

**A PRIORI ESTIMATES OF SMOOTHNESS OF SOLUTIONS
TO DIFFERENCE BELLMAN EQUATIONS
WITH LINEAR AND QUASI-LINEAR OPERATORS**

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ABSTRACT. A priori estimates for finite-difference approximations for the first and second-order derivatives are obtained for solutions of parabolic equations described in the title.

1. INTRODUCTION

The goal of this article is to prove a priori estimates for solutions of finite-difference approximations of parabolic Bellman equations with linear and quasi-linear operators. In the latter case the nonlinear operator defining the equation is still supposed to be convex with respect to the second-order derivatives of the unknown function. We present estimates for the finite-difference approximations of the first and second-order spatial derivatives. In particular, our results cover finite-difference approximations for degenerate quasi-linear parabolic equations. As far as we are aware these are the first results for such equations. The main parts of the linear and quasi-linear operators entering Bellman equations are assumed to be *linear* $a_k \Delta_k$ operators, that is, written as a linear combination of pure second-order derivatives in certain directions that are common to all operators. This assumption is always satisfied if the equation is uniformly nondegenerate and is generally necessary if we want to restrict ourselves to monotone difference approximations and meshes that are obtained from a fixed one by scaling (see more about it in Remark 2.4 below). Our results are valid for the usual Bellman equations and also for the optimal stopping and impulse control problems associated with them.

The motivation to obtain a priori estimates is the following. There is an approach suggested in [10], [11], and [12] to establishing the rate of convergence of u_h to u as $h \downarrow 0$, where u is the true solution, u_h the solution of finite-difference approximation of the same equation, and h typically is the mesh size. Two main ideas of this approach are that the original equation and its finite-difference approximation should play symmetric roles and that one can “shake the coefficients” of the equation in order to be able to mollify under the sign of the nonlinear operator.

For elliptic Bellman equations with constant coefficients and Lipschitz free terms the first idea led to the rate of convergence of order $h^{1/3}$, for generic finite-difference

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approximations and $h^{1/2}$ in the case of $a_k\Delta_k$ operators (see Remark 1.4 and Theorem 5.1 in [10], also see [2]). In contrast with the popular belief that assuming more smoothness of the data does not lead to better rates of convergence, it is proved in [5] that if the free terms are in $C^{1,1}$, then the rate is at least h for the constant coefficient $a_k\Delta_k$ case and h^2 for equations with better structure.

The second idea was introduced to treat equations with variable coefficients and led to quite satisfactory error bounds for $u - u_h$ from an “easy” side (depending on how the equation is written this can be either the upper or lower estimate of $u - u_h$). To get an estimate from the other side on the basis of the idea of symmetry between the approximating and the original equations one needed to solve the following problem:

(P) In the case of variable coefficients estimate how much the solution of the finite-difference equation loses in the process of shaking the equation.

In the absence of a solution of the problem (P) the idea of symmetry was still useful but only in obtaining some intermediate estimates (see, for instance, [2] and [11]) and various approaches to getting the error bounds from the “hard” side were developed. In addition to the above-cited papers the interested reader should consult [3], [4], and the references therein. Note that for generic finite-difference approximations, under the assumptions of Theorem 5.3 of [11] the result of [4] is the same $h^{1/3}$, but the result of Theorem 5.4 of [11] is improved from $h^{1/21}$ to $h^{1/7}$. The issue of solving the problem (P) for generic finite-difference approximations remains unsettled, and it is not clear how far off $h^{1/7}$ is from the true rate.

The problem (P) was recently reduced to the problem of estimating the modulus of continuity of approximate solutions and was solved in [13] for the $a_k\Delta_k$ case in which a sharp error bound of order $h^{1/2}$ was obtained. The idea of symmetry worked again as in the constant coefficients case. This activity was continued in [7], where for the first time equations in domains were treated, and in [6], where under various smoothness assumptions the rates $h^{1/2}$, h , and h^2 were obtained for *linear* degenerate equations of $a_k\Delta_k$ form. For the linear case the rate $h^{1/2}$ was earlier obtained in [8] by a method close to a method from [12] (Lemma 5.1 of [8] is a version of Theorem 2.1 of [12]). However, this method does not allow one to get rates h and h^2 .

The main technical result of [13] is the a priori estimate of the derivative of u_h with respect to x stated as Theorem 5.2 and proved by quite subtle estimates. It turns out that there is a much easier method to prove Theorem 5.2 of [13] which in addition carries over to much more general equations with quasi-linear operators and to obtaining estimates for the second-order finite differences of u_h . The method is almost as simple as the one used in [6] for linear equations.

We present this new method here and concentrate only on a priori estimates to keep the article within reasonable limits. Once the a priori estimates are obtained, one can follow familiar patterns to get error bounds in the various cases of linear or quasi-linear operators, degenerate or weakly nondegenerate or else uniformly nondegenerate, with C^1 or $C^{1,1}$ coefficients. In particular, we hope to obtain first estimates on the rate of convergence in the case of degenerate quasi-linear operators with Lipschitz coefficients. Our preliminary computations also show that under the assumptions of Theorem 2.12 the estimate $|u - u_h| \leq Nh^{2/3}$ holds in the elliptic case. These and some other possible applications indicated below of our results, we intend to develop in the future.

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2. SETTING AND MAIN RESULTS

Our first few results concern equations of the type

$$(2.1) \quad F(\delta_\tau^T u, \Delta_{h,\ell_k} u, \delta_{h,\ell_k} u, u) = 0,$$

where

$$(2.2) \quad \begin{aligned} &F(\phi, q_k, p_k, \psi) = F(\phi, q, p, \psi, t, x) \\ &= \sup_{\alpha \in A} [r^\alpha(t)\phi + \sum_{\substack{|k|=1 \\ k=1}}^{d_1} (a_k^\alpha(t, x)q_k + b_k^\alpha(t, x)p_k) - c^\alpha(t, x)\psi + f^\alpha(p, \psi, t, x)], \end{aligned}$$

$\delta_\tau^T u, \Delta_{h,\ell_k} u, \delta_{h,\ell_k} u$ are finite-difference approximations of the time derivative, the pure second-order derivative in the direction ℓ_k , and the first-order derivative in the direction ℓ_k , respectively. Detailed descriptions of the above objects now follow.

Let A be a separable metric space, $d, d_1 \geq 1$ integers, and let

$$r^\alpha = r^\alpha(t), \quad a_k^\alpha = a_k^\alpha(t, x), \quad b_k^\alpha = b_k^\alpha(t, x), \quad c^\alpha = c^\alpha(t, x)$$

be real-valued *bounded* functions of (α, t, x) defined on $A \times \mathbb{R} \times \mathbb{R}^d$ for $k = \pm 1, \dots, \pm d_1$. Also let some vectors $\ell_k \in \mathbb{R}^d$ be defined for $k = \pm 1, \dots, \pm d_1$ and let

$$(2.3) \quad T, h_0 \in (0, \infty), \quad \delta \in (0, 1], \quad K_0, K_1, K_2, K_3 \in [0, \infty), \quad m \in \mathbb{R}$$

be some constants fixed throughout the article. It is worth noting that $\ell_k, k = \pm 1, \dots, \pm d_1$, are *not* supposed to form a basis in \mathbb{R}^d or even generate \mathbb{R}^d . This becomes crucial when one proves the estimates of the first-order differences of solutions with respect to parameters on which the coefficients may depend. Notice also that the lengths of the ℓ_k 's can be different and some of them can be just zero (and we will use this possibility later). The constant T gives us the time interval $[0, T)$ on which the equation is investigated, h_0 “calibrates” the mesh-sizes in the x variable, and the constant δ will appear in various requirements of nondegeneracy. The constant K_0 is the most basic one; it is used in formulations of the very basic assumptions. The constant K_1 is used to control either the maximum magnitude of the solution or its oscillation. The constant K_2 will appear in our assumption on the growth of f with respect to the “gradient” of the solution (see Assumption 2.5 (ii), which looks very much like the one commonly used in the theory of quasi-linear PDEs. By the way, the author’s efforts to use Assumption 2.5 (iii), stated similarly, failed.) The constant K_3 is used to control various quantities having lesser impact on our results than those controlled by K_0, K_1, K_2 . Finally, the constant m is used to extract various results, which one gets in the theory of parabolic PDEs after replacing $u(t, x)$ with $u(t, x)e^{mt}$.

For any vector $l \in \mathbb{R}^d, \eta, \tau > 0$, and a function u , introduce

$$\begin{aligned} \delta_{\eta,l} u(x) &= \frac{u(x + \eta l) - u(x)}{\eta}, \quad \tau_T(t) = \tau \wedge (T - t)^+, \\ \delta_\tau^T u(t, x) &= \frac{u(t + \tau_T(t), x) - u(t, x)}{\tau}, \quad \delta_\tau u(t, x) = \frac{u(t + \tau, x) - u(t, x)}{\tau}, \\ \Delta_{\eta,l} u(x) &= \frac{u(x + \eta l) - 2u(x) + u(x - \eta l)}{\eta^2}, \end{aligned}$$

where the notation $a^\pm = (1/2)(|a| \pm a)$ is used. Observe that with the above definition of δ_τ^T equation (2.1) makes perfect sense for $t < T$ for functions $u(t, x)$ defined only for $t \leq T$. We do not need to extend u beyond T in order to compute the finite-difference approximation of its derivative in time for $t < T$.

Assumption 2.1. (i) The functions r^α , a_k^α , b_k^α , and c^α are continuous with respect to α ;

(ii) the functions b_k^α satisfy the Lipschitz condition with constant K_0 with respect to x ;

(iii) the function c^α satisfies the Lipschitz condition with constant K_3 with respect to x ;

(iv) we have

$$\ell_{-k} = -\ell_k, \quad a_{-k}^\alpha = a_k^\alpha, \quad |\ell_k| \leq K_0, \quad r_k^\alpha \geq 0, \quad a_k^\alpha \geq 0$$

(for all values of the arguments and k).

An important feature of Assumption 2.1 is that no control on the sizes of r^α , a_k^α , b_k^α , and c^α is imposed (however, remember that from the very beginning they are assumed to be bounded).

Assumption 2.2. For any unit $l \in \mathbb{R}^d$ and $\eta > 0$, we have

$$|\delta_{\eta,l} a_k^\alpha| \leq K_0(\sqrt{a_k^\alpha} + \eta).$$

Remark 2.1. It is easy to see that Assumption 2.2 is satisfied (with, perhaps, different K_0) if and only if $\sigma_k^\alpha := \sqrt{a_k^\alpha}$ satisfies the Lipschitz condition with constant K_0 with respect to x .

Indeed, the necessity follows after letting $\eta \downarrow 0$ and the sufficiency is a direct consequence of the formula

$$\delta_{\eta,l} a_k^\alpha = 2\sigma_k^\alpha \delta_{\eta,l} \sigma_k^\alpha + \eta(\delta_{\eta,l} \sigma_k^\alpha)^2.$$

Below we are also using the well-known fact that a continuous function $v(x)$ is Lipschitz continuous with constant K if and only if its generalized gradient $v_x = D_x v$ satisfies $|v_x| \leq K$ (a.e.).

Definition 2.2. Let B be a finite subset of \mathbb{R}^d and $p(x, y)$ a real-valued function on $\mathbb{R}^d \times \mathbb{R}^d$. For an $x_0 \in \mathbb{R}^d$ we say that the operator

$$(2.4) \quad S : u \rightarrow Su, \quad Su(x) = \sum_{y \in B \cup \{0\}} p(x, x+y)u(y)$$

respects the maximum principle at x_0 relative to B if, for any function $\phi(x)$ such that $\phi(x_0 + y) \geq \phi(x_0)$ for all $y \in B$, we have $Su(x_0) \geq 0$.

Obviously, the operators $\delta_{\eta,l}$ and $\Delta_{\eta,l}$ respect the maximum principle at any point relative to appropriate sets.

For $h > 0$ set

$$(2.5) \quad L_h^\alpha u = a_k^\alpha \Delta_{h,\ell_k} u + b_k^\alpha \delta_{h,\ell_k} u - c^\alpha u,$$

where and throughout the paper the summation convention is enforced. For each t the operator $L_h^\alpha = L_h^\alpha(t, x)$ can be considered as an operator on functions defined on \mathbb{R}^d .

Assumption 2.3. We have $a_k^\alpha \geq h_0(b_k^\alpha)^-$ (recall (2.3)).

Remark 2.3. It is easy to see that Assumption 2.3 implies that for $h \in (0, h_0]$, $t \in \mathbb{R}$, and $\alpha \in A$ the operator $L_h^\alpha(t, x) + c^\alpha(t, x)$ respects the maximum principle at any point x_0 relative to Λ_0 , where

$$(2.6) \quad \Lambda_0 := \{h\ell_k : k = \pm 1, \dots, \pm d_1\}.$$

In turn, provided that all ℓ_k are different, the said property of $L_h^\alpha(t, x) + c^\alpha(t, x)$ implies what is required in Assumption 2.3.

To satisfy Assumption 2.3 it is sufficient to require that $b_k^\alpha \geq 0$, in which case what we use is just an upwind discretization of the “transportation” term.

Remark 2.4. The operators L_h^α are natural approximations of the operator

$$(2.7) \quad L^\alpha u = a_k^\alpha \ell_k^i \ell_k^j u_{x^i x^j} + b_k^\alpha \ell_k^i u_{x^i} - cu$$

in the sense that $L_h^\alpha u \rightarrow L^\alpha u$ as $h \downarrow 0$ for all smooth u .

One may wonder how wide is the class of operators given in the usual form

$$(2.8) \quad Lu = a^{ij} u_{x^i x^j} + b^i u_{x^i}$$

which admit such a special approximation. We discuss this issue in Section 9.

Next, we describe the free term in the equation, which is given by a real-valued function

$$f^\alpha = f^\alpha(p, \psi, t, x)$$

defined on $A \times \mathbb{R}^{2d_1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$.

Assumption 2.4. The function f^α is bounded, f^α is continuous in α , continuous in (p, ψ, x) and, for any α and t , its generalized gradients $D_p f^\alpha$, $D_\psi f^\alpha$, and $D_x f^\alpha$ in p, ψ , and x , respectively, satisfy

$$|D_{p^k} f^\alpha| \leq K_0 \sqrt{a_k^\alpha}, \quad k = \pm 1, \dots, \pm d_1, \quad |D_\psi f^\alpha| \leq K_0, \quad |D_x f^\alpha| \leq K_3$$

for almost all $(p, \psi, x) \in \mathbb{R}^{2d_1} \times \mathbb{R} \times \mathbb{R}^d$.

For fixed $h, \tau > 0$ we consider the equation

$$(2.9) \quad \sup_{\alpha \in A} [r^\alpha \delta_\tau^T u + L_h^\alpha u + g^\alpha] = 0,$$

where

$$g^\alpha = g^\alpha(t, x) = f^\alpha(\delta_{h, \ell_k} u(t, x), u(t, x), t, x).$$

Observe that equation (2.9) takes the form (2.1). The presence of r^α in these equations allows us to treat the normalized Bellman equations (see [9]), which arise, for instance, in optimal stopping problems or problems with singular control.

Fix a vector $l \in \mathbb{R}^d$ with $|l| \leq K_0$ and a number $\eta \in (0, h]$. Set

$$h_i = h \quad \text{for } |i| = 1, \dots, d_1, \quad h_{d+1} = h_{-(d+1)} = \eta, \quad \ell_{d+1} = -\ell_{-(d+1)} = l,$$

$$(2.10) \quad \Lambda = \{h_1 \ell_{\pm 1}, \dots, h_{d+1} \ell_{\pm(d+1)}\}.$$

We treat Λ_0 as a list rather than a set with specified elements, even if $\ell_1 = \ell_2$ we include in the list this vector twice.

Observe that in (2.5) only $\ell_k \in \Lambda_0$ are involved. However, the method of “shaking” the coefficients requires estimates of difference derivatives in all directions and not only along the mesh. This is the reason why we introduce Λ . Set

$$\Lambda_n = \sum_1^n \Lambda = \{x : x = l_1 + \dots + l_n, l_1, \dots, l_n \in \Lambda\}, \quad \Lambda_\infty = \bigcup_n \Lambda_n,$$

$$(2.11) \quad \bar{\mathcal{M}}_T = \{(n\tau) \wedge T : n = 0, 1, \dots\} \times \Lambda_\infty, \quad \mathcal{M}_T = \bar{\mathcal{M}}_T \cap ([0, T] \times \mathbb{R}^d).$$

Fix a finite set $Q \subset \mathcal{M}_T$, assume that

$$(2.12) \quad Q|_0 := Q \cap (\{0\} \times \mathbb{R}^d) \neq \emptyset$$

and define

$$\bar{Q} = Q \cup \{(t + \tau_T(t), x) : (t, x) \in Q, t + \tau_T(t) = T\},$$

$$(2.13) \quad Q_1^o = \{(t, x) \in Q : t < T, (t + \tau_T(t), x) \in \bar{Q}, (t, x + \Lambda) \subset Q\},$$

$$\partial_1 Q = \bar{Q} \setminus Q_1^o.$$

Obviously, it may happen that $\bar{Q} = Q$. The subscript 1 is used above because later on we will need a “fatter” boundary $\partial_2 Q$.

Finally, define T' as the least $n\tau$, $n = 1, 2, \dots$, such that $n\tau \geq T$, recall that $m \in \mathbb{R}$ (see (2.3)) is a given fixed constant and introduce

$$\xi(t) = e^{mt}, \quad t < T, \quad \xi(T) = e^{mT'}, \quad \xi_{(+)} = \xi \vee 1, \quad \xi_{(-)} = \xi \wedge 1,$$

$$(2.14) \quad c_m = \frac{1 - e^{-m\tau}}{\tau}, \quad \lambda = \inf_{\alpha, t, x} [c^\alpha(t, x) + r^\alpha(t)c_m].$$

Introducing a discontinuous function $\xi(t)$ may look unnatural. However, what is important for us is that

$$(2.15) \quad \xi \delta_\tau^T u = e^{-m\tau} \delta_\tau^T (\xi u) - c_m(\xi u)$$

on \mathcal{M}_T for any $u = u(t, x)$.

Everywhere below in this section u is a given function on $\bar{\mathcal{M}}_T$ satisfying (2.9) in Q . In our first result no control on the sizes of r^α , a_k^α , b_k^α , and c^α is imposed.

Theorem 2.5. *Let $h \in (0, h_0]$. Then, under Assumptions 2.1 through 2.4 there are constants $N = N(d_1, K_0)$, $N^* = N^*(d_1, K_0, K_3)$ such that if $\lambda \geq N$, then on $Q|_0$,*

$$(2.16) \quad |\delta_{\eta, l} u| \leq N^* e^{m^+(T+\tau)} \left[1 + \max_Q |\xi_{(-)} u| + \max_{k, \partial_1 Q} (|\xi_{(-)} \delta_{h, \ell_k} u| + |\xi_{(-)} \delta_{\eta, l} u|) \right].$$

We prove this theorem in Section 4.

Remark 2.6. This theorem is similar to Theorem 5.2 of [13] and entails all the consequences derived from the latter in [13] and [7]. In particular, by using Theorem 5.6 of [13] and comparing the equations for u and $u(\tau + \cdot, \cdot)$ an estimate of $\delta_\tau u$ can be obtained if we require the data to have bounded derivatives in t .

Remark 2.7. Theorem 2.5 has an immediate application to elliptic equations. In that case u is independent of t , one can take $r^\alpha \equiv 0$, and use as large a negative m as one wishes without affecting λ . Then it is seen that in (2.16) the maximums over \bar{Q} and $\partial_1 Q$ reduce to the maximums over $Q|_0$ and $Q|_0 \cap \partial_1 Q$, respectively.

Our second result concerns Bellman equations with more general quasi-linear operators. This time (2.9) is assumed to be *uniformly nondegenerate* in the space generated by the ℓ_k 's. We will allow $f^\alpha(p, \psi, t, x)$ to grow quadratically with respect to p , and therefore no b_k^α are needed. The term $c^\alpha u$ also could be absorbed in f^α . However, we keep it, in order to state Theorem 2.11 in a simpler way.

Assumption 2.5. (i) The functions a_k^α also depend on ψ :

$$a_k^\alpha = a_k^\alpha(\psi, t, x)$$

and equation (2.1) holds in Q , where F is defined by (2.2) with $a_k^\alpha(\psi, t, x)$ in place of $a_k^\alpha(t, x)$. The functions $a_k^\alpha(\psi, t, x)$ are Lipschitz continuous in x with constant K_3 , Lipschitz continuous in ψ with a constant $\omega \in (0, \infty)$,

$$a_k^\alpha \geq \delta, \quad |k| \leq d_1, \quad c^\alpha \geq -K_3.$$

(ii) The function f^α is continuous in α , continuous in (p, ψ, x) , and for all values of the arguments, satisfying $|\psi| \leq K_1$ and $|p| \geq K_2$, it holds that

$$|f^\alpha| \leq \omega|p|^2 + K_3.$$

(iii) For each α and t the generalized gradients $D_p f^\alpha$, $D_\psi f^\alpha$, and $D_x f^\alpha$ of f^α with respect to p , ψ , and x , respectively, satisfy

$$|D_p f^\alpha| \leq \omega|p| + K_3, \quad |D_\psi f^\alpha| \leq \omega|p|^2 + K_3,$$

$$(2.17) \quad |D_x f^\alpha| \leq \omega|p|^3 + K_3$$

(a.e.) on the set $\{|\psi| \leq K_1\}$.

Remark 2.8. Clearly, Assumption 2.5 (ii) is satisfied with any $\omega > 0$ and appropriate $K_3(\omega)$ if

$$\sup_{\alpha, t, x, \psi} |f^\alpha(p, \psi, t, x)| = o(|p|^2)$$

as $|p| \rightarrow \infty$. This includes all functions affine in p provided that the coefficients are bounded. A similar situation occurs with Assumption 2.5 (iii).

Assumption 2.6. For a constant $C \geq 4$ depending only on d_1 , the exact value of which can be determined by examining the proof of Theorem 2.9, we have

$$(2.18) \quad CK_1(1 + K_1)\omega \leq \delta.$$

Theorem 2.9. *Let $b_k^\alpha \equiv 0$ and let Assumptions 2.1, 2.5, and 2.6 be satisfied. Assume that $|u| \leq K_1$ in \bar{Q} and $|\delta_{h, \ell_k} u| \leq K_3$ on $\partial_1 Q$ if $|k| \leq d_1$. Then in \bar{Q} ,*

$$|\delta_{h, \ell_k} u| \leq N = N(d_1, \delta, K_1, K_2, K_3), \quad |k| \leq d_1.$$

In particular, N is independent of T .

The proof of this theorem is given in Section 5.

Remark 2.10. If ω is large, we need K_1 to be small in order to satisfy (2.18), that is, we need u to be small. By replacing u with $u - \gamma$, where γ is any constant, we see that, actually, we need the oscillation of u rather than u itself to be small if ω is not. This restriction could be completely avoided if we proved an interior version of Theorem 2.9 and a priori Hölder continuity of u . It seems to the author that this is possible, but requires much more work.

Theorem 2.11. *Under the assumptions of Theorem 2.9 suppose that $|\delta_{\eta,l}u| \leq K_3$ on $\partial_1 Q$ and a_k^α are independent of ψ . Then there is a constant $N = N(d_1, \delta, K_1, K_2, K_3)$, such that if $\lambda \geq N$, then on $Q|_0$,*

$$|\delta_{\eta,l}u| \leq Ne^{m^+(T+\tau)}.$$

This is a simple corollary of Theorems 2.9 and 2.5 with $h_0 = h$ in the latter. Indeed, once we know that the values of $|\delta_{h,\ell_k}u|$ and $|u|$ are dominated by a constant, the behavior of $f^\alpha(p, \psi, t, x)$ for large $|p|$ becomes irrelevant and we can even multiply it by an appropriate cut-off function in such a way that the new f^α would satisfy Assumption 2.4 and u would still satisfy the new equation.

Our next result is about second-difference estimates.

Assumption 2.7. (i) The function f^α is independent of p and ψ .

(ii) For any $i, j = \pm 1, \dots, \pm(d_1 + 1)$ and ψ standing for any of the functions b_k^α, c^α , and f^α we have

$$(2.19) \quad |\delta_{h_j, \ell_j} \delta_{h_i, \ell_i} \psi| \leq K_3, \quad |\delta_{h_i, \ell_i} f^\alpha| \leq K_3, \quad |\delta_{h_j, \ell_j} \delta_{h_i, \ell_i} a_k^\alpha| \leq K_0 + K_3 \sqrt{a_k^\alpha}.$$

A typical case when the third inequality in (2.19) is satisfied occurs if $a_k^\alpha = (\sigma_k^\alpha)^2$, where σ_k^α is bounded and twice continuously differentiable.

In contrast with the above results in which no control on the magnitudes of $r^\alpha, a_k^\alpha, b_k^\alpha$, and c^α is required, this time we need the following.

Assumption 2.8. We have

$$\delta \leq \sup_{\alpha \in A} a_k^\alpha \leq K_0, \quad r^\alpha, |b_k^\alpha|, |c^\alpha|, |f^\alpha| \leq K_3.$$

The following assumption is about a special structure of the set of our basic vectors $\ell_k, k = \pm 1, \dots, \pm d_1$. For $d_1 = 2$ and the standard grid (generated by $\pm e_1, \pm e_2$) it means that this set contains all eight neighboring points of the origin on the grid.

Assumption 2.9. There exists an integer $1 \leq d_0 < d_1$ such that for the list

$$(2.20) \quad \mathcal{L} := \{h\ell_{\pm 1}, \dots, h\ell_{\pm d_0}\}$$

and any ℓ_k with $d_1 < |k| \leq d_1$ there exist $l_1, l_2 \in \mathcal{L}$ such that

$$l_1 \neq l_2, \quad l_1 \neq -l_2, \quad \ell_k = l_1 + l_2.$$

One may think that Assumption 2.9 excludes the equations with only one spatial variable, where it is natural to take $d_1 = 1$ and $\Lambda_0 = \{\ell_1, -\ell_1\}$. However, we do not require ℓ_k to be nonzero, and one can take Λ_0 to be $\{\ell_1, -\ell_1, \ell_2, -\ell_2, \ell_3, -\ell_3\}$ with $\ell_2 = 0$ and $\ell_3 = \ell_1$. In that case Assumption 2.9 is satisfied with $\mathcal{L} = \{\ell_1, -\ell_1, \ell_2, -\ell_2\}$. By the way the fact that now the origin is one of the ℓ_k in no way contradicts Assumption 2.8, because in that case $\delta_{h,\ell_k} \phi = 0$ and one can assign any value to a_k^α without changing the equation.

Define

$$Q_2^o = \{(t, x) \in Q : t < T, (t + \tau_T(t), x) \in \bar{Q}, (t, x + \Lambda_0 + \Lambda_0) \subset Q\},$$

$$\partial_2 Q = Q \setminus Q_2^o.$$

Here, naturally, $x + \Lambda_0 + \Lambda_0 = \{x + y + z : y, z \in \Lambda_0\}$.

Theorem 2.12. *Suppose that Assumptions 2.1, 2.2, 2.3, 2.7–2.9 are satisfied. Then there exists a constant $N = N(\delta, d_1, K_0)$ such that if $\lambda \geq N$, then in $Q|_0$ for $i, j = \pm 1, \dots, \pm d_1$ we have*

$$(2.21) \quad |\delta_{h,\ell_j} \delta_{h,\ell_i} u| \leq N^* e^{m^+(T+\tau)} R,$$

where $N^* = N^*(h_0, \delta, d_1, K_0, K_3)$,

$$R = 1 + \max_Q |\xi_{(-)} u| + \max_Q (\xi_{(-)} \delta_\tau^T u)^- \\ + \max_{|i|, |j| \leq d_1} \max_{\partial_2 Q} |\xi_{(-)} \delta_{h,\ell_i} \delta_{h,\ell_j} u| + \max_{|i| \leq d_1} \max_Q |\xi_{(-)} \delta_{h,\ell_i} u|.$$

This theorem is proved in Section 7 following a quite long Section 6 that contains the proof of Theorem 2.12 under additional assumptions.

Remark 2.13. To get “closed” estimates of $\delta_{h,\ell_j} \delta_{h,\ell_i} u$ we need to exclude $\delta_{h,\ell_i} u$ and $\delta_\tau^T u$ from R . This can be done by using Theorem 2.5 and the idea from Remark 2.6. Another situation when $\delta_\tau^T u$ drops out presents when u is independent of t , so that, actually, we are dealing with elliptic equations. We say more about this in the comments after Theorem 2.14

In the case of a_k^α independent of x , Assumption 2.9 is not needed.

Theorem 2.14. *Suppose that Assumptions 2.1, 2.2, 2.3, 2.7, and 2.8 are satisfied. Also assume that the a_k^α are independent of x , $|\delta_{h_i, \ell_i} b_k^\alpha| \leq K_3 \sqrt{a_k^\alpha}$, $|i| \leq d_1 + 1$, $|k| \leq d_1$, and $\lambda > 0$. Then in $Q|_0$ for $k = \pm 1, \dots, \pm d_1$ we have*

$$(2.22) \quad |\Delta_{h,\ell_k} u| \leq N^* e^{m^+(T+\tau)} R_0, \quad (\Delta_{\eta,l} u)^- \leq N^* e^{m^+(T+\tau)} R,$$

where $N^* = N^*(\lambda, h_0, \delta, d_1, K_3)$,

$$R_0 = 1 + \max_Q |\xi_{(-)} u| + \max_Q (\xi_{(-)} \delta_\tau^T u)^- \\ + \max_{|i| \leq d_1} \max_{\partial_1 Q} |\xi_{(-)} \Delta_{h_i, \ell_i} u| + \max_{|i| \leq d_1} \max_Q |\xi_{(-)} \delta_{h_i, \ell_i} u|,$$

and R is obtained from R_0 by taking $d_1 + 1$ in place of d_1 .

This theorem, proved in Section 8, is a direct generalization of the corresponding result from [5]: lower-order coefficients are allowed to depend on (t, x) and we consider parabolic equations. In connection with the latter observe that if $r^\alpha \equiv 0$ (elliptic case), then one can let $m \rightarrow -\infty$ and see that in the definitions of R_0 and R one can replace $\partial_1 Q$ with $Q|_0 \cap \partial_1 Q$.

3. SOME TECHNICAL TOOLS

For any $\eta \in \mathbb{R}^d$, $\nu \geq 0$ set

$$(3.1) \quad T_{\nu,\eta} \psi(x) := \psi(x + \nu \eta).$$

Lemma 3.1. *For any $\nu > 0$, $l_1, l_2 \in \mathbb{R}^d$, and functions $a(x), \psi(x)$,*

$$(3.2) \quad \begin{aligned} \delta_{\nu,l_1}(a\psi) &= (\delta_{\nu,l_1} a)\psi + (T_{\nu,l_1} a)\delta_{\nu,l_1} \psi = a\delta_{\nu,l_1} \psi + \psi\delta_{\nu,l_1} a + \nu(\delta_{\nu,l_1} a)\delta_{\nu,l_1} \psi, \\ \delta_{\nu,l_2} \delta_{\nu,l_1}(a\psi) &= a\delta_{\nu,l_2} \delta_{\nu,l_1} \psi + (\delta_{\nu,l_2} a)\delta_{\nu,l_1} \psi + (\delta_{\nu,l_1} a)\delta_{\nu,l_2} \psi \\ &\quad + h[\delta_{\nu,l_1} a + \delta_{\nu,l_2} a]\delta_{\nu,l_2} \delta_{\nu,l_1} \psi + (\delta_{\nu,l_2} \delta_{\nu,l_1} a)T_{h,l_1+l_2} \psi, \\ \Delta_{\nu,l_1}(a\psi) &= a\Delta_{\nu,l_1} \psi + \psi\Delta_{\nu,l_1} a + (\delta_{\nu,l_1} a)\delta_{\nu,l_1} \psi + (\delta_{\nu,-l_1} a)\delta_{\nu,-l_1} \psi. \end{aligned}$$

In particular,

$$\Delta_{\nu, l_1}(\psi^2) = 2\psi\Delta_{\nu, l_1}\psi + (\delta_{\nu, l_1}\psi)^2 + (\delta_{\nu, -l_1}\psi)^2.$$

This lemma is proved by straightforward computations (cf. [13]). In the following lemma we use Definition 2.2.

Lemma 3.2. *If an operator*

$$S\psi(x) = \sum_{y \in \Lambda \cup \{0\}} p(x, x + y)\psi(x + y)$$

respects the maximum principle at a point $x_0 \in \mathbb{R}^d$ relative to Λ and ψ is a function such that $\psi(x_0) \leq 0$, then $-S\psi \leq S(\psi^-)$ at x_0 . In particular, $\phi^- S\phi^- \geq -\phi^- S\phi$ at x_0 for any function ϕ .

This follows from the definition and the fact that $\psi + \psi^- \geq 0$ on $x_0 + \Lambda$ and $\psi + \psi^- = 0$ at x_0 .

The following lemma from [13] is used in the proof of Theorem 2.12.

Lemma 3.3. *Let ψ be a function on \mathbb{R}^d , $\nu > 0$. Then*

$$(3.3) \quad |\Delta_{\nu, \eta}\psi| \leq |\delta_{\nu, -\eta}((\delta_{\nu, \eta}\psi)^-)| + |\delta_{\nu, \eta}((\delta_{\nu, -\eta}\psi)^-)|.$$

4. PROOF OF THEOREM 2.5

We start with some preparations. From now on index k will run through $\{\pm 1, \dots, \pm d_1\}$ and i, j through $\{\pm 1, \dots, \pm(d_1 + 1)\}$. By N and N^* in this section we denote generic constants depending on the data as in the statement of the theorem. We use the notation (2.11) through (2.14) and introduce a few new objects. We need two constants ε and μ defined by

$$\varepsilon^{-1} - 2\varepsilon d_1 = 1, \quad 4\mu = (d_1 + 1)^{-1} \wedge \varepsilon.$$

Introduce

$$\Gamma = \{\gamma = (\gamma_i : i = \pm 1, \dots, \pm(d_1 + 1)) : \gamma_i \in [\varepsilon, \varepsilon^{-1}]\},$$

$$\delta_i = \delta_{h_i, \ell_i}, \quad P_\gamma \phi = \gamma_i \delta_i \phi, \quad v = \xi u, \quad \Delta_k = \Delta_{h_k, \ell_k}.$$

By using (2.15) we see that in Q ,

$$F(e^{-m\tau} \delta_\tau^T v - c_m v, \xi \Delta_k u, \xi \delta_k u, \xi u) = 0.$$

Also introduce

$$v_\gamma = P_\gamma v, \quad v_i = \delta_i v,$$

$$P_{\gamma\mu} \phi = v_\gamma^- P_\gamma \phi - \mu v_i \delta_i \phi,$$

$$W = \sum_i v_i^2, \quad V_{\gamma\mu} = [v_\gamma^-]^2 + \mu W.$$

Observe that

$$P_{\gamma\mu} v = -V_{\gamma\mu}.$$

Finally, let $(\gamma_0, t_0, x_0) \in \Gamma \times \bar{Q}$ be a point at which $V_{\gamma\mu}$ attains its maximum value over $\Gamma \times \bar{Q}$.

Theorem 4.1. *The assertions of Theorem 2.5 hold true if, in addition to its assumptions, $(t_0, x_0) \in Q_1^o$ and*

$$(4.1) \quad v_{\gamma_0}^-(t_0, x_0) \geq (1/2) \max_{\bar{Q}, i} |v_i|.$$

To prove this theorem we need an auxiliary result.

Lemma 4.2. *Assume (4.1). Then the operator $P_{\gamma_0\mu}$ respects the maximum principle at (t_0, x_0) , that is, for any function ϕ such that $\phi(x_0) \geq \phi(x_0 + \eta)$ for all $\eta \in \Lambda$, we have $P_{\gamma_0\mu}\phi(t_0, x_0) \leq 0$.*

Proof. Since $P_{\gamma\mu}1 = 0$ we may assume that $\phi(x_0) = 0$. Then at (t_0, x_0) ,

$$P_{\gamma_0\mu}\phi = (v_{\gamma_0}^- \gamma_{0i} - \mu v_i)\phi(x_0 + h_i \ell_i),$$

which is negative since $\phi(x_0 + h_i \ell_i) \leq 0$ and $v_{\gamma_0}^- \gamma_{0i} - \mu v_i \geq v_{\gamma_0}^- (\varepsilon - 2\mu) \geq 0$ ($\mu \leq \varepsilon/2$). The lemma is proved. \square

We also need the following construction. Notice that, if $(t_0, x_0) \in Q$, there is a sequence $\alpha_n \in A$ such that at (t_0, x_0) ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} [r^{\alpha_n} \delta_\tau^T u + a_k^{\alpha_n} \Delta_k u + b_k^{\alpha_n} \delta_k u - c^{\alpha_n} u + g^{\alpha_n}] \\ &= \sup_{\alpha \in A} [r^\alpha \delta_\tau^T u + a_k^\alpha \Delta_k u + b_k^\alpha \delta_k u - c^\alpha u + g^\alpha] = 0. \end{aligned}$$

Since the number of possible values of t for points in Q is finite, and the functions $a_k^\alpha(t, x)$, $b_k^\alpha(t, x)$, $c^\alpha(t, x)$, $f^\alpha(p, \psi, t, x)$ are uniformly continuous functions of (p, ψ, x) , there is a subsequence $\{n'\} \subset \{1, 2, \dots\}$ and functions \bar{r} , $\bar{a}_k(t, x)$, $\bar{b}_k(t, x)$, $\bar{c}(t, x)$, $\bar{f}(p, \psi, t, x)$ such that they satisfy our assumptions changed in an obvious way and

$$\begin{aligned} & (\bar{r}, \bar{a}_k(t, x), \bar{b}_k(t, x), \bar{c}(t, x), \bar{f}(p, \psi, t, x)) \\ &= \lim_{n' \rightarrow \infty} (r^{\alpha_{n'}}(t), a_k^{\alpha_{n'}}(t, x), b_k^{\alpha_{n'}}(t, x), c^{\alpha_{n'}}(t, x), f^{\alpha_{n'}}(p, \psi, t, x)) \end{aligned}$$

on Q for all p, ψ .

Obviously, for

$$\bar{g}(t, x) := \bar{f}(\delta_k u(t, x), u(t, x), t, x)$$

at (t_0, x_0) we have

$$\bar{r} \delta_\tau^T u + \bar{a}_k \Delta_k u + \bar{b}_k \delta_k u - \bar{c} u + \bar{g} = 0,$$

$$(4.2) \quad e^{-m\tau} \bar{r} \delta_\tau^T v + \bar{a}_k \Delta_k v + \bar{b}_k \delta_k v - (\bar{c} + \bar{r} c_m) v + \xi \bar{g} = 0$$

and, if $(t_0, x_0) \in Q_1^o$, then for any i ($= \pm 1, \dots, \pm(d_1 + 1)$), the shift operator T is introduced in (3.1)

$$(4.3) \quad T_{h_i, \ell_i} [e^{-m\tau} \bar{r} \delta_\tau^T v + \bar{a}_k \Delta_k v + \bar{b}_k \delta_k v - (\bar{c} + \bar{r} c_m) v + \xi \bar{g}] \leq 0,$$

where here and below for simplicity of notation we drop (t_0, x_0) in the arguments of functions we are dealing with.

Proof of Theorem 4.1. Set

$$\bar{L}_h^0 = \bar{a}_k \Delta_k + \bar{b}_k \delta_k, \quad \bar{L}_h = \bar{L}_h^0 - (\bar{c} + \bar{r} c_m).$$

By Assumption 2.3, Lemma 3.1, and Lemma 3.2 (with v_γ in place of ϕ)

$$\begin{aligned} 0 & \geq \bar{L}_h^0 V_{\gamma_0\mu} = 2v_{\gamma_0}^- \bar{L}_h^0 v_{\gamma_0}^- + 2\mu v_i \bar{L}_h^0 v_i + I_1 + \mu I_2 \\ & \geq -2v_{\gamma_0}^- \bar{L}_h^0 v_{\gamma_0} + 2\mu v_i \bar{L}_h^0 v_i + I_1 + \mu I_2, \end{aligned}$$

where

$$I_1 := 2\bar{a}_k(\delta_k v_{\gamma_0}^-)^2 + h\bar{b}_k(\delta_k v_{\gamma_0}^-)^2,$$

$$I_2 := 2\bar{a}_k \sum_i (\delta_k v_i)^2 + h\bar{b}_k \sum_i (\delta_k v_i)^2.$$

Since $2\bar{a}_k + h\bar{b}_k \geq \bar{a}_k$ we conclude

$$(4.4) \quad v_{\gamma_0}^- \bar{L}_h^0 v_{\gamma_0} - \mu v_i \bar{L}_h^0 v_i \geq (1/2)\mu\bar{a}_k \sum_i (\delta_k v_i)^2.$$

On the other hand, by (4.2) and (4.3) and Lemma 4.2 at (t_0, x_0) ,

$$(4.5) \quad P_{\gamma_0\mu}[e^{-m\tau}\bar{r}\delta_\tau^T v + \bar{L}_h v + \xi\bar{g}] \leq 0.$$

Owing to (4.4) we obtain

$$P_{\gamma_0\mu}\bar{L}_h v = v_{\gamma_0}^- \bar{L}_h^0 v_{\gamma_0} - \mu v_i \bar{L}_h^0 v_i + (\bar{c} + \bar{r}c_m)V_{\gamma_0\mu} + I_3 + I_4 + I_5$$

$$\geq \lambda V_{\gamma_0\mu} + (1/2)\mu\bar{a}_k \sum_i (\delta_k v_i)^2 + I_3 + I_4 + I_5,$$

where

$$I_3 := v_{\gamma_0}^- (P_{\gamma_0}\bar{a}_k)\Delta_k v + h_i v_{\gamma_0}^- \gamma_{0i}(\delta_i \bar{a}_k)\Delta_k v_i$$

$$- \mu v_i(\delta_i \bar{a}_k)\Delta_k v - \mu h_i v_i(\delta_i \bar{a}_k)\Delta_k v_i,$$

$$I_4 := [v_{\gamma_0}^- \gamma_{0i} - \mu v_i](\delta_i \bar{b}_k)T_{h_i, \ell_i} v_k,$$

$$I_5 := [v_{\gamma_0}^- \gamma_{0i} - \mu v_i](\delta_i \bar{c})T_{h_i, \ell_i} v.$$

Upon observing that

$$\Delta_k = -\delta_k \delta_{-k}, \quad h\Delta_k = \delta_k + \delta_{-k}, \quad h_i \leq h$$

and, by assumption (4.1),

$$h|\delta_k v_{-k}| \leq 2 \max_{\bar{Q}, i} |v_i| \leq 4v_{\gamma_0}^-, \quad |h^2 \Delta_k v_i| \leq 4 \max_{\bar{Q}} |v_i| \leq 8v_{\gamma_0}^-,$$

we find

$$|I_3| \leq Nv_{\gamma_0}^-(\sqrt{\bar{a}_k} + h)|\delta_k v_{-k}| + Nv_{\gamma_0}^- \sum_i (\sqrt{\bar{a}_k} + h)|h\Delta_k v_i|$$

$$\leq Nv_{\gamma_0}^- \sum_i \sqrt{\bar{a}_k} |\delta_k v_i| + N(v_{\gamma_0}^-)^2 \leq N(v_{\gamma_0}^-)^2 + (1/4)\mu\bar{a}_k \sum_i (\delta_k v_i)^2.$$

This yields

$$P_{\gamma_0\mu}\bar{L}_h v \geq \lambda V_{\gamma_0\mu} - N(v_{\gamma_0}^-)^2 + (1/4)\mu\bar{a}_k \sum_i (\delta_k v_i)^2 + I_4 + I_5.$$

Next, obviously,

$$|I_4| \leq N(v_{\gamma_0}^-)^2, \quad |I_5| \leq N^* v_{\gamma_0}^- \max_{\bar{Q}} |v|.$$

Therefore, and since $(v_{\gamma_0}^-)^2 \leq V_{\gamma_0\mu}$,

$$(4.6) \quad P_{\gamma_0\mu}\bar{L}_h v \geq (\lambda - N)V_{\gamma_0\mu} - N^* \max_{\bar{Q}} |v|^2 + (1/4)\mu\bar{a}_k \sum_i (\delta_k v_i)^2.$$

Now we deal with other terms in (4.5). Note that, since $(t_0 + \tau_T(t_0), x_0) \in \bar{Q}$,

$$0 \geq \delta_\tau^T V_{\gamma_0\mu} = 2v_{\gamma_0}^- \delta_\tau^T v_{\gamma_0}^- + 2\mu v_i \delta_\tau^T v_i + \tau(\delta_\tau^T v_{\gamma_0}^-)^2 + \mu\tau \sum_i (\delta_\tau^T v_i)^2,$$

so that (cf. Lemma 3.2)

$$0 \geq -v_{\gamma_0}^- \delta_\tau^T v_{\gamma_0} + \mu v_i \delta_\tau^T v_i = -P_{\gamma_0 \mu} \delta_\tau^T v.$$

By recalling that $r^\alpha \geq 0$, we find from (4.6) and (4.5) that

$$(4.7) \quad N^* \max_Q |v|^2 \geq (\lambda - N)V_{\gamma_0 \mu} + \xi P_{\gamma_0 \mu} \bar{g} + (1/4)\mu \bar{a}_k \sum_i (\delta_k v_i)^2.$$

By Assumption 2.4,

$$|\xi P_{\gamma_0 \mu} \bar{g}| \leq NV_{\gamma_0 \mu}^{1/2} \left(\sum_i \sqrt{\bar{a}_k} |\delta_k v_i| \right) + V_{\gamma_0 \mu}^{1/2} + N^* \xi.$$

It follows that the sum of the last two terms in (4.7) is greater than

$$-NV_{\gamma_0 \mu} - N^* \xi V_{\gamma_0 \mu}^{1/2} \geq -NV_{\gamma_0 \mu} - N^* \xi^2.$$

Hence

$$N^* (\max_Q |v|^2 + \xi^2) \geq (\lambda - N_1)V_{\gamma_0 \mu},$$

which for $\lambda - N_1 \geq 1$ shows that on $Q|_0$,

$$\begin{aligned} |\delta_{\eta, l} u|^2 &= |\delta_{\eta, l} v|^2 \leq \mu^{-1} V_{\gamma_0 \mu} \\ &\leq N^* (\max_Q |v|^2 + \xi^2) \leq N^* \xi_{(+)}^2(T) (\max_Q |\xi_{(-)} u|^2 + 1). \end{aligned}$$

This implies (2.16) and the theorem is proved. □

In light of this theorem to prove Theorem 2.5 we only need to show that in the case that the assumptions of Theorem 4.1 are not satisfied one can obtain the assertion of Theorem 2.5 differently. To do that we need two lemmas.

Lemma 4.3. *Take a function ϕ on \mathbb{R}^d and assume that*

$$\max_i |\delta_i \phi(x)| \leq \max_i |\delta_i \phi(0)|$$

for $x \in \Lambda$. Then

$$(4.8) \quad \max_i |\delta_i \phi(0)| \leq \max \left((P_\gamma \phi(x))^- : \gamma \in \Gamma, x \in \{0\} \cup \Lambda \right).$$

Proof. Take a j such that

$$\max_i |\delta_i \phi(0)| = |\delta_j \phi(0)|$$

and first assume that

$$(4.9) \quad |\delta_j \phi(0)| = -\delta_j \phi(0).$$

Then take $\gamma_j = \varepsilon^{-1}$ and $\gamma_i = \varepsilon$ for $i \neq j$. Since $-\delta_j \phi(0) \geq \delta_i \phi(0)$ and $-\delta_j \phi(0) \geq 0$, we have

$$\begin{aligned} P_\gamma \phi(0) &= \varepsilon^{-1} \delta_j \phi(0) + \varepsilon \sum_{i \neq j} \delta_i \phi(0) \\ &\leq (\varepsilon^{-1} - 2\varepsilon d_1) \delta_j \phi(0) = -(\varepsilon^{-1} - 2\varepsilon d_1) \max_i |\delta_i \phi(0)| \end{aligned}$$

and (4.8) follows since $\varepsilon^{-1} - 2\varepsilon d_1 = 1$.

If (4.9) is not satisfied, then

$$\max_i |\delta_i \phi(h_j \ell_j)| \leq |\delta_j \phi(0)| = \delta_j \phi(0) = -\delta_{-j} \phi(h_j \ell_j),$$

which combined with an obvious inequality between the extreme terms yields

$$\begin{aligned} \max_i |\delta_i \phi(h_j \ell_j)| &= -\delta_{-j} \phi(h_j \ell_j) = |\delta_{-j} \phi(h_j \ell_j)| \\ &= |\delta_j \phi(0)| = \max_i |\delta_i \phi(0)|. \end{aligned}$$

By the above argument applied to the point $h_j \ell_j$ in place of 0,

$$\begin{aligned} \min_{\Gamma} P_{\gamma} \phi(h_j \ell_j) &\leq (\varepsilon^{-1} - 2\varepsilon d_1) \delta_{-j} \phi(h_j \ell_j) \\ &= -(\varepsilon^{-1} - 2\varepsilon d_1) \max_i |\delta_i \phi(0)| = -\max_i |\delta_i \phi(0)|. \end{aligned}$$

The lemma is proved. □

Lemma 4.4. *Condition (4.1) is satisfied if $(t_0, x_0) \in Q_1^o$ and*

$$(4.10) \quad \max_{Q,i} v_i^2 = \max_{Q_1^o,i} v_i^2.$$

Indeed if (4.1) does not hold, then

$$\begin{aligned} \max_{\Gamma \times Q} [v_{\gamma}^{-}]^2 &\leq V_{\gamma_0 \mu}(t_0, x_0) \\ &< (2(d_1 + 1)\mu + 1/4) \max_{Q,i} v_i^2 = (2(d_1 + 1)\mu + 1/4) \max_{Q_1^o,i} v_i^2, \end{aligned}$$

which contradicts Lemma 4.3 since $\mu \leq 1/(4(d_1 + 1))$.

Proof of Theorem 2.5. If $(t_0, x_0) \in \partial_1 Q$, then in $Q|_0$,

$$|\delta_{\eta,l} u| = |\delta_{\eta,l} v| \leq \mu^{-1/2} V_{\gamma_0 \mu}^{1/2} \leq \mu^{-1/2} V_{\gamma_0 \mu}^{1/2}(t_0, x_0),$$

where the last term is obviously less than the right-hand side of (2.16). Furthermore, if $(t_0, x_0) \in Q_1^o$ but (4.1) is violated, then by Lemma 4.4,

$$\max_{Q,i} v_i^2 = \max_{\partial_1 Q,i} v_i^2$$

and one can use the above argument. Finally, in the remaining case, Theorem 4.1 is applicable. The theorem is proved. □

Remark 4.5. As in [5] one can relax conditions on λ by adding in its expression any large constant times $\min_k a_k^{\alpha}$ provided that $\Lambda = \Lambda_0$. This can be shown by considering maximum points of $V_{\gamma \mu} + \nu v^2$, where ν is a large constant. This would also allow us to relax the condition on $D_{\psi} f$ to $|D_{\psi} f| \leq K_0 + K_3 \min_k a_k^{\alpha}$.

5. PROOF OF THEOREM 2.9

Our goal is to show how to choose an appropriate $C = C(d_1)$ in (2.18). Below in this section, by N we denote generic constants depending only on $d_1, \delta, K_1, K_2, K_3$ but not ω .

First of all, observe that l does not enter either equation (2.9) or the statement of the theorem. It is involved, however, in the definition of $\partial_1 Q$, making it “fatter”. Because of that if we additionally assume that $l = 0$, the result will be stronger. Therefore, we assume that $l = 0$.

Set $m = 0$ and introduce $\varepsilon, \Gamma, v_{\gamma}, v_i, V_{\gamma \mu}, W, P_{\gamma}, P_{\gamma \mu}$ as in the beginning of Section 4. However, since $v = u$, we also write u_{γ}, u_i , and $U_{\gamma \mu}$ instead of v_{γ}, v_i , and $V_{\gamma \mu}$, respectively. Since $l = 0$, $u_{\pm(d_1+1)} = v_{\pm(d_1+1)} = 0$ and now there is no need to allow i to take the values $\pm(d_1 + 1)$. Therefore, we restrict i to the range

$\pm 1, \dots, \pm d_1$. As everywhere in the article, index k runs through $\pm 1, \dots, \pm d_1$. This time we take

$$8\mu = d_1^{-1} \wedge \varepsilon.$$

Introduce $\kappa \geq 0$ as a solution of

$$(5.1) \quad \max_{\Gamma \times \bar{Q}} [U_{\gamma\mu} + \kappa u^2] = 4\kappa K_1^2.$$

Observe that if $\kappa = 0$ is a solution of (5.1), then $u_k \equiv 0$ and the assertion of the theorem is trivial. Therefore, without losing generality we may assume that for $\kappa = 0$ the left-hand side of (5.1) is strictly greater than its right-hand side. Furthermore, as a function of κ the left-hand side is convex, increasing with Lipschitz constant not greater than K_1^2 . It follows that (5.1) has a unique solution $\kappa > 0$. After that we define (γ_0, t_0, x_0) as a point in $\Gamma \times \bar{Q}$ at which the maximum in (5.1) is attained. For simplicity of notation we drop the arguments (t_0, x_0) in what follows. We also use the abbreviated notation δ_i, Δ_k introduced in Section 4.

First we show that $[u_{\gamma_0}^-]^2$ is the main term in $U_{\gamma_0\mu} + \kappa u^2$.

Lemma 5.1. *Assume that (4.10) holds. Then*

$$(5.2) \quad [u_{\gamma_0}^-]^2 \geq 2\kappa K_1^2,$$

$$(5.3) \quad 2d_1\varepsilon^{-1} \max_i |u_i| \geq u_{\gamma_0}^- \geq (1/2) \max_{\bar{Q},i} |u_i|.$$

In particular $u_{\gamma_0} < 0$ and the operator $P_{\gamma_0\mu}$ respects the maximum principle at (t_0, x_0) .

Proof. First, notice that, owing to (4.10), Lemma 4.3, and the definitions of κ and (t_0, x_0) we have

$$\begin{aligned} 4\kappa K_1^2 &= [u_{\gamma_0}^-]^2 + \mu \sum_i u_i^2 + \kappa u^2 \leq [u_{\gamma_0}^-]^2 + 2d_1\mu \max_{\bar{Q},i} u_i^2 + \kappa K_1^2 \\ &\leq [u_{\gamma_0}^-]^2 + 2d_1\mu \max_{\Gamma \times \bar{Q}} [u_{\gamma}^-]^2 + \kappa K_1^2 \leq [u_{\gamma_0}^-]^2 + 2d_1\mu 4\kappa K_1^2 + \kappa K_1^2. \end{aligned}$$

This implies (5.2) since $8d_1\mu \leq 1$.

The first inequality in (5.3) is obvious. If the second one is wrong, then

$$\begin{aligned} \max_{\Gamma \times \bar{Q}} [u_{\gamma}^-]^2 &\leq U_{\gamma_0\mu} + \kappa u^2 < [2d_1\mu + 1/4] \max_{\bar{Q},i} u_i^2 + \kappa K_1^2 \\ &\leq [2d_1\mu + 1/4] \max_{\Gamma \times \bar{Q}} [u_{\gamma}^-]^2 + \kappa K_1^2 \leq (1/2) \max_{\Gamma \times \bar{Q}} [u_{\gamma}^-]^2 + \kappa K_1^2 \end{aligned}$$

contrary to (5.2). The last assertion of the lemma follows from Lemma 4.2. The lemma is proved. \square

Next, if $(t_0, x_0) \in \partial_1 Q$, then

$$4\kappa K_1^2 = U_{\gamma_0\mu} + \kappa u^2 \leq N' + \kappa K_1^2, \quad 3\kappa K_1^2 \leq N',$$

and $u_k^2(t, x) \leq (4/3)\mu^{-1}N'$ in \bar{Q} according to (5.1). In this case the assertion of the theorem is true. Similarly, as in the proof of Theorem 2.5 we get the result if (4.10) is violated. This justifies the first two assumptions in the following set which we impose:

$$(t_0, x_0) \in Q_1^o, \quad \max_{\bar{Q},k} u_k^2 = \max_{Q_1^o,k} u_k^2,$$

$$(5.4) \quad 4d_1\varepsilon^{-1}W^{1/2} \geq \max_{Q_{1,k}^0} |u_k|, \quad W \geq K_2^2 + 8K_1K_3\delta^{-1} + 1.$$

The third relation in (5.4) follows from Lemma 5.1, and the last assumption in (5.4) restricts us to the only nontrivial case in light of (5.3).

Next again owing to the fact that the number of points in Q is finite we can find some functions $\bar{r}(t)$, $\bar{a}_k(t, x)$, $\bar{c}(t, x)$, and $\bar{f}(p, \psi, r, t, x)$ satisfying Assumptions 2.1 and 2.5 and such that for

$$\bar{g}(t, x) := \bar{f}(u_k(t, x), u(t, x), t, x) - \bar{c}(t, x)u(t, x), \quad \bar{L}_h := \bar{a}_k\Delta_k$$

we have (recall that the arguments (t_0, x_0) are dropped)

$$(5.5) \quad \bar{r}\delta_\tau^T u + \bar{L}_h u + \bar{g} = 0, \quad T_{h,\ell_i}[\bar{r}\delta_\tau^T u + \bar{L}_h u + \bar{g}] \leq 0$$

for any i ($= \pm 1, \dots, \pm d_1$).

Now, as before,

$$(5.6) \quad \begin{aligned} 0 \geq \bar{L}_h[U_{\gamma_0\mu} + \kappa u^2] &\geq -2u_{\gamma_0}^- \bar{L}_h u_{\gamma_0} + 2\mu u_i \bar{L}_h u_i + 2\kappa u \bar{L}_h u \\ &+ 2\bar{a}_k[(\delta_k u_{\gamma_0}^-)^2 + \mu \sum_i (\delta_k u_i)^2 + \kappa(\delta_k u)^2]. \end{aligned}$$

Furthermore,

$$(5.7) \quad \begin{aligned} 0 \geq \delta_\tau^T [U_{\gamma_0\mu} + \kappa u^2] &= 2u_{\gamma_0}^- \delta_\tau^T u_{\gamma_0}^- + 2\mu u_i \delta_\tau^T u_i + 2\kappa u \delta_\tau^T u \\ &+ 2\tau[(\delta_\tau^T u_{\gamma_0}^-)^2 + \mu \sum_i (\delta_\tau^T u_i)^2 + \kappa(\delta_\tau^T u)^2] \\ &\geq -2u_{\gamma_0}^- \delta_\tau^T u_{\gamma_0} + 2\mu u_i \delta_\tau^T u_i + 2\kappa u \delta_\tau^T u. \end{aligned}$$

We multiply (5.7) by \bar{r} , add the result to (5.6), and use that $\bar{r} \geq 0$, $\bar{c} \geq -K_3$, $\bar{a}_k \geq \delta$, $|u| \leq K_1$, and

$$u(\bar{r}\delta_\tau^T u + \bar{L}_h u) = -u\bar{g} = \bar{c}u^2 - u(\bar{g} + \bar{c}u) \geq -K_3K_1 - u(\bar{g} + \bar{c}u),$$

where $|\bar{g} + \bar{c}u| \leq \omega W + K_3$ (recall (5.4)). Then we obtain

$$\begin{aligned} u(\bar{r}\delta_\tau^T u + \bar{L}_h u) &\geq -\omega K_1 W - 2K_1 K_3, \\ u_{\gamma_0}^- [\bar{r}\delta_\tau^T + \bar{L}_h] u_{\gamma_0} - \mu u_i [\bar{r}\delta_\tau^T + \bar{L}_h] u_i &\geq \delta\mu M + \kappa W(\delta - K_1\omega) - 2\kappa K_1 K_3, \end{aligned}$$

where

$$M := \sum_{i,k} (\delta_k u_i)^2.$$

Since $C \geq 4$ in Assumption 2.6, we have $K_1\omega \leq \delta/4$ which, along with (5.4), leads to

$$W(\delta - K_1\omega) - 2K_1 K_3 \geq (3/4)\delta W - 2K_1 K_3 \geq (\delta/2)W,$$

$$(5.8) \quad u_{\gamma_0}^- [\bar{r}\delta_\tau^T + \bar{L}_h] u_{\gamma_0} - \mu u_i [\bar{r}\delta_\tau^T + \bar{L}_h] u_i \geq \delta\mu M + (\delta/2)\kappa W.$$

On the other hand, owing to (5.4), (5.5), and Lemma 5.1,

$$(5.9) \quad P_{\gamma_0\mu}[\bar{r}\delta_\tau^T u + \bar{L}_h u + \bar{g}] \leq 0.$$

Here due to (5.8),

$$\begin{aligned} P_{\gamma_0\mu}[\bar{r}\delta_\tau^T u + \bar{L}_h u] &= u_{\gamma_0}^- [\bar{r}\delta_\tau^T + \bar{L}_h] u_{\gamma_0} - \mu u_i [\bar{r}\delta_\tau^T + \bar{L}_h] u_i + I \\ &\geq \delta\mu M + (1/2)\delta\kappa W + I, \end{aligned}$$

where

$$I := [u_{\gamma_0}^- \gamma_{0i} - \mu u_i](\delta_i \bar{a}_k)[\Delta_k u + h \Delta_k u_i].$$

Below by C we denote generic constants depending only on d_1 . It follows from the estimates

$$\begin{aligned} |\delta_i \bar{a}_k| &\leq K_3 + \omega |u_i|, & |h \Delta_k u_i| &= |\delta_k u_i + \delta_{-k} u_i| \leq 2M^{1/2}, \\ |\Delta_k u| &\leq M^{1/2}, & |u_{\gamma_0}^- \gamma_{0i} - \mu u_i| &\leq CW^{1/2}, \end{aligned}$$

that

$$|I| \leq CW^{1/2} M^{1/2} (K_3 + \omega W^{1/2}) \leq NW + (1/2)\delta\mu M + C\delta^{-1}\omega^2 W^2.$$

Hence,

$$(5.10) \quad P_{\gamma_0\mu}[\bar{r}\delta_\tau^T u + \bar{L}_h u] \geq (1/2)\delta\mu M + (1/2)\delta\kappa W - NW - C\delta^{-1}\omega^2 W^2.$$

To estimate $P_{\gamma_0\mu}\bar{g}$ recall that $\bar{c} \geq -N$, $U_{\gamma_0\mu} \geq 0$ and observe that

$$\begin{aligned} -P_{\gamma_0\mu}(\bar{c}u) &= -\bar{c}P_{\gamma_0\mu}u - [u_{\gamma_0}^- \gamma_{0k} - \mu u_k](T_{h,\ell_k} u)\delta_k \bar{c} \\ &= \bar{c}U_{\gamma_0\mu} - [u_{\gamma_0}^- \gamma_{0k} - \mu u_k](T_{h,\ell_k} u)\delta_k \bar{c} \geq -NW - NW^{1/2} \geq -NW, \end{aligned}$$

where the last inequality follows from (5.4). Furthermore,

$$h\delta_i(\bar{g} + \bar{c}u) = \bar{f}(T_{h,\ell_i} u_k, T_{h,\ell_i} u, t_0, x_0 + h\ell_i) - \bar{f}(u_k, u, t_0, x_0).$$

Owing to Assumption 2.5, (5.4), and the mean value theorem (this is the place, where one cannot assume that (2.17) holds only for large p)

$$\begin{aligned} |\delta_i(\bar{g} + \bar{c}u)| &\leq CM^{1/2}(\omega W^{1/2} + K_3) \\ &+ C[W^{1/2}(\omega W + K_3) + \omega W^{3/2} + K_3]. \end{aligned}$$

Note that the coefficients of $\delta_i(\bar{g} + \bar{c}u)$ in $P_{\gamma_0\mu}(\bar{g} + \bar{c}u)$ are dominated by $W^{1/2}$ and for any $\rho > 0$,

$$\omega M^{1/2}W \leq \rho M + \rho^{-1}\omega^2 W^2, \quad K_3 M^{1/2}W^{1/2} \leq \rho M + \rho^{-1}NW.$$

Therefore,

$$\begin{aligned} |P_{\gamma_0\mu}(\bar{g} + \bar{c}u)| &\leq (1/2)\delta\mu M + C[\delta^{-1}\omega^2 + \omega]W^2 + NW, \\ P_{\gamma_0\mu}\bar{g} &\geq -(1/2)\delta\mu M - C[\delta^{-1}\omega^2 + \omega]W^2 - NW \end{aligned}$$

and (5.9) and (5.10) yield

$$(1/2)\delta\kappa W \leq C[\delta^{-1}\omega^2 + \omega]W^2 + NW,$$

$$(5.11) \quad 2\delta\kappa \leq C_1[\delta^{-1}\omega^2 + \omega]W + N_1.$$

Since $W \leq \mu^{-1}4K_1^2\kappa$ (see (5.1)), we see that if C in Assumption 2.6 is such that

$$C \geq 4\mu^{-1}C_1,$$

then (recall that $C \geq 4$ and $\omega K_1 \leq \delta$)

$$C_1[\delta^{-1}\omega^2 + \omega]W \leq CK_1^2[\delta^{-1}\omega^2 + \omega]\kappa \leq C[K_1\omega + K_1^2\omega]\kappa \leq \delta\kappa.$$

In this case (5.11) allows us to conclude that $\kappa \leq \delta^{-1}N_1$ and we get the assertion of the theorem from (5.1). The theorem is proved.

6. CONDITIONAL ESTIMATES OF THE SECOND-ORDER DIFFERENCES

In this section we suppose that all the assumptions of Theorem 2.12 are satisfied apart from Assumptions 2.8 and 2.9. The notation in this section is somewhat different from Sections 4 and 5. Of course, we use our basic notation from Section 2; for instance, ξ and λ are defined in (2.14).

For $\varepsilon \in (0, 1]$ set (observe that now $\gamma_k = \gamma_{-k}$)

$$\Gamma(\varepsilon) = \{\gamma = (\gamma_k : k = \pm 1, \dots, \pm d_1) : \gamma_k = \gamma_{-k}, \varepsilon \leq \gamma_k \leq \varepsilon^{-1}, \forall k\}.$$

Fix a constant $\mu = \mu(d_1, \varepsilon) > 0$ such that

$$16d_1^2\mu \leq \varepsilon^2.$$

In this section the indices i, j, k, p, q run through $\{\pm 1, \dots, \pm d_1\}$. The main result of this section is the following.

Theorem 6.1. *Assume that*

$$(6.1) \quad 3\mu \max_Q \sum_{i,j} [(\xi \delta_{h,\ell_j} \delta_{h,\ell_i} u)^-]^2 \leq \max_{\Gamma(\varepsilon) \times Q} [(\xi \sum_i \gamma_i \Delta_{h,\ell_i} u)^-]^2.$$

Then there exists a constant $N = N(\varepsilon, \mu, d_1, K_0)$ such that if $\lambda \geq N$, then in $Q|_0$ for $i, j = \pm 1, \dots, \pm d_1$ we have

$$(6.2) \quad |\delta_{h,\ell_j} \delta_{h,\ell_i} u| \leq N^* e^{m^+(T+\tau)} (1 + \max_{\partial_2 Q, p, q} |\xi_{(-)} \delta_{h,\ell_p} \delta_{h,\ell_q} u| + \max_Q |\xi_{(-)} u| + \max_{Q, p} |\xi_{(-)} \delta_{h,\ell_p} u|),$$

where $N^* = N^*(h_0, \varepsilon, \mu, d_1, K_0, K_3)$.

Below in this section, by N and N^* we denote generic constants of the same type as in the theorem. As before, we use the abbreviated notation

$$\Delta_i = \Delta_{h,\ell_i}, \quad \delta_i = \delta_{h,\ell_i}.$$

Introduce $v = \xi u$ as in Section 4 and fix a constant $\nu \geq 1$. Set

$$\begin{aligned} P_\gamma &= \gamma_i \Delta_i, \quad v_\gamma = P_\gamma v, \quad v_i = \delta_i v, \quad v_{ij} = \delta_j \delta_i v, \\ P_{\gamma\mu\nu} \phi &= v_\gamma^- P_\gamma \phi + \mu v_{ij}^- \delta_j \delta_i \phi - \nu v_i \delta_i \phi, \\ W_1 &= \sum_i v_i^2, \quad W_2 = \sum_{i,j} [v_{ij}^-]^2, \quad V_{\gamma\mu\nu} = [v_\gamma^-]^2 + \mu W_2 + \nu W_1. \end{aligned}$$

Observe that this time again $P_{\gamma\mu\nu} v = -V_{\gamma\mu\nu}$ and also note that (6.1) is equivalent to the following:

$$(6.3) \quad 3\mu \max_Q W_2 \leq \max_{\Gamma(\varepsilon) \times Q} (v_\gamma^-)^2.$$

We introduce (γ_0, t_0, x_0) as a point in $\Gamma(\varepsilon) \times \bar{Q}$ maximizing $V_{\gamma\mu\nu}$ and first prove a few auxiliary results. Below, as usual, we drop the arguments (t_0, x_0) .

Lemma 6.2. (i) *For $(t, x) \in Q_1^o$ and any i, j ,*

$$(6.4) \quad |v_{ij}(t, x)| \leq \max_Q W_2^{1/2}.$$

(ii) *If (6.3) holds and $(t_0, x_0) \in Q$ and*

$$(6.5) \quad \nu \max_Q W_1 \leq \mu \max_Q W_2,$$

then at (t_0, x_0) ,

$$(6.6) \quad \mu \max_Q W_2 \leq [v_{\gamma_0}^-]^2, \quad \nu \max_Q W_1 \leq [v_{\gamma_0}^-]^2.$$

Furthermore, if additionally,

$$h\sqrt{\nu} \leq \varepsilon,$$

then the operator $P_{\gamma_0\mu\nu}$ respects the maximum principle at (t_0, x_0) relative to $\Lambda_0 + \Lambda_0$, that is, for any function ϕ such that $\phi(x_0) \geq \phi(x_0 + \eta)$ for all $\eta \in \Lambda_0 + \Lambda_0$, we have $P_{\gamma_0\mu\nu}\phi(t_0, x_0) \leq 0$.

Proof. (i) Obviously $v_{-i,j}^- \leq W_2^{1/2}$ on Q . Since (no summation in i) $T_{h,\ell_i}v_{-i,j}^- = v_{ij}^+$, we get $v_{ij}^+(t, x) \leq W_2^{1/2}(t, x + h\ell_i)$. This proves (i).

(ii) The second estimate in (6.6) follows from the first one and (6.5). Assuming that the first estimate in (6.6) does not hold, we obtain at (t_0, x_0) ,

$$\begin{aligned} V_{\gamma_0\mu\nu} &< 2\mu \max_Q W_2 + \nu \max_Q W_1 \leq 3\mu \max_Q W_2, \\ \max_{\Gamma(\varepsilon) \times Q} [v_{\gamma}^-]^2 &\leq V_{\gamma_0\mu\nu} < 3\mu \max_Q W_2, \end{aligned}$$

contrary to (6.3). This proves (6.6).

To prove the last assertion of the lemma we take a function with described properties and without loss of generality assume that $\phi(x_0) = 0$. We also note that

$$\begin{aligned} h^2\delta_i\delta_j\phi(x_0) &= \phi(x_0 + h\ell_i + h\ell_j) - \phi(x_0 + h\ell_i) - \phi(x_0 + h\ell_j) \\ &\quad + \phi(x_0) \leq -\phi(x_0 + h\ell_i) - \phi(x_0 + h\ell_j) \end{aligned}$$

and $\Delta_i\phi(x_0) \leq 0$. Therefore, as usual dropping the arguments (t_0, x_0) in $v\dots$, we infer from (6.6) that

$$\begin{aligned} h^2P_{\gamma_0\mu\nu}\phi(t_0, x_0) &\leq v_{\gamma_0}^- \sum_i (\phi(x_0 + h\ell_i) + \phi(x_0 - h\ell_i)) \\ &\quad - \mu v_{ij}^- (\phi(x_0 + h\ell_i) + \phi(x_0 + h\ell_j)) - h\nu v_i \phi(x_0 + h\ell_i) \\ &\leq v_{\gamma_0}^- [\varepsilon \sum_i (\phi(x_0 + h\ell_i) + \phi(x_0 - h\ell_i))] \\ &\quad - \sqrt{\mu} \sum_{i,j} (\phi(x_0 + h\ell_i) + \phi(x_0 + h\ell_j)) - h\sqrt{\nu} \sum_i \phi(x_0 + h\ell_i) \\ &= v_{\gamma_0}^- \sum_i \phi(x_0 + h\ell_i) [2\varepsilon - 2d_1\sqrt{\mu} - h\sqrt{\nu}]. \end{aligned}$$

The last expression is less than zero in light of the fact that $4d_1\sqrt{\mu} \leq \varepsilon$ and $h\sqrt{\nu} \leq \varepsilon$. The lemma is proved. \square

Remark 6.3. This lemma can be generalized to the case when the ℓ_k 's come with different h_k 's, but the h_k 's should be comparable. This is the reason why in Theorem 2.12 we do not include h_{d_1+1} and ℓ_{d_1+1} .

Set

$$\begin{aligned} Z_k &:= v_{\gamma_0}^- \Delta_k v_{\gamma_0} + \mu v_{ij}^- \Delta_k v_{ij} - \nu v_i \Delta_k v_i, \\ z_k &:= v_{\gamma_0}^- \delta_k v_{\gamma_0} + \mu v_{ij}^- \delta_k v_{ij} - \nu v_i v_{ki}, \\ R_k^\gamma &:= [\delta_k v_{\gamma_0}^-]^2, \quad R_k^\mu := \sum_{i,j} [\delta_k v_{ij}^-]^2, \quad R_k^\nu := \sum_i v_{ki}^2. \end{aligned}$$

These objects evaluated at (t_0, x_0) will be extensively used below in this section.

Lemma 6.4. *If $(t_0, x_0) \in Q_1^o$, then at (t_0, x_0) we have for any k ,*

$$(6.7) \quad \begin{aligned} 2Z_k &\geq -2v_{\gamma_0}^- \Delta_k v_{\gamma_0}^- - 2\mu v_{ij}^- \Delta_k v_{ij}^- - 2\nu v_i \Delta_k v_i \\ &\geq R_k^\gamma + R_{-k}^\gamma + \mu R_k^\mu + \mu R_{-k}^\mu + \nu R_k^\nu + \nu R_{-k}^\nu \geq 0. \end{aligned}$$

Furthermore, for any $\alpha \in A$ and $h \leq h_0/2$,

$$(6.8) \quad 4(a_k^\alpha Z_k + b_k^\alpha z_k) \geq 2a_k^\alpha Z_k + a_k^\alpha (R_k^\gamma + \mu R_k^\mu + \nu R_k^\nu).$$

Proof. The first inequality in (6.7) follows from Lemma 3.2. To prove the second one it suffices to observe that

$$\begin{aligned} 0 &\geq \Delta_k V_{\gamma_0 \mu \nu} = 2v_{\gamma_0}^- \Delta_k v_{\gamma_0}^- + 2\mu v_{ij}^- \Delta_k v_{ij}^- + 2\nu v_i \Delta_k v_i \\ &\quad + R_k^\gamma + R_{-k}^\gamma + \mu R_k^\mu + \mu R_{-k}^\mu + \nu R_k^\nu + \nu R_{-k}^\nu. \end{aligned}$$

Next, using that $h \leq h_0/2$, by (6.7) and Assumption 2.3 we get

$$\begin{aligned} 0 &\geq (a_k^\alpha \Delta_k + 2b_k^\alpha \delta_k) V_{\gamma_0 \mu \nu} = 2v_{\gamma_0}^- [a_k^\alpha \Delta_k + 2b_k^\alpha \delta_k] v_{\gamma_0}^- \\ &\quad + 2\mu v_{ij}^- [a_k^\alpha \Delta_k + 2b_k^\alpha \delta_k] v_{ij}^- + 2\nu v_i [a_k^\alpha \Delta_k + 2b_k^\alpha \delta_k] v_i \\ &\quad + 2a_k^\alpha [R_k^\gamma + \mu R_k^\mu + \nu R_k^\nu] + 2hb_k^\alpha [R_k^\gamma + \mu R_k^\mu + \nu R_k^\nu] \\ &\quad \geq -2v_{\gamma_0}^- [a_k^\alpha \Delta_k + 2b_k^\alpha \delta_k] v_{\gamma_0} \\ &\quad - 2\mu v_{ij}^- [a_k^\alpha \Delta_k + 2b_k^\alpha \delta_k] v_{ij} + 2\nu v_i [a_k^\alpha \Delta_k + 2b_k^\alpha \delta_k] v_i \\ &\quad + 2a_k^\alpha [R_k^\gamma + \mu R_k^\mu + \nu R_k^\nu] + 2hb_k^\alpha [R_k^\gamma + \mu R_k^\mu + \nu R_k^\nu] \\ &\quad \geq -2a_k^\alpha Z_k - 4b_k^\alpha z_k + a_k^\alpha [R_k^\gamma + \mu R_k^\mu + \nu R_k^\nu] \end{aligned}$$

and (6.8) follows. The lemma is proved. \square

In the following lemma we do the most important step in the proof of Theorem 6.1. Set

$$\bar{W}_1 := \max_Q W_1.$$

Lemma 6.5. *Under the assumptions of Theorem 6.1 there are constants N, N^* ,*

$$\nu = \nu^*(h_0, \varepsilon, \mu, d_1, K_0, K_3) \geq 1, \quad h^* = h^*(h_0, \varepsilon, \mu, d_1, K_0, K_3) > 0$$

such that $h^ \leq h_0/2$ and, if condition (6.5) is satisfied and $h \in (0, h^*]$ and $(t_0, x_0) \in Q_2^o$, then at (t_0, x_0) for any $\alpha \in A$ we have*

$$(6.9) \quad J := P_{\gamma_0 \mu \nu} (a_k^\alpha \Delta_k v + b_k^\alpha \delta_k v) \geq -N[v_{\gamma_0}^-]^2 - N^* \bar{W}_1.$$

Proof. We fix an $\alpha \in A$ and drop the superscript α for convenience. By (3.2) (no summation in k, i, j)

$$\begin{aligned} \delta_j \delta_i (b_k v_k) &= b_k \delta_k v_{ij} + (\delta_j b_k) v_{ki} + (\delta_i b_k) v_{kj} \\ &\quad + h[(\delta_i + \delta_j) b_k] \delta_k v_{ij} + (\delta_j \delta_i b_k) T_{h, \ell_i + \ell_j} v_k. \end{aligned}$$

Also by using (3.2) and the formulas $a_k = a_{-k}$ and $h\Delta_k = \delta_k + \delta_{-k}$ and summing with respect to k (but not in i, j) we get

$$\begin{aligned} \delta_j \delta_i (a_k \Delta_k v) &= a_k \Delta_k v_{ij} + (\delta_j a_k) \Delta_k v_i + (\delta_i a_k) \Delta_k v_j \\ &\quad + 2[(\delta_j + \delta_i) a_k] \delta_k v_{ij} + (\delta_j \delta_i a_k) T_{h, \ell_j + \ell_i} \Delta_k v. \end{aligned}$$

While applying this formula to Δ_i it is also useful to observe that

$$[(\delta_{-i} + \delta_i) a_k] \delta_k v_{i, -i} = -h(\Delta_i a_k) \Delta_i \delta_k v = -(\Delta_i a_k)(v_{ki} + v_{k, -i}).$$

Hence (recall that $\gamma_{0i} = \gamma_{0,-i}$ and $\ell_i = -\ell_{-i}$),

$$\begin{aligned} P_{\gamma_0}(a_k \Delta_k v + b_k \delta_k v) &= -\gamma_{0i} \delta_{-i} \delta_i (a_k \Delta_k v + b_k \delta_k v) \\ &= (a_k \Delta_k + b_k \delta_k) v_{\gamma_0} - 2\gamma_{0i} (\delta_{-i} a_k) \Delta_k v_i + 4\gamma_{0i} (\Delta_i a_k) v_{ki} \\ &\quad + \gamma_{0i} (\Delta_i a_k) \Delta_k v + 2\gamma_{0i} (\delta_i b_k) v_{ki} + \gamma_{0i} (\Delta_i b_k) v_k. \end{aligned}$$

Also, everywhere,

$$\begin{aligned} v_{ij}^- \delta_j \delta_i (a_k \Delta_k + b_k \delta_k) v &= v_{ij}^- (a_k \Delta_k + b_k \delta_k) v_{ij} + 2v_{ij}^- (\delta_j a_k) \Delta_k v_i \\ &\quad + h v_{ij}^- [(\delta_j + \delta_i) a_k] \Delta_k v_{ij} + v_{ij}^- (\delta_j \delta_i a_k) T_{h, \ell_j + \ell_i} \Delta_k v \\ &\quad + 2v_{ij}^- (\delta_j b_k) v_{ki} + 2h v_{ij}^- (\delta_i b_k) \delta_k v_{ij} + v_{ij}^- (\delta_j \delta_i b_k) T_{h, \ell_i + \ell_j} v_k. \end{aligned}$$

Therefore at (t_0, x_0) we have

$$J = a_k Z_k + b_k z_k + I_1 + \dots + I_4,$$

where

$$\begin{aligned} I_1 &= -2\gamma_{0i} v_{\gamma_0}^- (\delta_{-i} a_k) \Delta_k v_i + 2\mu v_{ij}^- (\delta_j a_k) \Delta_k v_i, \\ I_2 &= \mu h v_{ij}^- [(\delta_j + \delta_i) a_k] \Delta_k v_{ij}, \\ I_3 &= v_{\gamma_0}^- [4\gamma_{0i} (\Delta_i a_k) v_{ki} + \gamma_{0i} (\Delta_i a_k) \Delta_k v] \\ &\quad + \mu v_{ij}^- (\delta_j \delta_i a_k) T_{h, \ell_j + \ell_i} \Delta_k v, \\ I_4 &= 2v_{\gamma_0}^- \gamma_{0i} (\delta_i b_k) v_{ki} + v_{\gamma_0}^- \gamma_{0i} (\Delta_i b_k) v_k \\ &\quad + 2\mu v_{ij}^- (\delta_j b_k) v_{ki} + 2\mu h v_{ij}^- (\delta_i b_k) \delta_k v_{ij} + \mu v_{ij}^- (\delta_j \delta_i b_k) T_{h, \ell_i + \ell_j} v_k \\ &\quad - \nu v_i (\delta_i a_k) (\Delta_k v + 2v_{ki}) - \nu v_i (\delta_i b_k) T_{h, \ell_i} v_k. \end{aligned}$$

For $h \leq h_0/2$ it follows by Lemma 6.4 that

$$(6.10) \quad 4J \geq 2a_k Z_k + a_k (R_k^\gamma + \mu R_k^\mu + \nu R_k^\nu) + 4I_1 + \dots + 4I_4.$$

Estimating I_1 . Note that owing to (6.6),

$$|4I_1| \leq N v_{\gamma_0}^- (\sqrt{a_k} + h) \sum_i |\Delta_k v_i|$$

and by (3.3),

$$\begin{aligned} N v_{\gamma_0}^- \sqrt{a_k} \sum_i |\Delta_k v_i| &\leq N v_{\gamma_0}^- \sqrt{a_k} \sum_i [|\delta_{-k} v_{ki}^-| + |\delta_k v_{-k,i}^-|] \\ &\leq N (v_{\gamma_0}^-)^2 + (1/3) \mu a_k R_k^\mu. \end{aligned}$$

Furthermore, by the formula $h \Delta_k = \delta_k + \delta_{h, \ell_{-k}}$ and Lemma 6.2 we obtain

$$v_{\gamma_0}^- h \sum_{k,i} |\Delta_k v_i| \leq 2v_{\gamma_0}^- \sum_{k,i} |v_{ki}| \leq N v_{\gamma_0}^- \max_Q W_2^{1/2} \leq N (v_{\gamma_0}^-)^2.$$

Thus,

$$(6.11) \quad |4I_1| \leq N (v_{\gamma_0}^-)^2 + (1/3) \mu a_k R_k^\mu.$$

Estimating I_2 . Observe that

$$|4I_2| \leq N h v_{ij}^- (\sqrt{a_k} + h) |\Delta_k v_{ij}| \leq I_{21} + I_{22},$$

where (see Lemma 6.2 and recall that $(t_0, x_0) \in Q_2^0$)

$$I_{21} = N \sum_k h^2 v_{ij}^- |\Delta_k v_{ij}| = N \sum_k v_{ij}^- |(T_{h, \ell_k} - 2 + T_{h, \ell_{-k}}) v_{ij}| \leq N (v_{\gamma_0}^-)^2,$$

and by the formula $|\theta| = \theta + 2\theta^-$,

$$I_{22} = Nhv_{ij}^- \sqrt{a_k} |\Delta_k v_{ij}| = Nhv_{ij}^- \sqrt{a_k} \Delta_k v_{ij} + 2Nhv_{ij}^- \sqrt{a_k} (\Delta_k v_{ij})^- = Nh\sqrt{a_k} Z_k + Nhv_{ij}^- \sqrt{a_k} (\Delta_k v_{ij})^- + I_{23},$$

where

$$I_{23} = -Nh\sqrt{a_k} (v_{\gamma_0}^- \Delta_k v_{\gamma_0} - \nu v_i \Delta_k v_i).$$

Furthermore, by Lemma 3.2,

$$Nhv_{ij}^- \sqrt{a_k} (\Delta_k v_{ij})^- \leq Nh\sqrt{a_k} v_{ij}^- |\Delta_k v_{ij}^-| = N\sqrt{a_k} v_{ij}^- |(\delta_k + \delta_{-k}) v_{ij}^-|,$$

which is majorized by the right-hand side of (6.11).

To estimate I_{23} we use Lemma 3.2 to get

$$(6.12) \quad \begin{aligned} -Nh\sqrt{a_k} v_{\gamma_0}^- \Delta_k v_{\gamma_0} &= -N\sqrt{a_k} v_{\gamma_0}^- \delta_k v_{\gamma_0} \leq N\sqrt{a_k} v_{\gamma_0}^- \delta_k v_{\gamma_0}^- \\ &\leq N(v_{\gamma_0}^-)^2 + (1/3)a_k R_k^\gamma. \end{aligned}$$

Furthermore, by assumption (6.5),

$$Nh\nu v_i \sqrt{a_k} \Delta_k v_i = N\nu v_i \sqrt{a_k} v_{ki} \leq N(v_{\gamma_0}^-)^2 + (1/3)\nu a_k R_k^\nu.$$

It follows that

$$I_{23} \leq N(v_{\gamma_0}^-)^2 + (1/3)a_k [\delta_k v_{\gamma_0}^-]^2 + (1/3)\nu a_k R_k^\nu.$$

Hence

$$(6.13) \quad |4I_2| \leq N(v_{\gamma_0}^-)^2 + Nh\sqrt{a_k} Z_k + (1/3)a_k (R_k^\gamma + \mu R_k^\mu + \nu R_k^\nu).$$

Estimating I_3 . We use the following result of simple computations:

$$T_{h,\ell_j+\ell_i} \Delta_k v = -v_{k,-k} + v_{kj} + v_{-k,j} + v_{ki} + v_{-k,i} + h^2 \Delta_k v_{ij}.$$

This shows new terms entering I_3 . All of them, apart from the last one, are similar to the ones which are written explicitly in the definition of I_3 , and we show how to estimate only one of them. By Assumption 2.7 and Lemma 6.2 we have

$$|\mu v_{ij}^- (\delta_j \delta_i a_k) v_{k,-k}| \leq N(v_{\gamma_0}^-)^2 + N^* |v_{\gamma_0}^-| \sum_{k,i} \sqrt{a_k} |v_{ki}| \leq N(v_{\gamma_0}^-)^2 + N^* a_k R_k^\nu.$$

To estimate the remaining term in I_3 we proceed as in estimating I_2 . We have

$$\begin{aligned} &\mu h^2 v_{ij}^- (\delta_j \delta_i a_k) \Delta_k v_{ij} \\ &\leq Nh^2 v_{ij}^- (N + N^* \sqrt{a_k}) |\Delta_k v_{ij}| = I_{31} + I_{32}. \end{aligned}$$

Here by Lemma 6.2 and because $(t_0, x_0) \in Q_2^0$,

$$I_{31} = Nh^2 v_{ij}^- |\Delta_k v_{ij}| = Nv_{ij}^- |(T_{h,\ell_k} - 2 + T_{h,\ell_{-k}}) v_{ij}| \leq N(v_{\gamma_0}^-)^2.$$

Next,

$$I_{32} = N^* h^2 v_{ij}^- \sqrt{a_k} |\Delta_k v_{ij}| = I_{321} + I_{322},$$

with

$$\begin{aligned} I_{321} &= N_1^* h^2 \mu v_{ij}^- \sqrt{a_k} \Delta_k v_{ij}, \\ I_{322} &= N^* h^2 v_{ij}^- \sqrt{a_k} (\Delta_k v_{ij})^- \leq N^* h^2 v_{ij}^- \sqrt{a_k} |\Delta_k v_{ij}^-| \\ &= N^* h v_{ij}^- \sqrt{a_k} |\delta_k v_{ij}^- + \delta_{-k} v_{ij}^-| \leq N^* \bar{W}_1 + (1/3)\mu a_k R_k^\mu, \end{aligned}$$

where the last inequality is true since $h v_{ij}^- \leq 2\bar{W}_1^{1/2}$. Also observe that

$$I_{321} = N_1^* h^2 \sqrt{a_k} Z_k - N_1^* h^2 v_{\gamma_0}^- \sqrt{a_k} \Delta_k v_{\gamma_0} + N_1^* h^2 \nu v_i \sqrt{a_k} \Delta_k v_i,$$

where

$$N_1^* \nu h^2 v_i \sqrt{a_k} \Delta_k v_i = 2N_1^* \nu h v_i \sqrt{a_k} v_{ki} \leq N^* \nu^2 h^2 \bar{W}_1 + a_k R_k^\nu$$

and, according to (6.12) and the inequality $h|v_{\gamma_0}| \leq N\bar{W}_1^{1/2}$,

$$-N_1^* h^2 \sqrt{a_k} v_{\gamma_0}^- \Delta_k v_{\gamma_0} \leq N^* \bar{W}_1 + (1/3)a_k R_k^\gamma.$$

Therefore,

$$I_{321} \leq N_1^* h^2 \sqrt{a_k} Z_k + N^* (1 + \nu^2 h^2) \bar{W}_1 + (1/3)a_k R_k^\gamma + a_k R_k^\nu.$$

We can now specify h^* : we take

$$N_1^* h^* \leq 1 \quad \text{and} \quad h^* \leq h_0/2.$$

Then, for $h \leq h^*$,

$$\begin{aligned} I_{32} &\leq h\sqrt{a_k} Z_k + N^* \nu^2 \bar{W}_1 + (1/3)\mu a_k R_k^\mu + (1/3)a_k R_k^\gamma + a_k R_k^\nu, \\ |4I_3| &\leq h\sqrt{a_k} Z_k + N^* \nu^2 \bar{W}_1 + N(v_{\gamma_0}^-)^2 \\ &\quad + (1/3)a_k R_k^\gamma + (1/3)\mu a_k R_k^\mu + N^* a_k R_k^\nu. \end{aligned} \tag{6.14}$$

Estimating I_4 . By using Lemma 6.2 we easily see that

$$|4I_4| \leq N(v_{\gamma_0}^-)^2 + N^* \bar{W}_1 + \nu \bar{W}_1^{1/2} (N\sqrt{a_k} + N^* h) \sum_i |v_{ki}|,$$

where

$$\begin{aligned} N\nu \bar{W}_1^{1/2} \sqrt{a_k} \sum_i |v_{ki}| &\leq N\nu \bar{W}_1 + (1/3)\nu a_k R_k^\nu, \\ N^* \nu \bar{W}_1^{1/2} h \sum_{k,i} |v_{ki}| &\leq N^* \nu \bar{W}_1. \end{aligned}$$

It follows that

$$|4I_4| \leq N(v_{\gamma_0}^-)^2 + N^* \nu \bar{W}_1 + (1/3)\nu a_k R_k^\nu.$$

By combining this with (6.11), (6.13), and (6.14), recalling that $Z_k \geq 0$, and coming back to (6.10) we conclude that

$$(6.15) \quad 4J \geq (2a_k - N_1 h \sqrt{a_k}) Z_k - N(v_{\gamma_0}^-)^2 - N^* \nu^2 \bar{W}_1 + (\nu/3 - N_2^*) a_k R_k^\nu.$$

Now we specify $\nu = \nu^*$ by setting

$$\nu^* = 1 + 3N_2^*$$

and finish the argument as in [13]. Namely, if $2a_k - N_1 h \sqrt{a_k} \geq 0$ for a k , then we can drop the term on the right in (6.15) corresponding to this k because $Z_k \geq 0$. However, if $2a_k - N_1 h \sqrt{a_k} \leq 0$, then $\sqrt{a_k} \leq Nh$ and $|a_k - N_1 h \sqrt{a_k}| \leq Nh^2$, whereas

$$\begin{aligned} h^2 Z_k &= h^2 (v_{\gamma_0}^- \Delta_k v_{\gamma_0} + \mu v_{ij}^- \Delta_k v_{ij} - \nu v_i \Delta_k v_{ki}) \\ &\leq N \max_Q W_2 + N\nu \bar{W}_1^{1/2} \max_Q W_2^{1/2} \\ &\leq N \max_Q W_2 + N\nu^2 \bar{W}_1 \leq N(v_{\gamma_0}^-)^2 + N^* \bar{W}_1. \end{aligned}$$

This and (6.15) yield (6.9) and the lemma is proved. □

Proof of Theorem 6.1. Fix a constant ν according to Lemma 6.5 and first assume that $(t_0, x_0) \in \partial_2 Q$. Then

$$\sqrt{\mu} \max_Q W_2^{1/2} \leq V_{\gamma_0 \mu \nu}^{1/2}(t_0, x_0) \leq N^* (\max_{\partial_2 Q, i, j} |v_{ij}| + \max_{\partial_2 Q, i} |v_i|),$$

which by (6.4) yields a similar estimate for

$$\max_{Q_1^o, i, j} |v_{ij}|.$$

After that (6.2) is immediate (cf. the end of the proof of Theorem 4.1).

Therefore, in the rest of the proof we assume that

$$(t_0, x_0) \in Q_2^o.$$

Similarly, if (6.5) is violated, there is nothing to prove. Hence, we may assume that (6.5) holds. Finally, we may assume that $h \leq h^*$, where h^* is taken from Lemma 6.5 and further reduced if needed so as to satisfy $h^* \sqrt{\nu} \leq \varepsilon$. Indeed, if $h \geq h^*$, then in Q_1^o ,

$$|v_{ij}| \leq 2(h^*)^{-1} \max_Q |v_i|.$$

After justifying these additional assumptions, which allow us to use the assertions of Lemmas 6.2 and 6.5 as long as $h \leq h^*$, we construct functions $\bar{r}(t)$, $\bar{a}_k(t, x)$, $\bar{b}_k(t, x)$, $\bar{c}(t, x)$, $\bar{f}(p, \psi, t, x) = \bar{f}(t, x)$ as in Section 4 to get (4.2) and (4.3) satisfied. Then, since $(t_0, x_0) \in Q_2^o$ and (6.3) and (6.5) are valid and $h\sqrt{\nu} \leq \varepsilon$, by Lemma 6.2 at (t_0, x_0) we obtain

$$P_{\gamma_0 \mu \nu} (e^{-m\tau} \bar{r} \delta_\tau^T v + \bar{a}_k \Delta_k v + \bar{b}_k \delta_k v - (\bar{c} + \bar{r}c_m)v + \xi \bar{f}) \leq 0.$$

The fact that \bar{r} , \bar{a}_k , and \bar{b}_k are limits of some r^α , a_k^α , and b_k^α , allows us to assert that Lemma 6.5 holds with \bar{r} , \bar{a}_k , and \bar{b}_k in place of r^α , a_k^α , and b_k^α , respectively. Therefore,

$$(6.16) \quad -N(v_{\gamma_0}^-)^2 + P_{\gamma_0 \mu \nu} (e^{-m\tau} \bar{r} \delta_\tau^T v - (\bar{c} + \bar{r}c_m)v + \xi \bar{f}) \leq N^* \bar{W}_1.$$

Here $P_{\gamma_0 \mu \nu} \delta_\tau^T v \geq 0$ as right after (4.6). Furthermore,

$$P_{\gamma_0 \mu \nu} ((\bar{c} + \bar{r}c_m)v) = -(\bar{c} + \bar{r}c_m)V_{\gamma_0 \mu \nu} + I_1,$$

where I_1 is a linear combination of products of two types:

(i) $v_{\gamma_0}^-$ or v_{ij}^- times a difference operator applied to \bar{c} times either v or a first-order difference operator applied to v ; the second and third factors may be taken at a point different from (t_0, x_0) , but their coefficients in the linear combination are dominated by a constant N ;

(ii) v_i times a difference operator applied to \bar{c} times either v or a first-order difference operator applied to v ; these terms may be taken at a point different from (t_0, x_0) , but the coefficients of these terms are dominated by a constant N^* (recall that ν is entering $P_{\gamma \mu \nu}$).

Owing to Lemma 6.2, the absolute value of the linear combination of the products of type (i) is less than

$$N v_{\gamma_0}^- K_3 (\max_Q |v| + \bar{W}_1^{1/2}) \leq (v_{\gamma_0}^-)^2 + N^* (\max_Q |v|^2 + \bar{W}_1).$$

The absolute value of the linear combination of the products of type (ii) is clearly less than

$$N^* \bar{W}_1^{1/2} (\max_Q |v| + \bar{W}_1^{1/2}) \leq N^* (\max_Q |v|^2 + \bar{W}_1).$$

Now from the above estimates, (6.16), and the fact that $V_{\gamma_0\mu\nu} \geq (v_{\gamma_0}^-)^2$ and $\bar{c} + \bar{r}c_m \geq \lambda$ we conclude that

$$(6.17) \quad (\lambda - N)(v_{\gamma_0}^-)^2 + P_{\gamma_0\mu\nu}(\xi\bar{f}) \leq N^*(\max_Q |v|^2 + \bar{W}_1).$$

Finally, obviously

$$P_{\gamma_0\mu\nu}(\xi\bar{f}) \geq -N^*\xi(\max_{i,j} |v_{ij}| + \nu \max_i |v_i|) \geq -N^*\xi^2 - (v_{\gamma_0}^-)^2 - \bar{W}_1,$$

and we infer from (6.17) that

$$(\lambda - N_1)(v_{\gamma_0}^-)^2 \leq N^*(\max_Q |v|^2 + \bar{W}_1) + N^*\xi^2.$$

We set the constant N in the statement of the theorem to be $N_1 + 1$ and use Lemma 6.2 to conclude that in $Q_1^o \cap Q|_0$ for any i, j ,

$$|\delta_j \delta_i u| = |v_{ij}| \leq N v_{\gamma_0}^-(t_0, x_0) \leq N^*(\max_Q |v| + \bar{W}_1^{1/2}) + N^*\xi(t_0).$$

This implies (6.2) in $Q_1^o \cap Q|_0$. On the remaining part of $Q|_0$ estimate (6.2) is obvious and the theorem is proved. \square

7. PROOF OF THEOREM 2.12

We start with three auxiliary results. Everywhere in this section the assumptions of Theorem 2.12 are supposed to be satisfied. Recall that the set \mathcal{L} is introduced in (2.20).

Lemma 7.1. *For any function ϕ and $l_1, l_2 \in \Lambda_0$ we have*

$$(7.1) \quad \begin{aligned} |\delta_{h,l_1} \delta_{h,l_2} \phi(0)| &\leq 4 \max(|\Delta_{h,\ell_k} \phi(x)| : |k| \leq d_0, x \in (\Lambda_0 + \mathcal{L}) \cup \{0\}) \\ &+ 4 \max(|\Delta_{h,\ell_k} \phi(x)| : d_0 < |k| \leq d_1, x \in \Lambda_0 \cup \{0\}). \end{aligned}$$

Proof. Obviously we may assume that $h = 1$. Next, observe that

$$\begin{aligned} \delta_{1,l_1} \delta_{1,l_2} \phi(0) &= (1/2)[\Delta_{1,l_2} \phi(l_1) + \Delta_{1,l_1} \phi(l_2)] - (1/2)\Delta_{1,l_1-l_2} \phi(0), \\ \delta_{1,l_1} \delta_{1,l_1} \phi(0) &= \Delta_{h,l_1} \phi(l_1), \quad \delta_{1,l_1} \delta_{1,-l_1} \phi(0) = -\Delta_{h,l_1} \phi(0). \end{aligned}$$

It follows that if $l_1, l_2 \in \mathcal{L}$, then

$$\begin{aligned} |\delta_{h,l_1} \delta_{h,l_2} \phi(0)| &\leq \max(|\Delta_{h,\ell_k} \phi(x)| : |k| \leq d_0, x \in \mathcal{L} \cup \{0\}) \\ &+ \max(|\Delta_{h,\ell_k} \phi(0)| : d_0 < |k| \leq d_1). \end{aligned}$$

We substitute here $\phi(y + \cdot)$ in place of ϕ and use that $\Lambda_0 + \mathcal{L} \supset \mathcal{L}$ since $d_1 \geq 2$ and $\mathcal{L} = -\mathcal{L}$. Then we see that, if $y \in \Lambda_0 \cup \{0\}$ and $l_1, l_2 \in \mathcal{L}$, then

$$(7.2) \quad \begin{aligned} |\delta_{1,l_1} \delta_{1,l_2} \phi(y)| &\leq \max(|\Delta_{h,\ell_k} \phi(x)| : |k| \leq d_0, x \in (\Lambda_0 + \mathcal{L}) \cup \{0\}) \\ &+ \max(|\Delta_{h,\ell_k} \phi(x)| : d_0 < |k| \leq d_1, x \in \Lambda_0 \cup \{0\}). \end{aligned}$$

In case $l_1 = \zeta_1 + \zeta_2, l_2 = \eta_1 + \eta_2$ with $\zeta_1, \zeta_2, \eta_1, \eta_2 \in \mathcal{L}$ and $\zeta_1 \neq \zeta_2, \zeta_1 \neq -\zeta_2, \eta_1 \neq \eta_2, \eta_1 \neq -\eta_2$ either $\zeta_1 \neq \eta_1$ and $\zeta_1 \neq -\eta_1$ or $\zeta_1 \neq \eta_2$ and $\zeta_1 \neq -\eta_2$. The second possibility reduces to the first one by interchanging η_1 and η_2 . If the first possibility is realized, then we use the formula $\delta_{1,\eta+\zeta} = T_{1,\eta} \delta_{1,\zeta} + \delta_{1,\eta}$ to obtain

$$\begin{aligned} \delta_{1,l_1} \delta_{1,l_2} \phi(0) &= (T_{1,\zeta_1} \delta_{1,\zeta_2} + \delta_{1,\zeta_1})(T_{1,\eta_1} \delta_{1,\eta_2} + \delta_{1,\eta_1})\phi(0) \\ &= \delta_{1,\zeta_2} \delta_{1,\eta_2} \phi(\zeta_1 + \eta_1) + \delta_{1,\zeta_2} \delta_{1,\eta_1} \phi(\zeta_1) + \delta_{1,\zeta_1} \delta_{1,\eta_2} \phi(\eta_1) + \delta_{1,\zeta_1} \delta_{1,\eta_1} \phi(0). \end{aligned}$$

Here $\zeta_1 + \eta_1 \in \Lambda_0, \zeta_1, \eta_1 \in \Lambda_0$, and $0 \in \Lambda_0 \cup \{0\}$. Therefore, we get (7.1) from (7.2).

The remaining case that $l_1 \in \mathcal{L}$ and $l_2 = \eta_1 + \eta_2$ with η_i as above is taken care of by setting $\zeta_1 = 0$ in the above calculations. The lemma is proved. \square

Before stating the next lemma we remind the reader that the index k takes values in $\{\pm 1, \dots, \pm d_1\}$.

Lemma 7.2. *For any values of the arguments and $s > 0$ we have*

$$\sum_k q_k^+ \leq \frac{2d_1}{\delta} [s^{-1}F(s\phi, sq_k, sp_k, s\psi) + K_0 \sum_k q_k^- + K_3 (\sum_k |p_k| + |\psi| + \phi^- + s^{-1})].$$

Indeed, the expression in the brackets obviously is larger than

$$\sup_{\alpha \in A} a_k^\alpha(t, x)q_k^+,$$

which in turn is larger than δq_n^+ for each particular $n = \pm 1, \dots, \pm d_1$.

Below we use the notation $\Gamma(\varepsilon)$ and P_γ from Section 6.

Lemma 7.3. *Let $\theta \in (0, \varepsilon^{-1})$ and $\varepsilon \in (0, 1]$ be such that*

$$(7.3) \quad 2d_1K_0\kappa/\delta \leq 1/2, \quad \kappa := \varepsilon(\theta + \varepsilon)/(1 - \theta\varepsilon).$$

Let w, ψ, p_k be functions on Q and assume that

$$(7.4) \quad \max_{\Gamma(\varepsilon) \times Q} (\xi P_\gamma w)^- \leq \theta \max_Q \sum_k |\xi \Delta_{h, \ell_k} w|.$$

Then in Q we have

$$\sum_k |\xi \Delta_{h, \ell_k} w| \leq \frac{4d_1}{\delta} (1 + \kappa)I,$$

where

$$I = \max_Q [\xi F(\phi, \Delta_{h, \ell_k} w, p_k, \psi) + K_3 \xi (\sum_{|k| \leq d_1} |p_k| + |\psi| + \phi^- + 1)].$$

Proof. Set $\Phi^\pm = \sum_k (\xi \Delta_{h, \ell_k} w)^\pm$ and observe that due to (7.4),

$$\varepsilon \Phi^+ - \varepsilon^{-1} \Phi^- \geq -\theta \max_Q \Phi^+ - \theta \max_Q \Phi^-$$

in Q . Hence $(\theta + \varepsilon) \max_Q \Phi^+ \geq (\varepsilon^{-1} - \theta) \max_Q \Phi^-$, that is,

$$(7.5) \quad \max_Q \Phi^- \leq \kappa \max_Q \Phi^+.$$

By (7.5) and Lemma 7.2 with $s^{-1}\phi, s^{-1}\Delta_{h, \ell_k} w, s^{-1}p_k$, and $s^{-1}\psi$ in place of ϕ, q_k, p_k , and ψ , respectively, and $s^{-1} = \xi$ we find that in Q ,

$$\Phi^+ \leq \frac{2d_1}{\delta} [I + K_0\kappa \max_Q \Phi^+].$$

Upon taking the maximums over Q of both parts and taking into account (7.3) we get that $\Phi^+ \leq (4d_1/\delta)I$ in Q , which along with (7.5) yields the result. The lemma is proved. \square

Proof of Theorem 2.12. Here k, i, j run through $\pm 1, \dots, \pm d_1$. It is easy to see that one can find $\varepsilon = \varepsilon(\delta, d_1, K_0) \in (0, 1]$ and $\mu = \mu(\delta, d_1, K_0) > 0$ in such a way that the conditions:

$$16d_1^2\mu \leq \varepsilon^2, \quad 16d_1^2(3\mu)^{1/2} =: \theta < \varepsilon^{-1}$$

and (7.3) are satisfied. We choose and fix appropriate ε and μ .

If

$$(7.6) \quad \max_{\Gamma(\varepsilon) \times Q} (\xi P_\gamma u)^- \leq \theta \max_Q \sum_k |\xi \Delta_{h, \ell_k} u|,$$

then, by taking into account that u satisfies (2.1) in Q , from Lemma 7.3 we obtain that $|\xi \Delta_{h, \ell_k} u|$ are bounded in Q by the right-hand side of (2.21). By combining this with Lemma 7.1 we conclude that (2.21) is true in $Q|_0 \cap Q_2^o$ (notice that $(\Lambda_0 + \mathcal{L}) \cup \{0\} \subset \Lambda_0 + \Lambda_0$). Of course, (2.21) is obvious on $Q|_0 \cap \partial_2 Q$.

If (6.1) holds, then we get (2.21) from Theorem 6.1. In the remaining case both (7.6) and (6.1) are violated and

$$\begin{aligned} \theta H &:= \theta \max_Q \sum_k |\xi \Delta_{h, \ell_k} u| \\ &\leq \max_{\Gamma(\varepsilon) \times Q} (\xi P_\gamma u)^- \leq (3\mu)^{1/2} \max_Q \left(\sum_{i,j} |\xi \delta_{h, \ell_j} \delta_{h, \ell_i} u|^2 \right)^{1/2} \\ &\leq (3\mu)^{1/2} \max_{\partial_2 Q} \sum_{i,j} |\xi \delta_{h, \ell_j} \delta_{h, \ell_i} u| + (3\mu)^{1/2} \max_{Q_2^o} \sum_{i,j} |\xi \delta_{h, \ell_j} \delta_{h, \ell_i} u|. \end{aligned}$$

In light of Lemma 7.1 the last maximum over Q_2^o is less than $8d_1^2 H$. Hence

$$\theta H \leq N \max_{\partial_2 Q, i,j} |\xi \delta_{h, \ell_j} \delta_{h, \ell_i} u| + (1/2)\theta H, \quad H \leq N \max_{\partial_2 Q, i,j} |\xi \delta_{h, \ell_j} \delta_{h, \ell_i} u|,$$

and we can finish the proof of (2.21) in the same way as we did a few times before. The theorem is proved. \square

8. PROOF OF THEOREM 2.14

In the following lemma the assumption that $\sup_\alpha a_k^\alpha \geq \delta$ is not used. All other assumptions of Theorem 2.14 are supposed to hold. We use notation (2.10) and the notation from Section 6 with $d_1 + 1$ in place of d_1 and $\delta_i = \delta_{h_i, \ell_i}$, $\Delta_i = \Delta_{h_i, \ell_i}$. Here we take

$$\mu = 0$$

and show how to choose $\nu = \nu(\lambda, \varepsilon, d_1, K_3)$ in Lemma 8.1.

Lemma 8.1. *In $Q|_0$ for $\gamma \in \Gamma(\varepsilon)$ we have*

$$(8.1) \quad \begin{aligned} \left(\sum_i \gamma_i \Delta_{h_i, \ell_i} u \right)^- &\leq N^* e^{m^+(T+\tau)} \left(1 + \max_{Q, j} |\xi_{(-)} \delta_{h_j, \ell_j} u| \right. \\ &\quad \left. + \max_Q |\xi_{(-)} u| + \max_{\partial_1 Q, j} (\xi_{(-)} \Delta_{h_j, \ell_j} u)^- \right), \end{aligned}$$

where $N^* = N^*(\lambda, h_0, \varepsilon, d_1, K_3)$.

Proof. As many times before, if $(t_0, x_0) \in \partial_1 Q$, there is nothing to prove. Therefore, we assume that $(t_0, x_0) \in Q_1^0$. We may also assume that at (t_0, x_0) ,

$$\sum_i (\xi \delta_{h_i, \ell_i} u)^2 \leq \left[\left(\xi \sum_i \gamma_{0i} \Delta_{h_i, \ell_i} u \right)^- \right]^2.$$

Then the operator $P_{\gamma_0 0\nu}$ respects the maximum principle for $h\nu \leq 2\varepsilon$ (see the proof of Lemma 6.2 and recall that $\eta \leq h$).

Then as in the proof of Lemma 6.5 we obtain

$$(8.2) \quad 4J = 4P_{\gamma_0 0\nu}(a_k \Delta_k v + b_k \delta_k v) \geq a_k \nu R'_k + 4I_4$$

if $h \leq h_0/2$, where this time

$$I_4 = 2v_{\gamma_0}^- \gamma_{0i}(\delta_i b_k)v_{ki} + v_{\gamma_0}^- \gamma_{0i}(\Delta_i b_k)v_k - \nu v_i(\delta_i b_k)T_{h_i, \ell_i} v_k.$$

Since

$$v_{\gamma_0}^- \gamma_{0i} |(\delta_i b_k)v_{ki}| \leq N^* v_{\gamma_0}^- \sqrt{a_k} \sum_i |v_{ki}| \leq (\lambda/8)(v_{\gamma_0}^-)^2 + N^* a_k \sum_i v_{ki}^2$$

we get from (8.2) that, for $\nu = \nu^*(\lambda, \varepsilon, d_1, K_3)$, $h\nu^* \leq 2\varepsilon$, and $h \leq h_0/2$,

$$(8.3) \quad J \geq -(\lambda/2)(v_{\gamma_0}^-)^2 - N^* \bar{W}_1.$$

The rest is just a repetition of a part of the proof of Theorem 6.1 with obvious and great simplifications. The lemma is proved. \square

There is almost nothing else to do to finish the proof of Theorem 2.14. Indeed, (8.1) with d_1 in place of $d_1 + 1$ yields the first estimate in (2.22) as in the proof of Theorem 2.12. After getting estimates for $|\Delta_{h, \ell_k} u|$ the estimate of $(\Delta_{\eta, l} u)^-$ follows immediately from (8.1). The theorem is proved. \square

9. COMMENTS ON THE OPERATORS HAVING FORM (2.7)

We know (see, for instance, [5]) that if an operator L having form (2.7) admits an approximation with operators S_h of the form (2.4) respecting the maximum principle with hB in place of B and $\text{Span } B = \mathbb{R}^d$, then necessarily

$$Lu = a_k \ell_k^i \ell_k^j u_{x^i x^j} + b_k \ell_k^i u_{x^i}$$

with some $a_k, b_k \geq 0$ and $\ell_k \in B \cup (-B)$. A way to find such representations for $d = 2$ and given a^{ij} is suggested in [1].

The next natural issue is related to the smoothness of a_k, b_k if we are given that the a^{ij} are smooth. Recall that in Assumption 2.2 we need a_k^α to be at least Lipschitz continuous. Of course, this problem disappears if the a^{ij} are constant.

It is an easy and probably well-known fact that if $(a^{ij}) = (a^{ij}(t, x))$ is uniformly bounded and uniformly elliptic, then one can find d_1 and Λ_0 for which a_k can be chosen strictly positive and as smooth as a^{ij} are. The proof of this can be obtained from the fact that if we are given a closed convex polyhedron, then every point in the relative interior can be written as a convex combination of the extreme points with the coefficients > 0 which are infinitely differentiable functions of the point. By replacing L with $L + \varepsilon^2 \Delta$ one can approximate a possibly degenerate operator L with uniformly nondegenerate ones, so that there always exist a sequence of operators of the form (2.7) approximating L . Notice, however, that generally the set Λ_0 changes with ε . Nevertheless, one knows how to estimate the difference of solutions corresponding to $L + \varepsilon^2 \Delta$ and L (see, for instance, [9]) and between the solutions of the corresponding finite-difference approximations (see, for instance, Theorem 5.6 of [13] or Remark 2.6).

Another generic example is given by the so-called diagonally dominant matrices. For instance, take $d = 2$ and assume that $b \equiv 0$, the a^{ij} are twice continuously differentiable with respect to x , $a^{12} = a^{21}$, and $|a^{12}| \leq s$, where $s = a^{11} \wedge a^{22}$.

Set $\kappa = 1/3$ and take an infinitely differentiable, even, and convex function $\psi(t)$ on \mathbb{R} such that $\psi(y) = |y|$ for $|y| \geq \kappa$. Introduce

$$g = a^{12}s^{-1}, \quad h = s\psi(g), \quad 2\hat{a}^{1,\pm 2} = h \pm a^{12}, \quad 2\hat{a}^{ii} = a^{ii} - h,$$

where $i = 1, 2$ and $0 \cdot 0^{-1} := 0$. For other values of $i, j = \pm 1, \pm 2$ define \hat{a}^{ij} so that

$$\hat{a}^{ij} = \hat{a}^{ji}, \quad \hat{a}^{-i,-j} = \hat{a}^{ij}, \quad \hat{a}^{i,-i} = 0$$

and set

$$\ell_j = e_{|j|}\text{sign } j, \quad \ell_{ij} = \ell_i + \ell_j,$$

where e_1, e_2 are the basis vectors. Then simple manipulations yield

$$4Lu = \hat{a}^{ij} \ell_{ij}^k \ell_{ij}^r u_{x^k x^r}.$$

We now show that not only does L admit a representation as the sum of second-order directional derivatives with the directions independent of t, x but also the $\sqrt{\hat{a}^{ij}}$ are Lipschitz continuous in x . By the way, observe that obviously $\hat{a}^{ij} \geq 0$.

We are going to use that nonnegative and twice continuously differentiable functions are the squares of Lipschitz continuous functions. In particular, the a^{ii} and, consequently, $a^{11} \wedge a^{22}$ are the squares of Lipschitz continuous functions, $|a_x^{ii}| \leq N\sqrt{a^{ii}}$ and $|s_x| \leq N\sqrt{s}$. Furthermore, $a^{11} \pm a^{12}$ is nonnegative and twice continuously differentiable. Hence, it is the square of a Lipschitz continuous function. In particular,

$$|a_x^{11} \pm a_x^{12}| \leq N\sqrt{a^{11} \pm a^{12}}, \quad |a_x^{12}| \leq N\sqrt{a^{11}}, \quad |a_x^{12}| \leq N\sqrt{s},$$

and recalling that $|a^{12}| \leq s$ we find

$$|g_x| \leq |a_x^{12}|s^{-1} + |a^{12}| \cdot |s_x|s^{-2} \leq Ns^{-1/2}, \quad |h_x| \leq N\sqrt{s}.$$

Next, the function $\phi(y) := \psi(y) + y$ is smooth and nonnegative. Therefore

$$2|\hat{a}_x^{12}| = |\phi' g_x s + \phi s_x| \leq N\sqrt{\phi}\sqrt{s} + N\phi\sqrt{s} \leq N\sqrt{\phi}\sqrt{s} = N\sqrt{\hat{a}^{12}}.$$

A similar estimate holds for $|\hat{a}_x^{1,-2}|$ and $|\hat{a}_x^{ij}|$ if $i \neq j$.

On the set where $|a^{12}| > \kappa s$, we have $2\hat{a}^{11} = a^{11} - |a^{12}|$, so that by the above,

$$|\hat{a}_x^{11}| = |a_x^{11} - a_x^{12}\text{sign } a^{12}| \leq N\sqrt{a^{11} - a^{12}\text{sign } a^{12}} = N\sqrt{\hat{a}^{11}}.$$

Finally, on the set where $|a^{12}| < 2\kappa s$, it holds that $h \leq 2\kappa s$, $a^{11} - h \geq \kappa a^{11}$, and $|\hat{a}_x^{11}| \leq N\sqrt{a^{11}} \leq N\sqrt{\hat{a}^{11}}$. Similarly we get what we need for \hat{a}^{22} and the remaining \hat{a}^{ii} .

REFERENCES

- [1] Bonnans, J. F., Ottenwaelter, E., and Zidani, H., *A fast algorithm for the two dimensional HJB equation of stochastic control*, M2AN Math. Model. Numer. Anal., Vol. 38 (2004), No. 4, 723–735. MR2087732 (2005e:93165)
- [2] Barles, G., and Jakobsen, E.R., *On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations*, Mathematical Modelling and Numerical Analysis, ESAIM, M2AM, Vol. 36 (2002), No. 1, 33–54. MR1916291 (2003h:65142)
- [3] Barles, G., and Jakobsen, E.R., *Error bounds for monotone approximation schemes for Hamilton-Jacobi-Bellman equations*, SIAM J. Numer. Anal., Vol. 43 (2005), No. 2, 540–558. MR2177879
- [4] Barles, G., and Jakobsen, E.R., *Error bounds for monotone approximation schemes for parabolic Hamilton-Jacobi-Bellman equations*, preprint.
- [5] Dong, Hongjie and Krylov, N.V., *On the rate of convergence of finite-difference approximations for Bellman equations with constant coefficients*, Algebra i Analiz (St. Petersburg Math. J.), Vol. 17 (2005), No. 2, 108–132. MR2159586 (2006f:49050)

- [6] Dong, Hongjie and Krylov, N.V., *On the rate of convergence of finite-difference approximations for degenerate linear parabolic equations with C^1 and C^2 coefficients*, Electron. J. Diff. Eqns., Vol. 2005 (2005), No. 102, pp. 1-25. <http://ejde.math.txstate.edu> MR2162263 (2006i:35008)
- [7] Dong, Hongjie and Krylov, N.V., *On the rate of convergence of finite-difference approximations for parabolic Bellman equations with Lipschitz coefficients in cylindrical domains*, submitted to Applied Math. and Optimization.
- [8] Jakobsen, E. R., *On the rate of convergence of approximation schemes for Bellman equations associated with optimal stopping time problems*, Math. Models Methods Appl. Sci., Vol. 13 (2003), No. 5, 613-644. MR1978929 (2004g:49054)
- [9] Krylov, N.V., "Controlled diffusion processes", Nauka, Moscow, 1977 in Russian; English translation Springer, 1980. MR601776 (82a:60062)
- [10] Krylov, N.V., *On the rate of convergence of finite-difference approximations for Bellman's equations*, Algebra i Analiz, St. Petersburg Math. J., Vol. 9 (1997), No. 3, 245-256. MR1466804 (98h:49033)
- [11] Krylov, N.V., *Approximating value functions for controlled degenerate diffusion processes by using piece-wise constant policies*, Electronic Journal of Probability, Vol. 4 (1999), paper no. 2, 1-19, <http://www.math.washington.edu/~ejpecp/EjpVol4/paper2.abs.html>. MR1668597 (2000b:49056)
- [12] Krylov, N.V., *On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients*, Probab. Theory Relat. Fields, Vol. 117 (2000), No. 1, 1-16. MR1759507 (2001j:65134)
- [13] Krylov, N.V., *The rate of convergence of finite-difference approximations for Bellman equations with Lipschitz coefficients*, Applied Math. and Optimization, Vol. 52 (2005), No. 3, 365-399. MR2174020 (2006k:65219)

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