ON THE EMBEDDING PROBLEM
FOR $2^+ S_4$ REPRESENTATIONS

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Abstract. Let $2^+ S_4$ denote the double cover of $S_4$ corresponding to the element in $H^2(S_4, \mathbb{Z}/2\mathbb{Z})$ where transpositions lift to elements of order 2 and the product of two disjoint transpositions to elements of order 4. Given an elliptic curve $E$, let $E[2]$ denote its 2-torsion points. Under some conditions on $E$ elements in $H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), E[2]) \setminus \{0\}$ correspond to Galois extensions $N$ of $\mathbb{Q}$ with Galois group (isomorphic to) $S_4$. In this work we give an interpretation of the addition law on such fields, and prove that the obstruction for $N$ having a Galois extension $\tilde{N}$ with $\text{Gal}(\tilde{N}/\mathbb{Q}) \cong 2^+ S_4$ gives a homomorphism $s^+_4 : H^1(\text{Gal}(\mathbb{Q}), E[2]) \to H^2(\text{Gal}(\mathbb{Q}), \mathbb{Z}/2\mathbb{Z})$. As a corollary we can prove (if $E$ has conductor divisible by few primes and high rank) the existence of 2-dimensional representations of the absolute Galois group of $\mathbb{Q}$ attached to $E$ and use them in some examples to construct 3/2 modular forms mapping via the Shimura map to (the modular form of weight 2 attached to) $E$.

Introduction

The study of modular forms of weight 1 is equivalent to that of two-dimensional continuous faithful irreducible complex representations of $\text{Gal}_\mathbb{Q} := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (see [6]). Looking at their projectivization we have five different kinds: cyclic, dihedral, $A_4$, $S_4$, $A_5$ (see [7]). If the image is cyclic, then the original representation is abelian, hence reducible. The dihedral case corresponds to weight 1 modular forms which are a linear combination of theta series attached to binary quadratic forms. The “special ones” are the last three cases. They are constructed using different approaches (see [9] for algorithms to construct the $A_4$ and $A_5$ cases, and [10] for the $S_4$ case). To study the $S_1$ case, in [1] the next method is proposed: let $E$ be an elliptic curve over $\mathbb{Q}$ with negative discriminant (if the discriminant is positive the same method gives Maas forms), no 2-torsion points over $\mathbb{Q}$, and nontrivial Selmer $2$-group. The set $H^1(\text{Gal}(\mathbb{Q}), E[2]) \setminus \{0\}$ is in one to one correspondence with fields $N$ with Galois group $S_4$ over $\mathbb{Q}$ containing $\mathbb{Q}(E[2])$. The obstruction for $N$ having a field extension $\tilde{N}$ with Galois group over $\mathbb{Q}$ isomorphic to $2^+ S_4$ is an element in $H^2(\text{Gal}(\mathbb{Q}), \mathbb{Z}/2\mathbb{Z})$ (see [10] for a formula of the obstruction and [4], [5] for a method to compute a solution to the embedding problem when the obstruction is trivial). This induces a map $s^+_4 : H^1(\text{Gal}(\mathbb{Q}), E[2]) \setminus \{0\} \to H^2(\text{Gal}(\mathbb{Q}), \mathbb{Z}/2\mathbb{Z})$. The main result of this work is that if we define $s^+_4(0) = 0$, then $s^+_4$ is a group homomorphism.

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As a corollary, all elliptic curves (in the above conditions) of conductor \(2^r p^s\) with \(r, s \in \mathbb{N}_0\) and \(p\) a prime number such that the 2-Selmer group has rank at least two, have a 2-dimensional representation with Galois group \(2^3 S_4\) attached to them. We end this work with some examples of how by using these weight 1 modular forms one can try to construct weight 3/2 modular forms mapping via Shimura (see [17], Main Theorem) to the modular form (of weight two attached to) \(E\).

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1. Correspondence between \(H^1(\text{Gal}_Q, E[2])\) and fields

Let \(E\) be an elliptic curve over \(\mathbb{Q}\) with negative discriminant and no 2-rational points. The field \(L = \mathbb{Q}(E[2])\) is a Galois extension of \(\mathbb{Q}\) with Galois group \(S_3\).

**Definition.** Let \(M\) be a \(\text{Gal}_Q\)-module and \(p\) a prime of \(\mathbb{Z}\). Denote by \(I_p \subset \text{Gal}_Q\) the inertia group for \(p\). A cohomology class \(\phi \in H^1(\text{Gal}_Q, M)\) is said to be unramified at \(p\) if it is trivial in \(H^1(I_p, M)\).

Let \(S\) be a finite set of primes containing 2 and the primes dividing the conductor of \(E\) and denote \(H^1(\text{Gal}_Q, E[2], S)\) the cocycles unramified outside \(S\). Abusing notation we will denote \(H^1(\text{Gal}_Q, E[2], S)^\times\) the set \(H^1(\text{Gal}_Q, E[2], S) \setminus \{0\}\). By \(\mathcal{D}(K)\) we denote the discriminant of the field \(K\).

**Proposition 1.1.** The elements in \(H^1(\text{Gal}_Q, E[2], S)^\times\) are in one to one correspondence with fields \(N\) such that \(L \subset N\), \(\text{Gal}(N/\mathbb{Q}) = S_4\), and \(N/\mathbb{Q}\) is unramified outside \(S\). Furthermore, if \(K \subset N\) is a degree 4 extension of \(\mathbb{Q}\) whose normal closure is \(N\), then \(\mathcal{D}(K) = \mathcal{D}(L)\) in \(\mathbb{Q}^\times/(\mathbb{Q}^\times)^2\).

**Proof.** The correspondence is Proposition 1.1 of [11] (although they do not state the ramification condition). The main idea (that we will need later) is that if \(\phi\) is a nontrivial cocycle, then \(\phi|_{\text{Gal}_L}\) is a group homomorphism. If we denote \(N_{\phi}\) the fixed field of \(\ker(\phi)\), then \(L \subset N_{\phi}\) and \(\text{Gal}(N_{\phi}/L) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\) (the Klein group). The ramification condition follows from this isomorphism. For the last statement, note that since \(S_4\) has a unique normal subgroup of index 2 (namely \(A_4\)), there is a unique quadratic Galois subextension, namely \(\mathbb{Q}(\sqrt{\mathcal{D}(K)})\). Since \(\mathbb{Q}(\sqrt{\mathcal{D}(L)})\) is another quadratic Galois subextension, they must be equal. \(\square\)

1.1. Field addition interpretation. The group structure of \(H^1(\text{Gal}_Q, E[2])^\times\) induces a group structure on fields \(N\) satisfying the above condition. Using elementary field theory we will show a natural construction of such addition. It is an easy group theory exercise to check that \(S_4 \simeq S_3 \times (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})\) where, if we denote \(\{P_1, P_2, P_3\}\) the three nonzero elements of \(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\), the action is given by \(\sigma.P_i = P_{\sigma(i)}\).

**Lemma 1.2.** Let \(N_1, N_2\) be fields corresponding to elements in \(H^1(\text{Gal}_Q, E[2], S)^\times\), then \(N_1 \cap N_2 = L\).

**Proof.** Clearly \(L \subset N_1 \cap N_2\). Since \(N_1 \cap N_2\) is Galois over \(\mathbb{Q}\), \(\text{Gal}(N_1/N_1 \cap N_2)\) corresponds to a nonzero normal subgroup of \(S_4\) contained in \(\text{Gal}(N_1/L)\) (isomorphic to the Klein group), hence \(\text{Gal}(N_1/N_1 \cap N_2) \simeq \text{Gal}(N_1/L)\). \(\square\)
In particular, $N_1N_2$ is a Galois extension of $\mathbb{Q}$ with Galois group of order 96 and $\text{Gal}(N_1N_2/L) \simeq \text{Gal}(N_1/L) \oplus \text{Gal}(N_2/L) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

**Proposition 1.3.** $\text{Gal}(N_1N_2/Q) \simeq S_3 \ltimes \bigoplus_{i=1}^{2}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$, where if $u, v \in \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\sigma \in S_3$, $\sigma(u, v) = (\sigma.u, \sigma.v)$.

**Proof.** Let $K_i \subset N_i$ be the fixed field of the subgroup $S_3$ viewed a subgroup of $S_4 = \text{Gal}(N_i/Q)$ (fixing the fourth element). We have the diagram:

![Diagram](https://via.placeholder.com/150)

**Claim.** $\text{Gal}(N_1N_2/K_1K_2) \simeq \text{Gal}(L/Q)$. To see this, consider the sequence:

$$\text{Gal}(N_i/K_i) \hookrightarrow \text{Gal}(N_i/Q) \twoheadrightarrow \text{Gal}(L/Q)$$

where $\pi_i : \text{Gal}(N_i/Q) \to \text{Gal}(L/Q)$ is the restriction map (which is the same as the quotient by $\text{Gal}(N_i/L)$). Since $\text{Gal}(N_i/K_i) \cap \text{Gal}(N_i/L) = \{id\}$, the composition is an isomorphism.

Since $N_1K_2 = N_1N_2$, $\text{Gal}(N_1N_2/K_1K_2) \simeq \text{Gal}(N_1/K_1) \simeq \text{Gal}(L/Q)$ by the restriction map (respectively $\text{Gal}(N_1N_2/K_1K_2) \simeq \text{Gal}(N_2/K_2)$). From the sequence

$$\text{Gal}(N_1N_2/K_1K_2) \hookrightarrow \text{Gal}(N_1N_2/Q) \twoheadrightarrow \text{Gal}(L/Q)$$

given by restriction (where the composition is an isomorphism), we conclude that $\text{Gal}(N_1N_2/Q) \simeq \text{Gal}(L/Q) \ltimes \text{Gal}(N_1N_2/L)$ and comparing the action with that of $\text{Gal}(L/Q) \ltimes \text{Gal}(N_i/L)$, the result follows. □

**Proposition 1.4.** The group $G := S_3 \ltimes \sum_{i=1}^{4}(\mathbb{Z}/2\mathbb{Z})$ with the previous action has three normal subgroups of order 4.

**Proof.** Clearly the subgroups $\{0\} \ltimes (\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, 0), \{0\} \ltimes (0, 0, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ and $\{0\} \ltimes \{a, b, a, b\} : a, b \in \mathbb{Z}/2\mathbb{Z}$ are normal. Let $H \triangleleft G$ be any normal subgroup of order 4. Then $\Pi(H) \leq S_3$ with order 1 or 2. Since $S_3$ has no normal subgroups of order 2, $\Pi(H) = \{0\}$. The orbits of $S_3$ acting on $\{0\} \ltimes \sum_{i=1}^{4}(\mathbb{Z}/2\mathbb{Z})$ are:

- $\{0, 0, 0, 0\}$,
- $\{1, 0, 0, 0\}, (0, 1, 0, 0), (1, 1, 0, 0)$,
- $\{0, 0, 1, 0\}, (0, 0, 0, 1), (0, 0, 1, 1)$,
- $\{1, 0, 1, 0\}, (0, 1, 0, 1), (1, 1, 1, 1)$,
- $\{(1, 0, 0, 1), (0, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 0)\}$. □

**Corollary 1.5.** Let $\psi_i \in H^1(\text{Gal}_Q, E[2], S)^\times$ and $N_i$ the corresponding field. The cocycle $\psi_1 + \psi_2$, if nontrivial, corresponds to the field fixed by the third normal subgroup of order 4 in $\text{Gal}(N_1N_2/Q)$.

**Proof.** The morphisms $\psi_i|_{\text{Gal}_L} : \text{Gal}_L \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, satisfy $N_i = \ker(\psi_i|_{\text{Gal}_L})$. Clearly $\psi_1 + \psi_2$ is zero on $\text{Gal}_{N_1N_2}$, hence its kernel is a normal subgroup of order 4 in $\text{Gal}(N_1N_2/L)$ and the result follows from Proposition 1.4. □
The group

Lemma 1.6. To normal subfields

Remark. All normal subgroups of \( G \) have pairwise trivial intersection (corresponding to normal subfields \( K_1, K_2, \) and \( K_3 \)). If we define the subgroups:

- \( H_4 = \{0\} \times \{(0, 0, 0, 0), (1, 0, 0, 1), (0, 1, 1, 1), (1, 1, 1, 0)\} \),
- \( H_5 = \{0\} \times \{(0, 0, 0, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 0)\} \),

then all of these five subgroups have trivial pairwise intersection. They correspond to the other two (unique) subfields \( N_4 \) and \( N_5 \) of index 4 of \( N_1 N_2 \) with the property that \( N_i \cap N_j = L \) for all \( i \neq j \). Furthermore, \( N_4 \) and \( N_5 \) are Galois conjugates.

It is a nice exercise to prove that given any three order 4 subgroups of \( \bigoplus_{i=1}^4 \mathbb{Z}/2\mathbb{Z} \), having trivial pairwise intersection, there exists another two order 4 subgroups such that all of them have the same property.

1.2. Two coverings of \( S_4 \). We will consider cohomology groups with the trivial action. The central 2-extensions of \( S_4 \) correspond to elements in the group \( H^2(S_4, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), where the four groups are:

- \( S_4 \oplus \mathbb{Z}/2\mathbb{Z} \),
- \( 2^{\text{nd}} S_4 \), corresponding to the cup product of the signature with itself,
- \( 2^+ S_4 \) and \( 2^- S_4 \).

The group \( 2^+ S_4 \) is isomorphic to \( GL_2(F_4) \). A complete character table of the group \( 2^+ S_4 \) can be found in [7], Lemma 28.2.

Lemma 1.6. The group \( 2^+ S_4 \) has a subgroup isomorphic to \( S_3 \).

Proof. The subgroup of \( GL_2(F_3) \) spanned by \( \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \right\} \) is isomorphic to \( S_3 \), the isomorphism is given by sending the generators to the elements in \( S_3 \): id, \((1, 2)\) and \((1, 3)\), respectively. \( \square \)

Let \( s_i^+ \) denote the element in \( H^2(S_4, \mathbb{Z}/2\mathbb{Z}) \) corresponding to the group \( 2^+ S_4 \). Consider the projections from \( G \) to \( S_4 \):

- \( \Pi_1(\sigma, (x, y, z, w)) = (\sigma, (x, y)) \),
- \( \Pi_2(\sigma, (x, y, z, w)) = (\sigma, (z, w)) \),
- \( \Pi_3(\sigma, (x, y, z, w)) = (\sigma, (x + z, y + w)) \),

where \( \Pi_i \) maps \( G \) to Gal\((N_i/Q)\) with kernel \( H_i \), for \( i = 1, 2, 3 \) (the three normal subgroups of Proposition 1.4). The obstruction for the existence of \( K_i \), a field containing \( K_i \) and Galois group \( 2^+ S_4 \) is the element in the 2-Brauer group \( \Gamma^*(\Pi_i'(s_i^+)) \) where \( \Gamma: \text{Gal}_Q \to \text{Gal}(K_1 K_2/Q) \) is the restriction map. Our main theorem can be stated as follows.

Theorem 1.7. \( \Pi_1'(s_1^+) + \Pi_2'(s_2^+) + \Pi_3'(s_3^+) = 0 \).

From class field theory we know that the 2-Brauer group injects into the sum of its local components, i.e., \( H^2(\text{Gal}_Q, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow \bigoplus_i H^2(\text{Gal}_{Q_i}, \mathbb{Z}/2\mathbb{Z}) \) and the local components are isomorphic to \( \{\pm 1\} \). Let \( H^2(\text{Gal}_Q, \mathbb{Z}/2\mathbb{Z}, S) \) be the subgroup of the 2-Brauer group of elements with trivial image at the primes outside \( S \). If we extend \( s_i^+ \) to \( H^1(\text{Gal}_Q, E[2], S) \) by setting \( s_i^+(0) = 0 \), we get

Corollary 1.8. The map \( s_i^+ : H^1(\text{Gal}_Q, E[2], S) \to H^2(\text{Gal}_Q, \mathbb{Z}/2\mathbb{Z}, S) \) is a group homomorphism.

From Serre’s formula for the obstruction ([16], Theorem 1) it is clear that the image of \( s_i^+ \) is on this subgroup of the 2-Brauer group. The corollary is an immediate
consequence of Theorem 1.7 noting that the case when two fields are equal is trivial from the fact that the cohomology groups are 2-groups.

Proof (Theorem). Let $Z_2$ denote the group $\bigoplus_{i=1}^4 \mathbb{Z}/2\mathbb{Z}$ and consider the exact sequence
\[ 0 \to Z_2 \to G \to S_3 \to 0. \]

Using the inflation-restriction map we get an exact sequence:
\[ 0 \to H^3(S_3, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\text{Inf}} H^3(G, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\text{Res}} H^2(Z_2, \mathbb{Z}/2\mathbb{Z}). \]

Let $\psi := \Pi_1^+(s_4^1) + \Pi_2^+(s_4^1) + \Pi_3^+(s_4^1)$. Restricted to $Z_2$, $\Pi_1 + \Pi_2 + \Pi_3 = 0$, hence $\text{Res}(\psi) = 0$, i.e., $\psi$ is in the image of the inflation map. This implies that $\psi$ does not depend on representatives of the quotient map, in particular, it is determined by its values on $S_3 \times \{(0, 0, 0, 0)\} \times S_3 \times \{(0, 0, 0, 0)\}$. By Lemma 1.6 $S_3$ is a subgroup of $2^+ S_4$. Then $s_4^+((S_3, (0, 0)), (S_3, (0, 0))) = 0$ and the result follows. \hfill $\square$

Remark. The fact that the group $2^+ S_4$ has a subgroup isomorphic to $S_3$ is crucial for the map being a homomorphism. The same statement is false (in general) considering the maps between $H^1(\text{Gal}_{\mathbb{Q}}, E[2], S) \to H^2(\text{Gal}_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}, S)$ coming from the other groups $2^\text{det} S_4$ and $2^- S_4$.

Let $K_i$ be a degree 4 extensions of $\mathbb{Q}$ with normal closure $N_i$, let $Q_{N_i}$ be the quadratic form $\text{Tr}_{K_i/\mathbb{Q}}(x^2)$ and $W(Q_{N_i})$ its Witt invariant.

Corollary 1.9. $W(Q_{N_1}) = W(Q_{N_2}) + W(Q_{N_3}) + (2, \mathcal{D}(L))$ on $\text{Br}_2(\mathbb{Q})$.

Proof. This follows from Proposition 1.1 and Serre’s formula for the obstruction; see [16], Theorem 1. \hfill $\square$

Corollary 1.10. Let $E$ be an elliptic curve with conductor $2^r p^s$ with $r, s \in \mathbb{N}_0$ and 2-Selmer group of rank at least two. Then there exists a $2^+ S_4$ representation attached to $E$.

Proof. Since the Selmer group has rank at least 2, let $N_1$ and $N_2$ be two different fields corresponding to elements in $H^1(\text{Gal}_{\mathbb{Q}}, E[2], S)$, and $N_3 := N_1 + N_2$. Let $s_4^+(N_i)$ denote their obstruction. From the injection of the 2-Brauer group into its local components it is clear that the obstruction is characterized by the (finite set of) primes with $-1$ sign. Such a set has an even number of primes and is contained in the set $\{2, p\}$, hence if two elements have nonzero obstruction in the Brauer group, then the third one does. \hfill $\square$

2. Applications and examples

We give a brief summary of how to construct the weight 1 modular forms attached to the Galois group $2^+ S_4$. Let $K = \mathbb{Q}(x_1)$ be a degree four extension of $\mathbb{Q}$ with normal closure $N$, an extension with $S_4$ Galois group. By Theorem 1 of [16] the obstruction (for a lift with Galois group $2^+ S_4$) is trivial if and only if the quadratic form $\text{Tr}_{K/\mathbb{Q}}(x^2)$ is isomorphic (over $\mathbb{Q}$) to the form $x_1^2 + x_2^2 + 2x_3^2 + 2\mathcal{D}(K)x_4^2$. Furthermore, if $P$ is a transformation matrix sending one form to the other one, let

\[ \gamma := \det \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \]

\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 2\sqrt{\mathcal{D}(K)} & 2\sqrt{\mathcal{D}(K)} \end{pmatrix} \]

\[ + \text{id} \].
If \( \gamma \neq 0 \), all the solutions to the embedding problem are \( \tilde{N} = N(\sqrt{cR}) \) with \( c \in \mathbb{Q}^\times \) (see [5], Theorem 5). Furthermore, \( c \) can be chosen such that \( \tilde{N} \) is unramified outside \( S \) and has minimum ramification at the primes in \( S \). This choice of \( c \) gives a weight 1 modular, with character \( \left( \frac{C(K)}{N} \right) \) and minimum level. See [1], Proposition 2.3, Proposition 2.4 and Proposition 2.5 to compute the Fourier coefficients of the weight 1 modular form once \( \gamma \) is known.

We will give some examples of how this weight 1 modular form can be used to construct some “special” 3/2 modular forms. Given a weight 2 and level \( p \) modular form \( f \) (attached to an elliptic curve \( E \)), in [5] Gross gave a method to construct a weight 3/2 modular form (as linear combination of theta series) in the Kohnen space mapping to \( f \) via the Shimura map. If \( E \) has positive rank, the constructed weight 3/2 modular form is the zero form. For this elliptic curve we will show (in some examples) how using the weight 1 modular form coming from the solution of the obstruction problem (by Corollary 1.10 we know that such form exists if \( E \) has rank greater than 1) one can construct a nonzero weight 3/2 modular form mapping to \( f \) via the Shimura map. This construction has some limitations (from a computation point of view) as we will see later, but works on many cases (our approach is similar to that in [2]).

For \( n \in \mathbb{N} \), let

\[
\Theta_n(z) := \sum_{j=-\infty}^{+\infty} q^{nj^2}.
\]

Then the theta function \( \Theta_n(z) \) is a weight 1/2 modular form of level \( 4n \) and character \( \chi(z) \) (see [17] for the definition of modular forms of half integral weight). We denote \( \Theta(z) := \Theta_1(z) \) the classical theta series.

**Lemma 2.1.** Let \( g(z) \) be a weight 1, level \( n \) and character \( \left( \frac{-d}{\cdot} \right) \) modular form (with \( d > 0 \) and \( d \mid n \)). Then \( g(z)\Theta_d(z) \) is a modular form of weight 3/2, level \( \text{lcm}(n,4d) \), and trivial character.

**Proof.** Let \( M \in \Gamma_0(\text{lcm}(n,4d)) \), say \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \). Then:

- \( g(Mz) = (\gamma z + \delta) \left( \frac{-d}{\cdot} \right) g(z) \).
- \( \Theta_d(Mz)\Theta(z) = (\frac{d}{\cdot}) \Theta_d(z)\Theta(Mz) \).
- \( \Theta(Mz) = (\gamma z + \delta) \left( \frac{1}{\gamma} \right) \Theta(z) \).

Then \( g(Mz)\Theta_d(Mz)\Theta(z)^3 = g(z)\Theta_d(z)\Theta(Mz)^3 \), which is the definition of a weight 3/2 modular form of trivial character.

Now we compute the space of definite positive ternary quadratic forms whose theta series are modular forms with trivial character (this is equivalent to the forms having square discriminant), and add the product of the weight 1 modular form with some theta series. Applying the Hecke operators on this set we look for the eigenform corresponding to \( E \). In practice, the difference between the dimension of the space of weight 3/2 modular forms and that of the subspace spanned by theta series increases with the level (see [12] for some tables). Use of this method in general implies knowing too many Fourier coefficients of the constructed modular forms, which is impracticable. For this reason we use this method for the case \( n = 43 \), where the dimension of the whole space is almost the same as the subspace of theta series. If the weight 1 modular form \( g \) has level \( p \) and character \( \left( \frac{-p}{\cdot} \right) \), a
prime congruent to 3 (mod 4), \( g(4z) \) has level 4p and the same character, hence the product \( g(4z)\Theta_d(z) \) is in the Kohnen space of level 4p and trivial character, which (by \[11\], Theorem 2) is isomorphic to \( S_2(\Gamma_0(p)) \). In these cases we can construct the weight 3/2 modular forms for big values of \( p \).

**Remark.** Although the product of the constructed weight 1 modular form with \( \Theta_d(z) \) could lie on the space spanned by definite positive ternary quadratic forms, it is not the case for all examples we computed. The “reason” why this is expected is that representations with dihedral projective image gives weight 1 modular forms that are a linear combination of binary quadratic forms and multiplying them by \( \Theta_d(z) \) gives ternary quadratic forms, while our construction has a different projective image.

### 2.1. Examples

We use the method described above in some particular examples. The computations were done with the PARI/GP system \[13\].

**Notation.** The ternary quadratic forms will be denoted \( a_1, a_2, a_3, a_{23}, a_{13}, a_{12}, \) to express the form:

\[
Q(X_1, X_2, X_3) = a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_{23}X_2X_3 + a_{13}X_1X_3 + a_{12}X_1X_2.
\]

**2.1.1. Case 43:** The elliptic curve 43\( \mathcal{E} \) in Cremona’s table with equation \( y^2 + y = x^3 + x^2 \) has rank 1. A generator is \( P = (0,0) \) and it corresponds to the field \( K \) with equation \( P = x^4 - 2x - 1 \). By \[1\] we know that the embedding problem for this case is solvable, and a solution with minimal level is given by the element

\[
\gamma = 3(x_1^2x_2^2 - x_1x_2^2 + x_1^2x_2 + x_1x_2^2 + x_2^2) + x_1^3 - 2x_2^2 + 4x_2,
\]

where \( x_1, x_2 \) are roots of \( P \). The corresponding modular form has level 2\(43 \) and character \( \chi_{43}. \) All definite positive ternary quadratic forms of level 2\(43 \) and 2\(43 \) with trivial character are given in Tables 2.1 and 2.2, where we denote \( Q_{43,j} \) (respectively \( Q_{86,i} \)) the \( i \)-th form on the table of ternary quadratic forms of level 2\(43 \) (respectively of level 2\(43 \)). See \[20\] for interactive tables of definite positive ternary quadratic forms of a given level (the level is divided by 4).

Note that the cited tables are not complete, the forms listed are only the ones with the extra condition “omega” divides “delta”. The forms not satisfying this condition were supplied by Gonzalo Tornaría. See \[12\], Theorem 4 and Theorem 5, for the bijection between different genera.

Let \( \mathfrak{h} \) denote the upper half-plane plus the line \( \mathbb{P}^1(\mathbb{Q}) \).

**Theorem 2.2.** Let \( f \) be a nonzero modular form of weight \( k \) for \( \Gamma_0(N) \). Then the number of zeros of \( f \) in a fundamental domain for \( \mathfrak{h}/\Gamma_0(N) \) is

\[
\frac{|\text{SL}_2(\mathbb{Z})\Gamma_0(N)|k}{12}.
\]

**Proof.** This statement is essentially Theorem 8 of \[15\], p. 114. The statement there is stated in terms of \( \Gamma_1(N) \), but the argument works as well for \( \Gamma_0(N) \) and the result is the same.

The theta functions of the definite positive ternary quadratic forms of level 2\(43 \) and 2\(43 \) are not linearly independent. A basis is given by the theta functions of the forms:

\[
\{Q_{43,1}, Q_{43,2}, Q_{43,3}, Q_{43,4}, Q_{43,5}, Q_{43,6}, Q_{43,7}, Q_{43,8}, Q_{43,9}, Q_{43,10}, Q_{43,11}, Q_{86,1}, Q_{86,2}, Q_{86,3}, Q_{86,4}, Q_{86,5}, Q_{86,6}, Q_{86,7}, Q_{86,8}, Q_{86,9}, Q_{86,11}\}.
\]
Table 2.1. Coefficients of ternary quadratic forms, level $2^{2}43$.

<table>
<thead>
<tr>
<th>$Q_1$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_{23}$</th>
<th>$a_{13}$</th>
<th>$a_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_2$</td>
<td>4</td>
<td>11</td>
<td>43</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>6</td>
<td>9</td>
<td>10</td>
<td>−10</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>1</td>
<td>43</td>
<td>43</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Q_5$</td>
<td>2</td>
<td>22</td>
<td>43</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>3</td>
<td>29</td>
<td>29</td>
<td>−28</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$Q_7$</td>
<td>4</td>
<td>11</td>
<td>43</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

| $Q_8$ | 5     | 18    | 26    | 18      | 2       | 4       |
| $Q_9$ | 6     | 15    | 23    | 2       | 6       | 4       |
| $Q_{10}$ | 9   | 10    | 24    | 10      | 2       | 4       |
| $Q_{11}$ | 11  | 14    | 16    | −6      | 4       | 10      |
| $Q_{12}$ | 4   | 43    | 44    | 0       | 4       | 0       |
| $Q_{13}$ | 11  | 16    | 47    | 16      | 2       | 4       |
| $Q_{14}$ | 15  | 23    | 24    | 12      | 8       | 2       |

Table 2.2. Coefficients of ternary quadratic forms, level $2^{3}43$.

<table>
<thead>
<tr>
<th>$Q_1$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_{23}$</th>
<th>$a_{13}$</th>
<th>$a_{12}$</th>
</tr>
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<tbody>
<tr>
<td>$Q_2$</td>
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<td>115</td>
<td>−114</td>
<td>−2</td>
<td>−2</td>
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<tr>
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<td>43</td>
<td>88</td>
<td>0</td>
<td>−8</td>
<td>0</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>19</td>
<td>20</td>
<td>91</td>
<td>20</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>$Q_5$</td>
<td>4</td>
<td>87</td>
<td>87</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>15</td>
<td>24</td>
<td>92</td>
<td>24</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>$Q_7$</td>
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<td>23</td>
<td>95</td>
<td>−22</td>
<td>−14</td>
<td>−2</td>
</tr>
<tr>
<td>$Q_8$</td>
<td>16</td>
<td>44</td>
<td>47</td>
<td>4</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>$Q_9$</td>
<td>23</td>
<td>31</td>
<td>47</td>
<td>18</td>
<td>14</td>
<td>10</td>
</tr>
</tbody>
</table>

| $Q_{10}$ | 3   | 29    | 86    | 0       | 0       | −2      |
| $Q_{11}$ | 5   | 18    | 86    | 0       | 0       | −4      |
| $Q_{12}$ | 8   | 22    | 43    | 0       | 0       | −4      |
| $Q_{13}$ | 19  | 20    | 26    | −4      | −16     | −12     |
| $Q_{14}$ | 1   | 86    | 86    | 0       | 0       | 0       |
| $Q_{15}$ | 6   | 15    | 86    | 0       | 0       | −4      |
| $Q_{16}$ | 9   | 10    | 86    | 0       | 0       | −4      |
| $Q_{17}$ | 13  | 16    | 47    | 16      | 6       | 12      |
| $Q_{18}$ | 14  | 21    | 31    | 14      | 4       | 12      |

This follows from Theorem 2.2 (although the theorem is stated for integer weight modular forms, raising to the fourth power a half integer weight modular form we see that the same is true in any case) and the fact that $[\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N}(1 + \frac{1}{p})$ (see [13], Proposition 1.13). Then looking at their first 67 Fourier coefficients is enough to compute a basis. The space $S_{3/2}(2^{3}43)$ has dimension 25 (see [3], Theorem 2), hence there are 4 modular forms missing.

The definite positive ternary quadratic forms of level $2^{2}43$ lie in four different genera. The number of forms on each genus is $[3, 5, 3, 3]$, i.e., the first three ones lie on the same genus and so on. Since the Hecke operators preserve genera, we can compute their action on each genus and look for eigenforms. In the cases when there are three forms per genus the eigenforms are weight $3/2$ modular forms mapping via Shimura to (the weight two modular forms) $E_{86}$ (the weight two Eisenstein series of level 43) and $43B1$ (a two-dimensional abelian variety; see [19] for the labeling). In the remaining case the eigenvalues map to the forms $E_{86}$, $86A1$, and $86B1$ (the last ones are two-dimensional abelian varieties).

The forms of level $2^{3}43$ lie in four different genera as well, and the number of forms on each genus is $[3, 5, 5, 5]$. The Hecke eigenvalues on each genus map to the weight two modular forms (attached to the abelian varieties): $E_{86}$, $86B1$; $E_{86}$, $43B1$ and $86A1$; $E_{86}$, $86B1$ and $172B1$; $E_{86}$, $43B1$ and $86A1$, respectively.

If we look at the tables for modular forms of weight 2 and level 43, 86 and 172, we see that the only missing modular forms correspond to the elliptic curves $43A$ and $172A$ on Cremona’s table, respectively. The curve $172A$ has rank 1 and is given by the equation $y^2 = x^3 + x^2 − 13x + 15$. In this particular case, constructing weight
3/2 modular forms not lying on the space spanned by the theta series gives some of these missing 2 modular forms.

Let \( f(z) \) denote the weight 1 modular form of weight 1, level 2³13 and character \((\chi_4)\) associated to \( K \). By Lemma 2.1, \( f \Theta_{43} \) is a weight 3/2, level 2³13 and trivial character cusp form. Since its coefficients are not rational, we consider the two modular forms \( F_1(z) = \frac{1}{2}(f(z) + f(z)) \Theta_{43} \) and \( F_2(z) = \frac{i}{2}(f(z) - f(z)) \Theta_{43} \). These two forms do have rational coefficients and are linearly independent from the ternary theta functions. Looking at the 23 modular forms together and computing the Hecke operators on them we get the two eigenforms (we denote by \( \Theta \) the Theta function of the ternary quadratic form \( Q \)):

\[
\begin{align*}
(1) \quad G_{43A} &= -\frac{4}{5} \Theta_{Q_{43A}} + \frac{1}{2} \Theta_{Q_{43A}} + \Theta_{Q_{43A}} - \frac{1}{5} \Theta_{Q_{43A}} + 6 \Theta_{Q_{43A}} + \frac{5}{2} \Theta_{Q_{43A}} - 5 \Theta_{Q_{43A}} - 5 F_2, \\
(2) \quad G_{172A} &= -\frac{1}{2} \Theta_{Q_{172A}} - \frac{1}{2} \Theta_{Q_{172A}} + \Theta_{Q_{172A}} - \Theta_{Q_{172A}} + 3 \Theta_{Q_{172A}} + 3 \Theta_{Q_{172A}} - 6 \Theta_{Q_{172A}} - 2 \Theta_{Q_{172A}} + 4 \Theta_{Q_{172A}} + 4 \Theta_{Q_{172A}} - 2 \Theta_{Q_{172A}} - 4 \Theta_{Q_{172A}} - \frac{1}{2} F_1.
\end{align*}
\]

They are Hecke eigenforms, and they map by the Shimura map to the weight two modular forms associated to the elliptic curves 43A and 172A on Cremona’s table, respectively. The first 50 coefficients of their Fourier expansion are:

- \( G_{43A} = q^2 + q^3 + q^5 + 2q^6 + 3q^7 + 3q^9 + 2q^{10} + 3q^{11} + 3q^{12} + 2q^{13} + 2q^{14} + q^{15} + q^{16} + 2q^{17} + q^{18} + 2q^{19} + q^{20} + q^{21} + q^{22} + q^{23} + q^{24} + q^{25} + q^{26} + q^{27} + q^{28} + q^{29} + q^{30} + q^{31} + q^{32} + q^{33} + q^{34} + q^{35} + q^{36} + q^{37} + q^{38} + q^{39} + q^{40} + q^{41} + q^{42} + q^{43} + q^{44} + q^{45} + q^{46} + q^{47} + q^{48} + q^{49} + q^{50} \).

- \( G_{172A} = q + q^2 + q^3 + q^4 + q^5 + q^6 + q^7 + q^8 + q^9 + q^{10} + q^{11} + q^{12} + q^{13} + q^{14} + q^{15} + q^{16} + q^{17} + q^{18} + q^{19} + q^{20} + q^{21} + q^{22} + q^{23} + q^{24} + q^{25} + q^{26} + q^{27} + q^{28} + q^{29} + q^{30} + q^{31} + q^{32} + q^{33} + q^{34} + q^{35} + q^{36} + q^{37} + q^{38} + q^{39} + q^{40} + q^{41} + q^{42} + q^{43} + q^{44} + q^{45} + q^{46} + q^{47} + q^{48} + q^{49} + q^{50} \).

**Remark.** The constructed modular forms are not in Kohnen’s subspace. By Theorem 2 of [11] there exists a weight 3/2 modular form of level 2²33 mapping to the (weight two modular form) 43A, hence this is one of the missing forms. The other one maps to the form 43A also (\( S_{3/2} (2³13) \approx S_2 (2³43) \)) by the corollary of the Main Theorem of [21], i.e., there is no weight 3/2 modular form of level 344 in Kohnen’s subspace mapping to the form 172A while there is one outside Kohnen’s subspace.

**2.1.2. Case 563.** The elliptic curve 563A with the elliptic curve 563A is non-split at 2, i.e., \( y^2 + xy + y = x^3 + x^2 - 15x + 16 \) has rank 2. The points \( [2, -1] \) and \( [4, 4] \) are generators for the rational points. The field corresponding to the point \([2, -1]\) is given by the polynomial \( P = x^4 - 8x^3 + 19x^2 - 14x - 1 \). Its discriminant is \(-563\) and the obstruction is trivial for this field. A solution to the embedding problem is given by

\[
1126 \gamma = (57426x_3^2 - 408738x_2 - 155984)x_1^3 \\
+ (-434073x_2^2 + 2329098x_2 + 1542884)x_1^2 \\
+ (342834x_2^2 - 1089141x_2 - 4555297)x_1 \\
- 339522x_2^2 + 2651994x_2 - 6101295x_2 + 4078271,
\]

where \( x_1 \) and \( x_2 \) are roots of \( P \). Since \( E \) has discriminant \(-563\), by Corollary 2.7 of [8] we know the attached weight 1 modular form has level 2²563 and character \((\chi_4)\). By Theorem 2 of [13] we know that \( \gamma \) can be chosen such that the field \( \mathbb{Q}(\sqrt{7}) \) is given by \( \mathbb{Q}(x_0) \) where \( x_0 \) is a root of the polynomial:

\[
x_0^{24} - 3x_0^{23} - 9x_0^{22} + 22x_0^{21} + 55x_0^{20} - 68x_0^{19} - 212x_0^{18} + 85x_0^{17} + 467x_0^{16} - 34x_0^{15} \\
- 698x_0^{14} - 31x_0^{13} + 797x_0^{12} + 83x_0^{11} - 660x_0^{10} - 56x_0^9 \\
+ 420x_8 - 199x_6 + 32x_5 + 55x_4 - 20x_3 - 4x_2 + 3x + 1.
\]
## Table 2.3. Coefficients of ternary quadratic forms, level $2^2563$. 

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_{23}$</th>
<th>$a_{13}$</th>
<th>$a_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>4</td>
<td>563</td>
<td>564</td>
<td>0</td>
<td>-4</td>
<td>0</td>
</tr>
<tr>
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<td>751</td>
<td>751</td>
<td>-750</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>$Q_3$</td>
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<td>59</td>
<td>584</td>
<td>-52</td>
<td>-32</td>
<td>-14</td>
</tr>
<tr>
<td>$Q_5$</td>
<td>39</td>
<td>578</td>
<td>580</td>
<td>-48</td>
<td>-20</td>
<td>-38</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>44</td>
<td>52</td>
<td>563</td>
<td>0</td>
<td>0</td>
<td>-12</td>
</tr>
<tr>
<td>$Q_7$</td>
<td>47</td>
<td>48</td>
<td>575</td>
<td>48</td>
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<td>4</td>
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<tr>
<td>$Q_8$</td>
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<td>51</td>
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<td>14</td>
<td>48</td>
<td>28</td>
</tr>
<tr>
<td>$Q_9$</td>
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<td>52</td>
<td>576</td>
<td>52</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>$Q_{10}$</td>
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<td>-320</td>
<td>-4</td>
<td>-6</td>
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<td>563</td>
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<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>$Q_{12}$</td>
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<td>207</td>
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<td>-6</td>
<td>-10</td>
</tr>
<tr>
<td>$Q_{13}$</td>
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<td>143</td>
<td>567</td>
<td>6</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>$Q_{14}$</td>
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<td>119</td>
<td>596</td>
<td>-116</td>
<td>-16</td>
<td>-6</td>
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<td>584</td>
<td>84</td>
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</tr>
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<td>$Q_{17}$</td>
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</tr>
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<tr>
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<td>-4</td>
</tr>
<tr>
<td>$Q_{20}$</td>
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<td>6</td>
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<tr>
<td>$Q_{27}$</td>
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<td>107</td>
<td>191</td>
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<td>-14</td>
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<tr>
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<td>92</td>
<td>215</td>
<td>-84</td>
<td>-38</td>
<td>-24</td>
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<tr>
<td>$Q_{29}$</td>
<td>63</td>
<td>143</td>
<td>144</td>
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<td>16</td>
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</tr>
<tr>
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<td>32</td>
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</table>

Its discriminant is $-563^{13}$ (confirming that our choice of $\gamma$ gives an extension unramified at 2). All the Fourier coefficients of this weight 1 modular form can be computed as stated before except the one corresponding to the ramified prime. To compute $a_{563}$ we look at the inertia degree of 563 in $\mathbb{Q}(x_0)$ and since it is 2, it follows that $a_{563} = -1$.

Let $F_{563}(z)$ denote the weight 1 and level 563 modular form attached to this representation. The form $F_{563}(4z)\Theta_{563}(z)$ is in the Kohnen space with trivial character, whose space has dimension 48 (while the whole space has dimension 143). The form $f_{563}(z):=\frac{1}{2}(F_{563}(4z)+\overline{F}_{563}(4z))\Theta_{563}(z)$, has rational coefficients. From the 33 definite positive ternary quadratic forms of Table 2.3, the first 32 ones are linearly independent (by looking at their first 423 Fourier coefficients). The space spanned by their Theta series (which are weight 3/2 modular forms of level $2^2563$ and trivial character in the Kohnen space) and $f_{563}(z)$ is closed under the Hecke operators, and the form

$$F_{563A} = -11\Theta_{Q_1} - 2\Theta_{Q_2} + 44\Theta_{Q_3} - 8\Theta_{Q_4} - 30\Theta_{Q_5} + 16\Theta_{Q_6} - 38\Theta_{Q_7} + 34\Theta_{Q_8} - 2\Theta_{Q_9} + 15\Theta_{Q_{10}} + 2\Theta_{Q_{11}} + 22\Theta_{Q_{12}} - 22\Theta_{Q_{14}} + 4\Theta_{Q_{15}} - 7\Theta_{Q_{16}} - 18\Theta_{Q_{17}} + 29\Theta_{Q_{18}} + 26\Theta_{Q_{19}} + 24\Theta_{Q_{20}} + 14\Theta_{Q_{21}} - 6\Theta_{Q_{22}} + 28\Theta_{Q_{23}} - 28\Theta_{Q_{24}} - 34\Theta_{Q_{25}} - 46\Theta_{Q_{26}} + 22\Theta_{Q_{27}} + 8\Theta_{Q_{28}} - 14\Theta_{Q_{29}} + 20\Theta_{Q_{30}} - 42\Theta_{Q_{31}} + 13f_{563}$$

is an eigenform for the Hecke operators mapping via Shimura to the modular form (attached to the elliptic curve) $563A$. The first 50 coefficients of its Fourier expansion are:

$$-2q^3 + 2q^4 - 2q^7 + 2q^{11} - 2q^{16} + 4q^{23} + 2q^{27} + 4q^{28} + 2q^{39} - 2q^{40} + 2q^{47} + 4q^{48} + O(q^{51}).$$

**Remark.** The space of weight 2 cusp forms of level 563 has dimension 47. The space of Theta series maps via Shimura to the weight 2 Eisenstein series of level
563 and to (the modular form attached to the abelian variety) 563E1 (see [19] for
the notation). The reason why no other abelian variety is in the space of Theta
series is that all the other abelian varieties L-series’ vanish at the central value. By
Theorem 2 of [11] and the Corollary of [21] there are weight 3/2 modular forms of
level 2\textsuperscript{2}563 mapping to the abelian varieties 563B1, 563C1 and 563D1 of dimensions
3, 3 and 9, respectively. There is no reason for the space spanned by the Theta
series and \( f_{563} \) to be closed under the Hecke action, it could have given the whole
space (in which case it would not be feasible to do the computations since we would
need to compute too many Fourier coefficients of \( f_{563} \)). For the weight 3/2 modular
form \( \frac{F_{563}(4z)}{4} - \frac{F_{563}(4z)}{4} \Theta_{563}(z) \), we cannot compute its Hecke orbit.

2.1.3. Case 643: This case is similar to the previous one. The elliptic curve 643A
given by the equation \( y^2 + xy = x^3 - 4x + 3 \) has rank 2 and the points \([1,0]\) and
\([2,1]\) generate the rational points. Their obstruction is nontrivial. The sum is the
embedding problem is given by the element:

\[
\begin{align*}
643\gamma & = (-123456x_1^2 + 36008x_2 - 1376)x_1^1 + (79732x_2^2 - 70820x_2 + 21952)x_1^2 \\
& + (-9092x_1^2 - 101504x_2 + 51440)x_1 - 75964x_1^3 + 176272x_2^2 + 43724x_2 - 25540,
\end{align*}
\]

where \( x_1 \) and \( x_2 \) are roots of \( P \). The field \( \mathbb{Q}(\sqrt{7}) \) is given by \( \mathbb{Q}(x_0) \) where \( x_0 \) is a
root of the polynomial:

\[
\begin{align*}
x^{24} & - 5x^{23} + 11x^{22} - 8x^{21} - 10x^{20} + 23x^{19} + 9x^{18} - 86x^{17} + 171x^{16} - 121x^{15} \\
& - 212x^{14} + 636x^{13} - 504x^{12} - 156x^{11} + 766x^{10} - 1116x^9 + 1364x^8 - 1100x^7 \\
& + 697x^6 - 426x^5 + 227x^4 - 37x^3 + 25x^2 - 29x + 5.
\end{align*}
\]

This field has discriminant \(-643^{11}\), hence the weight 1 modular form has level
643 and character \((563i)\). Looking at the inertia, its Fourier coefficient \( a_{643} = -1 \).

Let \( F_{643}(z) \) denote the weight 1 and level 643 modular form attached to this
representation. The form \( F_{643}(4z)\Theta_{643}(z) \) is in the Kohnen space with trivial char-
acter, whose space has dimension 54 (while the whole space has dimension 163). The
form \( f_{643}(z) := \frac{1}{4}(F_{643}(4z) + F_{643}(4z))\Theta_{643}(z) \), has rational coefficients. The
space spanned by the Theta series of the definite positive ternary quadratic forms of
level \( 2^2643 \) (see Table 2.4) has dimension 29. A basis is given by choosing the
first 29 quadratic forms (this can be checked by looking at their first 484 Fourier
coefficients). The space \( \langle \Theta_{Q_i}, f_{643}(z) \rangle \) is inside Kohnen’s space and is closed under
the action of the Hecke operators. The form

\[
F_{643A} = -3\Theta_{Q_1} - \Theta_{Q_2} + 6\Theta_{Q_3} - 2\Theta_{Q_4} - \Theta_{Q_5} + 8\Theta_{Q_6} + \Theta_{Q_7} + 7\Theta_{Q_8} \\
+ 3\Theta_{Q_9} - 7\Theta_{Q_{10}} + \Theta_{Q_{11}} - 4\Theta_{Q_{12}} + 4\Theta_{Q_{13}} + 6\Theta_{Q_{14}} + 12\Theta_{Q_{15}} \\
- 11\Theta_{Q_{16}} + 6\Theta_{Q_{17}} - 4\Theta_{Q_{18}} - 3\Theta_{Q_{19}} - \Theta_{Q_{20}} - 9\Theta_{Q_{21}} + 2\Theta_{Q_{22}} \\
+ 2\Theta_{Q_{23}} - 8\Theta_{Q_{24}} - 12\Theta_{Q_{25}} + 2\Theta_{Q_{26}} + 6\Theta_{Q_{27}} + 4f_{643}
\]

is an eigenform for the Hecke operators mapping via Shimura to the modular form
(attached to the elliptic curve) 643A. The first 50 coefficients of its Fourier expand-
ion are:

\[
q^4 - q^7 + q^{15} - q^{16} + q^{23} - q^{24} + 2q^{28} + q^{31} - q^{36} + q^{40} + O(q^{51}).
\]
Table 2.4. Coefficients of ternary quadratic forms, level $2^2643$.

<table>
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<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_{12}$</th>
<th>$a_{13}$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_{12}$</th>
<th>$a_{13}$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_{12}$</th>
<th>$a_{13}$</th>
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<tbody>
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<td>0</td>
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<td>0</td>
<td>55</td>
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<td>328</td>
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<td>368</td>
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<td>327</td>
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<td>0</td>
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<td>335</td>
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<td>46</td>
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<tr>
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<td>112</td>
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<td>0</td>
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<td>131</td>
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<td>104</td>
<td>144</td>
<td>167</td>
<td>-60</td>
<td>-88</td>
</tr>
</tbody>
</table>

Remark. As in the previous example, the space of Theta series of level $2^2643$ map to the Eisenstein series of weight 2 and level 643 and to the Eisenstein form attached to the abelian variety 643C1. There is another abelian variety (labeled 643B1 in [19]) of dimension 24 whose L-series vanishes at the central value. A priori there is no reason for the action of the Hecke operators to be closed on the space spanned by the Theta series and $f_{643}$ (it could have given the whole space).

References


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