

CONTINUOUS INTERIOR PENALTY
hp-FINITE ELEMENT METHODS FOR ADVECTION
AND ADVECTION-DIFFUSION EQUATIONS

ERIK BURMAN AND ALEXANDRE ERN

ABSTRACT. A continuous interior penalty *hp*-finite element method that penalizes the jump of the gradient of the discrete solution across mesh interfaces is introduced. Error estimates are obtained for advection and advection-diffusion equations. The analysis relies on three technical results that are of independent interest: an *hp*-inverse trace inequality, a local discontinuous to continuous *hp*-interpolation result, and *hp*-error estimates for continuous L^2 -orthogonal projections.

1. INTRODUCTION

Interior penalty procedures for finite element methods utilizing continuous functions have been introduced in the pioneering works of Babuška and Zlámal [3] for the biharmonic operator and of Douglas and Dupont [12] for second-order elliptic and parabolic problems. The common feature of these methods consists of penalizing the jump of the gradient of the discrete solution at mesh interfaces, but the motivations behind [3] and [12] are different. The goal pursued in [3] was to weakly enforce C^1 -continuity. Because of a non-consistency in the formulation, a superpenalty procedure had to be applied, leading to suboptimal convergence rates. The subsequent work of Baker [4], valid for general $2m$ th order coercive operators, designed a consistent interior penalty method, utilizing *discontinuous* functions, that was shown to be optimally convergent.

The motivation behind the work of Douglas and Dupont was different, namely to keep *continuous* finite element methods because they were standard for elliptic problems and, at the same time, to cope with the difficulties encountered by such methods in problems where the first-order (advection) terms dominate the second-order (diffusion) terms. However, one of the main issues at stake, namely the robustness of the error estimates with respect to the cell Péclet numbers, was not addressed in [12]. This issue has only been addressed quite recently for linear finite elements, namely in the work of Burman and Hansbo in 2004 [9]. A unified framework for the convergence analysis of both continuous and nonconforming (Crouzeix–Raviart) finite elements with interior penalty, still in the linear case, has been proposed in [5], and a nonlinear shock-capturing term based on gradient jumps that rigorously

Received by the editor January 24, 2005 and, in revised form, March 25, 2006.

2000 *Mathematics Subject Classification*. Primary 65N30, 65N12, 65N15, 65D05, 65N35.

Key words and phrases. Continuous interior penalty, *hp*-finite element method, convection-diffusion, *hp*-interpolation and projection, *hp*-inverse trace inequality.

©2007 American Mathematical Society
Reverts to public domain 28 years from publication

guarantees a discrete maximum principle for advection-diffusion-reaction problems has been investigated in [6].

The goal of this paper is to present, for the first time, an hp -convergence analysis for a high-order Continuous Interior Penalty (CIP) finite element method applied to advection and (advection-dominated) advection-diffusion equations. The hp -version of the finite element method was introduced in the 1980s following the analysis presented by Babuška and Dorr [1] and by Babuška and Suri [2] for second-order elliptic problems. See also [25] for a review of theoretical background and applications of p - and hp -finite element methods. For advection and (advection-dominated) advection-diffusion equations, hp -finite element approximations have been investigated by Houston, Schwab, and Süli for continuous finite elements with streamline diffusion stabilization and for interior penalty discontinuous finite elements [18, 20, 19], leading, respectively, to the so-called hp -Streamline Diffusion (SD) method and the hp -Discontinuous Galerkin (DG) method. These are, to date, the two established methods for the hp -finite element approximation of advection and (advection-dominated) advection-diffusion equations.

The hp -CIP finite element method investigated in this paper penalizes the jump of the gradient of the discrete solution at mesh interfaces. One advantage with respect to both the hp -SD method and the hp -DG method is that the stabilization parameter is independent of the diffusion coefficient. This can be important in nonlinear problems where this coefficient depends on the discrete solution. The other advantage with respect to the hp -DG method is that the hp -CIP method requires less degrees of freedom (though only marginally less for high-order polynomials). The other advantage with respect to the hp -SD method is that the hp -CIP method leads to a single, symmetric stabilization term. In the hp -SD method, the stabilization terms involve couplings with the second-order term, the source term, and the time-derivative. This can pose problems when approximating stiff problems. The price to be paid for these advantages in the hp -CIP method is a slight sub-optimality (proportional to $p^{\frac{1}{4}}$) in the error estimate and, compared with the hp -SD method, a less compact discretization stencil. However, the hp -CIP method has optimal convergence properties in the diffusion-dominated regime, as opposed to the hp -DG method where a factor of $p^{\frac{1}{2}}$ is lost.

This paper is organized as follows. Section 2 introduces the discrete setting and the hp -CIP finite element method to approximate a first-order reaction-advection equation. Section 3 contains three technical results, that are of independent interest, for tensor-product finite elements. We prove an hp -inverse trace inequality, a local discontinuous to continuous hp -interpolation result, and hp -error estimates for continuous L^2 -orthogonal projections. Section 4 presents the convergence analysis of the method in the spirit of Strang's Second Lemma. Section 5 discusses the extension of the results derived in Sections 3 and 4 to simplicial finite elements. Section 6 investigates the extension of the results derived for tensor-product finite elements to advection-diffusion equations. Section 7 draws some conclusions.

2. CONTINUOUS INTERIOR PENALTY FINITE ELEMENT METHODS

Let Ω be an open bounded and connected set in \mathbb{R}^d , $d \leq 3$, with Lipschitz boundary $\partial\Omega$ and outer normal n , let $\beta \in [W^{1,\infty}(\Omega)]^d$ be a vector field, and let $\sigma \in L^\infty(\Omega)$. Let $f \in L^2(\Omega)$, let $\partial\Omega^\pm = \{x \in \partial\Omega; \pm\beta(x) \cdot n(x) > 0\}$, and consider

the problem

$$(1) \quad \begin{cases} \sigma u + \beta \cdot \nabla u = f, \\ u|_{\partial\Omega^-} = 0. \end{cases}$$

Define $W = \{w \in L^2(\Omega); \beta \cdot \nabla w \in L^2(\Omega)\}$. Provided $\partial\Omega^+$ and $\partial\Omega^-$ are well-separated [14] (see also [21]), functions in W have traces in $L^2(\partial\Omega; |\beta \cdot n|)$, that is, in the space of square integrable functions on $\partial\Omega$ for the surface measure $|\beta \cdot n|d\lambda$ where $d\lambda$ is the surface Lebesgue measure. Consider the operator $A : W \ni w \mapsto \sigma w + \beta \cdot \nabla w \in L^2(\Omega)$. Henceforth, it is assumed that there is $\sigma_0 > 0$ such that

$$(2) \quad \sigma - \frac{1}{2} \nabla \cdot \beta \geq \sigma_0, \quad \text{a.e. in } \Omega.$$

Then, letting $V = \{w \in W; w|_{\partial\Omega^-} = 0\}$, $A : V \rightarrow L^2(\Omega)$ is an isomorphism, i.e., (1) is well-posed; see, e.g., [15, 14].

Let \mathcal{K} be a subdivision of Ω into non-overlapping cells $\{\kappa\}$. For $\kappa \in \mathcal{K}$, h_κ denotes its diameter. Set $h = \max_{\kappa \in \mathcal{K}} h_\kappa$. Assume that (i) \mathcal{K} covers $\overline{\Omega}$ exactly, (ii) \mathcal{K} does not contain any hanging nodes, and (iii) \mathcal{K} is quasi-uniform in the sense that there exists a constant $\rho > 0$, independent of h , such that $\rho h \leq \min_{\kappa \in \mathcal{K}} h_\kappa$. Each $\kappa \in \mathcal{K}$ is assumed to be an affine image of the unit hypercube $\widehat{\kappa} = [-1, 1]^d$, i.e., $\kappa = T_\kappa(\widehat{\kappa})$. Let \mathcal{F} denote the set of interior faces ($(d - 1)$ -manifolds) of the mesh. For $F \in \mathcal{F}$, h_F denotes its diameter.

Let $p \geq 1$ and let $\mathbb{Q}_{p,d}(\widehat{\kappa})$ be the space of polynomials of degree at most p in each variable. Introduce the finite element spaces

$$(3) \quad V_h^p = \{v_h \in C^0(\overline{\Omega}); \forall \kappa \in \mathcal{K}, v_h|_\kappa \circ T_\kappa \in \mathbb{Q}_{p,d}(\widehat{\kappa})\},$$

$$(4) \quad W_h^p = \{w_h \in L^2(\Omega); \forall \kappa \in \mathcal{K}, w_h|_\kappa \circ T_\kappa \in \mathbb{Q}_{p,d}(\widehat{\kappa})\}.$$

For a subset $R \subset \Omega$, $(\cdot, \cdot)_R$ denotes the $L^2(R)$ -scalar product, $\|\cdot\|_R = (\cdot, \cdot)_R^{1/2}$ the corresponding norm, and $\|\cdot\|_{s,R}$ the $H^s(R)$ -norm. For $s \geq 1$, let $H^s(\mathcal{K})$ be the space of piecewise H^s functions. For $v \in H^2(\mathcal{K})$ and an interior face $F = \kappa_1 \cap \kappa_2$, where κ_1 and κ_2 are two distinct elements of \mathcal{K} with respective outer normals n_1 and n_2 , introduce the (scalar-valued) jump $[\nabla v \cdot n]_F = \nabla v|_{\kappa_1} \cdot n_1 + \nabla v|_{\kappa_2} \cdot n_2$ (the subscript F is dropped when there is no ambiguity). Similarly, for $v \in H^1(\mathcal{K})$, define the (scalar-valued) jump $[v]_F = v|_{\kappa_1} - v|_{\kappa_2}$ (the arbitrariness in the sign of $[v]_F$ can be avoided by considering the vector-valued jump $[v]_F = v|_{\kappa_1} n_1 + v|_{\kappa_2} n_2$; nothing that is stated hereafter depends on this arbitrariness).

On $H^1(\Omega) \times H^1(\Omega)$ define the standard Galerkin bilinear form

$$(5) \quad a(v, w) = ((\sigma - \nabla \cdot \beta)v, w)_\Omega - (v, \beta \cdot \nabla w)_\Omega + (\beta \cdot n v, w)_{\partial\Omega^+},$$

and on $H^q(\mathcal{K}) \times H^q(\mathcal{K})$, $q > \frac{3}{2}$, define the hp -CIP bilinear form

$$(6) \quad j(v, w) = \sum_{F \in \mathcal{F}} \frac{h_F^2}{p^\alpha} \beta_{\infty,F} ([\nabla v \cdot n], [\nabla w \cdot n])_F,$$

where $\beta_{\infty,F} := \|\beta \cdot n_F\|_{L^\infty(F)}$ and n_F is a unit normal vector to F (its orientation is irrelevant). Since $W^{1,\infty}(\Omega) \subset C^0(\overline{\Omega})$, the field β is continuous by assumption and, therefore, the quantity $\beta \cdot n_F$ is single-valued on all interior faces. The exponent α will be determined by the hp -convergence analysis in §4; see (41).

The hp -CIP finite element approximation to (1) consists of finding $u_h \in V_h^p$ such that

$$(7) \quad a(u_h, v_h) + j(u_h, v_h) = (f, v_h)_\Omega, \quad \forall v_h \in V_h^p.$$

For $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$, consider the norm

$$(8) \quad \|v\|_{a,j}^2 = \|\sigma_0^{\frac{1}{2}}v\|_{\Omega}^2 + \frac{1}{2}\|\beta \cdot n\|^{\frac{1}{2}}v\|_{\partial\Omega}^2 + j(v, v).$$

The well-posedness of the approximate problem (7) results from the following.

Lemma 2.1 (Coerciveness). *For all $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$, $a(v, v) + j(v, v) \geq \|v\|_{a,j}^2$.*

Proof. Straightforward verification using the divergence formula. □

3. TECHNICAL RESULTS

Henceforth, c denotes a generic constant, independent of p and h ; its actual value can change at each occurrence.

3.1. hp -trace inequalities. Let $\{g_j\}_{0 \leq j \leq p}$ be the Gauß–Lobatto nodes in the unit interval $[-1, 1]$. Set $I_{p,d} = \{0, \dots, p\}^d$ and $I_{p,d}^0 = \{1, \dots, p-1\}^d$. For a multi-index $(i) = (i_1, \dots, i_d) \in I_{p,d}$, the tensor-product Gauß–Lobatto node $a_{\widehat{\kappa},(i)}$ in the unit hypercube $\widehat{\kappa}$ is the point with coordinates equal to $(g_{i_1}, \dots, g_{i_d})$.

Let $\kappa \in \mathcal{K}$. Introduce the tensor-product Gauß–Lobatto nodes in K such that $a_{\kappa,(i)} = T_{\kappa}(a_{\widehat{\kappa},(i)})$ for all $(i) \in I_{p,d}$ and define the space

$$(9) \quad \mathbb{Q}_{p,d}^0(\kappa) = \{v \in \mathbb{Q}_{p,d}(\kappa); \forall (i) \in I_{p,d}^0, v(a_{\kappa,(i)}) = 0\}.$$

In other words, $\mathbb{Q}_{p,d}^0(\kappa)$ is the subspace of $\mathbb{Q}_{p,d}(\kappa)$ spanned by those polynomials that vanish at all the interior tensor-product Gauß–Lobatto nodes in κ . The following result is stated in arbitrary space dimension since it can be of independent interest.

Lemma 3.1. *The following trace and inverse trace inequalities hold:*

$$(10) \quad \forall v \in \mathbb{Q}_{p,d}(\kappa), \quad \|v\|_{\partial\kappa} \leq \left(\frac{p(p+1)}{2} \left(2 + \frac{1}{p}\right)^d \frac{\text{meas}(\partial\kappa)}{\text{meas}(\kappa)} \right)^{\frac{1}{2}} \|v\|_{\kappa},$$

$$(11) \quad \forall v \in \mathbb{Q}_{p,d}^0(\kappa), \quad \|v\|_{\kappa} \leq \left(\frac{2d}{p(p+1)} \left(2 + \frac{1}{p}\right)^{d-1} \frac{\text{meas}(\kappa)}{\text{meas}(\partial\kappa)} \right)^{\frac{1}{2}} \|v\|_{\partial\kappa},$$

with the convention that $\text{meas}(\partial\kappa) = 2$ if $d = 1$.

Proof. Let $\{\varpi_j\}_{0 \leq j \leq p}$ be the weights associated with the one-dimensional Gauß–Lobatto nodes. Recall that [11]

$$(12) \quad \varpi_j = \frac{2}{p(p+1)} \frac{1}{L_p^2(g_j)}, \quad j \in \{0, \dots, p\},$$

where L_p is the Legendre polynomial of degree p . For a multi-index $(i) = (i_1, \dots, i_d)$ in $I_{p,d}$, set $\varpi_{(i)} = \prod_{l=1}^d \varpi_{i_l}$. Recall that (see, e.g., [24])

$$(13) \quad \forall v \in \mathbb{Q}_{p,d}(\widehat{\kappa}), \quad \|v\|_{\widehat{\kappa}} \leq \left(\sum_{(i) \in I_{p,d}} \varpi_{(i)} v(a_{\widehat{\kappa},(i)})^2 \right)^{\frac{1}{2}} \leq \left(2 + \frac{1}{p}\right)^{\frac{d}{2}} \|v\|_{\widehat{\kappa}}.$$

(A) Proof of (10). Let $\widehat{v} \in \mathbb{Q}_{p,d}(\widehat{\kappa})$. Set $\widehat{F}_{\pm} = \{x_1 = \pm 1\}$. Using the first inequality in (13) in dimension $(d-1)$ on \widehat{F}_{\pm} yields

$$\|\widehat{v}\|_{\widehat{F}_+ \cup \widehat{F}_-}^2 \leq \sum_{(i') \in I_{p,d-1}} \varpi_{(i')} \widehat{v}(a_{\widehat{F}_+,(i')})^2 + \sum_{(i') \in I_{p,d-1}} \varpi_{(i')} \widehat{v}(a_{\widehat{F}_-,(i')})^2,$$

where $a_{\widehat{F}_\pm, (i')}$ are the tensor-product Gauß-Lobatto nodes on \widehat{F}_\pm . Using the fact that $\varpi_0 = \varpi_p = \frac{2}{p(p+1)}$ leads to

$$\|\widehat{v}\|_{\widehat{F}_+ \cup \widehat{F}_-}^2 \leq \frac{p(p+1)}{2} \sum_{\substack{(i) \in I_{p,d} \\ i_1 \in \{0,p\}}} \varpi_{(i)} \widehat{v}(a_{\widehat{\kappa},(i)})^2.$$

Hence, owing to the second inequality in (13) in dimension d , it is inferred that

$$\|\widehat{v}\|_{\widehat{F}_+ \cup \widehat{F}_-}^2 \leq \frac{p(p+1)}{2} \sum_{(i) \in I_{p,d}} \varpi_{(i)} \widehat{v}(a_{\widehat{\kappa},(i)})^2 \leq \frac{p(p+1)}{2} (2 + \frac{1}{p})^d \|v\|_{\widehat{\kappa}}^2.$$

Summing over all pairs of opposite faces yields

$$(14) \quad \|v\|_{\partial \widehat{\kappa}}^2 \leq d \frac{p(p+1)}{2} (2 + \frac{1}{p})^d \|v\|_{\widehat{\kappa}}^2.$$

Let $v \in \mathbb{Q}_{p,d}(\kappa)$. Then, $v \circ T_\kappa \in \mathbb{Q}_{p,d}(\widehat{\kappa})$. Use (14), the fact that $\text{meas}(\widehat{\kappa}) = \frac{1}{d} \text{meas}(\partial \widehat{\kappa})$, and a scaling argument to infer (10).

(B) Proof of (11). Let $\widehat{v} \in \mathbb{Q}_{p,d}^0(\widehat{\kappa})$. Using the first inequality in (13) in dimension d and the second inequality in (13) in dimension $(d-1)$ yields

$$\begin{aligned} \|\widehat{v}\|_{\widehat{\kappa}}^2 &\leq \sum_{(i) \in I_{p,d}} \varpi_{(i)} \widehat{v}(a_{\widehat{\kappa},(i)})^2 = \sum_{l=1}^d \sum_{\substack{(i) \in I_{p,d} \\ i_l \in \{0,p\}}} \varpi_{(i)} \widehat{v}(a_{\widehat{\kappa},(i)})^2 \\ &= \sum_{l=1}^d \frac{2}{p(p+1)} \sum_{\substack{(i) \in I_{p,d} \\ i_l \in \{0,p\}}} (\varpi_{i_1} \dots \varpi_{i_{l-1}} \varpi_{i_{l+1}} \dots \varpi_{i_p}) \widehat{v}(a_{\widehat{\kappa},(i)})^2 \\ &\leq \sum_{l=1}^d \frac{2}{p(p+1)} (2 + \frac{1}{p})^{d-1} \|\widehat{v}\|_{\{x_l = \pm 1\}}^2 = \frac{2}{p(p+1)} (2 + \frac{1}{p})^{d-1} \|\widehat{v}\|_{\partial \widehat{\kappa}}^2. \end{aligned}$$

Let $v \in \mathbb{Q}_{p,d}^0(\kappa)$. Then, $v \circ T_\kappa \in \mathbb{Q}_{p,d}^0(\widehat{\kappa})$. Conclude by proceeding as above. \square

While the trace inequalities in Lemma 3.1 are valid in arbitrary space dimension, the focus in this paper is set on the case where d is fixed, e.g., $d \leq 3$. This allows us to simplify the constants in the trace inequalities as follows:

$$(15) \quad \forall v \in \mathbb{Q}_{p,d}(\kappa), \quad \|v\|_{\partial \kappa} \leq c \left(\frac{p^2}{h_\kappa} \right)^{\frac{1}{2}} \|v\|_\kappa,$$

$$(16) \quad \forall v \in \mathbb{Q}_{p,d}^0(\kappa), \quad \|v\|_\kappa \leq c \left(\frac{h_\kappa}{p^2} \right)^{\frac{1}{2}} \|v\|_{\partial \kappa}.$$

Inequality (15) is derived in [25] for triangular and quadrangular reference elements; inequality (16) is, to the authors' knowledge, new. An important observation is that the inverse trace inequality (16) is optimal (asymptotically in p) with respect to the trace inequality (15).

3.2. Continuous hp -interpolation. The goal is to construct an operator $\mathcal{I}_{O_s} : W_h^p \rightarrow V_h^p$ endowed with a local hp -interpolation property.

Let $\kappa \in \mathcal{K}$. For a node ν in κ , set $\mathcal{K}_\nu = \{\kappa' \in \mathcal{K}; \nu \in \kappa'\}$; then, for $w_h \in W_h^p$, define $\mathcal{I}_{\text{Os}}w_h$ locally in κ by the value it takes at all the tensor-product Gauß-Lobatto nodes by setting

$$(17) \quad \mathcal{I}_{\text{Os}}w_h(\nu) = \frac{1}{\text{card}(\mathcal{K}_\nu)} \sum_{\kappa \in \mathcal{K}_\nu} w_h|_\kappa(\nu).$$

Clearly, $\mathcal{I}_{\text{Os}}w_h \in V_h^p$. The operator \mathcal{I}_{Os} is sometimes referred to as the Oswald interpolation operator; it has been considered in [5, 13, 16, 22]. Another hp -interpolation operator for non-smooth functions generalizing that of Clément and Scott–Zhang is analyzed in [23].

Lemma 3.2. *There exists c , independent of p and h , such that, for all $\kappa \in \mathcal{K}$, the following estimate holds:*

$$(18) \quad \forall w_h \in W_h^p, \quad \|w_h - \mathcal{I}_{\text{Os}}w_h\|_\kappa \leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[w_h]\|_F,$$

where $\mathcal{F}(\kappa) = \{F \in \mathcal{F}; F \cap \kappa \neq \emptyset\}$.

Proof. Let $w_h \in W_h^p$ and let $\kappa \in \mathcal{K}$. Set $\delta_h = (w_h - \mathcal{I}_{\text{Os}}w_h)|_\kappa$. For each tensor-product Gauß-Lobatto node $\nu \in \kappa$, let φ_ν denote the associated nodal basis function. By construction, δ_h vanishes at all the nodes located in the interior of κ . Hence,

$$\delta_h = \sum_{\nu \in \partial\kappa} \delta_h(\nu)\varphi_\nu.$$

(A) Define $\partial\kappa_{d-1}$ to be the set of $(d - 1)$ -manifolds in $\partial\kappa$ and for $m \in \{d - 2, \dots, 0\}$, define $\partial\kappa_m$ to be the set obtained by taking the intersection of any two distinct elements in $\partial\kappa_{m+1}$. For instance, for $d = 3$, $\partial\kappa_2$, $\partial\kappa_1$, and $\partial\kappa_0$ are, respectively, the set of faces, edges, and vertices of $\partial\kappa$. Partition the boundary nodes of κ as follows. For $m \in \{0, \dots, d - 1\}$, denote by \mathcal{V}_m the set of nodes in the interior of $\partial\kappa_m$; then, $\bigcup_{m=0}^{d-1} \mathcal{V}_m$ forms a partition of the boundary nodes of κ . Hence,

$$\delta_h = \sum_{\nu \in \mathcal{V}_{d-1}} \delta_h(\nu)\varphi_\nu + \sum_{m=0}^{d-2} r_m,$$

with

$$r_m = \sum_{\nu \in \mathcal{V}_m} \delta_h(\nu)\varphi_\nu.$$

Observe that for all $m \in \{0, \dots, d - 2\}$, $r_m \in \mathbb{Q}_{p,d}^0(\kappa)$ and

$$r_m \in \mathbb{Q}_{p,l}^0(\partial\kappa_l), \quad \forall l \in \{m + 1, \dots, d - 1\}.$$

(B) On each face $F \subset \partial\kappa$ and for each node ν located in the interior of F , equation (17) implies that $\delta_h(\nu) = \epsilon_F [w_h]_F(\nu)$ with $\epsilon_F = \pm \frac{1}{2}$. Hence,

$$\sum_{\nu \in \mathcal{V}_{d-1}} \delta_h(\nu)\varphi_\nu = \sum_{F \subset \partial\kappa} \psi_F,$$

with

$$\psi_F = \epsilon_F \sum_{\nu \in \overset{\circ}{F}} [w_h]_F(\nu)\varphi_\nu,$$

where $\overset{\circ}{F}$ denotes the interior of F . Since $\psi_F \in \mathbb{Q}_{p,d}^0(\kappa)$ and ψ_F vanishes on $\partial\kappa \setminus F$, the inverse trace inequality (16) yields

$$\|\psi_F\|_\kappa \leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \|\psi_F\|_{\partial\kappa} = c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \|\psi_F\|_F.$$

Hence,

$$(19) \quad \|\delta_h\|_\kappa \leq c \sum_{F \subset \partial\kappa} \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \|\psi_F\|_F + \sum_{m=0}^{d-2} \|r_m\|_\kappa := \text{(I)} + \text{(II)}.$$

The rest of the proof consists of estimating both terms in the right-hand side of (19). If $d = 1$, term (II) vanishes and $\psi_F|_F = \epsilon_F[w_h]_F$; hence, (19) readily yields (18).

(C) Assume $d = 2$. Let us first consider term (I). Let $F \subset \partial\kappa$. Since $\psi_F - \epsilon_F[w_h]_F$ is in $\mathbb{Q}_{p,1}^0(F)$, the inverse trace inequality (16) yields

$$(20) \quad \|\psi_F - \epsilon_F[w_h]_F\|_F \leq c \left(\frac{h_F}{p^2}\right)^{\frac{1}{2}} \|[w_h]_F\|_{\partial F},$$

since ψ_F vanishes on ∂F . Owing to the trace inequality (15),

$$(21) \quad \|[w_h]_F\|_{\partial F} \leq c \left(\frac{p^2}{h_F}\right)^{\frac{1}{2}} \|[w_h]_F\|_F.$$

Inequalities (20) and (21) imply

$$\|\psi_F - \epsilon_F[w_h]_F\|_F \leq c \|[w_h]_F\|_F.$$

Using the triangle inequality yields $\|\psi_F\|_F \leq c \|[w_h]_F\|_F$; hence,

$$(22) \quad \text{(I)} \leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \subset \partial\kappa} \|[w_h]_F\|_F.$$

Let us now consider term (II). Since $r_0 \in \mathbb{Q}_{p,2}^0(\kappa)$ and $r_0 \in \mathbb{Q}_{p,1}^0(F)$ for all $F \subset \partial\kappa$, using the inverse trace inequality (16) twice, yields

$$\|r_0\|_\kappa \leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \subset \partial\kappa} \|r_0\|_F \leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \subset \partial\kappa} \left(\frac{h_F}{p^2}\right)^{\frac{1}{2}} \|r_0\|_{\partial F}.$$

Observe that ∂F consists of two nodes and that for $\nu \in \partial F$,

$$r_0(\nu) = \sum_{F' \in \mathcal{F}_\nu} \eta_{F,F'} [w_h]_{F'}(\nu),$$

where \mathcal{F}_ν denotes the set of faces in the mesh containing the node ν and where $\eta_{F,F'} = \pm \frac{3}{8}$ if $F' \subset \partial\kappa$ and $\eta_{F,F'} = \pm \frac{1}{8}$ otherwise. Then, for all $F \subset \partial\kappa$, the following holds:

$$\begin{aligned} \|r_0\|_{\partial F} &\leq c \sum_{\nu \in \partial F} \sum_{F' \in \mathcal{F}_\nu} |[w_h]_{F'}(\nu)| \\ &\leq c \sum_{\nu \in \partial F} \sum_{F' \in \mathcal{F}_\nu} \left(\frac{p^2}{h_{F'}}\right)^{\frac{1}{2}} \|[w_h]_{F'}\|_{F'} \\ &\leq c \sum_{F'; F \cap F' \neq \emptyset} \left(\frac{p^2}{h_{F'}}\right)^{\frac{1}{2}} \|[w_h]_{F'}\|_{F'}, \end{aligned}$$

owing to the trace inequality (15). Hence,

$$\begin{aligned} \text{(II)} &\leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \subset \partial\kappa} \left(\frac{h_F}{p^2}\right)^{\frac{1}{2}} \sum_{F'; F \cap F' \neq \emptyset} \left(\frac{p^2}{h_{F'}}\right)^{\frac{1}{2}} \|[w_h]_{F'}\|_{F'} \\ &\leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[w_h]_F\|_F. \end{aligned}$$

The proof is complete.

(D) For $d = 3$, the proof proceeds similarly. Term (I) is estimated as for $d = 2$. Term (II) is equal to $\|r_0\|_\kappa + \|r_1\|_\kappa$. The quantity $\|r_0\|_\kappa$ is estimated as for $d = 2$ by using the inverse trace inequality (16) three times and the trace inequality (15) twice. To estimate $\|r_1\|_\kappa$, use twice the inverse trace inequality (16) to infer

$$\|r_1\|_\kappa \leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \subset \partial\kappa} \left(\frac{h_F}{p^2}\right)^{\frac{1}{2}} \sum_{E \subset \partial F} \|r_1\|_E,$$

where $E \subset \partial F$ means for all edges in ∂F . Straightforward algebra shows that

$$r_1|_E = \sum_{\nu \in \overset{\circ}{E}} \delta_h(\nu) \varphi_\nu = \sum_{\nu \in \overset{\circ}{E}} \sum_{F' \in \mathcal{F}_\nu} \eta_{F, F'} [w_h]_{F'}(\nu) \varphi_\nu,$$

where $\overset{\circ}{E}$ denotes the interior of E . Observe that the coefficient $\eta_{F, F'}$ is independent of ν . Let \mathcal{F}_E be the set of faces in the mesh that contain E . Then,

$$r_1|_E = \sum_{F' \in \mathcal{F}_E} \eta_{F, F'} \sum_{\nu \in \overset{\circ}{E}} [w_h]_{F'}(\nu) \varphi_\nu.$$

Hence,

$$\|r_1\|_E \leq c \sum_{F' \in \mathcal{F}_E} \left\| \sum_{\nu \in \overset{\circ}{E}} [w_h]_{F'}(\nu) \varphi_\nu \right\|_E \leq c \sum_{F' \in \mathcal{F}_E} (\|[w_h]_{F'}\|_E + \|\zeta_{h, F'}\|_E),$$

with $\zeta_{h, F'} = \sum_{\nu \in \overset{\circ}{E}} [w_h]_{F'}(\nu) \varphi_\nu - [w_h]_{F'}$. Since $\zeta_{h, F'} \in \mathbb{Q}_{p,1}^0(E)$, the inverse trace inequality (16) yields

$$\|\zeta_{h, F'}\|_E \leq c \left(\frac{h_E}{p^2}\right)^{\frac{1}{2}} \sum_{\nu \in \partial E} |[w_h](\nu)|.$$

The conclusion is now straightforward. □

Remark 3.1. The conclusion of Lemma 3.2 holds in arbitrary space dimension. This can be proven by proceeding as above for $d = 3$; the inverse trace inequality (16) is used d times and the trace inequality (15) is used $(d - 1)$ times.

Remark 3.2. hp -interpolation of discontinuous functions by continuous functions has been analyzed recently in the H^1 -seminorm in two space dimensions on triangular and quadrangular meshes [17]. The H^1 -seminorm estimate on quadrangles derived in [17], namely

$$\forall w_h \in W_h^p, \quad \|\nabla(w_h - \mathcal{I}_{\text{Os}} w_h)\|_\kappa \leq c \left(\frac{p^2}{h_\kappa}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[w_h]\|_F,$$

can be inferred from Lemma 3.2 above and the well-known inverse inequality

$$(23) \quad \|\nabla v_h\|_\kappa \leq c \frac{p^2}{h_\kappa} \|v_h\|_\kappa,$$

valid for all $v_h \in W_h^p$ (see, e.g., [10] for the proof of the p version of this inequality).

3.3. hp -error estimate for continuous $L^2(\Omega)$ -orthogonal projection. The purpose of this section is to investigate the approximation properties in the L^2 - and the H^1 -norms of the operator $\Pi_h : L^2(\Omega) \rightarrow V_h^p$ defined as the $L^2(\Omega)$ -orthogonal projector onto V_h^p .

First, we recall the following local hp -approximation property [10, 19]. Let $\Pi_h^* : L^2(\Omega) \rightarrow W_h^p$ be the $L^2(\Omega)$ -orthogonal projector onto W_h^p . Then, there is c , independent of p and h , such that for all $\kappa \in \mathcal{K}$ and all $w \in H^q(\mathcal{K})$, $q \geq 1$,

$$(24) \quad \|w - \Pi_h^* w\|_\kappa \leq c \left(\frac{h}{p}\right)^s \|w\|_{s,\kappa},$$

$$(25) \quad \|\nabla(w - \Pi_h^* w)\|_\kappa \leq cp^{\frac{1}{2}} \left(\frac{h}{p}\right)^{s-1} \|w\|_{s,\kappa},$$

with $s = \min(p+1, q)$. We now establish the global counterpart of (24)–(25) for the continuous $L^2(\Omega)$ -orthogonal projector Π_h .

Lemma 3.3. *There exists c , independent of p and h , such that for all $w \in H^q(\Omega)$, $q \geq 1$,*

$$(26) \quad \|w - \Pi_h w\|_\Omega \leq c \left(\frac{h}{p}\right)^s \|w\|_{s,\Omega},$$

$$(27) \quad \|\nabla(w - \Pi_h w)\|_\Omega \leq cp^{\frac{1}{2}} \left(\frac{h}{p}\right)^{s-1} \|w\|_{s,\Omega},$$

with $s = \min(p+1, q)$.

Proof. Let $w \in H^q(\Omega)$, $q \geq 1$.

(A) Proof of (26). Since $V_h^p \subset W_h^p$, the definition of Π_h and Π_h^* leads to

$$(\Pi_h w - \Pi_h^* w, v_h) = 0, \quad \forall v_h \in V_h^p.$$

Hence, $\Pi_h \Pi_h^* w = \Pi_h w$. Using the fact that $\mathcal{I}_{\text{Os}}(\Pi_h^* w) \in V_h^p$ yields

$$\|\Pi_h w - \Pi_h^* w\|_\Omega = \|\Pi_h(\Pi_h^* w) - \Pi_h^* w\|_\Omega \leq \|\mathcal{I}_{\text{Os}}(\Pi_h^* w) - \Pi_h^* w\|_\Omega.$$

Using Lemma 3.2, it is inferred that for all $\kappa \in \mathcal{K}$,

$$\begin{aligned} \|\mathcal{I}_{\text{Os}}(\Pi_h^* w) - \Pi_h^* w\|_\kappa &\leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[\Pi_h^* w]_F\|_F \\ &\leq c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[\Pi_h^* w - w]_F\|_F, \end{aligned}$$

since $[w]_F = 0$ by assumption. Recalling the fact [19] that there is c , independent of p and h , such that for all $\kappa \in \mathcal{K}$,

$$(28) \quad \|w - \Pi_h^* w\|_{\partial\kappa} \leq c \left(\frac{h_\kappa}{p}\right)^{s-\frac{1}{2}} \|w\|_{s,\kappa},$$

it is readily deduced that

$$(29) \quad \|\Pi_h w - \Pi_h^* w\|_\Omega \leq cp^{-\frac{1}{2}} \left(\frac{h}{p}\right)^s \|w\|_{s,\Omega}.$$

Conclude using the triangle inequality

$$\|w - \Pi_h w\|_\Omega \leq \|w - \Pi_h^* w\|_\Omega + \|\Pi_h^* w - \Pi_h w\|_\Omega,$$

and Equation (24).

(B) Proof of (27). Let $\kappa \in \mathcal{K}$. Using the inverse inequality (23) together with (29), it is inferred that

$$(30) \quad \left(\sum_{\kappa \in \mathcal{K}} \|\nabla(\Pi_h^* w - \Pi_h w)\|_\kappa^2\right)^{\frac{1}{2}} \leq cp^{\frac{1}{2}} \left(\frac{h}{p}\right)^{s-1} \|w\|_{s,\Omega}.$$

Conclude using the triangle inequality and Equation (25). □

Remark 3.3. Equation (29) shows that $\Pi_h w$ superconverges to $\Pi_h^* w$ by a factor of $p^{-\frac{1}{2}}$. This remarkable property allows us to compensate for the loss of one power of p in the inverse inequality (23) and thus to recover a sub-optimality factor of $p^{\frac{1}{2}}$ in (27), which is exactly the same as in (25).

4. CONVERGENCE ANALYSIS

The purpose of this section is to show how the results established in §3, namely the hp -inverse trace inequality, the local hp -interpolation result, and the hp -error estimate for the continuous $L^2(\Omega)$ -orthogonal projection, can be used to analyze the convergence of the hp -CIP finite element method introduced in §2.

Let u solve (1) and let u_h solve (7). Henceforth, it is assumed that the exact solution u is smooth enough, i.e., $u \in H^q(\Omega)$, $q > \frac{3}{2}$. Bounds on the approximation error $u - u_h$ are obtained in the spirit of Strang’s Second Lemma by establishing consistency and boundedness properties for the discrete setting. Recall that the discrete setting satisfies the stability property stated in Lemma 2.1.

Lemma 4.1 (Consistency). *Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve (1) and let u_h solve (7). Then, for all $v_h \in V_h^p$,*

$$(31) \quad a(u - u_h, v_h) + j(u - u_h, v_h) = 0.$$

Proof. Since $u \in H^q(\Omega)$, $q > \frac{3}{2}$, it is inferred that $[\nabla u \cdot n]_F = 0$ for all $F \in \mathcal{F}$. Hence, $j(u, v_h) = 0$ for all $v_h \in V_h^p$, whence (31) is readily deduced. □

For all $\kappa \in \mathcal{K}$, set $\beta_{\kappa,\infty} := \|\beta\|_{L^\infty(\Delta_\kappa)}$ where $\Delta_\kappa = \{\kappa' \in \mathcal{K}; \kappa \cap \kappa' \neq \emptyset\}$. Introduce the (semi-)norm

$$(32) \quad \|v\|_{h,\frac{1}{2}} = \left(\sum_{\kappa \in \mathcal{K}} h_\kappa^{-1} \beta_{\kappa,\infty} \|v\|_\kappa^2\right)^{\frac{1}{2}}.$$

Let $(V_h^p)^\perp = \{z \in L^2(\Omega); (z, v_h)_\Omega = 0, \forall v_h \in V_h^p\}$.

Lemma 4.2 (Boundedness). *There is c , independent of p and h , such that for all $z \in H^q(\mathcal{K}) \cap (V_h^p)^\perp$, $q > \frac{3}{2}$,*

$$(33) \quad \sup_{v_h \in V_h^p} \frac{a(z, v_h) + j(z, v_h)}{\|v_h\|_{a,j}} \leq \|z\|_{a,j} + c(p^2 h^{\frac{1}{2}} + p^{\frac{\alpha}{2}-1}) \|z\|_{h,\frac{1}{2}}.$$

Proof. The only term to estimate is $(z, \beta \cdot \nabla v_h)_\Omega$. Let β_h be the $L^2(\Omega)$ -orthogonal projection of β onto W_h^0 . Set $w_h = \beta_h \cdot \nabla v_h$. Observe that

$$(34) \quad (z, \beta \cdot \nabla v_h)_\Omega = (z, (\beta - \beta_h) \cdot \nabla v_h)_\Omega + (z, w_h)_\Omega.$$

(A) Since $\beta \in [W^{1,\infty}(\Omega)]^d$, the first term in the right-hand side of (34) is estimated as follows:

$$\begin{aligned} (z, (\beta - \beta_h) \cdot \nabla v_h)_\Omega &\leq \left(\sum_{\kappa \in \mathcal{K}} h_\kappa^{-1} \beta_{\kappa,\infty} \|z\|_\kappa^2 \right)^{\frac{1}{2}} \left(\sum_{\kappa \in \mathcal{K}} h_\kappa^3 \beta_{\kappa,\infty}^{-1} \|\beta\|_{W^{1,\infty}(\kappa)}^2 \|\nabla v_h\|_\kappa^2 \right)^{\frac{1}{2}} \\ &\leq cp^2 h^{\frac{1}{2}} \|z\|_{h,\frac{1}{2}} \|v_h\|_\Omega, \end{aligned}$$

owing to the inverse inequality (23).

(B) Let us estimate the second term in the right-hand side of (34). Using the fact that $z \in (V_h^p)^\perp$ yields

$$\begin{aligned} (z, w_h)_\Omega &= (z, w_h - \mathcal{I}_{\text{Os}} w_h)_\Omega \\ &\leq \left(\sum_{\kappa \in \mathcal{K}} h_\kappa^{-1} \beta_{\kappa,\infty} \|z\|_\kappa^2 \right)^{\frac{1}{2}} \left(\sum_{\kappa \in \mathcal{K}} h_\kappa \beta_{\kappa,\infty}^{-1} \|w_h - \mathcal{I}_{\text{Os}} w_h\|_\kappa^2 \right)^{\frac{1}{2}} \\ &\leq c \|z\|_{h,\frac{1}{2}} \left(\sum_{\kappa \in \mathcal{K}} \sum_{F \in \mathcal{F}(\kappa)} \beta_{\kappa,\infty}^{-1} \left(\frac{h_\kappa}{p} \right)^2 \|[w_h]\|_F^2 \right)^{\frac{1}{2}}, \end{aligned}$$

owing to Lemma 3.2. Observe that

$$(35) \quad \|[w_h]\|_F = \|[\beta_h \cdot \nabla v_h]\|_F \leq \|[(\beta - \beta_h) \cdot \nabla v_h]\|_F + \|[\beta \cdot \nabla v_h]\|_F.$$

Using the fact that $\beta \in [W^{1,\infty}(\Omega)]^d$, together with inequalities (15) and (23), the first term in the right-hand side of (35) is estimated as follows:

$$\begin{aligned} \|[(\beta - \beta_h) \cdot \nabla v_h]\|_F &\leq c \sum_{\kappa \supset F} h_\kappa \|\beta\|_{W^{1,\infty}(\kappa)} \|\nabla v_h|_\kappa\|_F \\ &\leq c \sum_{\kappa \supset F} h_\kappa \|\beta\|_{W^{1,\infty}(\kappa)} \left(\frac{p^2}{h_\kappa} \right)^{\frac{1}{2}} \|\nabla v_h\|_\kappa \\ &\leq c \sum_{\kappa \supset F} h_\kappa \|\beta\|_{W^{1,\infty}(\kappa)} \left(\frac{p^2}{h_\kappa} \right)^{\frac{1}{2}} \frac{p^2}{h_\kappa} \|v_h\|_\kappa \\ &\leq c \sum_{\kappa \supset F} \|\beta\|_{W^{1,\infty}(\kappa)} \frac{p^3}{h_\kappa^{\frac{1}{2}}} \|v_h\|_\kappa, \end{aligned}$$

where the notation $\kappa \supset F$ means for the two elements sharing F . Hence,

$$\sum_{\kappa \in \mathcal{K}} \sum_{F \in \mathcal{F}(\kappa)} \beta_{\kappa,\infty}^{-1} \left(\frac{h_\kappa}{p} \right)^2 \|[(\beta - \beta_h) \cdot \nabla v_h]\|_F^2 \leq cp^4 h \|v_h\|_\Omega^2.$$

Furthermore, notice that $|\beta \cdot \nabla v_h|_F = |\beta \cdot n_F| |\nabla v_h \cdot n|_F$ owing to the continuity of β and v_h . Hence,

$$\begin{aligned} \sum_{\kappa \in \mathcal{K}} \sum_{F \in \mathcal{F}(\kappa)} \beta_{\kappa, \infty}^{-1} \left(\frac{h_\kappa}{p}\right)^2 \|\beta \cdot \nabla v_h\|_F^2 &\leq c \sum_{F \in \mathcal{F}} \left(\frac{h_F}{p}\right)^2 \beta_{\infty, F} \|\nabla v_h \cdot n\|_F^2 \\ &\leq cp^{\alpha-2} j(v_h, v_h). \end{aligned}$$

Collecting the above estimates yields

$$(z, w_h)_\Omega \leq c \|z\|_{h, \frac{1}{2}} (p^2 h^{\frac{1}{2}} + p^{\frac{\alpha}{2}-1}) \|v_h\|_{a, j}.$$

The proof is complete. □

Lemma 4.3. *There is c , independent of p and h , such that*

$$(36) \quad \|u - u_h\|_{a, j} \leq c \|u - \Pi_h u\|_{a, j} + c(p^2 h^{\frac{1}{2}} + p^{\frac{\alpha}{2}-1}) \|u - \Pi_h u\|_{h, \frac{1}{2}}.$$

Proof. Owing to Lemmata 2.1 and 4.1,

$$\begin{aligned} \|\Pi_h u - u_h\|_{a, j}^2 &\leq a(\Pi_h u - u_h, \Pi_h u - u_h) + j(\Pi_h u - u_h, \Pi_h u - u_h) \\ &= a(\Pi_h u - u, \Pi_h u - u_h) + j(\Pi_h u - u, \Pi_h u - u_h). \end{aligned}$$

Since $\Pi_h u - u \in (V_h^p)^\perp$, Lemma 4.2 leads to

$$\|\Pi_h u - u_h\|_{a, j} \leq \|\Pi_h u - u\|_{a, j} + c(p^2 h^{\frac{1}{2}} + p^{\frac{\alpha}{2}-1}) \|\Pi_h u - u\|_{h, \frac{1}{2}}.$$

Conclude using the triangle inequality. □

Lemma 4.4. *There is c , independent of p and h , such that for all $w \in H^q(\Omega)$, $q > \frac{3}{2}$,*

$$(37) \quad \|w - \Pi_h w\|_{h, \frac{1}{2}} \leq cp^{-\frac{1}{2}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s, \Omega},$$

$$(38) \quad \|w - \Pi_h w\|_{a, j} \leq c(p^{\frac{1}{4}} + p^{2-\frac{\alpha}{2}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s, \Omega},$$

with $s = \min(p + 1, q)$.

Proof. Estimate (37) directly results from (26). To prove (38), first notice that owing to the standard trace inequality

$$(39) \quad \|v\|_{\partial\Omega} \leq c \|v\|_{\Omega}^{\frac{1}{2}} \|\nabla v\|_{\Omega}^{\frac{1}{2}},$$

valid for all $v \in H^1(\Omega)$, it is inferred from Lemma 3.3 that

$$\|w - \Pi_h w\|_{\partial\Omega} \leq cp^{\frac{1}{4}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s, \Omega}.$$

To control $j(w - \Pi_h w, w - \Pi_h w)$, observe that for all $\kappa \in \mathcal{K}$,

$$\begin{aligned} \|\nabla(w - \Pi_h w)\|_{\partial\kappa} &\leq \|\nabla(w - \Pi_h^* w)\|_{\partial\kappa} + \|\nabla(\Pi_h w - \Pi_h^* w)\|_{\partial\kappa} \\ &\leq \|\nabla(w - \Pi_h^* w)\|_{\partial\kappa} + ph_\kappa^{-\frac{1}{2}} \|\nabla(\Pi_h w - \Pi_h^* w)\|_{\kappa}, \end{aligned}$$

owing to the trace inequality (15). Recall the fact [19] that there is c , independent of p and h , such that for all $\kappa \in \mathcal{K}$,

$$(40) \quad \|\nabla(w - \Pi_h^* w)\|_{\partial\kappa} \leq cp \left(\frac{h_\kappa}{p}\right)^{s-\frac{3}{2}} \|w\|_{s, \kappa}.$$

As a result,

$$\begin{aligned} j(w - \Pi_h w, w - \Pi_h w) &\leq \sum_{F \in \mathcal{F}} h_F^2 p^{-\alpha} \beta_{\infty, F} \|[\nabla(w - \Pi_h w) \cdot n]\|_F^2 \\ &\leq c \sum_{\kappa \in \mathcal{K}} (h_\kappa^2 p^{-\alpha} \|\nabla(w - \Pi_h^* w)\|_{\partial\kappa}^2 + h_\kappa p^{2-\alpha} \|\nabla(\Pi_h w - \Pi_h^* w)\|_\kappa^2) \\ &\leq c \left(p^{4-\alpha} \left(\frac{h}{p}\right)^{2s-1} \|w\|_{s, \Omega}^2 + h p^{2-\alpha} \sum_{\kappa \in \mathcal{K}} \|\nabla(\Pi_h w - \Pi_h^* w)\|_\kappa^2 \right) \\ &\leq c p^{4-\alpha} \left(\frac{h}{p}\right)^{2s-1} \|w\|_{s, \Omega}^2, \end{aligned}$$

owing to (30). The conclusion is straightforward. \square

We are now in a position to state the main result of this section.

Theorem 4.1. *Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve (1) and let u_h solve (7). Take*

$$(41) \quad \alpha = \frac{7}{2}.$$

Then, there is c , independent of p and h , such that

$$(42) \quad \|u - u_h\|_{a,j} \leq c(p^{\frac{1}{4}} + p^{\frac{3}{2}} h^{\frac{1}{2}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s, \Omega},$$

with $s = \min(p+1, q)$. Hence, if $h \leq p^{-\frac{5}{2}}$, the following holds:

$$(43) \quad \|u - u_h\|_{a,j} \leq c p^{\frac{1}{4}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s, \Omega}.$$

Proof. Since $\frac{\alpha}{2} - 1 = \frac{3}{4}$, Lemma 4.3 yields

$$\begin{aligned} \|u - u_h\|_{a,j} &\leq c \|u - \Pi_h u\|_{a,j} + c(p^2 h^{\frac{1}{2}} + p^{\frac{3}{4}}) \|u - \Pi_h u\|_{h, \frac{1}{2}} \\ &\leq c \|u - \Pi_h u\|_{a,j} + c(p^{\frac{3}{2}} h^{\frac{1}{2}} + p^{\frac{1}{4}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s, \Omega}, \end{aligned}$$

owing to (37). Since $2 - \frac{\alpha}{2} = \frac{1}{4}$, (38) leads to

$$\|u - \Pi_h u\|_{a,j} \leq c p^{\frac{1}{4}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s, \Omega},$$

whence (42) is readily deduced. Finally, under the assumption that $h \leq p^{-\frac{5}{2}}$, (43) directly results from (42). \square

5. EXTENSION TO SIMPLICES

The goal of this section is to discuss the extension of the analysis presented in §§2–4 to simplicial finite elements. Several points differ with respect to the case of tensor-product finite elements. First, we prove that the inverse trace inequality cannot be optimal on simplices and that, at best, a factor $p^{\frac{1}{2}}$ is lost (see Lemma 5.2). This leaves open the question of an optimal hp -interpolation operator from discontinuous to continuous simplicial finite element spaces. Restricting ourselves to two space dimensions, we will assume that only the factor $p^{\frac{1}{2}}$ is lost in the inverse trace inequality; see Hypothesis 5.1. Furthermore, the convergence analysis relies on the

assumption that the local hp -approximation properties (24), (25), and (40) hold also on simplices; see Hypothesis 5.2. However, we will not assume that (28) holds.

Let \mathcal{K} be a subdivision of Ω into simplices $\{\kappa\}$. Assume that (i) \mathcal{K} covers $\overline{\Omega}$ exactly, (ii) \mathcal{K} does not contain any hanging nodes, and (iii) \mathcal{K} is locally quasi-uniform in the sense that there exists a constant $\rho > 0$, independent of h , such that $\rho \max_{E \subset \Delta_\kappa} h_E \leq \min_{E \subset \Delta_\kappa} h_E$ where $E \subset \Delta_\kappa$ means for all edges in the patch Δ_κ and h_E denotes the diameter of E . It is assumed that each $\kappa \in \mathcal{K}$ is an affine image of the canonical d -dimensional simplex in \mathbb{R}^d defined by $\widehat{\kappa} = \{(x_1, \dots, x_d) \in \mathbb{R}^d; |x_i| \leq 1; \sum_{i=1}^d x_i \leq 2 - d\}$.

Let $p \geq 1$ and let $\mathbb{P}_{p,d}(\widehat{\kappa})$ be the space of polynomials of total degree at most p . Introduce the finite element spaces

$$(44) \quad X_h^p = \{v_h \in C^0(\overline{\Omega}); \forall \kappa \in \mathcal{K}, v_h|_\kappa \in \mathbb{P}_{p,d}(\kappa)\},$$

$$(45) \quad Y_h^p = \{w_h \in L^2(\Omega); \forall \kappa \in \mathcal{K}, w_h|_\kappa \in \mathbb{P}_{p,d}(\kappa)\}.$$

5.1. Trace and inverse trace inequalities on simplices. The results presented in this section are stated in arbitrary space dimension since they can be of independent interest. First, recall the following trace inequality due to Warburton and Hesthaven [26].

Lemma 5.1. *Let $\kappa \in \mathcal{K}$. The following holds:*

$$(46) \quad \forall v \in \mathbb{P}_{p,d}(\kappa), \quad \|v\|_{\partial\kappa} \leq \left(\frac{(p+1)(p+d) \operatorname{meas}(\partial\kappa)}{d \operatorname{meas}(\kappa)} \right)^{\frac{1}{2}} \|v\|_\kappa.$$

To investigate inverse trace inequalities in $\mathbb{P}_{p,d}(\kappa)$, set $n_{p,d} = \dim(\mathbb{P}_{p,d})$ and let $\{\phi_i\}_{1 \leq i \leq n_{p,d}}$ be the canonical basis of $\mathbb{P}_{p,d}(\kappa)$. Let $\mathcal{A} = \{a_i\}_{1 \leq i \leq n_{p,d}}$ be a set of nodes in κ . The set \mathcal{A} is said to be *unisolvent* in $\mathbb{P}_{p,d}(\kappa)$ if the matrix with coefficients $(\phi_i(a_j))_{1 \leq i,j \leq n_{p,d}}$ is invertible. It is said to be *admissible* if it is unisolvent in $\mathbb{P}_{p,d}(\kappa)$ and if for all dimension $d' < d$ and for any simplex $\kappa' \subset \partial\kappa$ of dimension d' , the set of nodes in κ' is unisolvent in $\mathbb{P}_{p,d'}(\kappa')$. Given a unisolvent set of nodes \mathcal{A} , define

$$(47) \quad \mathbb{P}_{p,d}^{\mathcal{A}}(\kappa) = \{v \in \mathbb{P}_{p,d}(\kappa); \forall a_i \in \mathcal{A}, v(a_i) = 0 \text{ if } a_i \notin \partial\kappa\}.$$

Lemma 5.2. *For all unisolvent set of nodes \mathcal{A} in $\mathbb{P}_{p,d}(\kappa)$, the following holds:*

$$(48) \quad \inf_{v \in \mathbb{P}_{p,d}^{\mathcal{A}}(\kappa) \setminus \{0\}} \frac{\|v\|_{\partial\kappa}}{\|v\|_\kappa} \leq \left(\frac{2p+d \operatorname{meas}(\partial\kappa)}{d \operatorname{meas}(\kappa)} \right)^{\frac{1}{2}}.$$

Proof. Estimate (48) is established on the reference simplex and then mapped to an arbitrary element in \mathcal{K} .

(A) Choose the $L^2(\widehat{\kappa})$ -orthonormal basis of $\mathbb{P}_{p,d}(\widehat{\kappa})$ considered in [26]. Define the matrix \widehat{F} of order $n_{p,d}$ such that $(\widehat{V}, \widehat{F}\widehat{V})_{\mathbb{R}^{n_{p,d}}} = \|\widehat{v}\|_{\partial\widehat{\kappa}}^2$ for all $\widehat{v} \in \mathbb{P}_{p,d}(\widehat{\kappa})$, where \widehat{V} is the coordinate vector of \widehat{v} in the selected basis and where $(\cdot, \cdot)_{\mathbb{R}^{n_{p,d}}}$ denotes the Euclidean scalar product in $\mathbb{R}^{n_{p,d}}$. Clearly, \widehat{F} is symmetric positive semidefinite. The advantage of the basis selected in [26] is that \widehat{F} admits a block-structure with blocks having rank 1.

(B) To specify this structure, define $J_{p,d} = \{(i) \in I_{p,d}, \sum_{l=1}^d i_l \leq p\}$. The matrix \widehat{F} is then conveniently indexed by multi-indices (i) and (j) in $J_{p,d}$, and its entries

are given by

$$\widehat{F}_{(i)(j)} = \prod_{l=1}^{d-1} \delta_{i_l j_l} (-1)^{i_d} \left(\frac{2N_d(i) + d}{2} \right)^{\frac{1}{2}} (-1)^{j_d} \left(\frac{2N_d(j) + d}{2} \right)^{\frac{1}{2}},$$

where $N_d(i) = \sum_{l=1}^d i_l$. Let $r \in \{0, \dots, p\}$. With any $(i') \in I_{p,d-1}$ such that $\sum_{l=1}^{d-1} i'_l = r$, we can associate a block $\widehat{F}_{r,(i')}$ in \widehat{F} corresponding to the multi-indices $(i) = (i'_1, \dots, i'_{d-1}, i_d)$. In [26], it is proven that the block $\widehat{F}_{r,(i')}$ is of size $(p - r + 1)$, of rank 1, and its trace is equal to

$$\text{tr}(\widehat{F}_{r,(i')}) = \sum_{i_d=0}^{p-r} \frac{1}{2}(2r + 2i_d + d) = \frac{1}{2}(p - r + 1)(p + r + d).$$

(C) In [26], the largest eigenvalue of \widehat{F} is estimated whereas here we are interested in the lowest non-zero eigenvalue. Since the blocks $\widehat{F}_{r,(i')}$ are of rank 1, the spectrum of \widehat{F} is equal to

$$\sigma(\widehat{F}) = \left\{ \frac{1}{2}(p - r + 1)(p + r + d) \right\}_{0 \leq r \leq p} \cup \{0\}.$$

Since the product $(p - r + 1)(p + r + d)$ is monotonically decreasing in r , it is inferred that the smallest non-zero eigenvalue of \widehat{F} is $\frac{2p+d}{2}$. The conclusion is straightforward. \square

The result of Lemma 5.2, which is to the authors' knowledge new, has important consequences for the hp -convergence analysis on simplices since it implies that in space dimension $d \geq 2$, the inverse trace inequality on simplices cannot be optimal with respect to the trace inequality and that, at best, a factor $p^{\frac{1}{2}}$ is lost. In other words, for the inverse trace inequality to be optimal, one should prove that

$$(49) \quad \inf_{v \in \mathbb{P}_{p,d}^{\mathcal{A}}(\kappa) \setminus \{0\}} \frac{\|v\|_{\partial\kappa}}{\|v\|_{\kappa}} \geq c \left(p^2 \frac{\text{meas}(\partial\kappa)}{\text{meas}(\kappa)} \right)^{\frac{1}{2}},$$

for a constant c independent of p . Hence, the upper bound in (48) is too small by a factor proportional to $p^{\frac{1}{2}}$, and in the best situation where (48) is sharp, the inverse trace inequality is suboptimal by a factor of $p^{\frac{1}{2}}$ only.

The rest of the analysis presented in this section is restricted to triangles ($d = 2$) and relies on the following hypothesis, which means that (48) is sharp.

Hypothesis 5.1. *There exists an admissible set of nodes \mathcal{A}^* in $\mathbb{P}_{p,2}(\widehat{\kappa})$ such that*

$$(50) \quad \forall v \in \mathbb{P}_{p,2}^{\mathcal{A}^*}(\widehat{\kappa}), \quad \|v\|_{\widehat{\kappa}} \leq cp^{-\frac{1}{2}} \|v\|_{\partial\widehat{\kappa}}.$$

Henceforth, for all $\kappa \in \mathcal{K}$, $\mathbb{P}_{p,d}^0(\kappa)$ denotes the polynomial set $\mathbb{P}_{p,2}^{\mathcal{A}^*_{\kappa}}(\kappa)$ where \mathcal{A}^*_{κ} is the image of the set \mathcal{A}^* by the transformation mapping $\widehat{\kappa}$ onto κ . If d is not large with respect to p , estimate (46) and Hypothesis 5.1 lead to the following trace and inverse trace inequalities on triangles:

$$(51) \quad \forall v \in \mathbb{P}_{p,d}(\kappa), \quad \|v\|_{\partial\kappa} \leq c \left(\frac{p^2}{h_{\kappa}} \right)^{\frac{1}{2}} \|v\|_{\kappa},$$

$$(52) \quad \forall v \in \mathbb{P}_{p,d}^0(\kappa), \quad \|v\|_{\kappa} \leq c \left(\frac{h_{\kappa}}{p} \right)^{\frac{1}{2}} \|v\|_{\partial\kappa}.$$

Inequality (51) is derived in [25]. Inequality (52) is suboptimal by a factor $p^{\frac{1}{2}}$ with respect to (51). Inequality (52) should be compared with inequality (16) derived on tensor-product elements, the latter inequality being optimal with respect to the corresponding trace inequality (15).

5.2. hp -interpolation and projection on triangles. Let $\kappa \in \mathcal{K}$. For a node ν in \mathcal{A}_κ^* , set $\mathcal{K}_\nu = \{\kappa' \in \mathcal{K}; \nu \in \kappa'\}$; then, for $w_h \in Y_h^p$, define $\mathcal{I}_{\text{Os}}w_h$ locally in κ by the value it takes at all the nodes in \mathcal{A}_κ^* by setting

$$(53) \quad \mathcal{I}_{\text{Os}}w_h(\nu) = \frac{1}{\text{card}(\mathcal{K}_\nu)} \sum_{\kappa \in \mathcal{K}_\nu} w_h|_\kappa(\nu).$$

Clearly, $\mathcal{I}_{\text{Os}}w_h \in X_h^p$.

Lemma 5.3. *There exists c , independent of p and h , such that, for all $\kappa \in \mathcal{K}$, the following estimate holds:*

$$(54) \quad \forall w_h \in Y_h^p, \quad \|w_h - \mathcal{I}_{\text{Os}}w_h\|_\kappa \leq c \left(\frac{h_\kappa}{p}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[w_h]\|_F.$$

Proof. The proof is the same as that of Lemma 3.2 except that the trace and inverse trace inequalities (51) and (52) are used instead of (15) and (16). In dimension 2, the inverse trace inequality (52) is used once so that a factor of $p^{-\frac{1}{2}}$ is lost in (54) with respect to (18). \square

Remark 5.1. In dimension 3, the inverse trace inequality (52) has to be used twice so that a factor of p is lost.

To derive hp -approximation results for the $L^2(\Omega)$ -orthogonal projector onto X_h^p , we make henceforth the following assumption.

Hypothesis 5.2. *The $L^2(\Omega)$ -orthogonal projector onto X_h^p satisfies the hp -approximation results (24), (25), and (40).*

Lemma 5.4. *Let $\Pi_h : L^2(\Omega) \rightarrow X_h^p$ be the $L^2(\Omega)$ -orthogonal projector onto X_h^p . There exists c , independent of p and h , such that for all $w \in H^q(\Omega)$, $q \geq 1$,*

$$(55) \quad \|w - \Pi_h w\|_\Omega \leq cp^{\frac{1}{4}} \left(\frac{h}{p}\right)^s \|w\|_{s,\Omega},$$

$$(56) \quad \|\nabla(w - \Pi_h w)\|_\Omega \leq cp^{\frac{5}{4}} \left(\frac{h}{p}\right)^{s-1} \|w\|_{s,\Omega},$$

with $s = \min(p + 1, q)$.

Proof. The proof proceeds as that of Lemma 3.3 except that the bound (54) is used instead of (18) and that instead of (28), we use the inequality

$$\|w - \Pi_h^* w\|_{\partial\kappa} \leq cp^{\frac{1}{4}} \left(\frac{h_\kappa}{p}\right)^{s-\frac{1}{2}} \|w\|_{s,\kappa},$$

resulting from Hypothesis 5.2 and the standard trace inequality (39). \square

5.3. Convergence analysis on triangles. Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve (1) and let $u_h \in X_h^p$ solve

$$(57) \quad a(u_h, v_h) + j(u_h, v_h) = (f, v_h)_\Omega, \quad \forall v_h \in X_h^p.$$

The convergence analysis proceeds as in §4. Let $(X_h^p)^\perp = \{z \in L^2(\Omega); (z, v_h)_\Omega = 0, \forall v_h \in X_h^p\}$.

Lemma 5.5 (Boundedness). *There is c , independent of p and h , such that for all $z \in H^q(\mathcal{K}) \cap (X_h^p)^\perp$, $q > \frac{3}{2}$,*

$$(58) \quad \sup_{v_h \in X_h^p} \frac{a(z, v_h) + j(z, v_h)}{\|v_h\|_{a,j}} \leq \|z\|_{a,j} + c(p^2 h^{\frac{1}{2}} + p^{\frac{\alpha}{2} - \frac{1}{2}}) \|z\|_{h, \frac{1}{2}}.$$

Proof. The proof is similar to that of Lemma 4.2. However, on simplices, β_h can be chosen to be in X_h^1 . As a result, there is no need to invoke the triangle inequality (35), and it is directly inferred using (54) that

$$\sum_{\kappa \in \mathcal{K}} \sum_{F \in \mathcal{F}(\kappa)} \beta_{\kappa, \infty}^{-1} h_\kappa^2 p^{-1} \|[\beta_h \cdot \nabla v_h]\|_F^2 \leq c p^{\alpha-1} j(v_h, v_h).$$

The conclusion is straightforward. \square

Lemma 5.6. *There is c , independent of p and h , such that for all $w \in H^q(\Omega)$, $q > \frac{3}{2}$,*

$$(59) \quad \|w - \Pi_h w\|_{h, \frac{1}{2}} \leq c p^{-\frac{1}{4}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s, \Omega},$$

$$(60) \quad \|w - \Pi_h w\|_{a,j} \leq c(p^{\frac{3}{4}} + p^{\frac{11}{4} - \frac{\alpha}{2}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s, \Omega},$$

with $s = \min(p+1, q)$.

Proof. Similar to that of Lemma 4.4. \square

We are now in a position to state the main result of this section.

Theorem 5.1. *Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve (1) and let u_h solve (57). Take $\alpha = \frac{7}{2}$. Then, under Hypotheses 5.1 and 5.2, there is c , independent of p and h , such that*

$$(61) \quad \|u - u_h\|_{a,j} \leq c(p + p^{\frac{7}{4}} h^{\frac{1}{2}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s, \Omega},$$

with $s = \min(p+1, q)$. Hence, if $h \leq p^{-\frac{3}{2}}$, the following holds:

$$(62) \quad \|u - u_h\|_{a,j} \leq c p \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s, \Omega}.$$

Proof. Proceeding as in the proof of Theorem 4.1, it is inferred that

$$\|u - u_h\|_{a,j} \leq c \left(p^{\frac{3}{4}} + p^{\frac{11}{4} - \frac{\alpha}{2}} + p^{\frac{\alpha}{2} - \frac{3}{4}} + p^{\frac{7}{4}} h^{\frac{1}{2}}\right) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s, \Omega}.$$

Since $\alpha = \frac{7}{2}$, it is clear that $\frac{11}{4} - \frac{\alpha}{2} = \frac{\alpha}{2} - \frac{3}{4} = 1 \geq \frac{3}{4}$. The conclusion is straightforward. \square

6. ADVECTION-DIFFUSION PROBLEMS

The purpose of this section is to show that the hp -CIP bilinear form j defined by equation (6) is also suitable to approximate advection-diffusion problems in both the advection-dominated and the diffusion-dominated regimes. The discussion is restricted to finite element approximations on tensor-product meshes.

Consider the following PDE with mixed Neumann–Robin boundary conditions

$$(63) \quad \begin{cases} -\varepsilon\Delta u + \beta \cdot \nabla u + \sigma u = f, \\ (\beta u - \varepsilon \nabla u) \cdot n|_{\partial\Omega^-} = 0, \\ \varepsilon \nabla u \cdot n|_{\partial\Omega \setminus \partial\Omega^-} = 0, \end{cases}$$

where $\varepsilon > 0$ is a real parameter and where f , β , and σ satisfy the assumptions stated in §2. Without loss of generality, it is assumed that problem (63) is non-dimensionalized so that β is of order unity. Note that Neumann and Robin boundary conditions are natural outflow and inflow conditions, respectively, for advection-diffusion problems.

On $H^1(\Omega) \times H^1(\Omega)$ define the standard Galerkin bilinear form

$$(64) \quad a_\varepsilon(u, v) = \varepsilon(\nabla u, \nabla v)_\Omega + ((\sigma - \nabla \cdot \beta)u, v)_\Omega - (u, \beta \cdot \nabla v)_\Omega + (\beta \cdot nu, v)_{\partial\Omega^+}.$$

The finite element approximation to (63) consists of finding $u_h \in V_h^p$ such that

$$(65) \quad a_\varepsilon(u_h, v_h) + j(u_h, v_h) = (f, v_h)_\Omega, \quad \forall v_h \in V_h^p.$$

For $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$, consider the norm

$$(66) \quad \|v\|_{\varepsilon,j}^2 = \|\varepsilon^{\frac{1}{2}} \nabla v\|_\Omega^2 + \|\sigma_0^{\frac{1}{2}} v\|_\Omega^2 + \frac{1}{2} \|\beta \cdot n|^{\frac{1}{2}} v\|_{\partial\Omega}^2 + j(v, v).$$

The discrete problem (65) is clearly endowed with stability and consistency properties.

Lemma 6.1 (Coerciveness). *For all $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$, $a_\varepsilon(v, v) + j(v, v) \geq \|v\|_{\varepsilon,j}^2$.*

Lemma 6.2 (Consistency). *Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve (63) and let u_h solve (65). Then, for all $v_h \in V_h^p$,*

$$(67) \quad a_\varepsilon(u - u_h, v_h) + j(u - u_h, v_h) = 0.$$

Proposition 6.1 (Dominant advection). *Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve (63) and let u_h solve (65). Take $\alpha = \frac{7}{2}$. Assume $h \leq p^{-\frac{5}{2}}$. Then, there is c , independent of p , h , and ε , such that*

$$(68) \quad \|u - u_h\|_{\varepsilon,j} \leq c(\varepsilon^{\frac{1}{2}} p h^{-\frac{1}{2}} + p^{\frac{1}{4}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s,\Omega},$$

with $s = \min(p + 1, q)$. Hence, if $\varepsilon \leq hp^{-\frac{3}{2}}$, the following holds:

$$(69) \quad \|u - u_h\|_{\varepsilon,j} \leq cp^{\frac{1}{4}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s,\Omega}.$$

Proof. Proceed as in the proof of Theorem 4.1. The only additional term to estimate is $\|\varepsilon^{\frac{1}{2}} \nabla(u - \Pi_h u)\|_\Omega$. Use (27) to infer

$$\|\varepsilon^{\frac{1}{2}} \nabla(u - \Pi_h u)\|_\Omega \leq cp^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}.$$

The conclusion is straightforward. □

Proposition 6.2 (Dominant diffusion). *Let $u \in H^q(\Omega)$, $q \geq 2$, solve (63) and let u_h solve (65). Take $\alpha = \frac{7}{2}$. Then, there is c , independent of p , h , and ϵ , such that*

$$(70) \quad \|u - u_h\|_{\epsilon,j} \leq c(1 + \epsilon^{-\frac{1}{2}})(\epsilon^{\frac{1}{2}} + 1 + h^{\frac{1}{2}}p^{-\frac{3}{4}}) \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega},$$

with $s = \min(p + 1, q)$. Hence, if $\epsilon \geq \max(1, hp^{-\frac{3}{2}})$, the following holds:

$$(71) \quad \|u - u_h\|_{\epsilon,j} \leq c\epsilon^{\frac{1}{2}} \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}.$$

Proof. Recall the following interpolation result [2]: there exists $u_h^* \in V_h^p$ such that

$$(72) \quad \|u - u_h^*\|_{1,\Omega} \leq c \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}.$$

(A) Let us first prove that

$$(73) \quad j(u - u_h^*, u - u_h^*)^{\frac{1}{2}} \leq ch^{\frac{1}{2}}p^{-\frac{3}{4}} \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}.$$

Let $\Pi_{h,2}^* : H^2(\Omega) \rightarrow W_h^p$ be the H^2 -orthogonal projection onto W_h^p . Recall that $\Pi_{h,2}^*$ is endowed with optimal hp -approximation properties in the L^2 -, H^1 - and H^2 -norms; see [10]. Let $\kappa \in \mathcal{K}$ and let $F \subset \partial\kappa$. Owing to the trace inequality (15), it is inferred that

$$\begin{aligned} \|\nabla(u - u_h^*)|_{\kappa}\|_F &\leq \|\nabla(u - \Pi_{h,2}^*u)|_{\kappa}\|_F + \|\nabla(\Pi_{h,2}^*u - u_h^*)|_{\kappa}\|_F \\ &\leq c \left(\frac{h_{\kappa}}{p}\right)^{s-\frac{3}{2}} \|u\|_{s,\kappa} + cph_{\kappa}^{-\frac{1}{2}} \|\nabla(\Pi_{h,2}^*u - u_h^*)\|_{\kappa} \\ &\leq c \left(\frac{h_{\kappa}}{p}\right)^{s-\frac{3}{2}} \|u\|_{s,\kappa} + cph_{\kappa}^{-\frac{1}{2}} (\|\nabla(u - \Pi_{h,2}^*u)\|_{\kappa} + \|\nabla(u - u_h^*)\|_{\kappa}) \\ &\leq cp^{\frac{1}{2}} \left(\frac{h_{\kappa}}{p}\right)^{s-\frac{3}{2}} \|u\|_{s,\kappa} + cph_{\kappa}^{-\frac{1}{2}} \|\nabla(u - u_h^*)\|_{\kappa}. \end{aligned}$$

As a result,

$$\begin{aligned} j(u - u_h^*, u - u_h^*) &\leq \sum_{F \in \mathcal{F}} \frac{h_F^2}{p^{\frac{7}{2}}} \beta_{\infty,F} \|\nabla(u - u_h^*) \cdot \mathbf{n}\|_F^2 \\ &\leq c \sum_{\kappa \in \mathcal{K}} \frac{h_{\kappa}^2}{p^{\frac{7}{2}}} \left(p \left(\frac{h_{\kappa}}{p}\right)^{2s-3} \|u\|_{s,\kappa}^2 + p^2 h_{\kappa}^{-1} \|\nabla(u - u_h^*)\|_{\kappa}^2 \right), \end{aligned}$$

yielding (73).

(B) Since $\|v\|_{\epsilon,j} \leq c(\epsilon + 1)\|v\|_{1,\Omega} + j(v, v)$, it is inferred using (73) that

$$\|u - u_h^*\|_{\epsilon,j} \leq c(\epsilon^{\frac{1}{2}} + 1 + h^{\frac{1}{2}}p^{-\frac{3}{4}}) \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}.$$

Owing to Lemmata 6.1 and 6.2,

$$\begin{aligned} \|u - u_h\|_{\epsilon,j}^2 &\leq a_{\epsilon}(u - u_h, u - u_h^*) + j(u - u_h, u - u_h^*) \\ &\leq (1 + \epsilon^{-\frac{1}{2}}) \|u - u_h\|_{\epsilon,j} \|u - u_h^*\|_{\epsilon,j}. \end{aligned}$$

Hence,

$$\begin{aligned} \|u - u_h\|_{\varepsilon,j} &\leq (1 + \varepsilon^{-\frac{1}{2}}) \|u - u_h^*\|_{\varepsilon,j} \\ &\leq (1 + \varepsilon^{-\frac{1}{2}})(\varepsilon^{\frac{1}{2}} + 1 + h^{\frac{1}{2}} p^{-\frac{3}{4}}) \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}, \end{aligned}$$

yielding (70). The conclusion is straightforward. \square

Proposition 6.2 shows that in the diffusion-dominated regime, the hp -CIP bilinear form j need not be modified to guarantee optimal convergence in h and p . Furthermore, the above results indicate that the Péclet number to detect the regime transition is $\frac{h}{\varepsilon} p^{-\frac{3}{2}}$.

Remark 6.1. Following the ideas developed in [12], it is interesting to show that under mild additional assumptions on the approximability properties of a smooth subspace of the finite element space V_h^p , it is possible to derive in the diffusion-dominated regime, both energy-norm and L^2 -norm error estimates that are independent of the hp -CIP bilinear form j . Let $Z_h^p = V_h^p \cap C^1(\overline{\Omega})$ and assume that Z_h^p is endowed with the following optimal hp -approximation property: There is c , independent of p and h , such that for all $v \in H^s(\Omega)$, there is $z_h(v) \in Z_h^p$ such that

$$(74) \quad \|v - z_h(v)\|_{1,\Omega} \leq c \left(\frac{h}{p}\right)^{s-1} \|v\|_{s,\Omega},$$

where $s = \min(p+1, q) \geq 2$ and q is defined in Proposition 6.2. Property (74) is expected to hold when the polynomial order p is large enough. In the diffusion-dominated regime ($\varepsilon \sim 1$), the bilinear form a_ε is H^1 -coercive and bounded on $H^1(\Omega) \times H^1(\Omega)$ with constants that are of order unity. Furthermore, since $Z_h^p \subset C^1(\overline{\Omega})$, the following improved consistency property holds: For all $z_h \in Z_h^p$,

$$(75) \quad a_\varepsilon(u - u_h, z_h) = j(u_h, z_h) = 0,$$

whence it is readily inferred using (74) that

$$(76) \quad \|u - u_h\|_{1,\Omega} \leq c \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}.$$

Moreover, assuming that problem (63) has elliptic regularity and using a standard Aubin–Nitsche duality argument yields

$$(77) \quad \|u - u_h\|_{\Omega} \leq c \left(\frac{h}{p}\right)^s \|u\|_{s,\Omega}.$$

The h -version of the above error estimates can be found in [12].

7. CONCLUSIONS

The hp -CIP finite element method analyzed in this paper can be used to approximate efficiently advection and (advection-dominated) advection-diffusion equations on tensor-product meshes. The error estimates are suboptimal by a factor of $p^{\frac{1}{4}}$ in the advection-dominated regime, while optimality is recovered in the diffusion-dominated regime. In contrast, several questions remain open for simplicial finite elements. In particular, the existence of an admissible set of nodes yielding the best possible inverse trace inequality (48) and the sharpness of the estimates derived in

Lemmata 5.3 and 5.4 deserve further investigation. Finally, it is worthwhile to mention that the design of an hp -CIP finite element method for the Oseen equation is straightforward by combining the above results with those of [8] and that numerical experiments in one space dimension, including convergence rates and estimates of condition numbers, are reported in [7].

ACKNOWLEDGMENTS

The authors are thankful to the anonymous referees for constructive comments and to Professor Ch. Schwab for stimulating discussions. This work was partly supported by the GdR MoMaS (CNRS-2439, ANDRA, BRGM, CEA, EdF).

REFERENCES

1. I. Babuška and M.R. Dorr, *Error estimates for the combined h and p versions of the finite element method*, Numer. Math. **37** (1981), no. 2, 257–277. MR623044 (82h:65080)
2. I. Babuška and M. Suri, *The hp version of the finite element method with quasi-uniform meshes*, RAIRO, Math. Mod. Numer. Anal. **21** (1987), 199–238. MR0896241 (88d:65154)
3. I. Babuška and M. Zlámal, *Nonconforming elements in the finite element method with penalty*, SIAM J. Numer. Anal. **10** (1973), 863–875. MR0345432 (49:10168)
4. G.A. Baker, *Finite element methods for elliptic equations using nonconforming elements*, Math. Comp. **31** (1977), no. 137, 45–59. MR0431742 (55:4737)
5. E. Burman, *A unified analysis for conforming and nonconforming stabilized finite element methods using interior penalty*, SIAM J. Numer. Anal. **43** (2005), no. 5, 2012–2033 (electronic). MR2192329
6. E. Burman and A. Ern, *Stabilized Galerkin approximation of convection-diffusion-reaction equations: discrete maximum principle and convergence*, Math. Comp. **74** (2005), no. 252, 1637–1652 (electronic). MR2164090 (2006e:65211)
7. ———, *Continuous interior penalty hp -finite element methods for transport operators*, Numerical Mathematics and Advanced Applications (Berlin), Springer, 2006, Enumath 2005 Conf. Proc.
8. E. Burman, M. Fernández, and P. Hansbo, *Continuous interior penalty finite element method for the Oseen's equations*, SIAM J. Numer. Anal. **44** (2006), 1248–1274. MR2231863
9. E. Burman and P. Hansbo, *Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems*, Comput. Methods Appl. Mech. Engrg. **193** (2004), 1437–1453. MR2068903 (2005d:65186)
10. C. Canuto and A. Quarteroni, *Approximation results for orthogonal polynomials in Sobolev spaces*, Math. Comp. **38** (1982), 67–86. MR0637287 (82m:41003)
11. P.J. Davis and P. Rabinowitz, *Methods of numerical integration*, Computer Science and Applied Mathematics, Academic Press Inc., Orlando, FL, 1984. MR760629 (86d:65004)
12. J. Douglas Jr. and T. Dupont, *Interior penalty procedures for elliptic and parabolic Galerkin methods*, Computing Methods in Applied Sciences (Berlin) (R. Glowinski and J.-L. Lions, eds.), Lecture Notes in Physics, vol. 58, Springer-Verlag, 1976, pp. 207–216. MR440955 (55:13823)
13. L. El Alaoui and A. Ern, *Residual and hierarchical a posteriori error estimates for nonconforming mixed finite element methods*, M2AN Math. Model. Numer. Anal. **38** (2004), no. 6, 903–929. MR2108938 (2006a:65147)
14. A. Ern and J.-L. Guermond, *Discontinuous Galerkin methods for Friedrichs' systems. I. General theory*, SIAM J. Numer. Anal. **44** (2006), 753–778. MR2218968
15. K. O. Friedrichs, *Symmetric positive linear differential equations*, Comm. Pure Appl. Math. **11** (1958), 333–418. MR0100718 (20:7147)
16. R.H.W. Hoppe and B. Wohlmuth, *Element-oriented and edge-oriented local error estimators for non-conforming finite element methods*, Math. Model. Numer. Anal. **30** (2) (1996), 237–263. MR1382112 (97e:65124)
17. P. Houston, D. Schötzau, and Th.P. Wihler, *Energy norm a posteriori error estimation of hp -adaptive Discontinuous Galerkin methods for elliptic problems*, Tech. Report IMA Preprint Series 1985, Institute for Mathematics and its Applications, 2004.

18. P. Houston, Ch. Schwab, and E. Süli, *Stabilized hp-finite element methods for first-order hyperbolic problems*, SIAM J. Numer. Anal. **37** (2000), no. 5, 1618–1643 (electronic). MR1759909 (2001f:65135)
19. ———, *Discontinuous hp-finite element methods for advection-diffusion-reaction problems*, SIAM J. Numer. Anal. **39** (2002), no. 6, 2133–2163. MR1897953 (2003d:65108)
20. P. Houston and E. Süli, *Stabilised hp-finite element approximation of partial differential equations with nonnegative characteristic form*, Computing **66** (2001), no. 2, 99–119. MR1825801 (2002c:65211)
21. M. Jensen, *Discontinuous Galerkin methods for Friedrichs systems with irregular solutions*, Ph.D. thesis, University of Oxford, Oxford, UK, 2004.
22. O. Karakashian and F. Pascal, *A posteriori error estimates for a discontinuous Galerkin approximation of second order elliptic problems*, SIAM J. Numer. Anal. **41** (2003), no. 6, 2374–2399. MR2034620 (2005d:65192)
23. J. M. Melenk, *hp-interpolation of nonsmooth functions and an application to hp-a posteriori error estimation*, SIAM J. Numer. Anal. **43** (2005), no. 1, 127–155 (electronic). MR2177138
24. A. Quarteroni and A. Valli, *Numerical approximation of partial differential equations*, Springer Series in Computational Mathematics, vol. 23, Springer-Verlag, 1994. MR1299729 (95i:65005)
25. Ch. Schwab, *p- and hp-finite element methods*, Numerical Mathematics and Scientific Computation, The Clarendon Press Oxford University Press, New York, 1998. MR1695813 (2000d:65003)
26. T. Warburton and J.S. Hesthaven, *On the constants in hp-finite element trace inverse inequalities*, Comput. Methods Appl. Mech. Engrg. **192** (2003), 2765–2773. MR1986022 (2004d:65146)

INSTITUT D'ANALYSE ET CALCUL SCIENTIFIQUE (CMCS/IACS), ECOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE, STATION 8, CH-1015 LAUSANNE, SWITZERLAND

E-mail address: Erik.Burman@epfl.ch

CERMICS, ECOLE DES PONTS, PARISTECH, CHAMPS-SUR-MARNE, 77455 MARNE LA VALLÉE,
CEDEX 2, FRANCE

E-mail address: ern@cermics.enpc.fr