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THE EVALUATION OF κ_3

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ABSTRACT. Numerical evidence relevant to the evaluation of the constant κ_3 in the conjectural distribution of three-prime Carmichael numbers of Granville and Pomerance (2001) is summarised.

1. INTRODUCTION

Let $C_3(X)$ be the number of three-prime Carmichael numbers up to X. In §8 of [2] Granville and Pomerance conjecture that

$$C_3(X) \sim \tau_3 \frac{X^{\frac{1}{3}}}{(\log X)^3} \sim \frac{\tau_3}{27} \int_2^{X^{\frac{1}{3}}} \frac{dt}{(\log t)^3},$$

where $\tau_3 := \kappa_3 \lambda$, with $\lambda := \dfrac{243}{2} \displaystyle\prod_{p>3} \frac{1 - \frac{3}{p}}{\left(1 - \frac{1}{p}\right)^3} \simeq 77.1727$ and

$$\kappa_3 := \sum_{n \ge 1} \frac{(n,6)}{n^{\frac{4}{3}}} \prod_{\substack{p \mid n \\ p>3}} \frac{p}{p-3} \sum_{\substack{a<b<c, n=abc \\ a,b,c \text{ pairwise coprime}}} \delta_3(a,b,c) \prod_{\substack{p \nmid n \\ p>3}} \frac{p - \omega_{a,b,c}(p)}{p-3}$$

where $\delta_3(a,b,c) = 2$ if $a \equiv b \equiv c \not\equiv 0 \pmod 3$ and 1 otherwise and $\omega_{a,b,c}(p)$ is the number of distinct residues modulo p represented by a, b, c.

The infinite series for κ_3 converges exceedingly slowly, and Carl Pomerance invited the authors to attempt a better evaluation than that in the first preprint of [2]. This paper is a brief account of our computational work on this problem.

2. ALGORITHM AND IMPLEMENTATION

Let u_n be the general term of the above series for κ_3, write $\kappa := \kappa_3 = \sum_{n=1}^{\infty} u_n$, $\kappa(N) := \sum_{n=1}^{N} u_n$, and suppose $n = \prod_{j=1}^{k} p_j^{\alpha_j}$ is the prime factorisation of n, and $q_j := p_j^{\alpha_j}$. Clearly, if $k = 1$, then $u_n = 0$, and it is easy to show that the number of terms in the summation for u_n over (a,b,c) triples is $t_k := \dfrac{3^{k-1} - 1}{2}$. For each n to find the set S_k of all possible coprime triples (a,b,c) we used the following recursion: start with $S_2 := \{(1, q_1, q_2)\}$, and if $S_j := \bigcup_{i=1}^{i=t_j} \{(a_{ij}, b_{ij}, c_{ij})\}$, then $S_{j+1} = \left(\bigcup_{i=1}^{i=t_j} \{(a_{ij} q_{j+1}, b_{ij}, c_{ij}), (a_{ij}, b_{ij} q_{j+1}, c_{ij}), (a_{ij}, b_{ij}, c_{ij} q_{j+1})\}\right) \cup \{(1, q_{j+1}, \prod_{i=1}^{j} q_i)\}$

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until S_k is formed. (In general $a < b < c$ will not hold, but this is irrelevant for evaluation.)

For each (a, b, c), to evaluate the final infinite product in the expression for u_n we observe that when $\omega_{a,b,c}(p) = 3$ the corresponding factor in the product is 1, and that those p for which $\omega_{a,b,c}(p) = 1$ or 2 are the prime factors of $|a - b|, |b - c|$ or $|c - a|$, so the product is essentially finite; thus we found such p and $\omega_{a,b,c}(p)$, checking that $p \not| n$ and $p > 3$.

To minimise the effect of rounding errors when adding large numbers of small terms we used an array to retain all decimal digits of the computation (so for $\kappa(10^7)$, for example, we had 18 decimal places of which the first 7 or 8 might reasonably be expected to be correct).

Accordingly a program was devised in BASIC V. Using a RISC PC, running RISC OS 3.7 with 16 MB of RAM, $\kappa(10^7)$ was found and, discrepancies having been resolved, Granville and Pomerance encouraged us to advance. After several hundred hours of computing $\kappa(10^8)$ was reached: towards the end, each million increase in N took about 16 hours.

3. Statistics and evaluation

Table 1. The growth of $\kappa(N)$

N	$\kappa(N)$	N	$\kappa(N)$	N	$\kappa(N)$
10	0.782401	5×10^6	24.381023	7×10^7	25.642999
10^2	5.354251	10^7	24.787484	7.5×10^7	25.666883
10^3	11.469057	2×10^7	25.136199	8×10^7	25.688882
10^4	16.736742	3×10^7	25.316364	8.5×10^7	25.709247
10^5	20.593777	4×10^7	25.434477	9×10^7	25.728184
10^6	23.169099	5×10^7	25.520869	9.5×10^7	25.745867
2.5×10^6	23.908847	6×10^7	25.588228	10^8	25.762436

In §8 of [2] Granville and Pomerance give a heuristic argument to justify $\kappa = \kappa(N) + (\alpha + o(1)) \int_N^\infty \frac{(\log t)^2}{t^{\frac{4}{3}}} dt$, where α is a constant. We find

$$\int_N^\infty \frac{(\log t)^2}{t^{\frac{4}{3}}} dt = f(N) := \frac{3}{\sqrt[3]{N}} \{(\log N)^2 + 6 \log N + 18\},$$

so for given N we may regard this relationship as an equation in κ and α with known coefficients, but for which the constant $\kappa(N) + o(1)f(N)$ is not known precisely. If now we suppose that, for largish N, the $o(1)$ in the formula is negligible and may be discarded, then from two values N_i, N_j of N we can solve simultaneously to obtain a heuristic estimate for κ. Writing $\kappa(i, j)$ for the estimate thus obtained from $N_i = 10^7 i$ and $N_j = 10^7 j$ we get

$$\kappa(i, j) = \kappa(N_i) + \frac{f(N_i)(\kappa(N_j) - \kappa(N_i))}{f(N_i) - f(N_j)}.$$

Table 2 shows some values of $\kappa(i, j)$.

TABLE 2.

(i,j)	$\kappa(i,j)$	(i,j)	$\kappa(i,j)$	(i,j)	$\kappa(i,j)$	(i,j)	$\kappa(i,j)$
(1,2)	27.12610	(1,7)	27.11164	(3,10)	27.09903	(7.5,10)	27.09247
(1,3)	27.12114	(1,8)	27.11024	(4,10)	27.09687	(8,10)	27.09203
(1,4)	27.11779	(1,9)	27.10903	(5,10)	27.09526	(8.5,10)	27.09162
(1,5)	27.11528	(1,10)	27.10803	(6,10)	27.09398	(9,10)	27.09128
(1,6)	27.11386	(2,10)	27.10219	(7,10)	27.09292	(9.5,10)	27.09088

Ultra cautious extrapolation from Tables 1 and 2 would seem to justify $26 < \kappa < 27.09$, and various empirical approaches with no theoretical justification suggest a value near the upper end of this interval. For example, if we write $k_\mu := \kappa(2^{\mu-3}, 2^{\mu-2}), \delta_\mu := k_\mu - k_{\mu+1}$ and $\rho_\mu := \dfrac{\delta_{\mu+1}}{\delta_\mu}$, Table 1 enables us to calculate $k_1 = \kappa(0.25, 0.5) = 27.16985$, $k_2 = 27.14451, k_3 = 27.12610, k_4 = 27.11083, k_5 = 27.09862; \delta_1 = 0.02434, \delta_2 = 0.01941, \delta_3 = 0.01527, \delta_4 = 0.01221$; and $\rho_1 = 0.7975$, $\rho_2 = 0.7867, \rho_3 = 0.7996$. If now we arbitrarily assume that for $\mu \geq 4$ the series $\sum_{\mu=4}^{\infty} \delta_\mu$ behaves like a GP with common ratio ρ, then $\kappa = k_\infty = k_4 - \dfrac{\delta_4}{1-\rho}$, and any value of ρ satisfying $0.7814 < \rho < 0.8145$ gives $\kappa = 27.05$ correct to 2 decimal places.

We also pursued an empirical method which utilised our categorisation of terms u_n according to the number of consecutive zeroes after the decimal point before the first non-zero digit, which we needed to retain all decimal digits as mentioned above. Thus we found $s_i := \sum_{10^{-i-1} < u_n \leq 10^{-i}} u_n$ for $0 \leq i \leq 6$; for $i \leq 4$, s_i was complete with $n < 10^8$, but to find s_5 and s_6 we had to implement programs to find u_n with $n > 10^8$ and $6 \leq k = \omega(n) \leq 9$ up to various upper bounds for n, depending on k. We note *en passant* that forming the sum of $\kappa(10^8)$ and the extra terms thus found with $n > 10^8$ belonging to s_5, s_6 and s_7 gave a total of 25.817473, so certainly κ is greater than this. Then with $\alpha_i := \dfrac{s_i}{s_{i-1}}$ we get $s_0 = 4.952693, s_1 = 5.992839, s_2 = 4.965006, s_3 = 3.727265, s_4 = 2.586746, s_5 = 1.737140, s_6 = 1.138759$ and $\alpha_2 = 0.8284898, \alpha_3 = 0.7507070, \alpha_4 = 0.6940064, \alpha_5 = 0.6715544$ and $\alpha_6 = 0.6555362$. If now we *arbitrarily assume* that α_i continues to decrease as i increases, comparison with the appropriate GP's gives $\kappa < K_i := \sum_{j=0}^{i-2} s_j + \dfrac{s_{i-1}}{1-\alpha_i}, K_3 = 30.8619, K_4 = 28.0914, K_5 = 27.5135, K_6 = 27.2676$ and K_i is a decreasing sequence with $K_\infty = \kappa$.

A similar empirical approach based on $S_i := \kappa(10^i), \Delta_i := S_{i+1} - S_i$ and $r_i := \dfrac{\Delta_{i+1}}{\Delta_i}$ yields (*assuming* decreasing r_i) another decreasing sequence $K_i^* := S_i + \dfrac{\Delta_i}{1-r_i}$ with $K_3^* = 31.1398, K_4^* = 28.3437, K_5^* = 27.5245, K_6^* = 27.2397$ and $K_\infty^* = \kappa$.

These numerical discussions *suggest* sequences of approximations decreasing to κ. Even with $\kappa = \kappa_3$ at the top end of our above ultra cautious extrapolation range, we get $\tau_3 \doteq 2090$ rather than 2100, which is the 2-significant figure value taken by Granville and Pomerance in [2] based on the numerical evidence then available. Our best provisional estimate, $\kappa = 27.05$ (see above), gives $\tau_3 = 2087.5$. In [2] they define $\gamma := \gamma(X) = C_3(X) \,/ \int_2^{X^{\frac{1}{3}}} \dfrac{dt}{(\log t)^3}$ and state that $\gamma \to \tau_3$ from below as $X \to \infty$. Table 3 below (from Table 1 of [1]) shows γ approaching 2087.5 rather rapidly as N increases, where $X = 10^N$.

Table 3.

N	17	18	19	20	21	22	23	24
γ	1839	1899	1947	1984	2019	2047	2067	2081

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