ON THE LARGEST PRIME DIVISOR
OF AN ODD HARMONIC NUMBER

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ABSTRACT. A positive integer is called a *(Ore's) harmonic number* if its positive divisors have integral harmonic mean. Ore conjectured that every harmonic number greater than 1 is even. If Ore's conjecture is true, there exist no odd perfect numbers. In this paper, we prove that every odd harmonic number greater than 1 must be divisible by a prime greater than 10^5.

1. INTRODUCTION

Every perfect number is harmonic. Therefore, if Ore's conjecture, "all harmonic numbers other than 1 must be even" holds, then the famous conjecture "there does not exist an odd perfect number" also holds.

A positive integer n is said to be harmonic if the harmonic mean of its positive divisors

$$H(n) = \frac{n\tau(n)}{\sigma(n)}$$

is an integer, where $\tau(n)$ and $\sigma(n)$ denote the number and the sum of positive divisors of n, respectively. We call 1 the trivial harmonic number.

In advance of the innovation by Ore's harmonic numbers, Kanold [15] proved that the largest prime divisor p of any odd perfect number, if it exists, is greater than 60. The estimation of lower bound of p is further developed by Hagis and McDaniel [9] (resp. [10]) to $p > 10^4$ (resp. $p > 10^5$), by Hagis and Cohen [11] to $p > 10^6$, and by Jenkins [14] to $p > 10^7$.

Ore [21] defined and investigated harmonic numbers in 1948. He listed all harmonic numbers up to 10^4, and this list was extended by Garcia [6] to 10^7, by Cohen [3] to $2 \cdot 10^9$. Sorli [26] gave tables of harmonic numbers up to 10^{12} and of harmonic seeds (we recall the definition in this section) up to 10^{15}, and showed that any nontrivial odd harmonic number is greater than 10^{15}.

Kanold [16] showed the finiteness of the number $\#\{n \mid H(n) = k\}$ for any k. Goto and Shibata [7] gave the table of harmonic numbers n with $H(n) \leq 300$. Mills [17] showed that any nontrivial odd harmonic number, if it exists, has at least one component (prime power divisor) greater than 10^7. He also announced that one can extend the bound to 65531^2 by using a computer. Pomerance [22] announced that every harmonic number with two components is an even perfect number, and Edgar and Callan [5] published a proof of the fact.

In §3, we approach the largest prime divisor of an odd harmonic number using a concept of harmonic seeds, which is introduced by Cohen and Sorli [4].

Definition. Let d be a positive divisor of an integer n. If $d > 1$, d is said to be *proper*. If $(d, n/d) = 1$, we call d a *unitary divisor* of n, and n a *unitary multiple* of d. A harmonic number is called a *harmonic seed* if it does not have a smaller proper unitary divisor which is harmonic.

Theorem 1.1. *There are* 967 *nontrivial harmonic seeds whose prime divisors are all less than* 10^2. *These seeds are all even.*

The complete table of 967 seeds is available at `http://www.ma.noda.tus.ac.jp/u/tg/harmonic/seeds_100.pdf`. Table 7, in the last section, is the partial table which contains 118 harmonic seeds whose prime divisors are less than 50. In the table, seeds are ordered first according to their largest prime factor, and among those, according to the size of H.

Since every harmonic number is a unitary multiple of a certain harmonic seed, we immediately have the following corollary.

Corollary 1.2. *Every nontrivial odd harmonic number must be divisible by a prime greater than* 10^2.

In order to show Theorem 1.1, we need a computer, however, we can show Corollary 1.2 without a computer, by generalizing the approach of [25] and [2] (for details, see `http://www.ma.noda.tus.ac.jp/u/tg/harmonic/10e2.pdf`). Note that the corollary is an extension of the result of Kanold [15], who also may not use a computer.

Let $\mathcal{P}(N, M)$ (resp. $\mathcal{H}(N, M)$) be the set of perfect (resp. nontrivial harmonic) numbers whose N-th largest prime factor is less than M, and $\mathcal{O}$ the set of odd integers. Corollary 1.2 says that $\mathcal{H}(1, 10^2) \cap \mathcal{O}$ is empty. Iannucci [12] (resp. [13]) showed that $\mathcal{P}(2, 10^4) \cap \mathcal{O}$ (resp. $\mathcal{P}(3, 10^2) \cap \mathcal{O}$) is empty. Pomerance [23] remarked that the set $\mathcal{P}(N, M) \cap \mathcal{O}$ is finite for any positive integers N, M. In §3, we also prove the following fact, which is an extension of Pomerance's remark.

Theorem 1.3. *The sets* $\mathcal{H}(1, M)$ *and* $\mathcal{H}(N, M) \cap \mathcal{O}$ *are finite.*

For $N \geq 2$, $M \geq 2$, the set $\mathcal{H}(N, M)$ seems to be infinite since it contains even perfect numbers. In §4, we give an outline of the proof of the following theorem.

Theorem 1.4. *Every nontrivial odd harmonic number must be divisible by a prime greater than* 10^5.

2. Preliminaries

In this section, we recall some known facts about harmonic numbers and values of cyclotomic polynomials. Since τ and σ are multiplicative, we can express $H(n)$ as

$$H(n) = \prod_{i=1}^{r} \frac{p_i^{e_i} \tau(p_i^{e_i})}{\sigma(p_i^{e_i})}$$

when the canonical factorization of n is $\prod_{i=1}^{r} p_i^{e_i}$. Using the facts $\tau(p^e) = e + 1$ and $\sigma(p^e) = (p^{e+1} - 1)/(p - 1)$, we have

$$(1) \qquad H(n) \prod_{i=1}^{r} \prod_{\substack{d \neq 1, \\ d \mid (e_i + 1)}} \Phi_d(p_i) = \prod_{i=1}^{r} p_i^{e_i}(e_i + 1),$$

where $\Phi_d(x)$ denotes the d-th cyclotomic polynomial. The following facts about harmonic numbers are due to Ore, Garcia, or the second author. In this paper, $p^e \parallel n$ means that $p^e \mid n$ and $p^{e+1} \nmid n$. We denote by $\omega(n)$ the number of distinct prime factors of n.

Lemma 2.1 ([21]). *If n is a squarefree harmonic number, then $n = 1$ or 6.*

Lemma 2.2 ([6], [17], [7]). *If n is an odd harmonic number and $p^e \parallel n$, then $p^e \equiv 1 \pmod 4$.*

Lemma 2.3 ([8]). *Suppose that n is an odd harmonic number and $\omega(n) = k$. Then $n \leq (2^{4^k})^{2k(2k+1)}$. In particular, there exist only finitely many odd harmonic numbers n satisfying $\omega(n) = k$, for a given positive integer k.*

In the nineteenth century, values of cyclotomic polynomials, often called *cyclotomic numbers*, were studied by Kronecker, Sylvester, et al. Their works are described as Lemma 2.4 (see [20] or [24]). Throughout this section, the symbols a, d represent integers greater than 1, and p, q, k primes. If $k \nmid a$, we denote by $\mathrm{ord}_k(a)$ the order of $a \in (\mathbb{Z}/k\mathbb{Z})^\times$.

Lemma 2.4. *A prime k divides $\Phi_d(a)$ if and only if $d = k^c \mathrm{ord}_k(a)$ for some nonnegative integer c. Furthermore, if $k \mid \Phi_d(a)$ and $k \mid d$, then $k \parallel \Phi_d(a)$.*

From Lemma 2.4, we immediately obtain the following fact: if $k \mid \Phi_d(a)$, then $k \mid d$ or $k \equiv 1 \pmod d$. In the former (resp. latter) case, we say that k is *intrinsic* (resp. *primitive*). These terms were used by Murata and Pomerance [19].

The following lemma is known as Bang's theorem. The most famous proof is due to Birkhoff and Vandiver [1] (see also [24]). Motose [18] gave a simpler proof of the fact.

Lemma 2.5. *A cyclotomic number $\Phi_d(a)$ has no primitive prime factors if and only if $d = 2$, $a = 2^e - 1$, or $d = 6$, $a = 2$.*

3. Proofs of Theorems 1.1 and 1.3

In this paper, p, p_i, q, k will denote prime numbers.

Proposition 3.1. *Let $n = \prod_{i=1}^r p_i^{e_i}$ be a harmonic number greater than 6, and $p = \max(p_i)$. Then the following facts hold.*
  (a) *The largest prime divisor of the product $\prod_{i=1}^r (e_i + 1)$ is less than $p/2$.*
  (b) $\max(e_i) \leq p - 2$.

Proof. (a) Let q be the largest prime divisor of $\prod_{i=1}^r (e_i + 1)$. Then the left-hand side of (1) is divisible by $\Phi_q(p_i)$ for some p_i. Note that $q > 2$ from Lemma 2.1. By Lemma 2.5, $\Phi_q(p_i)$ has a primitive prime factor k, so that $k \equiv 1 \pmod q$. Assume that $q \geq p/2$. Then $k \geq 2q + 1 \geq p + 1$. The left-hand side of (1) is divisible by k, which is greater than $\max(p, q)$, a contradiction.

(b) Let $e = \max(e_i)$. Lemma 2.1 implies $e > 1$. Assume that $e = 5$ and the required inequality does not hold. Then it is necessary that $n = 2^{e_1} 3^{e_2} 5^{e_3}$ with $0 \leq e_i \leq 5$, $\max(e_i) = 5$, however, such an n is not harmonic. Hence we can assume that $e \neq 1, 5$, so $\Phi_{e+1}(p_j)$ has a primitive prime divisor k from Lemma 2.5. Assume that $e > p - 2$. Then we have $k \geq e + 2 > p$, which is contradictory to the equation (1). $\square$

TABLE 1.

p (largest prime)	3	5	7	11	13	17	19	23	29	31	37	41
# of seeds	1	1	3	0	5	2	19	0	0	37	10	19
p (largest prime)	43	47	53	59	61	67	71	73	79	83	89	97
# of seeds	21	0	0	0	89	143	84	97	26	0	127	283

It follows immediately from Proposition 3.1 that there exist only finitely many
harmonic numbers whose prime divisors are less than a given number. Since there
exist many such numbers, we consider only harmonic seeds. In order to show
Theorem 1.1, we use Garcia's method ([6]) and Proposition 3.1. For example,
suppose that n is an harmonic seed whose largest prime factor is 7. We demonstrate
that n is one of the following:

$$2^2 7, \quad 2^5 3 \cdot 7, \quad 2^5 3^3 5 \cdot 7.$$

If $p^e \parallel n$, then Proposition 3.1 (b) implies that $e \leq 5$, and (a) implies $e \neq 4$. Since
$H(7^5) = 7^5/(2^2 19 \cdot 43)$, it is impossible that $7^5 \parallel n$. If $p \in \{2, 3, 5, 7\}$, $e \in \{1, 2, 3, 5\}$
and the denominator of $H(p^e)$ has no prime factor greater than 7, then p^e is one of
the following:

$$2, \ 2^2, \ 2^3, \ 2^5, \ 3, \ 3^3, \ 5, \ 7, \ 7^3.$$

Suppose that $7 \parallel n$. Since $H(7) = 7/2^2$, it is necessary that $2^2 \mid n$. Considering
that $2^2 \parallel n$, we get a seed $2^2 7$. Suppose that $2^3 \parallel n$. Since $H(7 \cdot 2^3) = (2^3 7)/(3 \cdot 5)$,
it is necessary that $5 \parallel n$. Since $H(7 \cdot 2^3 5) = (2^3 5 \cdot 7)/3^2$, it is necessary that
$3^3 \parallel n$. We obtain $H(7 \cdot 2^3 5 \cdot 3^3) = (2^2 3 \cdot 7)/5$, however, we cannot eliminate 5 in
the denominator. We illustrate whole procedures as follows:

    7, (need to eliminate 2^2),
      2^2, (seed).
      2^3, 5, 3^3, (cannot eliminate 5).
      2^5, (need to eliminate 3),
        3, (seed).
        3^3, 5, (seed).
    7^3, (cannot eliminate 5^2).

Using this method and a computer, we can show Theorem 1.1. Table 1 gives the
number of seeds whose largest prime divisor is equal to p ($< 10^2$).

Proof of Theorem 1.3. The finiteness of $\mathcal{H}(1, M)$ is immediate from Proposition
3.1. Let $\pi(x)$ denote the number of primes less than x. If $n \in \mathcal{H}(N, M) \cap \mathcal{O}$, then
$\omega(n) < N + \pi(M)$. There are only finitely many such n from Lemma 2.3. $\square$

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Assume that n is a nontrivial odd har-
monic number whose canonical factorization is $\prod_{i=1}^{r} p_i^{e_i}$ with $\max(p_i) < 10^5$. Put
$\mathcal{N} = \{p_1^{e_1}, \ldots, p_r^{e_r}\}$. If $p^e \in \mathcal{N}$, then it is necessary that the following four conditions
hold (cf. Lemma 2.2, Proposition 3.1 (b) and equation (1)).

    (i) $3 \leq p < 10^5$,
    (ii) $1 \leq e < 10^5$,
    (iii) $p^e \equiv 1 \pmod{4}$,
    (iv) the integer $\prod_{d|(e+1), \, d\neq 1} \Phi_d(p)$ has no prime factors greater than 10^5.

TABLE 2.

i	0	1	2	3	4	5	6	7	8
$\#\mathcal{A}_i$	10621	8539	7600	7041	6590	6307	6127	5996	5929
i	9	10	11	12	13	14	15	16	17
$\#\mathcal{A}_i$	5779	5596	5458	5371	5297	5262	5255	5250	5250

Let $\mathcal{A}$ be the set of prime powers p^e satisfying these conditions. From (i) and (ii), the set $\mathcal{A}$ is finite, so we can determine $\mathcal{A}$ using a computer. In fact, $\#\mathcal{A} = 10621$ and all elements of $\mathcal{A}$ are given in `http://www.ma.noda.tus.ac.jp/u/tg/harmonic/10e5.nb` (using this file and Mathematica®, we also have other sets given in this section). In order to determine $\mathcal{A}$, TABLE 1 in [11] is useful. For example, suppose that $3^e \in \mathcal{A}$. According to the table, if $\Phi_q(3)$ has no prime factors greater than 10^5, then $q = 3, 5, 7, 11$ or 17. Hence condition (iv) implies that $e + 1$ has the form $3^{e(3)}5^{e(5)}7^{e(7)}11^{e(11)}17^{e(17)}$. Since $\Phi_{3^4}(3)$ has a prime factor greater than 10^5, it is necessary that $e(3) \leq 3$. Similarly, we have $e(5), e(7), e(11), e(17) \leq 1$. Checking all divisors of $3^3 5 \cdot 7 \cdot 11 \cdot 17$, we have $e + 1 = 3, 5, 7, 9, 11, 15, 17$ or 27. Therefore, all powers of 3 contained in $\mathcal{A}$ are $3^2, 3^4, 3^6, 3^8, 3^{10}, 3^{14}, 3^{16}$ and 3^{26}.

Next, we define subsets of $\mathcal{A}$ inductively. Let $\mathcal{A}_0 = \mathcal{A}$ and for $i \geq 1$,

$$\mathcal{A}_i = \left\{ p^e \; ; \; \begin{array}{l} k^r \parallel \prod_{d|(e+1),\, d \neq 1} \Phi_d(p) \text{ implies that} \\ \text{either (v) or (vi) holds} \end{array} \right\},$$

where conditions (v) and (vi) are given as

(v) there exists $k^s \in \mathcal{A}_{i-1}$ such that $s \geq r$,

(vi) there exists $q^f \in \mathcal{A}_{i-1}$ such that $k \mid (f + 1)$.

We easily see by equation (1) that $\bigcap \mathcal{A}_i \supset \mathcal{N}$. Using a computer, we can determine $\mathcal{A}_i$. Table 2 gives the cardinalities of $\mathcal{A}_i$.

Unfortunately, $\mathcal{A}_{16} = \mathcal{A}_{17}$, so that $\bigcap \mathcal{A}_i = \mathcal{A}_{16}$. Note that

$$\{q \; ; \; q \mid (e + 1) \text{ for some } p^e \in \mathcal{A}_{16}\} = \{2, 3, 5, 7\};$$

therefore, if a prime k divides the left-hand side of (1) and $k > 7$, then $k \mid n$. We now show the following claim.

Claim 1. $269^6, 7^4, 79^6, 31^2 \notin \mathcal{N}$.

Proof. The proof of $269^6 \notin \mathcal{N}$ is relatively easy. Assume that $269^6 \parallel n$. Since $2633 \mid \Phi_7(269)$, it follows that $2633 \mid n$. All powers of 2633 in $\mathcal{A}_{16}$ are 2633 and 2633^3. In both cases, $\Phi_2(2633)$ appears in the left-hand side of (1). Since $439 \mid \Phi_2(2633)$, it is necessary that $439 \mid n$, so that $439^2 \parallel n$. Since $211 \mid \Phi_7(269)$, it is necessary that $211^2 \parallel n$. We obtain

$$H(269^6 2633 \cdot 439^2 211^2) = \frac{7 \cdot 211 \cdot 269^6 \cdot 439}{3 \cdot 13 \cdot 31^3 \cdot 37 \cdot 43 \cdot 67 \cdot 631 \cdot 25229},$$

however, we cannot eliminate 31^3 in the denominator (note that 31^2 is the only power of 31 in $\mathcal{A}_{16}$). Therefore, we have $269^6 \notin \mathcal{N}$. For brevity, let us write this proof as follows:

$269^6, 2633, 439^2, 211^2$ $\boxed{31}$.

Next, we prove $7^4 \notin \mathcal{N}$ and $79^6 \notin \mathcal{N}$. Note that if $p^e \in \mathcal{A}_{16}$ and $7 \mid (e+1)$, then $p^e = 19^6$, 79^6 or 269^6. We now know $269^6 \notin \mathcal{N}$.

(proof of $7^4 \notin \mathcal{N}$)

$\quad 7^4$, 2801^2, 4933^2, 127^2, 5419^2, 43^2, 631^4, 46601, 863^2, 11^2, 41, $(313$ or $313^2)$,*
$\qquad 313$, 157, $(79^2$ or $79^6)$,
$\qquad\quad 79^2$, $\boxed{7}$ †.
$\qquad\quad 79^6$, 2017, $\boxed{1009}$.
$\qquad 313^2$, $(181^2$ or $181^3)$,
$\qquad\quad 181^2$, $(79^2$ or $79^6)$,
$\qquad\qquad 79^2$, $\boxed{7}$.
$\qquad\qquad 79^6$, 2017, $\boxed{1009}$.
$\qquad\quad 181^3$, $(19^6$ or $79^6)$,‡
$\qquad\qquad 19^6$, 70841, 11807^2, 2017, $\boxed{1009}$.
$\qquad\qquad 79^6$, 2017, $\boxed{1009}$.

(proof of $79^6 \notin \mathcal{N}$)

$\quad 79^6$, $(1289$ or $1289^2)$,
$\qquad 1289$, 43^2, 631^4, 46601, 1511^2, 863^2, 15217^2, 12799^2, 9001^2, 6067^2, 337, 281,
$\qquad 41$, 11^2, $(67^2$ or $67^4)$,
$\qquad\quad 67^2$, 7^6, 4733, 263^2, $(163^2$ or $163^4)$,
$\qquad\qquad 163^2$, 13, $\boxed{7}$.
$\qquad\qquad 163^4$, 1301, 13, $\boxed{7}$.
$\qquad\quad 67^4$, 26881, 13441, 761, 127^2, 5419^2, 2017, $\boxed{1009}$.
$\qquad 1289^2$, 127^2, 5419^2, 2017, $\boxed{1009}$.

The proof of $31^2 \notin \mathcal{N}$ is very long. The complete proof is available at `http://www.ma.noda.tus.ac.jp/u/tg/harmonic/31e2.pdf`.

(initial part of the proof of $31^2 \notin \mathcal{N}$)

$\quad 31^2$, $(331^2$ or $331^4)$,
$\qquad 331^2$, $(5233$ or $5233^2)$,
$\qquad\quad 5233$, $(2617$ or $2617^2)$,
$\qquad\qquad 2617$, 11^2, $(19^2$ or $19^6)$,
$\qquad\qquad\quad 19^2$, 127^2, 5419^2, $(313$ or $313^2)$,
$\qquad\qquad\qquad 313$, 157, 79^2, 43^2, 631^4, 46601, 1511^2, 863^2, 7^6, 4733, 263, $\boxed{7}$.
$\qquad\qquad\qquad 313^2$, $(181^2$ or $181^3)$,
$\qquad\qquad\qquad\quad 181^2$, 79^2, 43^2, 631^4, 46601, 1511^2, 863^2, 7^6, 4733, 263, $\boxed{7}$.
$\qquad\qquad\qquad\quad 181^3$, 16381^2, 72313, $(1237$ or $1237^2)$,
$\qquad\qquad\qquad\qquad 1237$, 619^2, $\boxed{19}$.
$\qquad\qquad\qquad\qquad 1237^2$, 1117, 43^2, 631^4, $\boxed{11}$.
$\qquad\qquad\quad 19^6$, 70841, 11807^2, 701, $(13^3$, 13^4 or $13^5)$,
$\qquad\qquad\qquad\qquad\vdots$

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*If $313^e \in \mathcal{A}_{16}$, then $313^e = 313$, 313^2, 313^3 or 313^5. However, we can deal with the cases of $313^3, 313^5$ similarly to the case of 313.

†Recall that a boxed 7 means that 7 cannot be eliminated. For $m = 7^4 2801^2 4933^2 127^2 \times 5419^2 43^2 31^4 46601 \cdot 863^2 11^2 41 \cdot 313 \cdot 157 \cdot 79^2$, we have $H(m) = x/(7^2 y)$ with $7 \nmid x$, $7 \nmid y$. In order to eliminate 7, we need 7^e or p^6, however, m has prime power divisors 7^4, 79^2. Hence we can use only 19^6, and cannot cancel 7^2.

‡In order to eliminate 7, we need 19^6 or 79^6.

TABLE 3.

p^e	$H(p^e)$
5	$5/3$
5^2	$(3 \cdot 5^2)/31$
5^3	$5^3/(3 \cdot 13)$
5^4	$5^5/(11 \cdot 71)$
5^5	$5^5/(3 \cdot 7 \cdot 31)$
5^7	$5^7/(3 \cdot 13 \cdot 313)$
5^9	$5^{10}/(3 \cdot 11 \cdot 71 \cdot 521)$
5^{11}	$5^{11}/(3 \cdot 7 \cdot 13 \cdot 31 \cdot 601)$
5^{14}	$(3 \cdot 5^{15})/(11 \cdot 31 \cdot 71 \cdot 181 \cdot 1741)$
5^{19}	$5^{20}/(3 \cdot 11 \cdot 13 \cdot 41 \cdot 71 \cdot 521 \cdot 9161)$
5^{29}	$5^{30}/(3 \cdot 7 \cdot 11 \cdot 31 \cdot 61 \cdot 71 \cdot 181 \cdot 521 \cdot 1741 \cdot 7621)$
3^4	$(3^4 5)/11^2$
3^{14}	$(3^{15}5)/(11^2 13 \cdot 4561)$
13^4	$(5 \cdot 13^4)/30941$
47^4	$(5 \cdot 47^4)/(11 \cdot 31 \cdot 14621)$
67^4	$(5 \cdot 67^4)/(761 \cdot 26881)$
109^4	$(5 \cdot 109^4)/(31 \cdot 191 \cdot 24061)$
163^4	$(5 \cdot 163^4)/(11 \cdot 31 \cdot 1301 \cdot 1601)$
757^4	$(5 \cdot 757^4)/(11 \cdot 191 \cdot 2521 \cdot 62081)$
953^4	$(5 \cdot 953^4)/(41 \cdot 1601 \cdot 2161 \cdot 5821)$
1447^4	$(5 \cdot 1447^4)/(4831 \cdot 16901 \cdot 53731)$
1697^4	$(5 \cdot 1697^2)/(11^2 41 \cdot 941 \cdot 1021 \cdot 1741)$
7727^4	$(5 \cdot 7727^4)/(11 \cdot 31 \cdot 181 \cdot 461 \cdot 2851 \cdot 43951)$
20749^4	$(5 \cdot 20749^4)/(11 \cdot 61 \cdot 211 \cdot 5381 \cdot 5801 \cdot 41941)$
28759^4	$(5 \cdot 28759^4)/(11 \cdot 41 \cdot 191 \cdot 521 \cdot 761 \cdot 2861 \cdot 7001)$
51637^4	$(5 \cdot 51637^4)/(11 \cdot 41 \cdot 641 \cdot 701 \cdot 751 \cdot 3761 \cdot 12421)$

One of the most difficult cases is that n is a unitary multiple of

$$m = 31^2 331^2 5233 \cdot 2617^2 11833 \cdot 193 \cdot 97^2 3169 \cdot 317 \cdot 7^6 4733 \cdot 263^2 109$$
$$\times \, 11^2 19^6 70841 \cdot 11807^2 701 \cdot 13^7 61 \cdot 409 \cdot 14281 \cdot 2017 \cdot 1009 \cdot 193^2$$
$$\times \, 149 \cdot 101 \cdot 17^3 41 \cdot 29^3 421 \cdot 211^2 37^7 10529 \cdot 137^2 89 \cdot 73 \cdot 53$$

with

$$H(m) = \frac{7 \cdot 11 \cdot 13 \cdot 17 \cdot 19^4 29 \cdot 37^3 137 \cdot 193 \cdot 211 \cdot 263 \cdot 331 \cdot 2617 \cdot 11807}{3^{18} 5^{15}}.$$

In order to eliminate 5, we need a prime power p^e with $p = 5$ or $5 \mid (e+1)$. Table 3 gives all such prime powers in $\mathcal{A}_{16}$. Since $13, 109$ divide m, we cannot use $13^4, 109^4$. If

$$p^e \in \{5^2, 5^5, 5^{11}, 5^{14}, 5^{29}, 47^4, 163^4, 7727^4\},$$

then 31 divides the denominator of $H(p^e)$, so we cannot use these prime powers. If

$$p^e \in \{3^4, 3^{14}, 757^4, 1697^4, 20749^4, 28759^4, 51637^4\},$$

TABLE 4.

i	0	1	2	3	4	5	6	7	8	9
$\#\mathcal{B}_i$	5249	4920	4599	4328	4143	3941	3781	3681	3451	3199
i	10	11	12	13	14	15	16	17	18	
$\#\mathcal{B}_i$	3051	2767	2649	2617	2580	2564	2552	2550	2550	

TABLE 5.

p^e	$H(p^e)$	p^e	$H(p^e)$
5	$5/3$	5^9	$5^{10}/(3 \cdot 11 \cdot 71 \cdot 521)$
5^3	$5^3/(3 \cdot 13)$	5^{19}	$5^{20}/(3 \cdot 11 \cdot 13 \cdot 41 \cdot 71 \cdot 521 \cdot 9161)$
5^4	$5^5/(11 \cdot 71)$	3^4	$(3^4 5)/11^2$
5^7	$5^7/(3 \cdot 13 \cdot 313)$		

then 11 divides the denominator of $H(p^e)$, so we can use at most one of these prime powers. Now, we must consider only 5^{19}. We can deal with this case as follows:

m, 5^{19}, (71^2 or 71^4),
   71^2, 5113, 2557^2, $\boxed{7}$,
   71^4, $\boxed{11}$.

In this way, we can complete a proof of Claim 1. $\hfill\square$

Now, we know $31^2 \notin \mathcal{N}$, so we define $\mathcal{B}_0 = \mathcal{A}_{16}\backslash\{31^2\}$ and for $i \geq 1$, define $\mathcal{B}_i$ similarly to $\mathcal{A}_i$. It is clear that $\bigcap \mathcal{B}_i \supset \mathcal{N}$. Table 4 gives the cardinalities of $\mathcal{B}_i$.

Hence we have $\bigcap \mathcal{B}_i = \mathcal{B}_{17}$. We now prove the following claim.

Claim 2. $19^6 \notin \mathcal{N}$.

Proof. If $19^6 \parallel n$, then it is necessary that n is a multiple of

$$m = 19^6 70841 \cdot 11807^2 2017 \cdot 1009 \cdot 701 \cdot 409 \cdot 101 \cdot 41 \cdot 13^3 \cdot 17^3 \cdot 29$$

with $H(m) = (17 \cdot 19^6 11807)/(3^8 5^5)$. In order to eliminate 5, we need a prime power p^e with $p = 5$ or $5 \mid (e+1)$. Table 5 gives all such prime powers in $\mathcal{B}_{17}$.

It is necessary that $5^4 \mid n$, however, we lead to a contradiction as follows:

5^4, 71^2, 5113, 2557^2, 11^2, 7^6, $\boxed{29}$.
5^7, 313^2, 181^3, 16381^2, 72313, 173, $\boxed{29}$.
5^9, 521, $\boxed{29}$.
5^{19}, 521, $\boxed{29}$.

Now, the proof of Claim 2 is completed. $\hfill\square$

We define $\mathcal{C}_0 = \mathcal{B}_{17}\backslash\{19^6\}$, and for $i \geq 1$, define $\mathcal{C}_i$ similarly to $\mathcal{A}_i$. It is clear that $\bigcap \mathcal{C}_i \supset \mathcal{N}$. Table 6 gives the cardinalities of $\mathcal{C}_i$. Hence we have $\bigcap \mathcal{C}_i = \phi$ and $\mathcal{N} = \phi$, so that the proof of Theorem 1.4 is completed.

TABLE 6.

i	0	1	2	3	4	5	6	7
$\#\mathcal{C}_i$	2549	2283	1713	1179	938	824	535	353
i	8	9	10	11	12	13	14	15
$\#\mathcal{C}_i$	279	226	202	170	125	26	3	0

5. Concluding remark

We have proved that the largest prime divisor of a nontrivial odd harmonic number is greater than 10^5. This is an extension of the result of Hagis and McDaniel [10]. It may be expected that one can raise this bound to 10^6 by the same method. If we define the sets $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$ similarly for the bound 10^6, then we have

$$\#\mathcal{A}_0 = 85126, \quad \bigcap \mathcal{A}_i = \mathcal{A}_{19}, \quad \#\mathcal{A}_{19} = 41124,$$

$$\bigcap \mathcal{B}_i = \mathcal{B}_{17}, \quad \#\mathcal{B}_{17} = 18306, \quad \mathcal{C}_{19} = \phi$$

(see the file `10e5.nb`). Therefore, we have to prove only $31^2 \notin \mathcal{N}$ and $19^6 \notin \mathcal{N}$. However, it seems to be difficult to prove $31^2 \notin \mathcal{N}$ by a similar method, since the set $\mathcal{A}_{19}$ is relatively large.

6. Appendix: Table of seeds

TABLE 7. All harmonic seeds whose prime divisors are less than 50

n	$H(n)$	n	$H(n)$
$2 \cdot 3$	2	$2^4 \cdot 31$	5
$2 \cdot 3^3 5$	6	$2^3 5^2 31$	10
$2^2 7$	3	$2^5 5^2 7 \cdot 31$	25
$2^5 3 \cdot 7$	8	$2^9 3^3 11 \cdot 31$	48
$2^5 3^3 5 \cdot 7$	24	$2^5 5^2 7^3 31$	49
$2 \cdot 3^2 7 \cdot 13$	9	$2^5 5^2 7^2 19 \cdot 31$	70
$2^3 3^2 5 \cdot 7 \cdot 13$	24	$2^9 7 \cdot 11^2 19 \cdot 31$	88
$2^2 5^3 7^3 13$	35	$2^7 3^3 5^2 17 \cdot 31$	96
$2^3 3^3 5^3 7 \cdot 13$	60	$2^9 3^2 7 \cdot 11 \cdot 13 \cdot 31$	120
$2^5 3^3 5^3 7^3 13$	168	$2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 31$	240
$2^2 3^2 5^3 7^3 13^3 17$	273	$2^{11} 5^2 7^2 13 \cdot 19 \cdot 31$	256
$2^5 3^5 5^3 7^3 13^3 17$	936	$2^9 3^2 7^2 11 \cdot 13 \cdot 19 \cdot 31$	336
$2^2 5 \cdot 7^2 19$	14	$2^3 3^2 5^5 7^2 13 \cdot 19 \cdot 31$	375
$2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19$	42	$2^5 3^3 5^5 7^2 19 \cdot 31$	375
$2^2 5^3 7^2 13 \cdot 19$	50	$2^5 3^2 5^5 7^3 13 \cdot 31$	375
$2 \cdot 3^5 5 \cdot 7^2 13 \cdot 19$	81	$2^9 3^2 5 \cdot 7^3 11 \cdot 13 \cdot 31$	392
$2^5 3^2 5 \cdot 7^2 13 \cdot 19$	96	$2^7 3 \cdot 5^2 7 \cdot 11^2 17 \cdot 19 \cdot 31$	484
$2 \cdot 3^3 5^3 7^2 13 \cdot 19$	105	$2^9 3^5 7^2 11 \cdot 13 \cdot 19 \cdot 31$	648
$2^3 3^4 5 \cdot 7 \cdot 11^2 19$	108	$2^7 3^2 5^2 7^2 13 \cdot 17 \cdot 19 \cdot 31$	672
$2^3 3^3 5^3 7^2 13 \cdot 19$	168	$2^9 3^5 5 \cdot 7^3 11 \cdot 13 \cdot 31$	756
$2^5 3^3 5^3 7^2 13 \cdot 19$	240	$2^{11} 3^3 5^5 7^3 13 \cdot 31$	960
$2^7 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19$	384	$2^7 3^4 5^2 7 \cdot 11^2 17 \cdot 19 \cdot 31$	1080
$2^2 3^2 5^3 7^2 13^3 17 \cdot 19$	390	$2^7 3^5 5^2 7^2 13 \cdot 17 \cdot 19 \cdot 31$	1296
$2^5 3 \cdot 5^3 7^3 11^2 13 \cdot 19$	484	$2^9 3 \cdot 5^3 7^3 11^2 13 \cdot 19 \cdot 31$	1540
$2^2 3^3 5^3 7^2 13^3 17 \cdot 19$	507	$2^7 3^5 5^5 7^3 13 \cdot 17 \cdot 31$	1800
$2 \cdot 3^5 5^3 7^2 13^3 17 \cdot 19$	585	$2^9 3^3 5^3 7^3 11^2 13 \cdot 19 \cdot 31$	2772
$2^3 3^4 5^3 7^3 11^2 13 \cdot 19$	756	$2^7 3^3 5^5 7^3 11^2 17 \cdot 19 \cdot 31$	3388
$2^3 3^5 5^3 7^2 13^3 17 \cdot 19$	936	$2^9 3^5 5 \cdot 7^2 11 \cdot 13^3 17 \cdot 19 \cdot 31$	4056
$2^5 3^4 5^3 7^3 11^2 13 \cdot 19$	1080	$2^7 3^4 5^5 7^3 11^2 17 \cdot 19 \cdot 31$	4200
$2^7 3^4 5^3 7^3 11^2 13 \cdot 17 \cdot 19$	2688	$2^7 3^2 5^5 7^3 11^2 13 \cdot 17 \cdot 19 \cdot 31$	4840
$2^5 3^4 5^3 7^3 11^2 13^3 17 \cdot 19$	4056	$2^9 3^5 5^3 7^3 11 \cdot 13^3 17 \cdot 31$	5460

TABLE 7 (continued)

n	$H(n)$	n	$H(n)$
$2^7 3^5 5^5 7^3 13 \cdot 17^3 29 \cdot 31$	6936	$2^9 3^7 5^3 7^3 11^2 13 \cdot 19 \cdot 31 \cdot 41$	10692
$2^9 3^5 5^3 7^2 11 \cdot 13^3 17 \cdot 19 \cdot 31$	7800	$2^7 3^7 5^5 7^2 17^3 19 \cdot 29 \cdot 31 \cdot 41$	13872
$2^9 3^2 5^3 7^3 11^2 13^3 17 \cdot 19 \cdot 31$	8008	$2^9 3^7 5^3 7^3 11 \cdot 13^3 17 \cdot 31 \cdot 41$	14196
$2^9 3^5 5^3 7^3 11^2 13^3 17 \cdot 19 \cdot 31$	15444	$2^9 3^7 5^3 7^2 11 \cdot 13^3 17 \cdot 19 \cdot 31 \cdot 41$	20280
$2^7 3^4 5^5 7^3 11^2 17^3 19 \cdot 29 \cdot 31$	16184	$2^7 3^7 5^5 7^3 13 \cdot 17 \cdot 19 \cdot 31^3 37 \cdot 41$	34596
$2^9 3^5 5^3 7^2 11 \cdot 13^3 17^3 19 \cdot 29 \cdot 31$	30056	$2^9 3^7 5^5 7^3 11 \cdot 13 \cdot 19 \cdot 31^3 37 \cdot 41$	41850
$2^9 3 \cdot 5^2 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31^3 37$	1550	$2^9 3^7 5^5 7^3 11 \cdot 13^3 17 \cdot 19 \cdot 31^3 37 \cdot 41$	157170
$2^9 3^3 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31^3 37$	1922	$2^5 3^2 5 \cdot 7^5 11 \cdot 13 \cdot 19 \cdot 43$	686
$2^9 3^3 5^2 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31^3 37$	2790	$2^5 3^5 5 \cdot 7^5 11 \cdot 13 \cdot 19 \cdot 43$	1323
$2^9 3 \cdot 5^2 7^3 11 \cdot 13 \cdot 19 \cdot 31^3 37$	3038	$2^5 3^3 5^3 7^5 11 \cdot 13 \cdot 19 \cdot 43$	1715
$2^9 3^2 5^2 7 \cdot 11 \cdot 13^3 17 \cdot 19 \cdot 31^3 37$	8060	$2^{11} 3 \cdot 5^2 7^5 11 \cdot 13 \cdot 19 \cdot 31 \cdot 43$	2744
$2^9 3^3 5^2 7 \cdot 11 \cdot 13^3 17 \cdot 19 \cdot 31^3 37$	10478	$2^7 3^2 5^2 7^5 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43$	4802
$2^9 3^3 5^5 7^3 11 \cdot 13 \cdot 19 \cdot 31^3 37$	10850	$2^5 3^7 5^3 7^5 11 \cdot 13 \cdot 19 \cdot 41 \cdot 43$	6615
$2^9 3^5 5^3 7^3 11 \cdot 13^3 17 \cdot 19 \cdot 31^3 37$	40362	$2^7 3^5 5^2 7^5 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43$	9261
$2^9 3^5 5^5 7^3 11 \cdot 13^3 17 \cdot 19 \cdot 31^3 37$	60450	$2^5 3^5 5^3 7^5 11 \cdot 13^3 17 \cdot 19 \cdot 43$	9555
$2^9 3^5 5^5 7^3 11 \cdot 13^3 17^3 19 \cdot 29 \cdot 31^3 37$	232934	$2^{11} 3^3 5^5 7^5 11 \cdot 13 \cdot 19 \cdot 31 \cdot 43$	9800
$2^9 3^7 7 \cdot 11 \cdot 31 \cdot 41$	324	$2^7 3^7 5^3 7^5 11 \cdot 13 \cdot 17 \cdot 19 \cdot 41 \cdot 43$	16464
$2 \cdot 3^7 5^3 7^2 13 \cdot 19 \cdot 41$	405	$2^7 3^5 5^5 7^5 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43$	18375
$2^7 3^7 5^2 7 \cdot 17 \cdot 31 \cdot 41$	648	$2^{11} 3^5 5^3 7^5 11 \cdot 13^3 17 \cdot 19 \cdot 43$	18816
$2^3 3^7 5^3 7^2 13 \cdot 19 \cdot 41$	648	$2^5 3^7 5^3 7^5 11 \cdot 13^3 17 \cdot 19 \cdot 41 \cdot 43$	24843
$2^5 3^7 5^3 7^3 13 \cdot 41$	648	$2^7 3^7 5^5 7^5 11 \cdot 17 \cdot 19 \cdot 31 \cdot 41 \cdot 43$	25725
$2^9 3^7 5 \cdot 7^2 11 \cdot 19 \cdot 31 \cdot 41$	1512	$2^{11} 3^7 5^5 7^5 11 \cdot 13 \cdot 19 \cdot 31 \cdot 41 \cdot 43$	37800
$2 \cdot 3^7 5^3 7^2 13^3 17 \cdot 19 \cdot 41$	1521	$2^{11} 3^5 5^5 7^5 11 \cdot 13^3 17 \cdot 19 \cdot 31 \cdot 43$	54600
$2^7 3^7 5^3 7^2 13 \cdot 17 \cdot 19 \cdot 41$	2304	$2^7 3^5 5^5 7^5 11 \cdot 13 \cdot 17^3 19 \cdot 29 \cdot 31 \cdot 43$	70805
$2^7 3^7 5^5 7^3 17 \cdot 31 \cdot 41$	2520	$2^7 3^7 5^5 7^5 11 \cdot 17^3 19 \cdot 29 \cdot 31 \cdot 41 \cdot 43$	99127
$2^7 3^7 5^5 7^2 17 \cdot 19 \cdot 31 \cdot 41$	3600	$2^7 3^5 5^5 7^5 11 \cdot 13^3 17^3 19 \cdot 29 \cdot 31 \cdot 43$	140777
$2^9 3^7 5^3 7^3 11 \cdot 13 \cdot 31 \cdot 41$	3780	$2^{11} 3^7 5^5 7^5 11 \cdot 13^3 17 \cdot 19 \cdot 31 \cdot 41 \cdot 43$	141960
$2^9 3^7 5^3 7^2 11 \cdot 13 \cdot 19 \cdot 31 \cdot 41$	5400	$2^{11} 3^5 5^5 7^5 11 \cdot 13^3 17^3 19 \cdot 29 \cdot 31 \cdot 43$	210392

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