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COMPUTATION OF THE p-PART OF THE IDEAL CLASS GROUP OF CERTAIN REAL ABELIAN FIELDS

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ABSTRACT. Under Greenberg's conjecture, we give an efficient method to compute the p-part of the ideal class group of certain real abelian fields by using cyclotomic units, Gauss sums and prime numbers. As numerical examples, we compute the p-part of the ideal class group of the maximal real subfield of $\mathbf{Q}(\sqrt{-f}, \zeta_{p^{n+1}})$ in the range $1 < f < 200$ and $5 \leq p < 100000$. In order to explain our method, we show an example whose ideal class group is not cyclic.

1. INTRODUCTION

Let K be a number field and p a prime number. Let K_∞ be the cyclotomic $\mathbf{Z}_p$-extension of K and K_n the subfield of K_∞ such that $[K_n : K] = p^n$. Further let A_n be the p-part of the ideal class group of K_n. Greenberg's conjecture claims that $\sharp A_n$ is bounded as $n \to \infty$ if K is totally real. We have not been able to find any counter-example to the conjecture. On the other hand, it has been verified for certain real abelian fields and some prime numbers by computer calculation (cf. [8, 16]).

In [11, 14], under Greenberg's conjecture, Kraft-Schoof and Ozaki gave a nice method to compute the p-part of the ideal class group of certain real abelian fields by using cyclotomic units. In the computation, we need to know whether a cyclotomic unit $c_n \in K_n$ is a p^{n+1}th power or not in K_n. As the degree of the minimal polynomial for c_n over $\mathbf{Q}$ gets larger, the computation of the minimal polynomial for $\sqrt[p^{n+1}]{c_n}$ becomes more difficult. In [15, 16], by using Gauss sums and prime numbers, we avoided the difficulty and gave an efficient method to compute the p-part of the ideal class number of certain real abelian fields. In this paper, combining them, we give an efficient method to compute the p-part of the ideal class group.

Following [15, 16], we give numerical examples of the p-part of the ideal class group of the maximal real subfield $K_{f,p}$ of $\mathbf{Q}(\sqrt{-f}, \zeta_p)$ in the range $1 < f < 200$ and $5 \leq p < 100000$. The first purpose of this computation is to verify Greenberg's conjecture for each case. In fact we verify the conjecture in the above range. Therefore we can make use of the method of [11, 14] to compute the structure

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of the p-part of the ideal class group. Let χ be the nontrivial Dirichlet character associated to $\mathbf{Q}(\sqrt{-f})$ and $\omega = \omega_p$ the Teichmüller character. Here we call $(p, \chi\omega^k)$ an exceptional pair if and only if $\chi\omega^k$ is even, $\chi\omega^k(p) \neq 1$, $\chi\omega^{1-k}(p) \neq 1$, and one of the following conditions is satisfied: $\nu_p(\chi\omega^k) > 0$, $v_p(L_p(1, \chi\omega^k)) > 1$, $v_p(L_p(0, \chi\omega^k)) > 1$, or $\tilde{\lambda}_p(\chi\omega^k) > 1$, where $\nu_p(\chi\omega^k)$ is the $\chi\omega^k$-part of the Iwasawa ν_p-invariant, v_p is the p-adic valuation such that $v_p(p) = 1$ and $\tilde{\lambda}_p(\chi\omega_k)$ is the degree of the Iwasawa polynomial for $\chi\omega^k$. The second purpose of the computation is to find exceptional pairs, as many as possible for large prime numbers in order to argue about their expected numbers (cf. [17, pp.158–159]). From our data, the actual numbers of exceptional pairs seem to be close to the expected numbers.

Following [1], we compute A_n for $f = 4 \cdot 14606$ and $p = 5$ (i.e., $K = K_{4\cdot 14606, 5}$):

$$\begin{cases} A_0 \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}, \\ A_n \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \text{ for } n \geq 1. \end{cases}$$

Since p splits in $\mathbf{Q}(\sqrt{-f})$, we need to modify some conditions in order to apply the criterion of [15]. We explain about the modification and difficulty in the following section.

2. A method of computation of A_n

Let F be an abelian field and p an odd prime number. For simplicity, we assume the following condition:

(C1) The exponent of $\mathrm{Gal}(F/\mathbf{Q})$ divides $p - 1$.

Let $K = F(\zeta_p)$ and $A_n = A_n(K)$ be the p-part of the ideal class group of $K_n = F(\zeta_{p^{n+1}})$. Let D_n be the subgroup of A_n consisting of classes which contain an ideal all of whose prime factors lie above p. Set $A'_n = A_n/D_n$. Let M_n be the maximal abelian extension of K_n unramified outside p, L_n the maximal unramified abelian extension of K_n, and L'_n the maximal unramified abelian extension of K_n in which every prime divisor above p splits completely. Set $X_n = \mathrm{Gal}(L_n/K_n)$ and $X'_n = \mathrm{Gal}(L'_n/K_n)$. By the class field theory, we have $A_n \simeq X_n$ and $A'_n \simeq X'_n$. Set $L_\infty = \bigcup L_n$, $L'_\infty = \bigcup L'_n$, $X_\infty = \mathrm{Gal}(L_\infty/K_\infty)$ and $X'_\infty = \mathrm{Gal}(L'_\infty/K_\infty)$.

Set $\Delta = \mathrm{Gal}(K_\infty/\mathbf{Q}_\infty) \simeq \mathrm{Gal}(K_0/\mathbf{Q})$. Let ψ be a Dirichlet character of Δ and $e_\psi = \frac{1}{\sharp\Delta}\sum_{\delta\in\Delta}\psi(\delta)\delta^{-1} \in \mathbf{Z}_p[\Delta]$. For a $\mathbf{Z}_p[\Delta]$-module A, we denote $e_\psi A$ by A^ψ. Let $\lambda_p(\psi)$, $\mu_p(\psi)$ and $\nu_p(\psi)$ (resp. $\lambda'_p(\psi)$, $\mu'_p(\psi)$ and $\nu'_p(\psi)$) be the Iwasawa invariants associated to A_n^ψ (resp. ${A'}_n^\psi$), i.e.,

$$\sharp A_n^\psi = p^{\lambda_p(\psi)n + \mu_p(\psi)p^n + \nu_p(\psi)} \quad (\text{resp. } \sharp {A'}_n^\psi = p^{\lambda'_p(\psi)n + \mu'_p(\psi)p^n + \nu'_p(\psi)})$$

for all sufficiently large integers n. By Ferrero-Washington's theorem in [4], we have $\mu_p(\psi) = \mu'_p(\psi) = 0$ for all p and ψ.

Assume that ψ is even. The Iwasawa polynomial $g_\psi(T) \in \mathbf{Z}_p[T]$ for the p-adic L-function is defined as follows. Let $L_p(s, \psi)$ be the p-adic L-function constructed in [12]. Let f_0 be the least common multiple of p and f_ψ the conductor of ψ. By [9, §6], there uniquely exists $G_\psi(T) \in \mathbf{Z}_p[[T]]$ satisfying $G_\psi((1+f_0)^{1-s} - 1) = L_p(s, \psi)$ for all $s \in \mathbf{Z}_p$ if $\psi \neq \psi^0$. By [4], it was proved that p does not divide $G_\psi(T)$. Therefore, by the p-adic Weierstrass preparation theorem, we can uniquely write $G_\psi(T) = g_\psi(T)u_\psi(T)$, where $g_\psi(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$ and $u_\psi(T)$ is an invertible element of $\mathbf{Z}_p[[T]]$. Similarly we can define $g^*_\psi(T) \in \mathbf{Z}_p[T]$

from $G_\psi^*(T) \in \mathbf{Z}_p[[T]]$ satisfying $G_\psi^*((1+f_0)^s-1) = L_p(s,\psi)$. Put $\tilde{\lambda}_p(\psi) = \deg g_\psi(T) = \deg g_\psi^*(T)$.

Let $\gamma \in \Gamma = \mathrm{Gal}(\bigcup \mathbf{Q}(\zeta_{f_n})/\mathbf{Q}(\zeta_{f_0})) \simeq \mathrm{Gal}(K_\infty/K_0)$ be a generator of Γ such that $\zeta_{f_n}^\gamma = \zeta_{f_n}^{1+f_0}$ for all $n \geq 0$ and $f_n = f_0 p^n$. As usual, we can identify the complete group ring $\mathbf{Z}_p[[\Gamma]]$ with the formal power series ring $\Lambda = \mathbf{Z}_p[[T]]$ by $\gamma = 1+T$. By this identification, we can consider a $\mathbf{Z}_p[[\Gamma]]$-module as a Λ-module. Set $\omega_n = (1+T)^{p^n}-1$ and $\nu_{m,n} = \omega_m/\omega_n$ for $m \geq n \geq 0$. For a finitely generated torsion Λ-module A, we define the Iwasawa polynomial $\mathrm{char}_\Lambda(A)$ to be the characteristic polynomial of the action T on $A \otimes \mathbf{Q}_p$ (cf. [17, §13]). By Mazur-Wiles' theorem in [13], $\mathrm{char}_\Lambda(X^{\psi^{-1}\omega}) = g_\psi^*(T)$.

Let $\mathfrak{p}$ be a prime ideal of K over p and $\mathfrak{p}_n$ the unique prime ideal of K_n over $\mathfrak{p}$. Denote by $K_{\mathfrak{p}_n}$ the completion of K_n at $\mathfrak{p}_n$, and by $\mathcal{U}_{\mathfrak{p}_n}$ the group of principal units of $K_{\mathfrak{p}_n}$. Put $\mathcal{V}_{\mathfrak{p}_n} = \bigcap_{m\geq n} N_{m,n}\mathcal{U}_{\mathfrak{p}_n}$, $N_{m,n}$ denoting the norm map from $K_{\mathfrak{p}_m}$ to $K_{\mathfrak{p}_n}$. We set

$$\mathcal{U}_n = \prod_{\mathfrak{p}|p} \mathcal{U}_{\mathfrak{p}_n} \text{ and } \mathcal{V}_n = \prod_{\mathfrak{p}|p} \mathcal{V}_{\mathfrak{p}_n},$$

where $\mathfrak{p}$ runs over all prime ideals of K over p.

Let E_n' be the group of units ε of K_n satisfying $\varepsilon \equiv 1 \bmod \mathfrak{p}_n$ for all $\mathfrak{p}_n|p$. Denote by C_n the subgroup of $K_n^\times$ generated by all the units

$$N_{\mathbf{Q}(\zeta_{f_n})/K_n}(1-\zeta_{f_n})^u, \quad u \in \mathbf{Z}[\mathrm{Gal}(K_n/\mathbf{Q})]^0,$$

where X^0 is the augmentation ideal of the group ring X. Denote, respectively, $\mathcal{E}_n$ and $\mathcal{C}_n$ as the closures of the images of E_n' and $C_n' = C_n \cap E_n'$ under the diagonal embedding $d_n : E_n' \to \mathcal{U}_n$.

From now on, we also assume the following condition:

$$\text{(C2)} \qquad \psi(p) \neq 1.$$

Set $\psi^* = \psi^{-1}\omega$ and $\omega_0^* = T - f_0$. Then we have the following facts (see [5, Theorem 1, 2]):

Fact 1.

$$\mathcal{U}_n^\psi = \mathcal{V}_n^\psi.$$

If $\psi^*(p) \neq 1$,

$$\begin{array}{ccc} \mathcal{U}_n^\psi & \simeq & \Lambda/(\omega_n) \\ \cup & & \cup \\ \mathcal{C}_n^\psi & \simeq & (g_\psi(T),\omega_n)/(\omega_n). \end{array}$$

If $\psi^*(p) = 1$,

$$\begin{array}{ccc} \mathcal{U}_n^\psi/\mathbb{T}_n & \simeq & \Lambda/(\omega_n) \\ \cup & & \cup \\ \mathcal{C}_n^\psi\mathbb{T}_n/\mathbb{T}_n & \simeq & (\tilde{g}_\psi(T),\omega_n)/(\omega_n), \end{array}$$

where $\mathbb{T}_n = \mathrm{Tor}_{\mathbf{Z}_p}\mathcal{U}_n^\psi \simeq \Lambda/(\omega_0^*,\omega_n)$ and $\tilde{g}_\psi(T) = g_\psi(T)/\omega_0^*$.

We also have the following fact on E_n' and $\mathcal{E}_n$, which follows from the Leopoldt conjecture for (K_n,p) (cf. [17, §5.5]):

Fact 2. The inclusion $d_n : E_n' \to \mathcal{E}_n$ induces an isomorphism

$$(E_n'/{E_n'}^{p^a})^\psi \simeq (\mathcal{E}_n/{\mathcal{E}_n}^{p^a})^\psi$$

for any $a \geq 0$. Therefore $\mathcal{E}_n^\psi$ has no nontrivial torsion element.

Our computation is based on the following theorem:

Theorem 1 (Kraft-Schoof, Ozaki)**.** *Assume* (C1) *and* (C2)*. Greenberg's conjecture holds for A_n^ψ if and only if there exists an integer n_0 such that $(\mathcal{E}_n/\mathcal{C}_n)^\psi$ is stabilized for $n \geq n_0$. Then*

$$A_n^\psi \simeq (\mathcal{E}_n/\mathcal{C}_n)^\psi$$

for all $n \geq n_0$.

We give an outline of a proof for convenience of the readers.

Proof. By (C1) and (C2), we obtain the following exact sequence for $m \geq n$ (cf. [15, §2]):

$$0 \to H^1(\Gamma_n, E_m)^\psi \to A_n^\psi \to (A_m^{\Gamma_n})^\psi \to H^2(\Gamma_n, E_m)^\psi \to 0,$$

where E_n is the group of units of K_n and $\Gamma_n = \Gamma^{p^n}$. If Greenberg's conjecture holds for A_n^ψ, we can take m and n ($m \geq n$) such that $N_{m,n} : A_m^\psi \simeq A_n^\psi$ and that $i_{n,m} : A_n^\psi \to (A_m^{\Gamma_n})^\psi$ is a zero map, where $i_{n,m}$ is the induced map by the natural inclusion $k_n \hookrightarrow k_m$ (see [6, Proposition 1]). Since we have $H^2(\Gamma_n, E_m)^\psi \simeq (E_n/N_{m,n}E_m)^\psi$,

$$0 \to A_m^\psi \to (E_n/N_{m,n}E_m)^\psi \to 0.$$

Further, we have $C_n = N_{m,n}C_m \subseteq N_{m,n}E_m$ and $\sharp A_m^\psi = \sharp((E_m/C_m)(p))^\psi = \sharp((E_n/C_n)(p))^\psi$ for $n \geq n_0$ by Mazur-Wiles' theorem, where $A(p)$ is the p-part of the finite abelian group A. Therefore we have $A_n^\psi \simeq A_m^\psi \simeq ((E_n/C_n)(p))^\psi \simeq ((E_m/C_m)(p))^\psi \simeq (\mathcal{E}_m/\mathcal{C}_m)^\psi$. □

Let $L_n(\psi^*)$ be the fixed subfield of L_n by $\bigoplus_{\chi \neq \psi^*} X_n^\chi$. In a similar way, we define $M_n(\psi^*)$, $L_\infty(\psi^*)$, etc. For an ideal $\mathfrak{L}$ of K_n, set $\sigma_{\mathfrak{L}}^{\psi^*} = \left(\frac{L_n(\psi^*)/K_n}{\mathfrak{L}}\right) \in (X_n/\bigoplus_{\chi \neq \psi^*} X_n^\chi) \simeq X_n^{\psi^*}$, where $(\frac{*}{*})$ is the Artin symbol. In order to calculate $(\mathcal{E}_n/\mathcal{C}_n)^\psi$, we use the following lemma:

Lemma 1. *For $k \leq n+1$, if $c_n \in C'_n$ satisfies*

$$\text{(A)} \qquad d_n(c_n) \in (\mathcal{U}_n^{p^k})^\psi \mathcal{C}_n^{p^k},$$

then $\sqrt[p^k]{c_n} \in L_n(\psi^*)$. Further assume that

$$\text{(B)} \qquad X_n^{\psi^*} \textit{ is generated by } \sigma_{\mathfrak{L}_i}^{\psi^*} \textit{ for } \mathfrak{L}_i \nmid p \textit{ and } 1 \leq i \leq r.$$

Then

$$c_n \in {E'_n}^{p^k} \textit{ if and only if } (c_n \bmod \mathfrak{L}_i)_{1 \leq i \leq r} \in \prod_{i=1}^{r} (\mathcal{O}_{K_n}/\mathfrak{L}_i)^{p^k}.$$

Proof. Since $\sqrt[p^k]{c_n} \in M_n(\psi^*)$, (A) implies the former assertion. By (B), $K_n(\sqrt[p^k]{c_n}) = K_n$ if and only if the splitting field of $\mathfrak{L}_i$ in $L_n(\psi^*)/K_n$ includes $K_n(\sqrt[p^k]{c_n})$ for every i, i.e., c_n is a p^kth power at $\mathfrak{L}_i$. Therefore we obtain the latter assertion. □

In [15], when $\psi^*(p) \neq 1$, we gave explicit conditions for (A) and (B) by using cyclotomic units, Gauss sums and prime numbers. When $\psi^*(p) = 1$, ω_0 divides $g_\psi^*(T)$. For $n = 0$, we can obtain full information from Gauss sums of a subfield of K_0 (cf. [1]). However, for $n \geq 1$, we cannot directly obtain full information on $A_n^{\psi^*}$ from Gauss sums (see [7, §4] and the last example of section 3). So we will replace the conditions (A) and (B) with ($\tilde{\text{A}}$) and ($\tilde{\text{B}}$) in Lemma 4.

Let us write the Kummer pairing:

$$X_\infty^{\psi^*} \times W_\infty^{\psi} \to \mu_{p^\infty} = \bigcup \langle \zeta_{p^n} \rangle,$$

where W_∞ is the subgroup of $K_\infty^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p$ which corresponds to X_∞ via Kummer theory. Let $\mathrm{Ker}_{X_\infty^{\psi^*}} \omega_0$ be the subgroup of $X_\infty^{\psi^*}$ consisting of all elements annihilated by ω_0. Set $\tilde{X}_\infty^{\psi^*} = X_\infty^{\psi^*}/\mathrm{Ker}_{X_\infty^{\psi^*}} \omega_0$. By Ferrero-Greenberg's theorem in [3], ω_0 does not divide $\tilde{g}_\chi^*(T) = g_\psi^*(T)/\omega_0$. Hence we have

$$\varphi : X_\infty^{\psi^*} \hookrightarrow \bigoplus_{i=1}^{s} \Lambda/(g_i^*(T)) \oplus \Lambda/(\omega_0),$$

where $\prod_{i=1}^s g_i^*(T) = \tilde{g}_\psi^*(T)$. Let π be the projection from $\bigoplus_{i=1}^s \Lambda/(g_i^*(T)) \oplus \Lambda/(\omega_0)$ to $\bigoplus_{i=1}^s \Lambda/(g_i^*(T))$. Then $\tilde{X}_\infty^{\psi^*} \simeq (\varphi(X_\infty^{\psi^*})(\Lambda/(\omega_0)))/(\Lambda/(\omega_0)) \simeq \pi(\varphi(X_\infty^{\psi^*}))$. Hence $\tilde{X}_\infty^{\psi^*}$ has no nontrivial finite submodule (cf. [10, Theorem 18]) and $\mathrm{char}_\Lambda(\tilde{X}_\infty^{\psi^*}) = \tilde{g}_\psi^*(T)$. Set $\tilde{W}_\infty^\psi = \omega_0^* W_\infty^\psi$. Then we have the following Kummer pairing:

$$\tilde{X}_\infty^{\psi^*} \times \tilde{W}_\infty^{\psi} \to \mu_{p^\infty}.$$

When $\psi^*(p) = 1$, we consider the above paring. In order to obtain elements in $\tilde{W}_\infty^\psi$ satisfying ($\tilde{\mathrm{A}}$) in Lemma 4, we use the following lemma:

Lemma 2. *For integers n and k, we set*

$$C_{n,k} = \langle [c_n] \in (C_n' {E_n'}^{p^k}/{E_n'}^{p^k})^\psi | d_n(c_n) \in (\mathcal{U}_n^{p^k} \mathbb{T}_n)^\psi \mathcal{C}_n^{p^k} \rangle.$$

Let a be the minimum integer such that $p^a \in (\omega_0^*, \tilde{g}_\psi(T))$. Then

$$p^a C_{n,k} \subseteq \omega_0^* C_{n,k} \subseteq C_{n,k}.$$

Assume that Greenberg's conjecture holds for A_n^ψ. Then

$$\bigcup_{n,k} C_{n,k} = \tilde{W}_\infty^\psi.$$

Proof. By Ferrero-Greenberg's theorem, ω_0^* does not divide $\tilde{g}_\psi(T)$. Therefore the minimum integer a exists. Write $p^a = \omega_0^* a(T) + \tilde{g}_\psi(T) b(T)$ for $a(T),\ b(T) \in \Lambda$. Then we have $d_n(c_n)^{p^a} = d_n(c_n)^{\omega_0^* a(T)} d_n(c_n)^{\tilde{g}_\psi(T) b(T)} \in \mathcal{C}_n^{\omega_0^*} \mathcal{C}_n^{p^k} \mathbb{T}_n$. Since $\mathcal{C}_n \cap \mathbb{T}_n = \{1\}$, we have $p^a[c_n] \in \omega_0^* C_{n,k}$. Since $\mathbb{T}_n = \mathbb{T}_{n+k}^{p^k}$, $d_{n+k}(c_n)$ is a p^kth power in $\mathcal{U}_{n+k}$ for $[c_n] \in C_{n,k}$. Therefore we have $C_{n,k} \subseteq W_\infty^\psi$ and $p^a C_{n,k} \subseteq \tilde{W}_\infty^\psi$. Let

$$C_{n,k}' = \langle [c_n]' \in (C_n'/{C_n'}^{p^k})^\psi | d_n(c_n) \in (\mathcal{U}_n^{p^k} \mathbb{T}_n)^\psi \mathcal{C}_n^{p^k} \rangle.$$

By Fact 1, for sufficiently large integers n, $C_{n,k}' \simeq (\mathbf{Z}/p^k\mathbf{Z})^{\tilde{\lambda}(\psi)}$. If Greenberg's conjecture holds, $\sharp(\mathcal{E}_n/\mathcal{C}_n)^\psi$ is bounded. Hence $C_{n,k}$ has a subgroup which is isomorphic to $(\mathbf{Z}/p^{k-k'}\mathbf{Z})^{\tilde{\lambda}(\psi)}$, where $k' \le k$ is a constant integer. Further the natural map $i_{n,m} : C_{n,k} \to C_{m,k+m-n}$ ($[c_n] \mapsto [{c_n}^{p^{m-n}}]$) is injective. Therefore $i_{n,m}(C_{n,k}) \subseteq p^a C_{m,k+m-n}$ for sufficiently large integers m. Since $\bigcup C_{n,k} \simeq (\mathbf{Q}_p/\mathbf{Z}_p)^{\tilde{\lambda}(\psi)} \simeq \tilde{W}_\infty^\psi$, we have the equality. ☐

Let $\tilde{L}_\infty(\psi^*)$ be the fixed subfield of $L_\infty(\psi^*)$ by $\mathrm{Ker}_{X_\infty^{\psi^*}}\omega_0$. Set $\tilde{L}_n(\psi^*) = L_n(\psi^*) \cap \tilde{L}_\infty(\psi^*)$ and $\tilde{X}_n^{\psi^*} = \mathrm{Gal}(\tilde{L}_n(\psi^*)/K_n) \simeq \tilde{A}_n^{\psi^*}$. Then we can write

$$\tilde{X}_n^{\psi^*} \simeq \tilde{X}_\infty^{\psi^*}/\nu_{n,0}\tilde{Y}_\infty^{\psi^*}$$

for a submodule $\tilde{Y}_\infty^{\psi^*}$ of $\tilde{X}_\infty^{\psi^*}$ (see [17, Lemma 13.15]).

Let $\mathfrak{m} = (p, T)$ be the maximal ideal of Λ. In order to find ideals $\mathfrak{L}_i$ satisfying ($\tilde{\mathrm{B}}$) in Lemma 4, we use the following lemma.

Lemma 3. *Let X be a finitely generated torsion Λ-module which has no nontrivial finite Λ-submodule. Assume that ω_0 does not divide $\mathrm{char}_\Lambda(X)$. Then for any Λ-submodules X' and Z of X such that $Z \subseteq \mathfrak{m}\omega_0 X$, $(\omega_0 X + Z)/Z = (\omega_0 X' + Z)/Z$ holds if and only if $X = X'$.*

Proof. Since $\mathfrak{m}(\omega_0 X) \supseteq Z$, we have $\omega_0 X = \omega_0 X'$ by Nakayama's lemma. For any element $x \in X$, there exists $x' \in X'$ such that $\omega_0 x = \omega_0 x'$. Since $\omega_0 : X \to X$ ($x \mapsto \omega_0 x$) is injective, we have $x = x'$ and $X = X'$. □

In order to calculate $(\mathcal{E}_n/\mathcal{C}_n)^\psi$ for $\psi^*(p) = 1$, we use the following lemma, which can be proved in a similar way to Lemma 1:

Lemma 4. *Assume that Greenberg's conjecture holds for A_n^ψ. For $k \leq n+1$, if $c_n \in C_n'$ satisfies*

$$(\tilde{\mathrm{A}}) \qquad d_n(c_n) \in (\mathcal{U}_n^{p^k}\mathbb{T}_n)^\psi \mathcal{C}_n^{p^k},$$

then $\sqrt[p^k]{c_n} \in \tilde{L}_{n'}(\psi^*)$ for some n'. Further assume that

$$(\tilde{\mathrm{B}}) \qquad \tilde{X}_{n'}^{\psi^*} \text{ is generated by } \tilde{\sigma}_{\mathfrak{L}_i}^{\psi^*} \text{ for } \mathfrak{L}_i \nmid p \text{ and } 1 \leq i \leq r.$$

Then

$$c_n \in {E_n'}^{p^k} \text{ if and only if } (c_n \bmod \mathfrak{L}_i)_{1\leq i \leq r} \in \prod_{i=1}^{r} (\mathcal{O}_{K_{n'}}/\mathfrak{L}_i)^{p^k}.$$

By Mazur-Wiles' theorem, we can obtain $\sharp A_n^{\psi^*} = \sharp X_n^{\psi^*} = \sharp(X_\infty^{\psi^*}/\nu_{n,0}Y_\infty^{\psi^*})$ from generalized Bernoulli numbers. However it is difficult to compute $\sharp\tilde{X}_n^{\psi^*}$ because it is difficult to determine $\tilde{Y}_\infty^{\psi^*}$ in $\tilde{X}_\infty^{\psi^*}$. Therefore, in general, we have a difficulty in checking ($\tilde{\mathrm{B}}$) by our method.

3. Numerical examples of ideal class groups

Let χ be an *odd* primitive quadratic Dirichlet character, f_χ the conductor of χ, and p an odd prime number. Set $F = F_\chi = \mathbf{Q}(\sqrt{-f_\chi})$ and $K = \mathbf{Q}(\sqrt{-f_\chi}, \zeta_p)$. Let k be an odd integer with $3 \leq k \leq p-2$. Then $\chi\omega^k$ is an even character. For a pair $(p, \chi\omega^k)$, we set the following condition:

$$(\mathrm{C}) \qquad \chi\omega^k(p) \neq 1 \text{ and } \chi\omega^{1-k}(p) \neq 1.$$

If $\chi\omega^k(p) \neq 1$, we have $\lambda_p(\chi\omega^k) = \lambda_p'(\chi\omega^k)$ and $\nu_p(\chi\omega^k) = \nu_p'(\chi\omega^k)$. In the range $1 < f_\chi < 200$, $5 \leq p < 100000$ and odd integers k with $3 \leq k \leq p-2$, there are 14085400622 pairs of $(p, \chi\omega^k)$ satisfying (C). Among them, 296975 pairs satisfy $\tilde{\lambda}_p(\chi\omega^k) = 1$, 43 pairs $\tilde{\lambda}_p(\chi\omega^k) = 2$, and two pairs $\tilde{\lambda}_p(\chi\omega^k) = 3$. By the method of [8], we verified Greenberg's conjecture, i.e., $\lambda_p(\chi\omega^k) = 0$ for each of them. Moreover, we checked $\nu_p(\chi\omega^k) \leq 2$ by the method of [16]. In the above range, 44 pairs do not satisfy (C). For these cases, we also checked that $\lambda_p(\chi\omega^k) =$

$\nu_p(\chi\omega^k) = 0$ by the method of [8]. Further, using the following lemma, we verified that $\lambda_p(\chi\omega) = \nu_p(\chi\omega) = 0$ for all f_χ and p in the above range.

Proposition 1. *If A_0^χ is trivial, then $A_n^{\chi\omega}$ is trivial for every $n \geq 0$.*

Proof. Assume that $A_n^{\chi\omega}$ is not trivial for some n. Then we have $\deg g^*_{\chi\omega}(T) = \deg g_{\chi\omega}(T) \geq 1$. Hence, if $\chi(p) \neq 1$, we have $v_p(\sharp A_0^\chi) = v_p(g^*_{\chi\omega}(0)) \geq 1$. If $\chi(p) = 1$, then $\chi\omega(p) \neq 1$. In this case, by the class field theory (see [8, Lemma 3]), $A_n^{\chi\omega} \neq \{0\}$ implies $A_0^{\chi\omega} \neq \{0\}$. Let $\mathfrak{a} \in c \in A_0^{\chi\omega}$ such that $\mathfrak{a}^p = (\alpha)$ for $\alpha \in K$. Further there exists $\varepsilon \in E_0' \setminus {E_0'}^p$ such that $[\varepsilon] \in (E_0'/{E_0'}^p)^{\chi\omega}$. Then we have $\sqrt[p]{\alpha}$, $\sqrt[p]{\varepsilon} \in M_0(\chi)$ and $\mathrm{Gal}(K(\sqrt[p]{\alpha}, \sqrt[p]{\varepsilon})/K) \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$. Since $(\mathcal{U}_0/\mathcal{U}_0^p)^\chi \simeq \mathbf{Z}/p\mathbf{Z}$, there exists a nontrivial unramified abelian p-extension of K contained in $M_0(\chi)$. Therefore, by the class field theory, A_0^χ is not trivial. $\square$

We obtain the following computational result:

Proposition 2. *Let $K_{f_\chi,p}$ be the maximal real subfield of $\mathbf{Q}(\sqrt{-f_\chi}, \zeta_p)$. $\lambda_p(K_{f_\chi,p}) = 0$ for all $1 < f_\chi < 200$ and $5 \leq p < 100000$. Exactly,*

$$\begin{array}{lll}
A_n(K_{f_\chi,p}) = \{0\} & \text{for } n \geq 0 & \text{and } (f_\chi, p) \text{ which does not appear in Table 1,} \\
A_n(K_{f_\chi,p}) \simeq \mathbf{Z}/p\mathbf{Z} & \text{for } n \geq 0 & \text{and } (f_\chi, p) \neq (136, 11) \text{ in Table 1,} \\
A_n(K_{f_\chi,p}) \simeq \begin{cases} \mathbf{Z}/p\mathbf{Z} & \text{for } n = 0, \\ \mathbf{Z}/p^2\mathbf{Z} & \text{for } n \geq 1, \end{cases} & & \text{and } (f_\chi, p) = (136, 11).
\end{array}$$

TABLE 1. $\nu_p(\chi\omega^k) = 1$ (2 for the $*$-marked case)

f_χ	p	k	f_χ	p	k	f_χ	p	k	f_χ	p	k
4	379	317	11	79	55	11	173	161	15	4909	2173
19	37	17	19	41	11	19	2251	1953	20	20261	19403
23	193	175	31	131	115	31	821	275	40	97	83
51	557	457	51	6553	3593	55	41189	2099	67	433	409
71	17	3	79	45943	18175	79	17	9	84	10133	9805
88	33049	9069	91	37	21	91	7069	3293	103	17	3
103	67	15	104	17837	285	116	4363	3845	120	4177	2253
127	67	53	131	853	127	136	54547	6417	136	11	3 *
139	4451	2233	148	23	13	152	863	617	152	3019	2319
155	12377	9137	163	79	55	167	797	245	187	79	63

TABLE 2. $v_p(a_0(\chi\omega^k)) = 2$ (3 for the $*$-marked case)

f_χ	p	k	f_χ	p	k	f_χ	p	k	f_χ	p	k
4	1381	609	11	17	7	15	31	5	19	2699	1579
23	2521	2473	39	11	3	40	19	15	43	71	57
47	373	53	52	83	79	52	241	51	79	7	5 *
79	41	5	79	4651	3373	84	31	25	88	70141	56107
103	7	3	104	3637	1487	115	1381	357	116	11	3
116	827	745	119	31	3	120	127	65	127	19	7
131	37	19	131	251	61	131	16267	11043	131	39569	13871
136	32869	6721	139	109	91	159	167	133	167	41	13
168	11	3	168	1087	475	179	19	15	179	2161	1605
187	17	15	199	19	9						

TABLE 3. $v_p(b_0(\chi\omega^k)) = 2$

f_χ	p	k	f_χ	p	k	f_χ	p	k	f_χ	p	k
3	257	101	7	173	97	19	52067	13617	19	71353	1597
19	2711	41	20	193	27	39	1187	349	39	9007	7117
43	757	123	51	107	27	71	46829	27893	71	1933	1275
79	43	25	79	269	107	79	2417	1389	84	59	41
88	19	11	91	277	99	91	1511	279	116	503	123
119	23	19	120	107	31	120	421	9	123	19	11
123	149	83	127	59183	29151	127	11	7	127	563	311
127	1409	517	131	349	53	131	2833	2047	139	349	53
143	19	5	159	359	245	167	71	19	183	1277	753
184	3119	2533	187	71	37	191	151	33	199	53	19

TABLE 4. $\tilde{\lambda}_p(\chi\omega^k) = 2$ (3 for the $*$-marked cases)

f_χ	p	k	f_χ	p	k	f_χ	p	k	f_χ	p	k
7	3613	1147	8	23	11	11	173	43	15	11	7
15	17681	5641	24	41	13	39	1289	211	39	26633	14233
43	127	105	47	5393	3265	51	4261	2867	52	19	17
52	2081	993	56	11	3	59	52313	44231	67	79337	30221
67	263	217	71	89	43	79	307	299	83	677	169
84	11593	4589	103	17	3	115	20357	16409	116	283	89
120	73	33	123	5	3 *	131	127	97	136	7	3
136	4919	2571	139	47	35 *	139	439	183	139	653	217
151	13	9	151	251	129	151	727	419	163	937	919
163	2441	1059	167	43	19	167	1031	445	168	7	3
168	733	117	168	1361	205	184	5	3	187	103	87
199	227	173									

TABLE 5. The χ-irregularity index density

r	n_r	n'_r	the density	the density'
0	360567	360726.71	0.60642302	0.60669164
1	180605	180279.99	0.30375222	0.30320562
2	44967	45035.78	0.07562817	0.07574385
3	7387	7499.17	0.01242389	0.01261256
4	959	936.41	0.00161290	0.00157491
5	86	93.53	0.00014463	0.00015730
6	9	7.78	0.00001513	0.00001309
7	0	0.55	0.00000000	0.00000093

From these tables, we can obtain concrete information on the higher K-groups of the ring of integers of $\mathbf{Q}(\sqrt{-f_\chi})$ (see [16, §4]).

Let us call a pair of integers (p, k) a χ-irregular pair if p is a prime, k is an odd integer satisfying $3 \leq k \leq p-2$, p divides $a_0(\chi\omega^k) = L_p(1, \chi\omega^k)$ (or $b_0(\chi\omega^k) = L_p(0, \chi\omega^k)$), and $(p, \chi\omega^k)$ satisfies (C). Further we define the χ-irregularity index $r_p(\chi)$ by

$$r_p(\chi) = \sharp\{(p,k) | (p,k) \text{ is a } \chi\text{-irregular pair}\}.$$

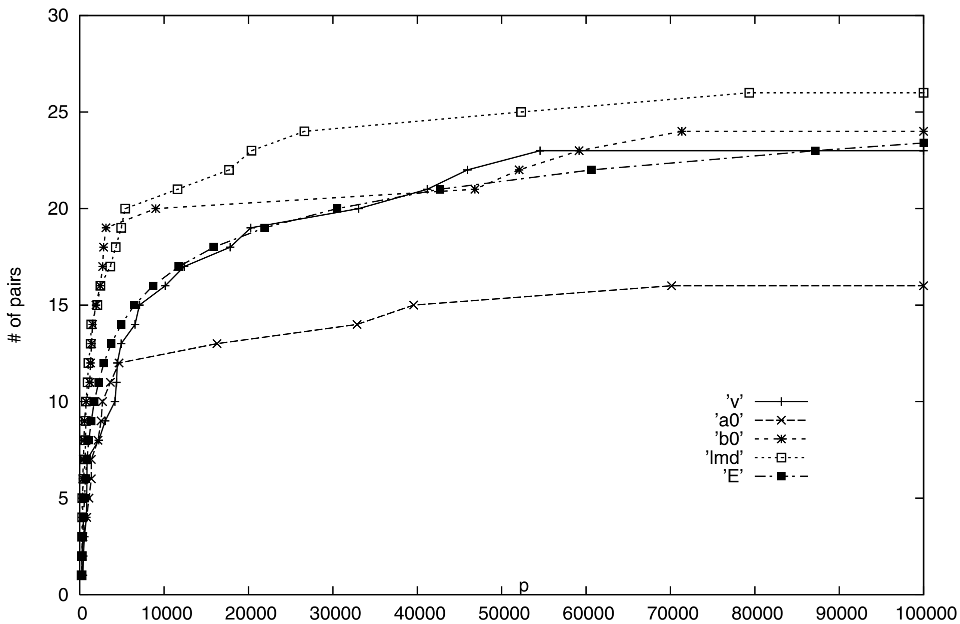

FIGURE 1. Exception pairs (odd quadratic, $1 < f < 200$, $200 < p < 100000$)

We call a prime number p χ-irregular if $r_p(\chi) > 0$. Let $m_p(\chi)$ be the number of even integers k with $3 \leq k \leq p-2$ such that $(p, \chi\omega^k)$ satisfies (C). We define

$$n_r = \sum_{(\chi,p) \text{ s.t. } r_p(\chi)=r} 1$$

and

$$n'_r = \sum_{\chi,p} {}_{m_p(\chi)}C_r \left(\frac{1}{p}\right)^r \left(\frac{p-1}{p}\right)^{m_p(\chi)-r},$$

where χ runs over all odd quadratic characters with $1 < f_\chi < 200$, and p runs all prime numbers with $5 \leq p < 100000$. The distribution of the indices of χ-irregularity is given in Table 5. The actual numbers n_r seem to be close to the expected numbers n'_r (cf. [2] and [17, p. 63]).

In Figure 1, we compare the actual numbers of exceptional pairs with the expected numbers in the range $200 < p < 100000$. Set

$$\begin{aligned}
&\nu(x) = \sharp\{(p, \chi\omega^k) | 200 < p < x,\ \chi\omega^k\text{: even},\ k \neq 1,\ \nu_p(\chi\omega^k) \geq 1\},\\
&a_0(x) = \sharp\{(p, \chi\omega^k) | 200 < p < x,\ \chi\omega^k\text{: even},\ k \neq 1,\ a_0(\chi\omega^k) \geq 2\},\\
&b_0(x) = \sharp\{(p, \chi\omega^k) | 200 < p < x,\ \chi\omega^k\text{: even},\ k \neq 1,\ b_0(\chi\omega^k) \geq 2\},\\
&lmd(x) = \sharp\{(p, \chi\omega^k) | 200 < p < x,\ \chi\omega^k\text{: even},\ k \neq 1,\ \tilde{\lambda}_p(\chi\omega^k) \geq 2\},\\
&E(x) = \left[\sharp\{\chi\} \sum_{200<p<x,\ p\text{:prime}} \frac{p-3}{2}\frac{1}{p^2}\right],
\end{aligned}$$

where $[*]$ is the Gauss symbol and χ runs over all odd quadratic characters with $1 < f_\chi < 200$.

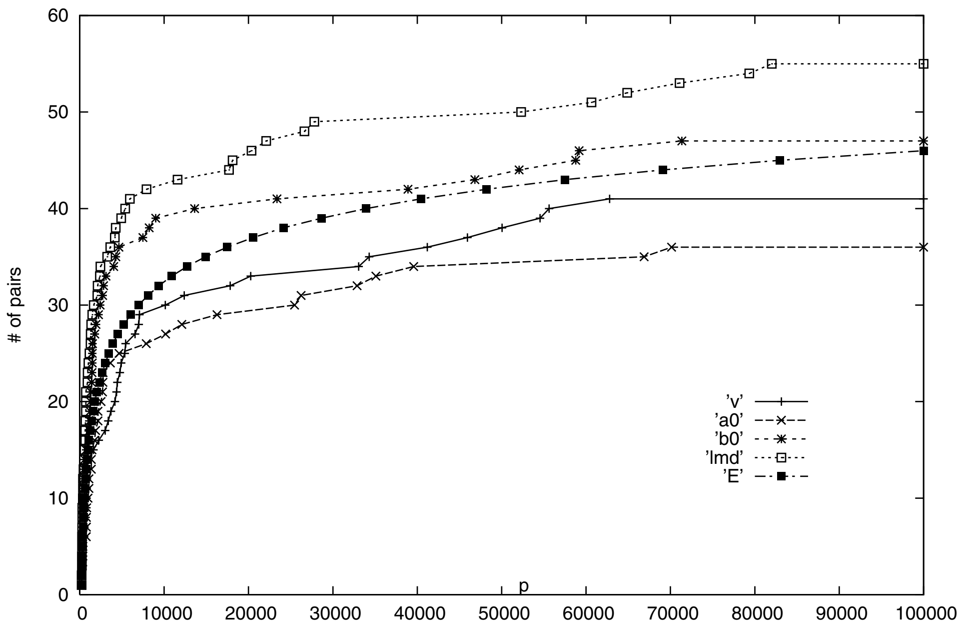

FIGURE 2. Exceptional pairs (quadratic, $1 < f < 200$, $200 < p < 100000$)

In order to increase the number of samples, we combine the above data with that in [16], and obtain Figure 2. Set

$$\begin{aligned}
&\nu(x) = \sharp\{(p, \chi\omega^k) | 200 < p < x,\ \chi\omega^k\text{: even},\ k \neq 0, 1,\ \nu_p(\chi\omega^k) \geq 1\},\\
&a_0(x) = \sharp\{(p, \chi\omega^k) | 200 < p < x,\ \chi\omega^k\text{: even},\ k \neq 0, 1,\ a_0(\chi\omega^k) \geq 2\},\\
&b_0(x) = \sharp\{(p, \chi\omega^k) | 200 < p < x,\ \chi\omega^k\text{: even},\ k \neq 0, 1,\ b_0(\chi\omega^k) \geq 2\},\\
&lmd(x) = \sharp\{(p, \chi\omega^k) | 200 < p < x,\ \chi\omega^k\text{: even},\ k \neq 0, 1,\ \tilde{\lambda}_p(\chi\omega^k) \geq 2\},\\
&E(x) = \left[\sharp\{\chi\} \sum_{200<p<x,\ p\text{:prime}} \frac{p-3}{2}\frac{1}{p^2}\right],
\end{aligned}$$

where χ runs over all quadratic characters with $1 < f_\chi < 200$.

From our data, the actual numbers seem to be close to the expected numbers. Even for large p, it might be possible that the actual numbers are near to the expected numbers.

Finally we give an example such that A_n is not cyclic. In [1], Aoki-Fukuda showed that

$$A_0^{\chi\omega} \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}, \quad A_0^{\chi\omega^3} \simeq \{0\}$$

for $(f_\chi, p) = (4 \cdot 14606, 5)$ by using cyclotomic units of $\mathbf{Q}(\zeta_{f_0 l_i})$ ($l_1 = 11251$ and $l_2 = 22501$). By our method (using cyclotomic units and Gauss sums of $\mathbf{Q}(\zeta_{f_n})$ for $n \leq 2$), we show the above and

$$A_n^{\chi\omega} \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}, \quad A_n^{\chi\omega^3} \simeq \{0\}$$

for $n \geq 1$. First we have

$$\begin{cases} g_{\chi\omega}(T) \equiv \omega_0^*(T^2 + 2380T + 2025) \bmod p^5, \\ g_{\chi\omega}^*(T) \equiv \omega_0(T^2 + 1305T + 2150) \bmod p^5, \end{cases} \begin{cases} g_{\chi\omega^3}(T) = 1, \\ g_{\chi\omega^3}^*(T) = 1. \end{cases}$$

Hence we immediately obtain the triviality of $A_n^{\chi\omega^3}$. For $\psi = \chi\omega$, we have $\psi(p) \neq 1$ and $\psi^*(p) = 1$.

Cyclotomic units.
$\mathbf{n = 0}$

$$\mathcal{C}_0^\psi \simeq (\omega_0, p^2)/(\omega_0),$$
$$\mathcal{E}_0^\psi \simeq (\mathcal{E}_0^\psi)' \subseteq \Lambda/(\omega_0).$$

Hence $(\mathcal{E}_0/\mathcal{C}_0)^\psi$ is a subgroup of $\Lambda/(\omega_0, p^2) \simeq \mathbf{Z}/p^2\mathbf{Z}$.
$\mathbf{n = 1}$

$$\mathcal{C}_1^\psi \simeq (\tilde{g}_\psi(T), pT, p^3)/(\omega_1).$$

Let $l_1 = 1 + 12f_1p = 87636001$ and $l_2 = 1 + 22f_1p = 160666001$. By studying the image of $\mathcal{C}_1^\psi$ in $\prod_{\mathfrak{L}_i | l_1 l_2}(\mathcal{O}_{K_1}/\mathfrak{L}_i)$, we have

$$\mathcal{E}_1^\psi \simeq (\mathcal{E}_1^\psi)' \subseteq (\tilde{g}_\psi(T), T, p)/(\omega_1).$$

Hence $(\mathcal{E}_1/\mathcal{C}_1)^\psi$ is a subgroup of $(\tilde{g}_\psi(T), T, p)/(\tilde{g}_\psi(T), pT, p^3) \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$.
$\mathbf{n = 2}$

$$\mathcal{C}_2^\psi \simeq (\tilde{g}_\psi(T), p^2T, p^4)/(\omega_2).$$

Let $l_1' = 1 + 8f_2p = 292120001$ and $l_2' = 1 + 14f_2p = 511210001$. By studying the image of $\mathcal{C}_2^\psi$ in $\prod_{\mathfrak{L}_i' | l_1' l_2'}(\mathcal{O}_{K_2}/\mathfrak{L}_i')$, we have

$$\mathcal{E}_2^\psi \simeq (\mathcal{E}_2^\psi)' \subseteq (\tilde{g}_\psi(T), pT, p^2)/(\omega_2).$$

Hence $(\mathcal{E}_2/\mathcal{C}_2)^\psi$ is a subgroup of $(\tilde{g}_\psi(T), pT, p^2)/(\tilde{g}_\psi(T), p^2T, p^4) \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$. This implies Greenberg's conjecture for A_n^ψ.

By computation of Gauss sums, we will show that $\mathcal{E}_1^\psi \simeq (\tilde{g}_\psi(T), T, p)/(\omega_1)$. Hence we have $\sharp(\mathcal{E}_1^\psi/\mathcal{C}_1^\psi) = p^3$, $\sharp(\mathcal{E}_2^\psi/\mathcal{C}_2^\psi) \geq p^3$, and $\mathcal{E}_2^\psi \simeq (\tilde{g}_\psi(T), pT, p^2)/(\omega_2)$. By this isomorphism, $\mathrm{Ker}(A_0 \to A_2)^\psi \simeq H^1(\Gamma_0, E_2)^\psi \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ (see the proof of Theorem 1). Therefore we have $A_0^\psi \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ and $A_n^\psi \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ for $n \geq 1$ (cf. [15, Theorem1]).

Gauss sums. Since $p\omega_0 \in (\tilde{g}_\psi^*(T), \omega_1)$, the exponent of $\omega_0 X_\infty^{\psi^*}/\omega_1 X_\infty^{\psi^*} \simeq \omega_0 \tilde{X}_\infty^{\psi^*}/\omega_1 \tilde{X}_\infty^{\psi^*}$ is p. Therefore, the exponent of $\omega_0 \tilde{A}_1^{\psi^*}$ is at most p. We will show that $\omega_0 \tilde{A}_1^{\psi^*} \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ by using Gauss sums and prime numbers.

Set $h(T) = 21T^4 + 17T^3 + 9T^2 + 5T + 15$. Then we have

$$h(T)g_\psi^*(T) \equiv p\omega_0 \bmod (\omega_1, p^2).$$

Let $\mathbf{e}_{\psi^*,m} \in \mathbf{Z}[\Delta]$ such that $\mathbf{e}_{\psi^*,m} \equiv e_{\psi^*} \bmod p^m$. For $\mathfrak{L}_i | l_1 l_2$, let $g_1(\mathfrak{L}_i)$ be the Gauss sum of K_1 which satisfies

$$(g_1(\mathfrak{L}_i)^{\mathbf{e}_{\psi^*,m}}) = \mathfrak{L}_i^{f_\chi \theta_1 \mathbf{e}_{\psi^*,m}},$$

where $\theta_1 \in \mathbf{Q}[\mathrm{Gal}(K_1/\mathbf{Q})]$ is the Stickelberger element (see [7, pp. 42-45] for details). Hence for any integer $m \geq 1$, there exists $g_m' \in K_1$ such that

$$(g_1(\mathfrak{L}_i)^{\mathbf{e}_{\psi^*,1}}) = \mathfrak{L}_i^{f_\chi \theta_1 \mathbf{e}_{\psi^*,m}}({g_m'}^{p}).$$

Since $G_\psi^*(T) \equiv \mathbf{e}_{\psi^*,m}\theta_1 \bmod (p^m, \omega_1)$, we have

$$(g_1(\mathfrak{L}_i)^{\mathbf{e}_{\psi^*,1}h(T)}) = \mathfrak{L}_i^{\omega_0 p u(T) \mathbf{e}_{\psi^*,m}} \mathfrak{L}_i^{p^m v \mathbf{e}_{\psi^*,m}}({g_m'}^{ph(T)})$$

for $u(T) \in \Lambda^\times$ and $v \in \mathbf{Z}_p[\mathrm{Gal}(K_1/\mathbf{Q})]$. Let $l_1 = 1 + 11f_1 = 16066601$, $l_2 = 1 + 14f_1 = 20448401$, $l_1^* = 1 + 4(2f_1l_1l_2) = 383888095771419568480 1$ and $l_2^* = 1 + 7(2f_1l_1l_2) = 671804167599984244840 1$. By studying the images of $g_1(\mathfrak{L}_i)^{\mathbf{e}_{\psi^*,1}h(T)}$ in $\prod_{\mathfrak{L}_j^*|l_1^*l_2^*}(\mathcal{O}_{\mathbf{Q}(\zeta_{l_1l_2f_1})}/\mathfrak{L}_j^*)$, we conclude that the classes of $\mathfrak{L}_i^{\omega_0\mathbf{e}_{\psi^*,m}}$ for $\mathfrak{L}_i|l_1l_2$ generate a subgroup of $\tilde{A}_1^{\psi^*}$ whose quotient is isomorphic to $\mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$. Since $\sharp(\omega_0\tilde{X}_\infty^{\psi^*}/\omega_1\tilde{X}_\infty^{\psi^*}) = p^2$, this happens only when $\nu_{1,0}\tilde{Y}_\infty^{\psi^*} = \omega_1\tilde{X}_\infty^{\psi^*}$, i.e. $\tilde{Y}_\infty^{\psi^*} = \omega_0\tilde{X}_\infty^{\psi^*}$. By Lemma 3 ($X = \tilde{X}_\infty^{\psi^*}$ and $Z = \omega_1\tilde{X}_\infty^{\psi^*}$) and the class field theory, we have $\langle\tilde{\sigma}_{\mathfrak{L}_i}^{\mathbf{e}_{\psi^*,m}}\rangle_{\mathfrak{L}_i|l_1l_2} = \tilde{X}_1^{\psi^*}$. By Lemma 4 ($n = n' = 1$) and the image of $\mathcal{C}_1^\psi$ in $\prod_{\mathfrak{L}_i|l_1l_2}(\mathcal{O}_{K_1}/\mathfrak{L}_i)$, we obtain $\mathcal{E}_1^\psi \simeq (\tilde{g}_\psi(T), T, p)/(\omega_1)$.

We used thirty personal computers for three months to make the tables in this section. The programs were written in UBASIC and C, in which the GNU MP library was included. For the last example, it took a few minutes to calculate cyclotomic units modulo prime ideals, and thirty minutes to calculate Gauss sums modulo prime ideals on one PC (CPU: Pentium IV, 3.6GHz, RAM 2GB). In [1], it took 6 hours and 42 minutes to compute A_0 by using Alpha 21264, 667MHz, RAM 4GB.

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