MATHEMATICS OF COMPUTATION
Volume 76, Number 259, July 2007, Pages 1695–1696
S 0025-5718(07)02024-8
Article electronically published on March 9, 2007

4[35-XX, 49-XX]—*Variational analysis in Sobolev and BV spaces*, by Hedy Attouch, Giuseppe Buttazzo, and Gérard Michaille, SIAM, Philadelphia, PA, 2006, xii+634 pp., softcover, List US$140.00/Member US$98.00, ISBN 0-89871-600-4

This book has two main objectives.

The first one, which is oriented more toward students, is to provide to readers the basic tools and methods of variational analysis and optimization in infinite dimensional spaces together with classical applications to PDE problems. The authors follow this objective in the first part of the book, Chapters 1 through 9, where the classical Sobolev spaces are used. The exposition of the material is self-contained with an attempt to introduce each new development from various perspectives (historical, theoretical-analytical, numerical, etc.).

The second objective, which is oriented more toward researchers, and corresponds to the second part of the book (Chapters 10 through 16), is to present new trends in variational analysis and some of the most recent developments and applications. The exposition takes place in the new spaces $BV(\Omega)$ and $SBV(\Omega)$.

Part I reflects the development of the famous programm of D. Hilbert throughout the twentieth century. The basic elements of variational analysis are introduced which allow one to solve the classical problems and related ones, such as the Dirichlet, the Neumann or the Stokes problems.

Chapter 2 introduces the weak solutions method, weak topologies and weak convergences. An exposition of such basic abstract variational principles as the Lax-Milgram theorem, the Galerkin method, minimization problems, convex minimization theorems and Eckeland's ε-variational principle is given in Chapter 3. The authors show how weak topologies, reflexivity, and convexity properties come naturally into play.

Chapter 4 describes in a self-contained way elements of geometric measure theory, in particular, the important notation of Hausdorff measure and both the Lebesgue measure on an open set in $\mathbb{R}^N$ and surface measures which are used, for example, in the definition of the space trace of Sobolev spaces. The generalized differential calculus of distribution theory and the generalized integration theory by Lebesgue are the basis for introduction in Chapter 5 of the classical Sobolev spaces. Now, having all necessary abstract ingredients, the authors describe in Chapter 6 various model examples including Dirichlet, Neumann, mixed, and some other problems.

The next two chapters round up the classical exposition of variational methods by introducing two of the most powerful methods (which are used also as efficient numerical tools) for variational problems, namely, the finite element method (Chapter 7) and the spectral method (Chapter 8).

Chapter 9 provides a description of such important concepts of convex variational analysis such as the Fenchel duality, subdifferential calculus as well as multipliers in the context of duality.

In recent years, variational methods have been developed to study a number of problems of modern technology, like image processing, shape optimization, composite materials and others. To solve these problems the classical framework of variational analysis must be enlarged and the second part of the book describes some of it's extensions.

The practical problems mentioned above require the introduction of new functional spaces permitting discontinuities of the solution. Chapter 10 contains a self-contained presentation of spaces $BV(\Omega)$ (the space of functions with bounded variations) and $SBV(\Omega)$ (the subspace of $BV(\Omega)$ including functions whose first distributional derivatives are bounded measures with no Cantor part) which extend the theory of Sobolev spaces.

Chapter 11 deals with the question of relaxation of functionals in calculus of variations. The next chapter introduces the concept of Γ-convergence which has been successfully applied to a variety of approximation and perturbation problems in calculus of variations and mechanics (homogenization of composite materials, materials with many small holes and porous media, thin structures and others).

Lower semicontinuity for integral functionals is the key tool used to apply the direct methods of the calculus of variations. This tool is considered in Chapter 13 and plays an important role in the modeling of large deformations in mechanics and plasticity, as described in Chapter 14. The coercivity property is an other important aspect of the direct method. Variational problems with a lack of coercivity are studied in Chapter 15.

An introduction to shape optimization problem given in Chapter 16 is the last topic of the book which shows the powerfulness and the limitations of direct methods of variational analysis.

The list of 229 references and Index round up the book.

The book is a comprehensive guide for anyone who works in the field of variational analysis and PDEs beginning with undergraduate students and including experienced researchers.

The book can be useful also for applied scientists working in the fields of geometry, mechanics, elasticity, pseudoplasticity, fracture mechanics, computer vision, and others.

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