right\|_{L^\infty(Q_T)} \to 0 \quad \textit{and} \quad \left\|v_{\Delta x} - v\right\|_{L^p(Q_T)} \to 0, \quad \textit{for any } p \in [1,4].$$

Proof. The proof consists only in noting that the assumptions of Lemma 3.7 hold for the limit triplet (v,u,w). □

6. The case $v_0 \in L^1 \cap L^2$

In this section we treat the pure $L^1 \cap L^2$ case. We make no assumption about the sign of the initial data v_0 and assume simply that

$$v_0 \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+). \tag{6.1}$$

The space L^2 is the natural one for the Hunter–Saxton equation, whereas L^1 is as before a convenient replacement of the compact support condition used in [17, 18].

To handle sign changing solutions, we need to modify the numerical schemes. In addition, the convergence analysis becomes more complicated. The modification of the schemes concerns the discretization of the transport term uv_x in (1.5), which must account for a "velocity" u that may be both positive and negative. Moreover, this discretization must be "compatible" with the equation $v = u_x$ in (1.5).

Instead of giving the details for all the numerical schemes, we have chosen to focus on the modification of the semi-discrete scheme from Section 3.

We begin by stating the modified version of the semi-discrete scheme. Let $\{(v_j(t), u_j(t))\}_{j\in\mathbb{N}_0}$ satisfy the system of ordinary differential equations

$$\begin{aligned} &\dot v_j + (u_j \vee 0)\, D_- v_j + (u_{j+1}\wedge 0)\, D_+ v_j = -\frac{1}{2}(v_j)^2, \qquad D_+ u_j = v_j, \\ &v_j|_{t=0} = v_j^0, \qquad u_0(t) = 0, \end{aligned} \tag{6.2}$$

where we have used the notation $(a\wedge b) = \min\{a,b\}$, $(a\vee b) = \max\{a,b\}$. The scheme (6.2) holds for $j = 0,\dots,J_{\Delta x} = J/(\Delta x^2)$ for some large constant J. As for the other schemes, for convenience we define $u_{-1} = v_{-1} = 0$. Moreover, by the definition of the scheme,

$$u_j(t) = \Delta x \sum_{i=0}^{j-1} v_i(t) \quad \text{ for } j = 1, 2, \dots,$$

and $v_0(t) = u_1(t)/\Delta x$ for any $t > 0$.

Regarding the compatibility mentioned above, the variable sign scheme (6.2) is set up such that the following identity always holds:

$$D_+\,(u_j \vee 0) + D_-\,(u_{j+1}\wedge 0) = D_+ u_j = v_j,$$

which is important for the convergence analysis.

As in Section 3, we let $\{v_j^0\}_{j\in\mathbb{N}_0}$ be sequence of discrete initial data chosen such that

$$v^0_{\Delta x}(x) = \sum_{j\in\mathbb{N}_0} v_j^0 \mathbf{1}_{I_j}(x) \tag{6.3}$$

converges to the initial function v_0 in $L^2(\mathbb{R}^+)$ as $\Delta x \to 0$, and as before we introduce the pointwise defined functions

$$v_{\Delta x}(x,t) = \sum_{j\in\mathbb{N}_0} v_j(t)\mathbf{1}_{I_j}(x), \qquad u_{\Delta x}(x,t) = \int_0^x v_{\Delta x}(y,t)\,dy. \tag{6.4}$$

For later use, let us write our scheme (6.2) on conservative form. To this end, first note that

$$\begin{aligned} u_j D_- v_j &= (u_j\vee 0)\, D_- v_j + (u_j \wedge 0)\, D_- v_j \\ &= (u_j\vee 0)\, D_- v_j + (u_j\wedge 0)\, D_+ v_{j-1}. \end{aligned}$$

Using this and the discrete Leibniz rule (2.1) we find that

$$\begin{aligned} (u_j\vee 0)\, D_- v_j + (u_{j+1}\wedge 0)\, D_+ v_j &= u_j D_- v_j + D_-\left[(u_{j+1}\wedge 0)\, D_+ v_j\right]\Delta x \\ &= D_-\,(u_j v_j) - (v_{j-1})^2 + D_-\left[(u_{j+1}\wedge 0)\, D_+ v_j\right]\Delta x. \end{aligned}$$

Hence, the conservative version of the scheme (6.2) reads

$$\begin{aligned} \dot v_j + D_-\,(u_j v_j) &= (v_{j-1})^2 - \frac{1}{2}(v_j)^2 - \Delta x D_-\left[(u_{j+1}\wedge 0)\, D_+ v_j\right] \\ &= \frac{1}{2}(v_j)^2 - \Delta x D_-\,(v_j)^2 - \Delta x D_-\left[(u_{j+1}\wedge 0)\, D_+ v_j\right]. \end{aligned} \tag{6.5}$$

In Lemma 6.2 we show that $v_{\Delta x}(\,\cdot\,, t)$ is bounded in $L^2(\mathbb{R}^+)$. As for the scheme in Section 3, this implies that we do not only have local (in time) existence of a C^1 solution to the ordinary differential equation (6.2), but that a C^1 solution exists for any positive t.

We now prove an Oleĭnik-type (one-sided Lipschitz) estimate. In this section the Oleĭnik-type estimate will be of crucial importance for the convergence analysis.

Lemma 6.1. *Set* $\bar{v}_{\Delta x}(t) = \max_{j\in\mathbb{N}_0} v_j(t)$*. Then for* $t > 0$ *and* $j \in \mathbb{N}_0$ *we have*

$$v_j(t) \leq \bar{v}_{\Delta x}(t) \leq \frac{2\bar{v}_{\Delta x}(0)}{t\bar{v}_{\Delta x}(0)+2} \leq \frac{2}{t}. \tag{6.6}$$

Proof. If $v_k(s) \geq v_{k\pm 1}(s)$ for some k and s, then

$$D_- v_k(s) \geq 0 \quad \text{and} \quad D_+ v_k(s) \leq 0.$$

Using this in (6.2) for k and s, we find that

$$\dot{v}_k(s) \leq -\frac{1}{2} v_k^2(s).$$

The function $\bar{v}_{\Delta x}(t)$ is continuous and differentiable almost everywhere, since differentiability fails at most at a countable number of times. At all points of differentiability $\bar{v}_{\Delta x}$ satisfies

$$\dot{\bar{v}}_{\Delta x}(t) \leq -\frac{1}{2}\bar{v}_{\Delta x}^2(t).$$

By the Gronwall inequality we have that

$$\bar{v}_{\Delta x}(t) \leq \frac{2\bar{v}_{\Delta x}(0)}{\bar{v}_{\Delta x}(0)t+2} \leq \frac{2}{t}. \tag{6.7}$$

□

Let f be a twice continuously differentiable function. Using the scheme (6.2) and the discrete chain rule (2.4) we find

$$\frac{d}{dt} f(v_j) + (u_j \vee 0)\, D_- f(v_j) + (u_{j+1} \wedge 0)\, D_+ f(v_j) + I_{\Delta x, j}(f) = -\frac{1}{2}(v_j)^2 f'(v_j), \tag{6.8}$$

where $I_{\Delta x,j}(f)$ is the numerical dissipation associated with the upwind nature of the scheme, which takes the form

$$I_{\Delta x,j}(f) = \frac{\Delta x}{2}\left((u_j \vee 0)\, f''\left(\xi_j^-\right)(D_- v_j)^2 - (u_{j+1} \wedge 0)\, f''\left(\xi_j^+\right)(D_+ v_j)^2 \right), \tag{6.9}$$

with $\xi_j^\pm$ being a number between v_j and $v_{j\pm 1}$.

Starting off from (6.8), we derive some basic a priori estimates for $v_{\Delta x}, u_{\Delta x}$, most notably a uniform L^2 estimate for $v_{\Delta x}$.

Lemma 6.2. *Suppose* (6.1) *holds. For any* $t > 0$ *there holds*

$$\begin{aligned}
\|v_{\Delta x}(\,\cdot\,,t)\|_{L^2(\mathbb{R}^+)} &\leq \|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}, \\
\|v_{\Delta x}(\,\cdot\,,t)\|_{L^1(\mathbb{R}^+)} &\leq \|v_{\Delta x}(\,\cdot\,,0)\|_{L^1(\mathbb{R}^+)} + \frac{t}{2}\|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}^2, \\
\|u_{\Delta x}(\,\cdot\,,t)\|_{L^\infty(\mathbb{R}^+)} &\leq \|v_{\Delta x}(\,\cdot\,,0)\|_{L^1(\mathbb{R}^+)} + \frac{t}{2}\|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}^2.
\end{aligned}$$

Proof. Multiplying (6.8) by Δx and summing over j yields, after doing summation by parts on the "transport terms",

$$\frac{d}{dt}\Delta x \sum_{j\in\mathbb{N}_0} f(v_j) + I_{\Delta x}(f) = \Delta x \sum_{j\in\mathbb{N}_0} v_j\big(f(v_j) - \frac{1}{2}v_j f'(v_j)\big), \tag{6.10}$$

where we have assumed that $f(0) = 0$ and $I_{\Delta x}(f) := \Delta x \sum_{j\geq 0} I_{\Delta x,j}(f)$. If f is such that $f'' \geq 0$, then $I_{\Delta x}(f) \geq 0$. In particular, for $f(v) = v^2$ we find, by integrating from 0 to t

$$\|v_{\Delta x}(\,\cdot\,,t)\|_{L^2(\mathbb{R}^+)} \leq \|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}\,, \tag{6.11}$$

which proves the L^2 estimate, but also shows that

$$0 \leq \int_0^t I_{\Delta x}(v^2)\,dt \leq C \tag{6.12}$$

for a constant C independent of Δx. Next we choose $f(v) = |v|$ in (6.10). Then, via an approximation argument that we omit, (6.10) yields

$$\frac{d}{dt}\Delta x \sum_{j\in\mathbb{N}_0} |v_j| \leq \frac{\Delta x}{2} \sum_{j\in\mathbb{N}_0} \operatorname{sign}(v_j)\,|v_j|^2 \leq \frac{\Delta x}{2} \sum_{j\in\mathbb{N}_0} |v_j|^2\,,$$

which proves the L^1 estimate. The L^∞ estimate can established as in the proof of Lemma 3.2. □

The next lemma contains an improved integrability estimate showing that $v_{\Delta x}$ is uniformly bounded in L^p for any $p \in [1,3)$. This estimate is important, as it prevents $v_{\Delta x}^2$ from exhibiting concentrations as $\Delta x \to 0$. Our proof makes use of the one-sided Lipschitz bound in Lemma 6.1 and the $L^1 \cap L^2$ bound in Lemma 6.2.

Lemma 6.3. *Suppose* (6.1) *holds. Then there exists a finite constant C such that*

$$\|v_{\Delta x}\|_{L^p(Q_T)} \leq C, \qquad p \in [2,3). \tag{6.13}$$

The constant C depends on T, p, and the $L^1 \cap L^2$ norm of v_0, but not on Δx.

Proof. Fix any $\kappa \in [0,1)$. For any $t \in [0,T]$, let $\mathcal{N}(t)$ denote the set of indices $j \in \mathbb{N}_0$ such that $v_j(t) < 0$ and $\mathcal{P}(t)$ denote the set of indices $j \in \mathbb{N}_0$ such that $v_j(t) > 0$. We start by writing

$$\|v_{\Delta x}\|_{L^{2+\kappa}(Q_T)}^{2+\kappa} = \int_0^T \Delta x \sum_{j\in\mathbb{N}_0} |v_j|^{2+\kappa}\,dt = I_+ + I_-,$$

where

$$I_+ = \int_0^T \Delta x \sum_{j\in\mathcal{P}(t)} |v_j|^{2+\kappa}\,dt, \qquad I_- = \int_0^T \Delta x \sum_{j\in\mathcal{N}(t)} |v_j|^{2+\kappa}\,dt.$$

In view of Lemmas 6.1 and 6.2,

$$I_+ \leq \int_0^T \Delta x \sum_{j\in\mathcal{P}(t)} |v_j|^2 \left(\frac{2}{t}\right)^\kappa dt \leq \|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}^2 \frac{2^\kappa T^{1-\kappa}}{1-\kappa}. \tag{6.14}$$

It remains to estimate I_-. Choosing $f(v) = |v|^{1+\kappa}$ in (6.10) yields

$$\frac{d}{dt}\Delta x \sum_{j\in\mathbb{N}_0} |v_j|^{1+\kappa} \leq \frac{1-\kappa}{2}\Delta x \sum_{j\in\mathbb{N}_0} \operatorname{sign}(v_j)\,|v_j|^{2+\kappa}\,.$$

We have

$$\int_0^T \Delta x \sum_{j\in\mathbb{N}_0} \operatorname{sign}(v_j)\,|v_j|^{2+\kappa}\,dt = I_+ - I_-.$$

Therefore

$$\begin{aligned} I_- &\le I_+ + \frac{2}{1-\kappa}\left[\|v_{\Delta x}(\,\cdot\,,0)\|_{L^{1+\kappa}}^{1+\kappa} - \|v_{\Delta x}(\,\cdot\,,T)\|_{L^{1+\kappa}}^{1+\kappa}\right] \\ &\le \|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}^2 \frac{2^\kappa T^{1-\kappa}}{1-\kappa} + \frac{2}{1-\kappa}\|v_{\Delta x}(\,\cdot\,,0)\|_{L^{1+\kappa}}^{1+\kappa}, \end{aligned}$$

where (6.14) was used to derive the second inequality.

From the bounds just obtained for $I_\pm$, we conclude that

$$\begin{aligned} \int_0^T \Delta x \sum_{j\in\mathbb{N}_0} |v_j|^{2+\kappa}\,dt &\le 2^{1+\kappa}\frac{T^{1-\kappa}}{1-\kappa}\|v_{\Delta x}(\,\cdot\,,0)\|_{L^2(\mathbb{R}^+)}^2 \\ &\qquad + \frac{2}{1-\kappa}\|v_{\Delta x}(\,\cdot\,,0)\|_{L^{1+\kappa}}^{1+\kappa}. \end{aligned} \tag{6.15}$$

Since v_0 belongs to $L^{1+\kappa}$ by interpolation, we deduce that there is a finite constant C, depending on T,κ and the $L^1\cap L^2$ norm of v_0 but not Δx, such that

$$\int_0^T \Delta x \sum_{j\in\mathbb{N}_0} |v_j|^{2+\kappa}\,dt \le C^{1+\kappa},$$

which concludes the proof of the lemma. □

Using the three previous lemmas we can prove some basic convergence results.

Lemma 6.4. *Suppose* (6.1) *holds. Extracting subsequences if necessary, we have the following convergence results as $\Delta x \to 0$:*

$$\begin{aligned} &u_{\Delta x} \to u \text{ uniformly in } [0,R]\times[0,T] \text{ for each } R>0, \text{ pointwise in } \overline{Q_T}, \\ &\text{and the limit } u \text{ belongs to } W^{1,p}(\overline{Q_T}) \text{ for any } p\in[1,3); \end{aligned} \tag{6.16}$$

$$\begin{aligned} &v_{\Delta x} = \partial_x u_{\Delta x} \rightharpoonup \partial_x u =: v \text{ in } L^p(Q_T) \text{ for any } p\in[1,3); \\ &\text{and } v_{\Delta x} = \partial_x u_{\Delta x} \overset{\star}{\rightharpoonup} \partial_x u =: v \text{ in } L^\infty((0,T);L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)); \end{aligned} \tag{6.17}$$

$$(v_{\Delta x})^2 \rightharpoonup \overline{v^2} \text{ in } L^p(Q_T) \text{ for any } p\in[1,3/2); \tag{6.18}$$

$$\begin{aligned} &u_{\Delta x}v_{\Delta x} \rightharpoonup uv \text{ in } L^p(Q_T) \text{ for any } p\in[1,3); \\ &\text{and } u_{\Delta x}v_{\Delta x} \overset{\star}{\rightharpoonup} uv \text{ in } L^\infty((0,T);L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)). \end{aligned} \tag{6.19}$$

Proof. By Lemma 6.3, the function $\partial_x u_{\Delta x} = v_{\Delta x}$ is uniformly bounded in $L^{2+\kappa}(Q_T)$. Next, we bound $\partial_t u_{\Delta x}$ as follows:

$$\begin{aligned} \frac{d}{dt}u_j &= \Delta x \sum_{i=0}^{j-1} \dot v_i \\ &= \Delta x \sum_{i=0}^{j-1}\left[-D_-(u_i v_i) + \frac{1}{2}(v_i)^2 - \Delta x D_-(v_i)^2 - \Delta x D_-\left[(u_{i+1}\wedge 0)\,D_+ v_i\right]\right] \\ &= -u_{j-1}v_{j-1} - \Delta x v_{j-1}^2 - (u_j\wedge 0)(v_j - v_{j-1}) + \frac{\Delta x}{2}\sum_{i=0}^{j-1}(v_i)^2. \end{aligned}$$

Hence, thanks to Lemma 6.2, we can find a constant C_1, independent of Δx, such that $\left|\frac{d}{dt}u_j\right| \le C_1(|v_{j-1}|+|v_j|+1)$. Fix any $R>0$ and let J be an integer such that

$J\Delta x \le R$. Then

$$\Delta x \sum_{j=0}^{J} \left| \frac{d}{dt} u_j \right|^{2+\kappa} \le C_2 + \|v_{\Delta x}(\,\cdot\,,t)\|^{2+\kappa}_{L^{2+\kappa}(\mathbb{R}^+)} \le C_3,$$

where C_2, C_3 depend on R but not on Δx, and thus $u_{\Delta x}$ is uniformly bounded in $W^{1,2+\kappa}([0,R]\times[0,T])$, which is compactly embedded into $C^{0,\ell}([0,R]\times[0,T])$, with $\ell = 1 - 2/(2+\kappa)$. Consequently, there is a continuous function $u\colon \overline{Q_T} \to \mathbb{R}$ such that, up to extracting a subsequence if necessary,

$$u_{\Delta x} \to u \text{ uniformly on } [0,R]\times[0,T] \text{ and pointwise in } \overline{Q_T} \text{ as } \Delta x \to 0.$$

Now (6.16) follows from a standard diagonal argument as $R \to \infty$.

Finally, (6.17) and (6.18) are consequences of (6.13), while (6.19) is a consequence of (6.16) and (6.17). □

In the remaining part of this section the aim is to improve the weak convergence of $\{v_{\Delta x}\}_{\Delta x>0}$ to strong convergence. As in the previous sections, the idea is to derive a transport equation for the evolution of the nonnegative defect measure $\overline{v^2} - v^2$; thus if it is zero at time $t = 0$, then it will continue to be zero at later times $t > 0$. The proof is, however, complicated by the fact that we do not have a uniform bound on $v_{\Delta x}$ from below but merely (6.6), and that we only have uniform L^p bounds on $v_{\Delta x}$ for $p < 3$. For these reasons we decompose the function $f(v) = v^2$ into its increasing part f^+ and its decreasing part f^-, and then work with appropriate truncations $f_R^{\pm}$ of the functions $f^{\pm}$. This strategy was implemented first by Zhang and Zheng [18] in their proof of existence of a dissipative solution, and we will herein adapt this stategy to our numerical scheme. We commence by defining:

$$\begin{aligned}
f^{\pm}(v) &= \frac{1}{2}(0 \vee \pm v)^2, \qquad v \in \mathbb{R},\\
f_R^+(v) &= \begin{cases} 0, & \text{for } v < 0,\\ \frac{1}{2}v^2, & \text{for } v \in [0,R],\\ Rv - \frac{1}{2}R^2, & \text{for } v > R, \end{cases}\\
f_R^-(v) &= \begin{cases} -Rv - \frac{1}{2}R^2, & \text{for } v < -R,\\ \frac{1}{2}v^2, & \text{for } v \in [-R,0],\\ 0, & \text{for } v > 0, \end{cases}\\
f_R(v) &= f_R^-(v) + f_R^+(v).
\end{aligned}$$

In the next lemma, we derive the system of equations satisfied by the limit triplet $(v, u, \overline{v^2})$, as well as certain renormalizations of this system.

Lemma 6.5. *The limit triplet $(v,u,\overline{v^2})$ from Lemma 6.4 satisfies*

$$v_t + (uv)_x = \frac{1}{2}\overline{v^2}, \qquad u_x = v \tag{6.20}$$

in the sense of distributions on Q_T and

$$v \in C([0,T]; L^p(\mathbb{R}^+)), \quad \lim_{t\to 0} \|v(\,\cdot\,,t) - v_0\|_{L^p(\mathbb{R}^+)} = 0, \tag{6.21}$$

for any $p \in [1,2)$. Moreover,

$$v \in C_+([0,T]; L^2(\mathbb{R}^+)), \quad \lim_{t\to 0} \|v(\,\cdot\,,t) - v_0\|_{L^2(\mathbb{R}^+)} = 0, \tag{6.22}$$

where $C_+([0,T];L^2(\mathbb{R}^+))$ denotes the set of functions in $L^\infty((0,T);L^2(\mathbb{R}^+))$ that are right-continuous in time on $[0,T]$ with values in $L^2(\mathbb{R}^+)$. In addition,

$$\lim_{t\to 0}\int_{\mathbb{R}^+}\overline{v^2}(x,t)\,dx=\int_{\mathbb{R}^+}v_0(x)^2\,dx. \tag{6.23}$$

For any $R>0$, the renormalized equations

$$\big(f_R^\pm(v)\big)_t+\big(uf_R^\pm(v)\big)_x=(f_R^\pm)'(v)\big(\frac{1}{2}\overline{v^2}-v^2\big)+vf_R^\pm(v) \tag{6.24}$$

hold in the sense of distributions on Q_T. Moreover, (6.21) *and* (6.22) *hold with v, v_0 replaced by $f_R^\pm(v), f_R^\pm(v_0)$, respectively.*

Proof. Set $\varphi_j(t)=\frac{1}{\Delta x}\int_{I_j}\varphi(x,t)\,dx$ for $\varphi\in C_c^\infty(Q_T)$. Using the scheme (6.5), we find

$$\begin{aligned}\int_0^T\Delta x\sum_{j\in\mathbb{N}_0}\Big[v_j\varphi_j'+u_jv_jD_+\varphi_j+\frac{1}{2}(v_j)^2\varphi_j\Big]\,dt\\
&=-\int_0^T(\Delta x)^2\sum_{j\in\mathbb{N}_0}(v_j)^2D_+\varphi_j\,dt\\
&\quad-\int_0^T\Delta x\sum_{j\in\mathbb{N}_0}(u_{j+1}\wedge 0)\,D_+v_jD_+\varphi_j\,dt\\
&=:E_2+E_2.\end{aligned}$$

We have that

$$|E_1|\le\Delta xT\,\|\varphi_x\|_{L^\infty(Q_T)}\,\|v_0\|^2_{L^2(\mathbb{R}^+)}\to 0\quad\text{as }\Delta x\to 0.$$

For the second term we write

$$\begin{aligned}|E_2|&=\Delta x\int_0^T\sum_{j\in\mathbb{N}_0}\sqrt{\Delta x}\sqrt{-(u_{j+1}\wedge 0)}\,|D_+\varphi_j|\,\sqrt{\Delta x}\sqrt{-(u_{j+1}\wedge 0)}\,|D_+v_j|\,\,dt\\
&\le\sqrt{\Delta x}\sqrt{\|u_{\Delta x}\|_{L^\infty(Q_T)}}\,\|\varphi_x\|_{L^2(Q_T)}\sqrt{I_{\Delta x}(v^2)},\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and the notation

$$I_{\Delta x}(v^2)=\int_0^T\Delta x\sum_{j\in\mathbb{N}_0}\Big[(u_j\vee 0)\,(D_-v_j)^2-(u_{j+1}\wedge 0)\,(D_+v_j)^2\Big]\le C;$$

cf. (6.8) and (6.9). The bound comes from (6.12). Hence $E_2\to 0$ as $\Delta x\to 0$.

Now we have

$$\begin{aligned}\int_0^T \Delta x \sum_{j\in\mathbb{N}_0} \left[v_j\varphi_j' + u_jv_jD_+\varphi_j - \frac{1}{2}(v_j)^2\varphi_j \right] dt \\ = \iint_{Q_T} v_{\Delta x}\varphi_t + u_{\Delta x}v_{\Delta x}\varphi_x - \frac{1}{2}v_{\Delta x}^2\varphi \, dx\, dt \\ + \underbrace{\int_0^T \sum_{j\in\mathbb{N}_0}\int_{I_j} u_jv_j\left(D_+\varphi_j - \varphi_x\right)\,dx\,dt}_{E_3} \\ + \underbrace{\int_0^T \sum_{j\geq 0}\int_{I_j} \left(u_j - u_{\Delta x}\right)v_j\varphi_x\,dt}_{E_4}.\end{aligned}$$

Clearly,

$$\iint_{Q_T} v_{\Delta x}\varphi_t\,dx\,dt \to \iint_{Q_T} v\varphi_t\,dx\,dt,$$
$$\iint_{Q_T} u_{\Delta x}v_{\Delta x}\varphi_x\,dx\,dt \to \iint_{Q_T} uv\varphi_x\,dx\,dt,$$

and

$$\iint_{Q_T} \frac{1}{2}v_{\Delta x}^2\varphi\,dx\,dt \to \iint_{Q_T} \frac{1}{2}\overline{v^2}\varphi\,dx\,dt.$$

It remains to show that terms E_3 and E_4 tend to zero as $\Delta x \to 0$. Regarding E_3,

$$|E_3| \leq \Delta x \left\|u_{\Delta x}\right\|_{L^\infty(Q_T)} \left\|v_{\Delta x}\right\|_{L^1(Q_T)} \left\|\varphi_{xx}\right\|_{L^\infty(Q_T)} \leq C(T)\Delta x,$$

where we have used the L^1 estimate in Lemma 6.2. Consequently, $E_3 \to 0$ as $\Delta x \to 0$. Regarding E_4, for $x \in I_j$ we have

$$u_j - u_{\Delta x} = \left(x_{j+1/2} - x\right)v_j,$$

and therefore

$$\begin{aligned}|E_4| &\leq \int_0^T \sum_{j\in\mathbb{N}_0}\int_{I_j} |x - x_j|\,(v_j)^2\,|\varphi_x|\,dx\,dt \\ &\leq \Delta x \left\|\varphi_x\right\|_{L^\infty(Q_T)} \left\|v_0\right\|_{L^2(\mathbb{R}^+)}^2 T \leq C(T)\Delta x,\end{aligned}$$

where we have also used the L^2 estimate in Lemma 6.2. Hence $I_2 \to 0$ as $\Delta x \to 0$. This concludes the proof of the first part of (6.20). The second part is already contained in (6.17).

The two statements in (6.21) follow from arguments that are standard in the theory of renormalized solutions (see, for example, [18]) and also from the definition of the numerical scheme. Let us now prove (6.22) and (6.23). Here the arguments are also rather standard, but we include them for completeness. By (6.20) and [12, Appendix C] it is not hard to see that $v(\,\cdot\,,t) \rightharpoonup v_0$ in $L^2(\mathbb{R})$ as $t \to 0$, so that by the weak lower semicontinuity of norms we have on one hand

$$\int_{\mathbb{R}^+} v_0(x)^2\,dx \leq \liminf_{t\to 0}\int_{\mathbb{R}^+} v(x,t)^2\,dx. \tag{6.25}$$

On the other hand, by the L^2 estimate in Lemma 6.2,

$$\int_{\mathbb{R}^+} \overline{v^2}(x,t)\,dx \le \int_{\mathbb{R}^+} v_0(x)^2\,dx, \qquad \text{for any } t>0,$$

so that

$$\limsup_{t\to 0} \int_{\mathbb{R}^+} \overline{v^2}(x,t)\,dx \le \int_{\mathbb{R}^+} v_0(x)^2\,dx. \tag{6.26}$$

Clearly, (6.25) and (6.26) imply (6.23) and the second part of (6.22). To prove the first part of (6.22) apply the above argument for any $t \in [0,T]$ (not just $t=0$).

Let us prove (6.24). Since $u_x = v$, we also have that

$$v_t + uv_x = \frac{1}{2}w - v^2 \tag{6.27}$$

holds in the sense of distributions, where we have reverted to the notation $w = \overline{v^2}$. Set $v_\varepsilon = v \star \omega_\varepsilon$, $w_\varepsilon = \overline{v^2} \star \omega_\varepsilon$, where ω_ε is a standard mollifier. Then according to the DiPerna–Lions folklore lemma v^ε solves

$$v_t^\varepsilon + uv_x^\varepsilon = \frac{1}{2}w^\varepsilon - (v^\varepsilon)^2 + r^\varepsilon,$$

where $r^\varepsilon \to 0$ in $L^p(Q_T)$ for all $p \in [1, 3/2)$. This equation can now be multiplied by $(f_R^\pm)'(v^\varepsilon)$ to yield

$$\left(f_R^\pm(v^\varepsilon)\right)_t + u\left(f_R^\pm(v^\varepsilon)\right)_x = \left(f_R^\pm\right)'(v^\varepsilon)\,\frac{1}{2}w^\varepsilon - \left(f_R^\pm\right)'(v^\varepsilon)\,(v^\varepsilon)^2 + \left(f_R^\pm\right)'(v^\varepsilon)\,r^\varepsilon.$$

Now since $\left|(f_R^\pm)'(v^\varepsilon)\right| \le R$, we infer that

$$\left(f_R^\pm\right)'(v^\varepsilon)\,r^\varepsilon \to 0 \text{ in } L^1(Q_T) \text{ as } \varepsilon \to 0.$$

Therefore, (6.24) will follow by first using $u_x = v$ and then sending ε to zero. The final claim of the lemma is obvious since $\left|(f_R^\pm)'\right|$ is bounded by R. □

Let $\overline{f^\pm(v)}$ denote the weak limits of $\{f^\pm(v_{\Delta x})\}_{\Delta x>0}$. Hence, up to extracting subsequences if necessary, as $\Delta x \to 0$

$$f^\pm(v_{\Delta x}) \rightharpoonup \overline{f^\pm(v)} \text{ in } L^p(Q_T) \text{ for any } p \in [1,3/2),$$

and $f^\pm(v) \le \overline{f^\pm(v)}$ a.e. in Q_T. Similarly let $\overline{f_R^\pm(v)}$ denote the weak limits of $\left\{f_R^\pm(v_{\Delta x})\right\}_{\Delta x>0}$. Hence, up to extracting subsequences if necessary, as $\Delta x \to 0$

$$f_R^\pm(v_{\Delta x}) \rightharpoonup \overline{f_R^\pm(v)} \text{ in } L^p(Q_T) \text{ for any } p \in [1,3)$$
$$\text{and } f_R^\pm(v_{\Delta x}) \overset{\star}{\rightharpoonup} \overline{f_R^\pm(v)} \text{ in } L^\infty([0,T]; L^2(\mathbb{R}^+)),$$

where the *same* extracted subsequences work for any $R>0$, Moreover, there holds the inequality $f_R^\pm(v) \le \overline{f_R^\pm(v)}$ a.e. in Q_T.

In the next next lemma we derive transport equations for $\overline{f_R^\pm(v)}$. Below we denote by

$$\overline{vf_R^\pm(v) - \frac{1}{2}v^2(f_R^\pm)'(v)}$$

the weak limits in $L^p(Q_T)$ for any $p \in [1,3/2)$ of the sequences

$$\left\{v_{\Delta x} f_R^\pm(v_{\Delta x}) - \frac{1}{2}v_{\Delta x}^2 (f_R^\pm)'(v_{\Delta x})\right\}_{\Delta x>0}.$$

For later use, let us collect some useful formulas.

Remark 6.6. For each $v \in \mathbb{R}$, the following formulas hold:

$$\begin{aligned}
f_R(v) &= \frac{1}{2}v^2 - \frac{1}{2}(R - |v|)^2 \mathbf{1}_{(-\infty,-R)\cup(R,\infty)}(v), \\
f_R'(v) &= v + (R - |v|) \operatorname{sign} v \mathbf{1}_{(-\infty,-R)\cup(R,\infty)}(v), \\
f_R^+(v) &= \frac{1}{2}(v_+)^2 - \frac{1}{2}(R - v)^2 \mathbf{1}_{(R,\infty)}(v), \\
(f_R^+)'(v) &= v_+ + (R - v)\mathbf{1}_{(R,\infty)}(v), \\
f_R^-(v) &= \frac{1}{2}(v_-)^2 - \frac{1}{2}(R + v)^2 \mathbf{1}_{(-\infty,-R)}(v), \\
(f_R^-)'(v) &= v_- - (R + v)\mathbf{1}_{(-\infty,-R)}(v).
\end{aligned}$$

Introducing the notation $v_- = (0 \wedge v)$ and $v_+ = (0 \vee v)$ for $v \in \mathbb{R}$, the following formulas are obvious:

$$\left.\begin{aligned}
v &= v_+ + v_- = \overline{v_+} + \overline{v_-}, \\
v^2 &= (v_+)^2 + (v_-)^2, \\
\overline{v^2} &= \overline{(v_+)^2} + \overline{(v_-)^2},
\end{aligned}\right\} \qquad \text{a.e. on } Q_T.$$

Lemma 6.7. *For any $R > 0$, the equations*

$$\left(\overline{f_R^\pm(v)}(v)\right)_t + \left(u\overline{f_R^\pm(v)}(v)\right)_x = \overline{v f_R^\pm(v) - \frac{1}{2}v^2 (f_R^\pm)'(v)}, \qquad u_x = v, \tag{6.28}$$

hold in the sense of distributions on Q_T and

$$\lim_{t\to 0} \int_{\mathbb{R}^+} \left[\overline{f_R^\pm(v)}(x,t) - f_R^\pm(v_0(x))\right] dx = 0. \tag{6.29}$$

Proof. Similar to the derivation of (6.5), we can prove that a conservative version of the scheme (6.8) for any twice differentiable function $f(v_j)$ reads

$$\begin{aligned}
&\frac{d}{dt} f(v_j) + D_- \left(u_j f(v_j)\right) + I_{\Delta x, j}(f) \\
&\quad = v_j f(v_j) - \frac{1}{2}(v_j)^2 f'(v_j) - \Delta x\, v_j D_- f(v_j) - \Delta x\, D_- \left((u_{j+1} \wedge 0)\, D_+ f(v_j)\right),
\end{aligned} \tag{6.30}$$

where the numerical dissipation term $I_{\Delta x,j}(f)$ is defined in (6.9). Choosing $f = f_R^\pm$ in (6.30) and using the convexity of $f_R^\pm$, it follows that

$$\begin{aligned}
\frac{d}{dt} f_R^\pm(v_j) + D_- \left(u_j f_R^\pm(v_j)\right) &\le v_j f_R^\pm(v_j) - \frac{1}{2}(v_j)^2 (f_R^\pm)'(v_j) \\
&\quad - \Delta x\, v_j D_- f_R^\pm(v_j) - \Delta x\, D_- \left((u_{j+1} \wedge 0)\, D_+ f_R^\pm(v_j)\right).
\end{aligned} \tag{6.31}$$

When we send $\Delta x \to 0$ in (6.31) we can proceed as in the proof of Lemma 6.5, since

$$\left|f_R^\pm(v_j)\right| \le R\,|v_j|, \qquad \left|D_+ f_R^\pm(v_j)\right| \le R\,|D_+ v_j|.$$

This concludes the proof of (6.28).

Next we prove (6.29). By Lemma 6.5, and specifically (6.21), it is sufficient to establish

$$\lim_{t\to 0} \int_{\mathbb{R}^+} \left[\overline{f_R^\pm(v)}(x,t) - f_R^\pm(v(x,t))\right] dx = 0. \tag{6.32}$$

Then observe that

$$\overline{f_R(v)} - f_R(v) = \frac{1}{2}\left(\overline{v^2} - v^2\right) - \left(\overline{\frac{1}{2}v^2 - f_R(v)} - \left[\frac{1}{2}v^2 - f_R(v)\right]\right).$$

Since f_R and $\frac{1}{2}v^2 - f_R(v)$ are convex functions,

$$0 \le \overline{f_R(v)} - f_R(v) \le \frac{1}{2}\big(\overline{v^2} - v^2\big),$$

which, combined with (6.22) and (6.23), yields

$$\lim_{t\to 0}\int_{\mathbb{R}^+}\left[\overline{f_R(v)}(x,t) - f_R(v(x,t))\right]dx = 0.$$

Since $\overline{f_R^{\pm}(v)} - f_R^{\pm}(v) \le \overline{f_R(v)} - f_R(v)$, we conclude that (6.32) holds. $\square$

Remark 6.8. Observe that because of the dissipation in our numerical scheme, we cannot claim any continuity of $[0,T] \ni t \mapsto \overline{f_R^{\pm}(v)}(\,\cdot\,,t)$ as an object taking values in some Lebesgue space, not even when the Lebesgue space is equipped with the weak topology. However, it possesses a right-continuity property that can be used to make sense to the initial data; cf. (6.29).

The purpose of the next three lemmas is to deduce that $\overline{v^2} = v^2$ a.e., which will imply the desired strong convergence. Since we do not have a lower bound on $v_{\Delta x}$, we decompose into positive and negative parts, and use truncations of the negative part. The main step is to derive transport equations for the defect measures $\overline{f^+(v)} - f^+(v)$ and $\overline{f_R^-(v)} - f_R^-(v)$.

Lemma 6.9. *For a.e. $t \in (0,T)$, there holds*

$$\int_{\mathbb{R}^+}\left[\overline{f^+(v)}(x,t) - f^+(v(x,t))\right]dx \le 0. \tag{6.33}$$

Proof. From Lemma 6.5, and equations (6.24) and (6.20), we deduce for each $R > 0$ the transport inequality

$$\begin{aligned}\left(\overline{f_R^+(v)} - f_R^+(v)\right)_t &+ \left(u\left[\overline{f_R^+(v)} - f_R^+(v)\right]\right)_x \\ &\le \left[\overline{v f_R^+(v)} - v f_R^+(v)\right] - \frac{1}{2}\left[\overline{v^2 (f_R^+)'(v)} - v^2 (f_R^+)'(v)\right] \\ &\quad - \frac{1}{2}\left(\overline{v^2} - v^2\right)(f_R^+)'(v),\end{aligned}\tag{6.34}$$

which holds in the sense of distributions on Q_T. As f_R^+ is increasing,

$$-\frac{1}{2}(\overline{v^2} - v^2)(f_R^+)'(v) \le 0. \tag{6.35}$$

Moreover, for each $v \in \mathbb{R}$ we have the identity

$$v f_R^+(v) - \frac{1}{2}v^2 (f_R^+)'(v) = -\frac{R}{2}v(R-v)\mathbf{1}_{(R,\infty)}(v),$$

which implies

$$\overline{v f_R^+(v)} - \frac{1}{2}\overline{v^2 (f_R^+)'(v)} = -\frac{R}{2}\overline{v(R-v)\mathbf{1}_{(R,\infty)}(v)},$$

and hence

$$vf_R^+(v) - \frac{1}{2}v^2(f_R^+)'(v) = \overline{vf_R^+(v)} - \frac{1}{2}\overline{v^2(f_R^+)'(v)} = 0, \tag{6.36}$$

in $\Omega_{R,T} = \mathbb{R}^+ \times \left(\frac{2}{R}, T\right)$ (i.e., whenever $R > \frac{2}{T}$). In view of (6.34)–(6.36), the following transport inequality holds in the sense of distributions on $\Omega_{R,T}$:

$$\left(\overline{f^+(v)} - f^+(v)\right)_t + \left(u\left[\overline{f^+(v)} - f^+(v)\right]\right)_x \leq 0, \tag{6.37}$$

for $t > 2/R$. Along the same lines as in the proof of Lemma 3.7, we conclude from (6.37) that

$$\begin{aligned}\int_{\mathbb{R}^+}\left[\overline{f^+(v)}(x,t) - f^+(v(x,t))\right]dx \\ \leq \int_{\mathbb{R}^+}\left[\overline{f_R^+(v)}(x,\tfrac{2}{R}) - f_R^+\big(v(x,\tfrac{2}{R})\big)\right]dx,\end{aligned} \tag{6.38}$$

for a.e. $t > \frac{2}{R}$. Now, by appropriately sending $R \to \infty$ in (6.38) and using (6.29) or (6.32), we obtain the desired result (6.33). □

Lemma 6.10. *Fix any $R > 0$. For a.e. $t \in (0,T)$,*

$$\begin{aligned}\int_{\mathbb{R}^+}\left[\overline{f_R^-(v)}(x,t) - f_R^-(v(x,t))\right]dx \leq{}& \frac{R^2}{2}\int_0^t\int_{\mathbb{R}^+}\overline{(R+v)\mathbf{1}_{(-\infty,-R)}(v)}\,dx\,ds \\ &- \frac{R^2}{2}\int_0^t\int_{\mathbb{R}^+}(R+v)\mathbf{1}_{(-\infty,-R)}(v)\,dx\,ds \\ &+ R\int_0^t\int_{\mathbb{R}^+}\left[\overline{f_R^-(v)} - f_R^-(v)\right]dx\,ds \\ &+ \frac{R}{2}\int_0^t\int_{\mathbb{R}}\left[\overline{(v_+)^2} - (v_+)^2\right]dx\,ds.\end{aligned} \tag{6.39}$$

Proof. From Lemma 6.5, and equations (6.24) and (6.20), we deduce the transport inequality

$$\begin{aligned}\left(\overline{f_R^-(v)} - f_R^-(v)\right)_t &+ \left(\gamma u\left[\overline{f_R^-(v)} - f_R^-(v)\right]\right)_x \\ &\leq \left[\overline{vf_R^-(v)} - vf_R^-(v)\right] - \frac{1}{2}\left[\overline{v^2(f_R^-)'(v)} - v^2(f_R^-)'(v)\right] \\ &\quad - \frac{1}{2}\left(\overline{v^2} - v^2\right)(f_R^-)'(v),\end{aligned} \tag{6.40}$$

which holds in the sense of distributions on Q_T. Since $-R \leq (f_R^-)' \leq 0$,

$$-\frac{1}{2}\left(\overline{v^2} - v^2\right)(f_R^-)'(v) \leq \frac{R}{2}\left(\overline{v^2} - v^2\right). \tag{6.41}$$

One can easily check that

$$\begin{aligned}vf_R^-(v) - \frac{1}{2}v^2(f_R^-)'(v) &= -\frac{R}{2}(R+v)\mathbf{1}_{(-\infty,-R)}(v), \\ \overline{vf_R^-(v)} - \frac{1}{2}\overline{v^2(f_R^-)'(v)} &= -\frac{R}{2}\overline{v(R+v)\mathbf{1}_{(-\infty,-R)}(v)}.\end{aligned} \tag{6.42}$$

Inserting (6.41) and (6.42) into (6.40) yields the transport inequality

$$\begin{aligned}&\left(\overline{f_R^-(v)} - f_R^-(v)\right)_t + \left(u\left[\overline{f_R^-(v)} - f_R^-(v)\right]\right)_x \\ &\quad\le \frac{R}{2}v(R+v)\mathbf{1}_{(-\infty,-R)}(v) - \frac{R}{2}\overline{v(R+v)\mathbf{1}_{(-\infty,-R)}(v)} + \frac{R}{2}\left(\overline{v^2} - v^2\right),\end{aligned}$$

which holds in the sense of distributions on Q_T. As in proof of Lemma 3.7, we conclude from this that for a.e. $t \in (0,T)$ the inequality

$$\begin{aligned}\int_{\mathbb{R}^+} &\left[\overline{f_R^-(v)}(x,t) - f_R^-(v(x,t))\right] dx \\ &\le \frac{R}{2}\int_0^t\int_{\mathbb{R}^+} v(R+v)\mathbf{1}_{(-\infty,-R)}(v)\,dx\,ds \\ &\quad - \frac{R}{2}\int_0^t\int_{\mathbb{R}^+} \overline{v(R+v)\mathbf{1}_{(-\infty,-R)}(v)}\,dx\,ds \\ &\quad + \frac{R}{2}\int_0^t\int_{\mathbb{R}^+}\left[\overline{v^2} - v^2\right] dx\,ds\end{aligned} \tag{6.43}$$

holds. One can check that

$$\begin{aligned}\overline{f_R^-(v)} - f_R^-(v) &= \frac{1}{2}\overline{(v_-)^2} - \frac{1}{2}(v_-)^2 \\ &\quad + \frac{1}{2}(R+v)^2\mathbf{1}_{(-\infty,-R)}(v) - \frac{1}{2}\overline{(R+v)^2\mathbf{1}_{(-\infty,-R)}(v)}.\end{aligned}$$

Hence, by (6.43),

$$\begin{aligned}\int_{\mathbb{R}^+}\left[\overline{f_R^-(v)}(x,t) - f_R^-(v(x,t))\right] dx &\le -\frac{R}{2}\int_0^t\int_{\mathbb{R}^+}\overline{v(R+v)\mathbf{1}_{(-\infty,-R)}(v)}\,dx\,ds \\ &\quad + \frac{R}{2}\int_0^t\int_{\mathbb{R}^+} v(R+v)\mathbf{1}_{(-\infty,-R)}(v)\,dx\,ds \\ &\quad + R\int_0^t\int_{\mathbb{R}^+}\left[\overline{f_R^-(v)} - f_R^-(v)\right] dx\,ds \\ &\quad - \frac{R}{2}\int_0^t\int_{\mathbb{R}^+}(R+v)^2\mathbf{1}_{(-\infty,-R)}(v)\,dx\,ds \\ &\quad + \frac{R}{2}\int_0^t\int_{\mathbb{R}^+}\overline{(R+v)^2\mathbf{1}_{(-\infty,-R)}(v)}\,dx\,dt \\ &\quad + \frac{R}{2}\int_0^t\int_{\mathbb{R}^+}\left[\overline{(v_+)^2} - (v_+)^2\right] dx\,ds.\end{aligned}$$

Finally, applying the identity $\frac{R}{2}(R+v)^2 - \frac{R}{2}v(R+v) = \frac{R^2}{2}(R+v)$ twice yields (6.39). □

Lemma 6.11. *There holds the equality*

$$\overline{v^2} = v^2 \text{ a.e. in } Q_T. \tag{6.44}$$

Proof. Adding (6.33) and (6.39) gives for a.e. $t \in (0,T)$

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^+}\Big[(\overline{(v_+)^2}-(v_+)^2)+(\overline{f_R^-(v)}-f_R^-(v))\Big]\,dx \\
&\qquad \le \frac{R^2}{2}\int_0^t\int_{\mathbb{R}^+}\overline{(R+v)\mathbf{1}_{(-\infty,-R)}(v)}\,dx\,ds \\
&\qquad\quad - \frac{R^2}{2}\int_0^t\int_{\mathbb{R}^+}(R+v)\mathbf{1}_{(-\infty,-R)}(v)\,dx\,ds \\
&\qquad\quad + R\int_0^t\int_{\mathbb{R}^+}\Big[\overline{f_R^-(v)}-f_R^-(v)\Big]\,dx\,ds \\
&\qquad\quad + \frac{R}{2}\int_0^t\int_{\mathbb{R}^+}\Big[\overline{(v_+)^2}-(v_+)^2\Big]\,dx\,ds.
\end{aligned}\tag{6.45}$$

By the formulas

$$\begin{aligned}
v_+ + (f_R^-)'(v) &= v-(R+v)\mathbf{1}_{(-\infty,-R)}(v),\\
\overline{v_+} + \overline{(f_R^-)'(v)} &= v-\overline{(R+v)\chi_{(-\infty,-R)}(v)}
\end{aligned}$$

and the convexity of the map $\mathbb{R}\ni v\mapsto v_+ + (f_R^-)'(v)$, we infer

$$\begin{aligned}
0 &\le [\overline{v_+}-v_+] + \Big[\overline{(f_R^-)'(v)}-(f_R^-)'(v)\Big]\\
&= (R+v)\mathbf{1}_{(-\infty,-R)}(v)-\overline{(R+v)\mathbf{1}_{(-\infty,-R)}(v)}.
\end{aligned}$$

Since $\mathbb{R}\ni v\mapsto (R+v)\mathbf{1}_{(-\infty,-R)}(v)$ is concave,

$$\overline{(R+v)\mathbf{1}_{(-\infty,-R)}(v)}-(R+v)\mathbf{1}_{(-\infty,-R)}(v)\le 0 \quad \text{a.e. in } Q_T.$$

Inserting this into (6.45) yields for a.e. $t\in(0,T)$

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^+}\Big[(\frac{1}{2}\overline{(v_+)^2}-\frac{1}{2}(v_+)^2)+(\overline{f_R^-(v)}-f_R^-(v))\Big]\,dx\\
&\le R\int_0^t\int_{\mathbb{R}^+}\Big[(\frac{1}{2}\overline{(v_+)^2}-\frac{1}{2}(v_+)^2)+(\overline{f_R^-(v)}-f_R^-(v))\Big]\,dx\,ds,
\end{aligned}$$

so that by Gronwall's inequality we conclude that

$$\int_{\mathbb{R}^+}\Big[(\frac{1}{2}\overline{(v_+)^2}-\frac{1}{2}(v_+)^2)+(\overline{f_R^-(v)}-f_R^-(v))\Big]\,dx=0 \quad \text{for a.e. } t\in Q_T.$$

By Fatou's lemma we can send $R\to\infty$, with the result that

$$\int_{\mathbb{R}^+}\Big[\overline{v^2}(x,t)-(v(x,t))^2\Big]\,dx=0 \quad \text{for a.e. } t\in(0,T).$$

This concludes the proof of (6.44). □

Let us summarize our findings in the main convergence theorem.

Theorem 6.12. *Let v_0 be a function satisfying* (6.1). *Define the semi-discrete finite difference approximation $(v_{\Delta x},u_{\Delta x})$ for Δx positive using* (6.4), (6.3), *and* (6.2). *Then $\{(v_{\Delta x},u_{\Delta x})\}_{\Delta x>0}$ converges to a dissipative solution (v,u) of* (1.5) *in the sense of Definition* 1.2. *More precisely, as $\Delta x\to 0$*

$$\|u_{\Delta x}-u\|_{L^\infty(Q_T)}\to 0, \qquad \|v_{\Delta x}-v\|_{L^p(Q_T)}\to 0 \quad \textit{for any } p\in[1,3).$$

Proof. Equipped with Lemmas 6.1, 6.2, 6.3, 6.4, 6.5, and in particular 6.11, the proof is similar to that of Theorem 3.8. □

Remark 6.13. In addition to the properties stated in Theorem 3.8, the proof also shows that the limits u, v possess the following properties:

$$\begin{aligned}
&u \in W^{1,p}(\overline{Q_T}) \text{ for all } p \in [1,3),\\
&v \in L^\infty((0,T); L^2(\mathbb{R}^+)) \text{ and } v \in L^p(Q_T) \text{ for all } p \in [1,3),\\
&v \in C([0,T]; L^p(\mathbb{R}^+)) \text{ for all } p \in [1,2),\\
&v \in C_+([0,T]; L^2(\mathbb{R}^+)).
\end{aligned}$$

7. Numerical examples

In order to test our schemes in practice, we compared them with two other schemes, the first order Engquist–Osher scheme proposed in [6] and a central scheme which is an adaptation of schemes presented in [10]. We have no convergence proofs for these schemes. The Engquist–Osher scheme is a scheme that works directly with the u variable, that is, the scheme is based on discretizing (1.2), and is given by

$$D_+^t u_j^n + D_- f^{\mathrm{EO}}\left(u_{j+1}^n, u_j^n\right) = \frac{1}{2\Delta x}\sum_{i=0}^{j}\left(u_i^n - u_{i-1}^n\right)^2, \tag{7.1}$$

where we have set $u_{-1}^n = u_0^n = 0$, and f^{EO} denotes the Engquist–Osher flux

$$f^{\mathrm{EO}}(u_1, u_2) = \frac{1}{2}\left[((u_1 \wedge 0))^2 + ((u_2 \vee 0))^2\right].$$

Of course, if $v \geq 0$, then $f^{\mathrm{EO}}(u_1, u_2) = u_2^2/2$. To calculate the v variable, we set

$$v_j^n = D_- u_j^n, \quad j = 0, 1, \dots.$$

The central scheme we use is formally second order and is defined as

$$\begin{aligned}
\tilde{u}_j^n &= \mathrm{MM}_\theta\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right),\\
s_j^n &= s_{j-1}^n + \frac{1}{4\Delta x}\left[\left(\tilde{u}_{j-1}^n\right)^2 + \left(\tilde{u}_j^n\right)^2\right], \quad j > 0, \quad s_0^n = 0,\\
u_j^{n+1/2} &= u_j^n - \frac{\lambda}{2}u_j^n \tilde{u}_j^n - \frac{\Delta t}{2} s_j^n,
\end{aligned} \tag{7.2}$$

where MM_θ denotes the limiter

$$\mathrm{MM}_\theta(a, b, c) = \mathrm{MM}\left(c - b, \frac{c-a}{2}, b - a\right)$$

with

$$\mathrm{MM}(a_1, a_2, \dots) = \begin{cases} \min_j\{a_j\}, & \text{if } a_j > 0 \text{ for all } j,\\ \max_j\{a_j\} & \text{if } a_j < 0 \text{ for all } j,\\ 0 & \text{otherwise.}\end{cases}$$

Next let

$$\begin{aligned}
\tilde{u}_j^{n+1/2} &= \mathrm{MM}_\theta\left(u_{j-1}^{n+1/2}, u_j^{n+1/2}, u_{j+1}^{n+1/2}\right),\\
s_j^{n+1/2} &= s_{j-1}^{n+1/2} + \frac{1}{4\Delta x}\left[\left(\tilde{u}_{j-1}^{n+1/2}\right)^2 + \left(\tilde{u}_j^{n+1/2}\right)^2\right], \quad j > 0, \quad s_0^{n+1/2} = 0,
\end{aligned}$$

and set

$$\Delta u_j = \frac{1}{2}\left(u^n_{j+1} - u^n_{j-1}\right) - \frac{1}{8}\left(\tilde{u}^n_{j-1} - 2\tilde{u}^n_j + \tilde{u}^n_{j+1}\right)$$
$$- \frac{\lambda}{2}\left(\left(u^{n+1/2}_{j-1}\right)^2 - 2\left(u^{n+1/2}_j\right)^2 + \left(u^{n+1/2}_{j+1}\right)^2\right) + \Delta t\left(s^{n+1/2}_{j+1} - s^{n+1/2}_{j-1}\right),$$
$$\tilde{u}_{j+1/2} = \mathrm{MM}\left(\Delta u_j, \Delta u_{j+1}\right), \quad j = 0, 1, 2, \dots.$$

Finally we can define u^{n+1}_j by

$$(7.3)\quad u^{n+1}_j = \frac{1}{4}\left(u^n_{j-1} + 2u^n_j + u^n_{j+1}\right) - \frac{1}{16}\left(\tilde{u}^n_{j+1} - \tilde{u}^n_{j-1}\right)$$
$$- \frac{\lambda}{4}\left(\left(u^{n+1/2}_{j+1}\right)^2 - \left(u^{n+1/2}_{j-1}\right)^2\right) - \frac{1}{8}\left(\tilde{u}_{j+1/2} - \tilde{u}_{j-1/2}\right) + \Delta t s^{n+1/2}_j.$$

For completeness we define

$$(7.4)\qquad v^{n+1}_j = \frac{1}{\Delta x}\tilde{u}^n_j.$$

As a test case we consider the problem with the exact solution given by

$$(7.5)\qquad v(x,t) = \begin{cases} \frac{2}{t+1} & \text{for } 0 \le x \le (t+1)^2, \\ 0 & \text{otherwise,} \end{cases}$$

where $t \ge 0$. We use the initial value $v(x,0)$ and calculate the approximations at $t = 1$.

We have calculated the approximations for x in the interval $[0,5]$, where necessary, we have defined v^n_{-1} by linear interpolation, and set $u^n_0 = 0$. For the semi-discrete scheme we used a standard fourth order Runge–Kutta scheme to integrate in time. Figure 1 shows the approximations to $v(x,1)$ calculated by the various schemes with $\Delta x = 5/64$. In each figure the exact solution is indicated by a broken line. At this level of discretization, it seems that the explicit scheme performs "best". However, we tested the convergence of all the schemes, and this produced Table 1, which shows the L^2 errors, or more precisely

$$100 \cdot \frac{\sum_j \left(v(x_j,1) - v^N_j\right)^2}{\sum_j v(x_j,1)^2},$$

where $t_N = 1$. We use a discretization $\Delta x = 5 \cdot 2^{-k}$, where $k = 4, 5, \dots, 11$. Indeed it seems that the explicit scheme produces the smallest errors, but both the

TABLE 1. The relative L^2 errors in the v variable for the various schemes.

k	Semi	Implicit	Explicit	EO	Central
4	29.3	30.4	41.6	42.2	37.0
5	22.9	28.0	22.4	27.8	22.3
6	20.6	26.2	9.5	21.0	18.5
7	16.8	21.4	8.4	15.8	14.1
8	13.8	17.6	8.6	12.4	11.4
9	11.8	15.1	5.7	9.9	9.1
10	10.3	12.9	4.7	8.3	7.8
11	8.6	10.8	3.9	6.7	6.2

Engquist–Osher and the central scheme work with the u variable, and then use a first-order differentiation to find v. If we measure the L^∞ error in the u variable instead, i.e.,

$$100 \cdot \frac{\max_j \left|u(x_j,1) - u_j^N\right|}{\max_j |u(x_j,1)|},$$

we get Table 2. From Table 2 we see that for the u variable the results produced by the explicit scheme and the central scheme are comparable, a somewhat surprising result.

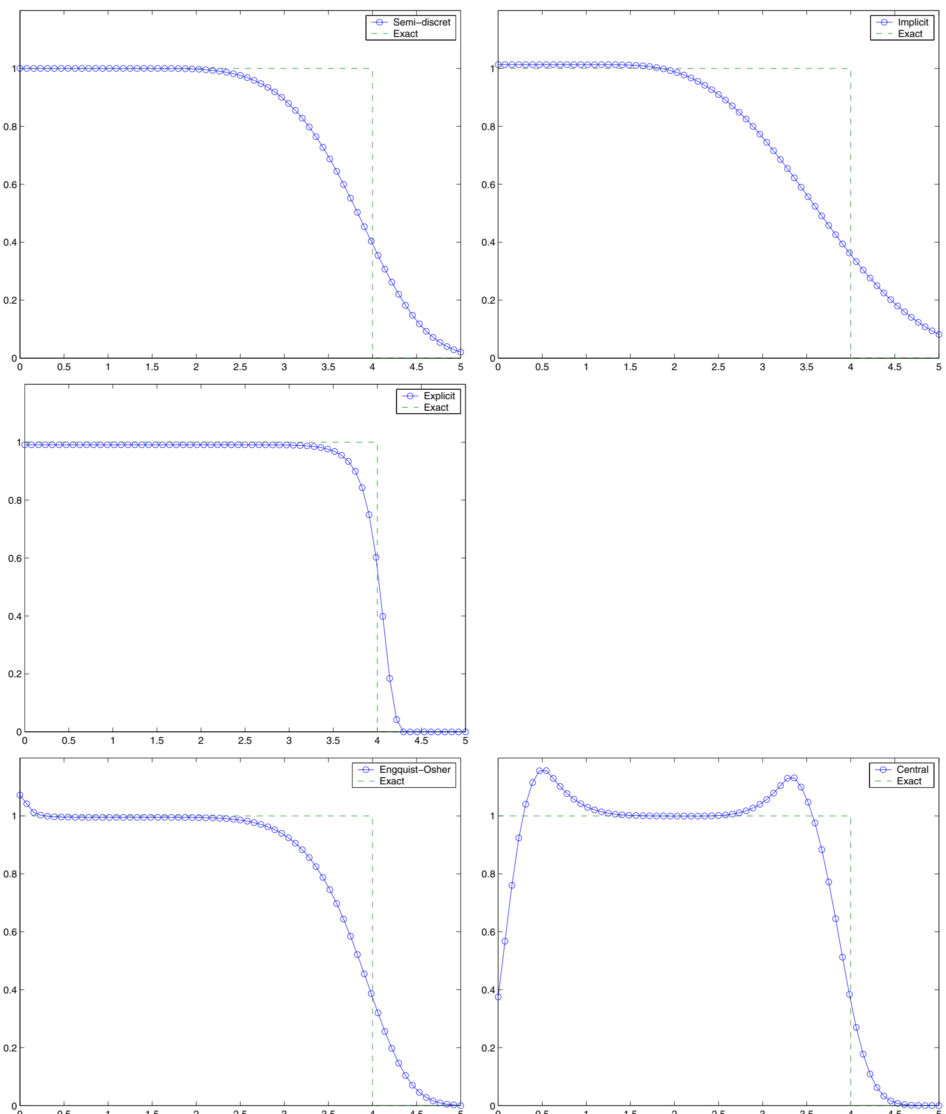

FIGURE 1. The approximations to (7.5) for $t = 1$.

TABLE 2. The relative L^∞ errors in the u variable for the various schemes.

k	Semi	Implicit	Explicit	EO	Central
4	6.5	11.3	17.5	18.7	15.0
5	7.4	12.3	6.8	6.7	5.4
6	8.1	12.1	2.1	3.0	2.0
7	5.5	8.6	1.1	2.1	1.3
8	3.8	6.1	0.8	1.5	1.2
9	3.0	4.7	0.5	1.4	0.5
10	2.3	3.5	0.5	1.2	0.3
11	1.6	2.5	0.3	0.8	0.2

If we solve (6.2) numerically by the forward Euler scheme, we get the following numerical scheme:

$$\begin{aligned} v_j^{n+1} &= v_j^n - \Delta t\left(\left(u_j^n \vee 0\right) D_- v_j^n + \left(u_{j+1}^n \wedge 0\right) D_- v_{j+1}^n\right) + \frac{\Delta t}{2}\left(v_j^n\right)^2 \\ u_j^{n+1} &= \Delta x \sum_{i=1}^{j-1} v_i^{n+1}, \end{aligned} \tag{7.6}$$

with the boundary condition $u_0^n = 0$. We call this the variable sign scheme. Note that this amounts to an explicit version of the scheme analyzed in Section 6, and we have *not* been able to show any convergence properties of the scheme defined by (7.6). Nevertheless, it seems to work well in practice. As a test example we used the exact solution defined by

$$v(x,t) = \frac{-2}{2-t}\mathbf{1}_{\{x<(2-t)^2\}}. \tag{7.7}$$

This solution is called a *negative kink-wave*. For $t > 2$ it formally continues as a positive kink wave. In this case the L^2 norm of v is constant, so that this is the conservative solution. We may however also continue the solution past $t = 2$ by setting $v(x,t) = 0$ for $t > 2$. This would then be the dissipative solution. We have tested the Engquist–Osher scheme, the second-order central scheme, and our scheme defined in Section 6 for this example. In all the computations we have used $\Delta x = 1 \cdot 10^{-9}$. In Figure 2 we show a contour plot of the computed $v(x,t)$ for $(x,t) \in [0,1.1] \times [0,3]$. We see that the Engquist–Osher scheme produces an approximation which does not seem close either to the conservative or to the dissipative solution. The central scheme and the variable sign scheme produce approximations that seem close to the dissipative solution.

It is also interesting to plot the L^2 norm of the approximate solutions as functions of time. We show this in Figure 3. Here we have plotted the L^2 norm of the three approximations as functions of t for $t \in [0,3]$. We see that the variable sign scheme is the only scheme that gives a nonincreasing L^2 norm in this case. Based on this experiment, we guess that of the three schemes considered, the variable sign scheme would be easiest to analyze, since the analysis in the case where the L^2 norm can increase is probably much more difficult.

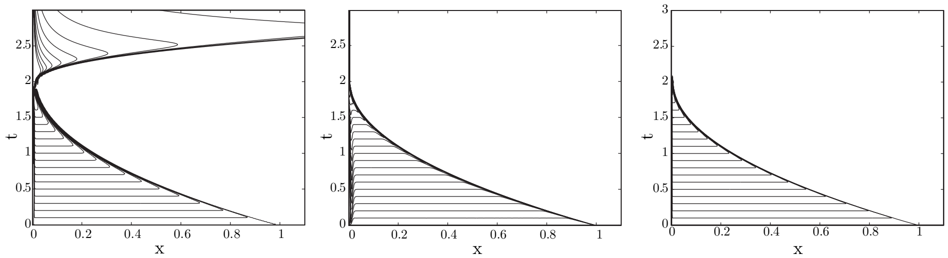

FIGURE 2. Contour plots of the approximations to (7.7) for $(x, t) \in [0, 1.1] \times [0, 3]$. Left: The Engquist–Osher scheme. Center: The central scheme. Right: The variable sign scheme.

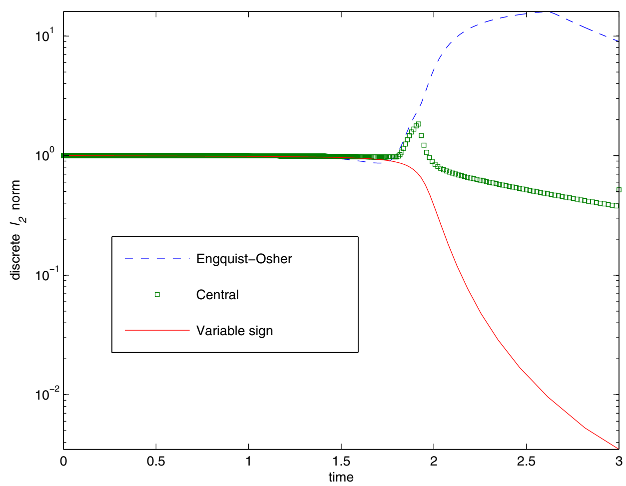

FIGURE 3. The L^2 norm of the approximate solution as a function of time.

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