

## $C^1$ SPLINE WAVELETS ON TRIANGULATIONS

RONG-QING JIA AND SONG-TAO LIU

ABSTRACT. In this paper we investigate spline wavelets on general triangulations. In particular, we are interested in  $C^1$  wavelets generated from piecewise quadratic polynomials. By using the Powell-Sabin elements, we set up a nested family of spaces of  $C^1$  quadratic splines, which are suitable for multiresolution analysis of Besov spaces. Consequently, we construct  $C^1$  wavelet bases on general triangulations and give explicit expressions for the wavelets on the three-direction mesh. A general theory is developed so as to verify the global stability of these wavelets in Besov spaces. The wavelet bases constructed in this paper will be useful for numerical solutions of partial differential equations.

### 1. INTRODUCTION

In this paper we investigate spline wavelets on general triangulations. In particular, we are interested in  $C^1$  wavelets generated from piecewise quadratic polynomials.

In [7] and [8] Chui and Wang used B-splines to construct compactly supported semi-orthogonal wavelets. Their results were extended to compactly supported box spline wavelets. See the work of Chui, Stöckler, and Ward [6], Jia and Micchelli [20], and Riemenschneider and Shen [30]. For a comprehensive study on box splines, we refer the reader to the book [1] by de Boor, Höllig, and Riemenschneider. Recently, compactly supported wavelet bases for Sobolev spaces were studied by Lorentz and Oswald [26], and by Jia, Wang, and Zhou [21].

It has been a challenging problem to construct spline wavelets on general triangulations. For piecewise linear functions on general triangulations, Yserentant [38] introduced the so-called hierarchical bases in the finite element application to second-order elliptic boundary value problems. In [36] Vassilevski and Wang modified the hierarchical basis functions by using some projections on each level, yielding a basis of approximate wavelets. In the same spirit, Stevenson [34, 35] constructed the three-point wavelet bases and established  $H^1$ -stability of the bases. For piecewise quadratic polynomials on general triangulations, Liu [24] constructed wavelet bases which are  $H^1$ -stable.

The wavelets given in [34, 35] are  $H^1$ -stable; but it is still an open question whether these wavelets are  $L_2$ -stable. In [12] and [13] Floater and Quak were able to construct semi-orthogonal wavelets of continuous piecewise linear functions on general triangulations. Certainly, semi-orthogonal wavelets are  $L_2$ -stable. In [15] Hardin and Hong further considered orthogonal wavelets of continuous piecewise

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linear functions. See the survey paper [14] of Goodman and Hardin for recent progress in the study of multivariate spline wavelets.

The wavelets discussed in the preceding two paragraphs are continuous, but not continuously differentiable. Recently,  $C^1$  spline wavelets have attracted attention of many researchers in the areas of splines and wavelets. For the univariate case, using Hermite cubic splines, in [19] we constructed a pair of  $C^1$  wavelets with nice properties. In particular, the construction of boundary wavelets is remarkably simple. The wavelet basis given in [19] was used to solve the Sturm-Liouville equation with the Dirichlet boundary equation. The condition number of the corresponding stiffness matrix was shown to be very small. Our work significantly improved the earlier results of Xu and Shann [37] on Galerkin-wavelet methods for two-point boundary value problems.

Now let us consider  $C^1$  spline wavelets on general triangulations of polygonal domains in  $\mathbb{R}^2$ . Using the Powell-Sabin elements (see [29]), Oswald in [27] introduced hierarchical bases and demonstrated that the hierarchical bases are sub-optimal for the condition numbers of the corresponding discretization matrices. But the hierarchical bases are not truly  $H^2$ -stable. In [9] Davydov and Petrushev constructed hierarchical sequences of  $C^1$  spline bases on multilevel triangulations and investigated nonlinear approximation of such redundant systems. Recently, using  $C^1$  cubic splines of Lagrange type, Davydov and Stevenson [10] constructed  $C^1$  hierarchical Riesz bases for  $H^\mu$  with  $1 < \mu < 5/2$ .

In this paper we shall employ the Hermite interpolation property of the Powell-Sabin elements to construct  $C^1$  spline wavelets on general triangulations. These wavelet bases will be shown to be  $H^2$ -stable. In the process we shall establish a general theory for wavelet bases in Besov spaces, which will be useful for future research.

Here is an outline of the paper. In Section 2 we review some basic properties of Besov spaces as well as Bernstein type inequalities for spline functions on polygonal domains. In Section 3, using the Powell-Sabin elements, we formulate bases for the concerned spaces of spline functions and establish Jackson type inequalities. In Section 4 we discuss multiresolution analysis of Besov spaces and related norm equivalence. In Section 5 we develop a general theory for wavelet bases in Besov spaces. Finally, in Section 6, we construct  $C^1$  wavelet bases on general triangulations and show that the wavelet bases are  $H^2$ -stable. In particular, for the three-direction mesh, the wavelets are explicitly given.

It is expected that the  $C^1$  spline wavelets constructed in this paper will have applications to numerical solutions of partial differential equations. These wavelet bases are particularly suitable for the biharmonic equation (see the discussion in [27] and [10]). The condition numbers of the corresponding discretization matrices will be uniformly bounded. Our wavelet bases could have applications to a wide range of problems in numerical analysis, such as numerical solutions of integral equations and operator equations, and singular perturbation problems. See the related work of Chen, Micchelli and Xu [2], Liu and Xu [25], and Shen and Lin [32]. The refinable spline functions discussed in this paper also could have applications to computer graphics and multi-level data representation. See the related work of Chui and Jiang [4, 5] on surface subdivision schemes.

2. BESOV SPACES ON POLYGONAL DOMAINS

In this section we first introduce some notation that will be needed later. Then we review basic properties of Besov spaces. At the end of the section we establish a Bernstein type inequality for spline functions on polygonal domains.

Let  $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{R}$  denote the set of positive integers, integers, and real numbers, respectively. For a positive integer  $s$ ,  $\mathbb{R}^s$  denotes the  $s$ -dimensional Euclidean space with the inner product given by

$$x \cdot y := x_1y_1 + \dots + x_sy_s, \quad \text{for } x = (x_1, \dots, x_s) \text{ and } y = (y_1, \dots, y_s) \in \mathbb{R}^s.$$

Consequently, the norm of a vector  $x \in \mathbb{R}^s$  is given by  $|x| := (x \cdot x)^{1/2}$ .

Let  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . An element of  $\mathbb{N}_0^s$  is called a multi-index. The length of a multi-index  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_0^s$  is given by  $|\alpha| := \alpha_1 + \dots + \alpha_s$ . For  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_0^s$  and  $x = (x_1, \dots, x_s) \in \mathbb{R}^s$ , define

$$x^\alpha := x_1^{\alpha_1} \dots x_s^{\alpha_s}.$$

The function  $x \mapsto x^\alpha$  ( $x \in \mathbb{R}^s$ ) is called a monomial and its (total) degree is  $|\alpha|$ . A polynomial is a linear combination of monomials. In other words, a polynomial  $q$  has a representation  $\sum_\alpha c_\alpha x^\alpha$ , where the coefficients  $c_\alpha$  are real numbers and  $c_\alpha \neq 0$  only for finitely many  $\alpha$ . The degree of  $q$  is defined to be  $\deg q := \max\{|\alpha| : c_\alpha \neq 0\}$ . For an integer  $k \geq 0$ , we use  $\Pi_k$  to denote the linear space of all polynomials of degree at most  $k$ .

For a vector  $y = (y_1, \dots, y_s) \in \mathbb{R}^s$ , we use  $D_y$  to denote the differential operator given by

$$D_y f(x) := \lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t}, \quad x \in \mathbb{R}^s.$$

Let  $e_1, \dots, e_s$  be the unit coordinate vectors in  $\mathbb{R}^s$ . For  $j = 1, \dots, s$ , we write  $D_j$  for  $D_{e_j}$ . For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_s)$ ,  $D^\alpha$  stands for the differential operator  $D_1^{\alpha_1} \dots D_s^{\alpha_s}$ .

Now let  $\Omega$  be a (Lebesgue) measurable subset of  $\mathbb{R}^s$ . Suppose  $f$  is a (real-valued) measurable function on  $\Omega$ . For  $1 \leq p < \infty$ , let

$$\|f\|_{p,\Omega} := \left( \int_\Omega |f(x)|^p dx \right)^{1/p}.$$

For  $p = \infty$ , let  $\|f\|_{\infty,\Omega}$  be the essential supremum of  $|f|$  on  $\Omega$ . When  $\Omega = \mathbb{R}^s$ , we omit the reference to  $\Omega$ . By  $L_p(\Omega)$  ( $1 \leq p \leq \infty$ ) we denote the linear space of all functions  $f$  on  $\Omega$  such that  $\|f\|_{p,\Omega} < \infty$ . Equipped with the norm  $\|\cdot\|_{p,\Omega}$ ,  $L_p(\Omega)$  becomes a Banach space. For  $p = 2$ ,  $L_2(\Omega)$  is a Hilbert space with the inner product given by

$$\langle f, g \rangle := \int_\Omega f(x)g(x) dx, \quad f, g \in L_2(\Omega).$$

Suppose  $\Omega$  is a (nonempty) open subset of  $\mathbb{R}^s$ . Let  $C(\Omega)$  be the linear space of all continuous functions on  $\Omega$ . For an integer  $r \geq 0$ , we use  $C^r(\Omega)$  to denote the linear space of all  $r$  times continuously differentiable functions on  $\Omega$ . For  $1 \leq p \leq \infty$ , we use  $W_p^r(\Omega)$  to denote the Sobolev space consisting of all functions  $f \in L_p(\Omega)$  such that  $D^\alpha f \in L_p(\Omega)$  for all  $|\alpha| \leq r$ .

The Fourier transform of a function  $f$  in  $L_1(\mathbb{R}^s)$  is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^s.$$

The Fourier transform can be naturally extended to functions in  $L_2(\mathbb{R}^s)$ . For  $\mu > 0$ , we denote by  $H^\mu(\mathbb{R}^s)$  the space of all functions  $f \in L_2(\mathbb{R}^s)$  such that the semi-norm

$$|f|_{H^\mu(\mathbb{R}^s)} := \left( \int_{\mathbb{R}^s} |\hat{f}(\xi)|^2 |\xi|^{2\mu} d\xi \right)^{1/2}$$

is finite. For a nonempty open subset  $\Omega$  of  $\mathbb{R}^s$ , we define

$$H^\mu(\Omega) := \{f|_\Omega : f \in H^\mu(\mathbb{R}^s)\}.$$

For  $x, y \in \mathbb{R}^s$ , we use  $[x, y]$  to denote the line segment  $\{(1-t)x + ty : 0 \leq t \leq 1\}$ . For  $y \in \mathbb{R}^s$ , let  $\Omega_y$  denote the set  $\{x \in \Omega : [x - y, x] \subset \Omega\}$ . The **modulus of continuity** of a function  $f \in L_p(\Omega)$  for  $1 \leq p < \infty$ , or  $f \in C(\Omega)$  for  $p = \infty$ , is defined by

$$\omega(f, h)_p := \sup_{|y| \leq h} \|\nabla_y f\|_{p, \Omega_y}, \quad h > 0,$$

where  $\nabla_y$  denotes the difference operator given by  $\nabla_y f(x) = f(x) - f(x - y)$ ,  $x \in \Omega_y$ . For a positive integer  $m$ , the  $m$ th **modulus of smoothness** of  $f$  is defined by

$$\omega_m(f, h)_p := \sup_{|y| \leq h} \|\nabla_y^m f\|_{p, \Omega_{my}}, \quad h > 0.$$

See the work of Johnen and Scherer [22] on the equivalence of the  $K$ -functional and moduli of smoothness.

Let  $m \in \mathbb{N}$  and  $y \in \mathbb{R}^s$ . Then the following inequality is valid for  $1 \leq p \leq \infty$ :

$$(2.1) \quad \|\nabla_y^m f\|_{p, \Omega_{my}} \leq \|D_y^m f\|_{p, \Omega} \quad \forall f \in W_p^m(\Omega).$$

Indeed, for  $f \in W_p^1(\Omega)$ , the relation

$$\nabla_y f(x) = \int_0^1 D_y f(x - ty) dt$$

is true for almost every  $x \in \Omega_y$ . Applying the Minkowski inequality to the above integral, we see that (2.1) is valid for  $m = 1$ . Consequently, (2.1) can be verified by induction on  $m$ .

Suppose  $f \in W_p^r(\Omega)$  for  $1 \leq p < \infty$  or  $f \in C^r(\Omega)$  for  $p = \infty$ , where  $r \in \mathbb{N}_0$ . For  $m > r$  and  $y \in \mathbb{R}^s$ , by (2.1) we have

$$\begin{aligned} \|\nabla_y^m f\|_{p, \Omega_{my}} &= \|\nabla_y^r \nabla_y^{m-r} f\|_{p, \Omega_{my}} \leq \|D_y^r \nabla_y^{m-r} f\|_{p, \Omega_{(m-r)y}} \\ &= \|\nabla_y^{m-r} D_y^r f\|_{p, \Omega_{(m-r)y}}. \end{aligned}$$

Hence, there exists a positive constant  $B$  such that

$$(2.2) \quad \omega_m(f, h)_p \leq Bh^r \sum_{|\alpha|=r} \omega_{m-r}(D^\alpha f, h)_p, \quad h > 0.$$

For  $\mu > 0$  and  $1 \leq p, q \leq \infty$ , the Besov space  $B_{p,q}^\mu(\Omega)$  is the collection of those functions  $f \in L_p(\Omega)$  for which the following semi-norm is finite:

$$|f|_{B_{p,q}^\mu} := \begin{cases} \left( \int_0^\infty [t^{-\mu} \omega_m(f, t)_p]^q \frac{1}{t} dt \right)^{1/q}, & \text{for } 1 \leq q < \infty, \\ \sup_{t>0} \{t^{-\mu} \omega_m(f, t)_p\}, & \text{for } q = \infty, \end{cases}$$

where  $m$  is the least integer greater than  $\mu$ . It is easily seen that

$$|f|_{B_{p,q}^\mu(\Omega)} \approx \begin{cases} \left( \sum_{j \in \mathbb{Z}} [2^{j\mu} \omega_m(f, 2^{-j})_p]^q \right)^{1/q}, & \text{for } 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} \{ 2^{j\mu} \omega_m(f, 2^{-j})_p \}, & \text{for } q = \infty. \end{cases}$$

The norm for  $B_{p,q}^\mu(\Omega)$  is

$$\|f\|_{B_{p,q}^\mu(\Omega)} := \|f\|_{L_p(\Omega)} + |f|_{B_{p,q}^\mu(\Omega)}.$$

For  $x \in \mathbb{R}^s$  and  $\varepsilon > 0$ , by  $B_\varepsilon(x)$  we denote the ball  $\{y \in \mathbb{R}^s : |y - x| < \varepsilon\}$ . Let  $\Omega$  be an open subset of  $\mathbb{R}^s$ . Its boundary is denoted by  $\partial\Omega$ . We say that  $\Omega$  is a Lipschitz-graph domain (see [22]) if there exists a finite collection of open sets  $\{U_j\}$ , a corresponding collection of cones  $\{C_j\}$  (all congruent to a fixed finite cone  $C$ ), and a real number  $\varepsilon > 0$  such that for each  $x \in \partial\Omega$ ,  $B_\varepsilon(x)$  is contained in some  $U_j$ , and that  $x + C_j \subset \Omega$  for each  $x \in U_j \cap \Omega$ . For a Lipschitz-graph domain  $\Omega$ , if we replace  $\omega_m$  by  $\omega_n$  for some  $n > m$ , then we obtain an equivalent norm for the Besov space  $B_{p,q}^\mu(\Omega)$ .

It was shown by Sharpley [31] that  $\Omega$  is a Lipschitz-graph domain if and only if  $\partial\Omega$  is minimally smooth (see [33, p. 189] for the definition). Thus, the extension theorem is valid for Sobolev spaces on such a domain (see Chapter VI of [33]). It is well known that  $H^\mu(\mathbb{R}^s) = B_{2,2}^\mu(\mathbb{R}^s)$  for  $\mu > 0$  (see Chapter V of [33]). Therefore, for  $0 < \mu < \infty$ ,

$$H^\mu(\Omega) = B_{2,2}^\mu(\Omega).$$

Now let  $\mathcal{T}$  be a finite or countable collection of triangles in  $\mathbb{R}^2$ . The intersection of any two triangles in  $\mathcal{T}$  is empty, or a common vertex, or a common edge of them. Let  $\Omega$  be the union of the triangles in  $\mathcal{T}$ . Then  $\Omega$  is a polygonal domain in  $\mathbb{R}^2$  and  $\mathcal{T}$  is a triangulation of  $\Omega$ . The domain  $\Omega$  could be bounded or unbounded. But  $\Omega$  is always assumed to be a Lipschitz-graph domain.

A vertex of a triangle in  $\mathcal{T}$  is said to be a vertex of  $\mathcal{T}$ . A vertex  $v$  is called an interior vertex, if  $v$  is in the interior of  $\Omega$ ; otherwise, it is called a boundary vertex. An edge of a triangle in  $\mathcal{T}$  is said to be an edge of  $\mathcal{T}$ . An edge  $e$  is called an interior edge, if the interior of  $e$  is included in the interior of  $\Omega$ ; otherwise, it is called a boundary edge.

A direction is assigned to each edge  $e$  so that  $e$  becomes a vector in  $\mathbb{R}^2$ . If  $e = (a, b)$ , then  $e^\perp := (b, -a)$  is a vector perpendicular to  $e$ . We call  $D_{e^\perp}$  the normal derivative with respect to  $e$ .

The length of an edge  $e$  is denoted by  $|e|$ . The supremum of the length of the edges of  $\mathcal{T}$  is called the mesh size of  $\mathcal{T}$ . Let  $h$  be the mesh size of  $\mathcal{T}$ . We assume that  $\mathcal{T}$  is quasi-uniform, that is, there exists a positive constant  $M$  such that  $Mh \leq |e| \leq h$  for all edges  $e$  of  $\mathcal{T}$ . We also assume that there exists some  $\theta > 0$  such that every angle of the triangles in  $\mathcal{T}$  is greater than or equal to  $\theta$ . Consequently, the number of edges with a common vertex is bounded.

Let  $v$  be a vertex of  $\mathcal{T}$ . For  $N = 1, 2, \dots$ , the stars  $\text{St}_\mathcal{T}^N(v)$  are defined as follows. Let  $\text{St}_\mathcal{T}(v) = \text{St}_\mathcal{T}^1(v)$  be the union of the triangles of  $\mathcal{T}$  with  $v$  as a vertex. For  $N > 1$ , let  $\text{St}_\mathcal{T}^N(v)$  be the union of the triangles of  $\mathcal{T}$  that intersect  $\text{St}_\mathcal{T}^{N-1}(v)$ .

**Lemma 2.1.** *Let  $k$  be a nonnegative integer. Suppose that  $\tau$  is a triangle in  $\mathbb{R}^2$ ,  $\theta$  is the minimum of the angles of  $\tau$  and  $h$  is the maximum length of the edges of  $\tau$ . Then there exist two positive constants  $C_1$  and  $C_2$  depending only on  $k$  and  $\theta$  such*

that

$$(2.3) \quad C_1 h^{2/p} \|f\|_{\infty, \tau} \leq \|f\|_{p, \tau} \leq C_2 h^{2/p} \|f\|_{\infty, \tau} \quad \forall f \in \Pi_k|_{\tau} \text{ and } 1 \leq p \leq \infty.$$

Moreover, for  $\alpha \in \mathbb{N}_0^2$  with  $|\alpha| = r \leq k$ , there exists a positive constant  $C$  depending only on  $k$  and  $\theta$  such that

$$(2.4) \quad \|D^\alpha f\|_{p, \tau} \leq C h^{-r} \|f\|_{p, \tau} \quad \forall f \in \Pi_k|_{\tau} \text{ and } 1 \leq p \leq \infty.$$

*Proof.* Let  $T$  be the triangle in  $\mathbb{R}^2$  with vertices  $(0, 0)$ ,  $(0, 1)$ , and  $(1, 0)$ . Then  $\Pi_k|_T$  is a finite dimensional space. Since any two norms on a finite dimensional space are equivalent, there exist positive constants  $K_1$  and  $K_2$  such that

$$(2.5) \quad K_1 \|g\|_{\infty, T} \leq \|g\|_{p, T} \leq K_2 \|g\|_{\infty, T} \quad \forall g \in \Pi_k|_T \text{ and } 1 \leq p \leq \infty.$$

Moreover, for  $\alpha \in \mathbb{N}_0^2$  there exists a positive constant  $K_\alpha$  such that

$$(2.6) \quad \|D^\alpha g\|_p \leq K_\alpha \|g\|_p \quad \forall g \in \Pi_k|_T \text{ and } 1 \leq p \leq \infty.$$

There exists an affine transform  $A$  on  $\mathbb{R}^2$  such that  $A$  maps  $T$  one-to-one and onto  $\tau$ . The Jacobian determinant of  $A$ , denoted  $J(A)$ , is a constant. We have

$$|J(A)| = \frac{\text{area}(\tau)}{\text{area}(T)} = 2 \text{area}(\tau).$$

Consequently,  $M_1 h^2 \leq |J(A)| \leq M_2 h^2$  for some positive constants  $M_1$  and  $M_2$  depending only on  $\theta$ . Let  $f \in \Pi_k|_{\tau}$ , and let  $g(x) = f(Ax)$  for  $x \in T$ . Then  $g \in \Pi_k|_T$ . We have  $f(x) = g(A^{-1}x)$ ,  $x \in \tau$ . Hence,

$$\|f\|_{p, \tau} = |J(A)|^{1/p} \|g\|_{p, T}, \quad 1 \leq p \leq \infty.$$

This together with (2.5) verifies (2.3).

Suppose  $\alpha \in \mathbb{N}_0^2$  and  $|\alpha| = r$ , where  $0 \leq r \leq k$ . Since  $f(x) = g(A^{-1}x)$  for  $x \in \tau$ , there exists a positive constant  $B_\alpha$  such that

$$\|D^\alpha f\|_{p, \tau} \leq B_\alpha \|A^{-1}\|^r |J(A)|^{1/p} \|D^\alpha g\|_{p, T};$$

but  $\|A^{-1}\| \leq M_0/h$  for some constant  $M_0 > 0$ . Moreover, it follows from (2.6) that

$$|J(A)|^{1/p} \|D^\alpha g\|_{p, T} \leq K_\alpha |J(A)|^{1/p} \|g\|_{p, T} = K_\alpha \|f\|_{p, \tau}.$$

The above estimates tell us that (2.4) holds true for a positive constant  $C$  depending only on  $k$  and  $\theta$ . □

For a positive integer  $k$ , let  $S_k(\mathcal{T})$  denote the space of all splines of degree  $k$  on  $\mathcal{T}$ . In other words,  $s \in S_k(\mathcal{T})$  if and only if, on each triangle  $\sigma$  in  $\mathcal{T}$ ,  $s$  agrees with a polynomial of degree at most  $k$ . For  $r = 0, 1, \dots, k - 1$ , let

$$S_k^r(\mathcal{T}) := S_k(\mathcal{T}) \cap C^r(\Omega).$$

The following lemma gives a Bernstein type inequality for spline functions on a polygonal domain. See [11] and [16] for related results.

**Lemma 2.2.** *Let  $\mathcal{T}$  be a triangulation of a polygonal domain  $\Omega$  in  $\mathbb{R}^2$ , and let  $\theta$  be the infimum of the angles of  $\mathcal{T}$ . Suppose that there exists a positive constant  $M$  such that  $Mh \leq |e| \leq h$  for every edge  $e$  of  $\mathcal{T}$ . Let  $f$  be a function in  $S_k^r(\mathcal{T})$  such that its support is contained in  $N$  triangles of  $\mathcal{T}$ . Then for  $1 \leq p \leq \infty$  and  $0 < \mu < r + 1 + 1/p$ ,*

$$(2.7) \quad |f|_{B_{p,p}^\mu(\Omega)} \leq C h^{-\mu} \|f\|_{p, \Omega},$$

where  $C$  is a constant depending on  $\theta$ ,  $M$  and  $N$  but independent of  $f$  and  $h$ .

*Proof.* First, consider the case  $p = \infty$ . Let us estimate  $\omega_m(f, t)_\infty$ , where  $m \geq r + 1 > \mu$  and  $t > 0$ . Note that  $\omega_m(f, t)_\infty \leq 2^m \|f\|_{\infty, \Omega}$ . Hence, for  $t \geq h$  we have

$$\omega_m(f, t)_\infty \leq (h^{-\mu} 2^m \|f\|_{\infty, \Omega}) t^\mu.$$

Suppose  $t < h$ . By (2.2) there exists a constant  $B > 0$  such that

$$(2.8) \quad \omega_m(f, t)_\infty \leq B t^r \sum_{|\alpha|=r} \omega_{m-r}(D^\alpha f, t)_\infty \leq B t^r \sum_{|\alpha|=r} 2^{m-r-1} \omega(D^\alpha f, t)_\infty.$$

For  $|\alpha| = r$ ,  $D^\alpha f$  belongs to  $W_\infty^1(\Omega)$ . For  $y \in \mathbb{R}^2$  we have

$$\|\nabla_y D^\alpha f\|_{\infty, \Omega_y} \leq \sup_{\tau \in \mathcal{T}} \|D_y D^\alpha f\|_{\infty, \tau}.$$

If  $|y| \leq t$ , then it follows from (2.4) that

$$\|D_y D^\alpha f\|_{\infty, \tau} \leq B_1 t h^{-r-1} \|f\|_{\infty, \tau},$$

where  $B_1$  is a constant independent of  $h$  and  $f$ . Consequently,

$$\omega(D^\alpha f, t)_\infty \leq B_1 t h^{-r-1} \|f\|_{\infty, \Omega}.$$

This together with (2.8) tells us that there exists a constant  $C > 0$  such that

$$\omega_m(f, t)_\infty \leq C(t/h)^{r+1} \|f\|_{\infty, \Omega} \leq C(t/h)^\mu \|f\|_{\infty, \Omega} = (C h^{-\mu} \|f\|_{\infty, \Omega}) t^\mu.$$

Therefore, (2.7) is valid for  $p = \infty$ .

It remains to prove (2.7) for the case  $1 \leq p < \infty$ . The proof is based on the norm equivalence

$$|f|_{B_{p,p}^\mu(\Omega)} \approx \left( \sum_{j \in \mathbb{Z}} [2^{j\mu} \omega_m(f, 2^{-j})_p]^p \right)^{1/p},$$

where  $m \geq k + 1 > r + 1 + 1/p$ . Let  $j_0$  be the integer such that  $h \leq 2^{-j_0} < 2h$ . We shall estimate the above sum for  $j \leq j_0$  and  $j > j_0$  separately.

For  $j \leq j_0$  we have  $\omega_m(f, 2^{-j})_p \leq 2^m \|f\|_{p, \Omega}$ . Hence,

$$\sum_{j=-\infty}^{j_0} [2^{j\mu} \omega_m(f, 2^{-j})_p]^p \leq [2^m \|f\|_{p, \Omega}]^p (2^{j_0 \mu})^p \sum_{j=-\infty}^{j_0} (2^{-\mu p})^{j_0-j}.$$

Consequently, there exists a positive constant  $C$  such that

$$\left( \sum_{j=-\infty}^{j_0} [2^{j\mu} \omega_m(f, 2^{-j})_p]^p \right)^{1/p} \leq C h^{-\mu} \|f\|_{p, \Omega}.$$

Let us consider the case  $j > j_0$ . By (2.2) there exists a constant  $B$  such that

$$\omega_m(f, 2^{-j})_p \leq B(2^{-j})^{r+1} \sum_{|\alpha|=r+1} \omega_{m-r-1}(D^\alpha f, 2^{-j})_p.$$

By our assumption, there exists a subcollection  $\mathcal{S}$  of  $N$  triangles in  $\mathcal{T}$  such that  $f$  is supported in  $\bigcup_{\tau \in \mathcal{S}} \tau$ . Let  $Y_j$  be the set of those points in  $\Omega$  whose distance to one of the edges of some triangle  $\tau \in \mathcal{S}$  is less than  $(m - r - 1)2^{-j}$ . If  $x \in \Omega \setminus Y_j$  and  $|y| \leq 2^{-j}$ , then

$$\nabla_y^{m-r-1}(D^\alpha f)(x) = 0 \quad \text{for } |\alpha| = r + 1,$$

since  $m > k$  and  $D^\alpha f|_\tau \in \Pi_{k-r-1}|_\tau$  for each triangle  $\tau \in \mathcal{T}$ . This shows that

$$\omega_{m-r-1}(D^\alpha f, 2^{-j})_p \leq 2^{m-r-1} \|D^\alpha f\|_{\infty, \Omega} (\text{area}(Y_j))^{1/p};$$

but there exists a constant  $M_1 > 0$  such that  $\text{area}(Y_j) \leq M_1 2^{-j}h$ . Moreover, there exists some  $\tau \in \mathcal{S}$  such that  $\|D^\alpha f\|_{\infty, \Omega} = \|D^\alpha f\|_{\infty, \tau}$ . By Lemma 2.1, for  $|\alpha| = r + 1$ , there exist positive constants  $K_1$  and  $K_2$  such that

$$\|D^\alpha f\|_{\infty, \tau} \leq K_1 h^{-(r+1)} \|f\|_{\infty, \tau} \leq K_2 h^{-(r+1)} h^{-2/p} \|f\|_{p, \tau}.$$

Combining the above estimates together, we obtain

$$\omega_m(f, 2^{-j})_p \leq C_1 (2^{-j})^{r+1} h^{-(r+1)} h^{-2/p} (2^{-j}h)^{1/p} \|f\|_{p, \Omega},$$

where  $C_1$  is a constant independent of  $h$  and  $f$ . Consequently,

$$\sum_{j=j_0+1}^{\infty} [2^{j\mu} \omega_m(f, 2^{-j})_p]^p \leq [C_1 h^{-(r+1+1/p)} \|f\|_{p, \Omega}]^p \sum_{j=j_0+1}^{\infty} 2^{-j(r+1+1/p-\mu)p}.$$

Since  $\mu < r + 1 + 1/p$ , the series on the right-hand side of the above inequality is a geometric series with its ratio less than 1. Hence, there exists a constant  $C_2 > 0$  such that

$$\sum_{j=j_0+1}^{\infty} 2^{-j(r+1+1/p-\mu)p} \leq C_2^p (2^{-(j_0+1)(r+1+1/p-\mu)p}) \leq [C_2 h^{r+1+1/p-\mu}]^p,$$

where the fact  $2^{-(j_0+1)} \leq h$  has been used to derive the last inequality. Therefore,

$$\left( \sum_{j=j_0+1}^{\infty} [2^{j\mu} \omega_m(f, 2^{-j})_p]^p \right)^{1/p} \leq C h^{-\mu} \|f\|_{p, \Omega},$$

where  $C = C_1 C_2$ . The proof of the lemma is complete. □

### 3. BASES OF PIECEWISE QUADRATIC POLYNOMIALS

In this section we review basic properties of the Powell-Sabin elements. Stable bases are formulated for the concerned spaces of quadratic splines. Approximation properties of such spaces are discussed.

Let  $\sigma$  be a triangle in  $\mathbb{R}^2$ . Suppose  $A_1, A_2$ , and  $A_3$  are the vertices of  $\sigma$ . Let  $B_1, B_2, B_3$  be the middle point of the line segment  $A_2A_3, A_3A_1$  and  $A_1A_2$ , respectively.

The triangle  $\sigma$  is divided into four smaller triangles:  $\triangle A_1B_2B_3, \triangle A_2B_1B_3, \triangle A_3B_1B_2$ , and  $\triangle B_1B_2B_3$ . We use  $\delta_4(\sigma)$  to denote the collection of these four triangles.

The lines  $A_1B_1, A_2B_2$ , and  $A_3B_3$  intersect at the barycenter  $O$  of  $\sigma$ . The triangle  $\sigma$  is divided into 6 smaller triangles:  $\triangle OA_1B_3, \triangle OB_3A_2, \triangle OA_2B_1, \triangle OB_1A_3, \triangle OA_3B_2$ , and  $\triangle OB_2A_1$ . We use  $\delta_6(\sigma)$  to denote the collection of these six triangles.

The line segments  $A_1B_1, A_2B_2, A_3B_3, B_1B_2, B_2B_3$ , and  $B_3B_1$  divide  $\sigma$  into twelve smaller triangles. We use  $\delta_{12}(\sigma)$  to denote the collection of these twelve triangles. Obviously,  $\delta_{12}(\sigma)$  is a subdivision of both  $\delta_4(\sigma)$  and  $\delta_6(\sigma)$ . Moreover,  $\delta_6(\delta_4(\sigma))$  is a subdivision of  $\delta_{12}(\sigma)$ .

Figure 1 shows the 6-split and 12-split of a triangle.

Splines in  $S_2^1(\delta_6(\sigma))$  are called the first type Powell-Sabin elements, and splines in  $S_2^1(\delta_{12}(\sigma))$  are called the second type Powell-Sabin elements. It is known that a function  $f \in S_2^1(\delta_6(\sigma))$  is uniquely determined by its function values and gradients at the vertices of  $\sigma$ . Moreover, if the function values of  $f$  and its gradients vanish on both vertices  $A_1$  and  $A_2$ , then  $f$  together with its normal derivative vanishes on the edge  $A_1A_2$ . A function  $g \in S_2^1(\delta_{12}(\sigma))$  is uniquely determined by its function



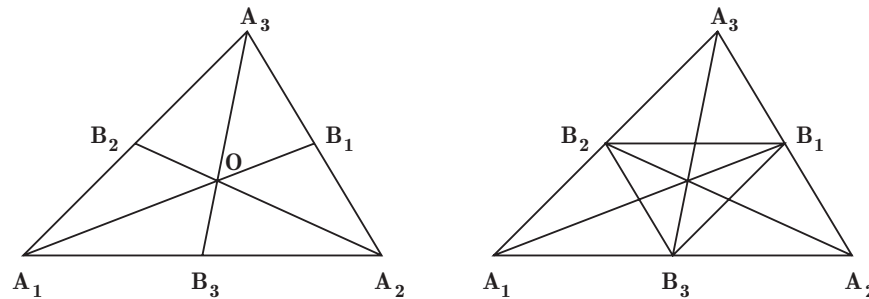


FIGURE 1. 6-split and 12-split of a triangle

values and gradients at the vertices of  $\sigma$  and its normal derivatives at the midpoints of edges of  $\sigma$ ; see the original paper [29] of Powell and Sabin for details. Also, see the paper [3] of Chui and He for discussions on splines based on the Powell-Sabin elements. In [23], Lai and Schumaker gave some generalizations of the Powell-Sabin elements.

Two triangles  $\sigma$  and  $\tau$  are said to be neighboring, if  $\sigma \cap \tau$  is a common edge of  $\sigma$  and  $\tau$ . Suppose  $\sigma$  and  $\tau$  are two neighboring triangles and  $f$  is a function defined on  $\sigma \cup \tau$ . If  $f|_{\sigma} \in S_2^1(\delta_6(\sigma))$  and  $f|_{\tau} \in S_2^1(\delta_6(\tau))$ , it is not always true that  $f \in C^1(\sigma \cup \tau)$ ; but  $f$  indeed lies in  $C^1(\sigma \cup \tau)$  provided  $\sigma \cup \tau$  forms a parallelogram. If  $f|_{\sigma} \in S_2^1(\delta_{12}(\sigma))$  and  $f|_{\tau} \in S_2^1(\delta_{12}(\tau))$ , and in addition, the normal derivatives of  $f|_{\sigma}$  and  $f|_{\tau}$  at the middle point of the common edge are equal, then  $f \in C^1(\sigma \cup \tau)$ .

The above discussion motivates us to consider combinations of the first type and second type of the Powell-Sabin elements. A triangle  $\sigma$  is called **regular** if  $\sigma$  and its any neighboring triangle  $\tau$  form a parallelogram. Otherwise,  $\sigma$  is called **irregular**. All the edges of regular triangles are said to be regular, and all the other edges are said to be irregular.

Let  $\mathcal{T}$  be a quasi-uniform triangulation of a polygonal domain  $\Omega$  in  $\mathbb{R}^2$  with  $V$  being the set of its vertices. We use  $\delta_4(\mathcal{T})$  to denote the collection  $\bigcup_{\sigma \in \mathcal{T}} \delta_4(\sigma)$ . Let  $\mathcal{T}_r$  be the collection of all regular triangles of  $\mathcal{T}$ , and  $\mathcal{T}_i$  the collection of all irregular triangles of  $\mathcal{T}$ . Let

$$\mathcal{T}^* := \left( \bigcup_{\sigma \in \mathcal{T}_r} \delta_6(\sigma) \right) \cup \left( \bigcup_{\sigma \in \mathcal{T}_i} \delta_{12}(\sigma) \right).$$

Figure 2 shows a triangulation of a pentagon.

Let  $\mathcal{T}$  denote the triangulation, and let  $\mathcal{T}_1 = \delta_4(\mathcal{T})$  and  $\mathcal{T}_2 = \delta_4(\mathcal{T}_1)$ . A triangle of  $\mathcal{T}_2$  is irregular only if one of its edges is a part of an interior edge of  $\mathcal{T}$ . These triangles are shown as shaded. All the other triangles of  $\mathcal{T}_2$  are regular.

A spline  $s \in S_2^1(\mathcal{T}^*)$  is said to be **admissible** if for every irregular edge  $e$  of  $\mathcal{T}$ , the normal derivative of  $s$  at its middle point is equal to the average of the normal derivatives of  $s$  at its two end points. Let  $\tilde{S}_2^1(\mathcal{T})$  denote the linear space of all admissible splines in  $S_2^1(\mathcal{T}^*)$ . Hermite interpolation is permitted in the space  $\tilde{S}_2^1(\mathcal{T})$ . More precisely, for given sequences  $(c_{v0})_{v \in V}$ ,  $(c_{v1})_{v \in V}$ , and  $(c_{v2})_{v \in V}$  of real numbers, there exists a unique spline  $s \in \tilde{S}_2^1(\mathcal{T})$  such that

$$s(v) = c_{v0}, \quad D_1 s(v) = c_{v1}, \quad D_2 s(v) = c_{v2}, \quad \forall v \in V.$$

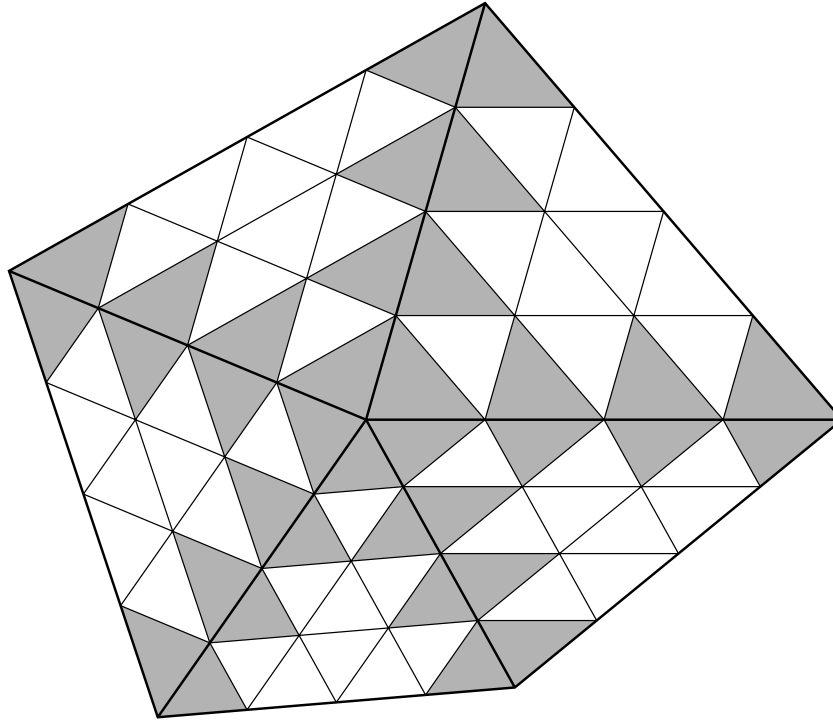


FIGURE 2. A triangulation of a pentagon

For a vertex  $v$  of  $\mathcal{T}$ , let  $\varphi_{v,0}$ ,  $\varphi_{v,1}$ , and  $\varphi_{v,2}$  be the unique splines in  $\tilde{S}_2^1(\mathcal{T})$  such that

$$\begin{bmatrix} \varphi_{v,0}(v) & D_1\varphi_{v,0}(v) & D_2\varphi_{v,0}(v) \\ \varphi_{v,1}(v) & D_1\varphi_{v,1}(v) & D_2\varphi_{v,1}(v) \\ \varphi_{v,2}(v) & D_1\varphi_{v,2}(v) & D_2\varphi_{v,2}(v) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\varphi_{v,i}(w) = 0, \quad D_1\varphi_{v,i}(w) = 0, \quad D_2\varphi_{v,i}(w) = 0,$$

for all  $w \in V \setminus \{v\}$  and  $i = 0, 1, 2$ . Clearly, each  $\varphi_{v,i}$  is supported on  $\text{St}_{\mathcal{T}}(v)$ . For  $v \in V$  and  $i = 0, 1, 2$ , let

$$\phi_{v,i} := \varphi_{v,i} / \|\varphi_{v,i}\|_{2,\Omega}.$$

Thus, each  $\phi_{v,i}$  is so normalized that  $\|\phi_{v,i}\|_{2,\Omega} = 1$ .

Suppose  $\sigma$  is a triangle in  $\mathcal{T}$  with three vertices  $v_0$ ,  $v_1$ , and  $v_2$ . For  $i, j = 0, 1, 2$ , let

$$\phi_{v_i,\sigma,j} := \phi_{v_i,j}|_{\sigma}.$$

Clearly,  $\phi_{v_i,\sigma,j}$  ( $i, j = 0, 1, 2$ ) are linearly independent functions in  $L_2(\sigma)$ . Hence, there exist unique functions  $\tilde{\phi}_{v_k,\sigma,l}$  ( $k, l = 0, 1, 2$ ) in the linear span of  $\{\phi_{v_i,\sigma,j} : i, j = 0, 1, 2\}$  such that

$$\langle \tilde{\phi}_{v_k,\sigma,l}, \phi_{v_i,\sigma,j} \rangle = \begin{cases} 1 & \text{if } k = i \text{ and } l = j, \\ 0 & \text{otherwise.} \end{cases}$$

For  $v \in V$  and  $i \in \{0, 1, 2\}$ , we define  $\tilde{\phi}_{v,i}$  to be the function on  $\Omega$  such that for each triangle  $\sigma$  of  $\mathcal{T}$ ,

$$\tilde{\phi}_{v,i}|_{\sigma} := \frac{1}{n_v} \tilde{\phi}_{v,\sigma,i},$$

where  $n_v$  is the number of triangles incident to  $v$ . Then we have

$$(3.1) \quad \langle \tilde{\phi}_{v,i}, \phi_{w,j} \rangle = \begin{cases} 1 & \text{if } v = w \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\tilde{\phi}_{v,i}$  ( $i = 0, 1, 2$ ) are piecewise quadratic polynomials and well defined except for the points lying on the edges of  $\mathcal{T}$ . Moreover,  $\tilde{\phi}_{v,i}$  ( $i = 0, 1, 2$ ) are supported on  $\text{St}_{\mathcal{T}}(v)$ . These functions are not necessarily continuous, but they belong to  $L_p(\Omega)$  for  $1 \leq p \leq \infty$ .

Suppose that  $\mathcal{T}$  is a quasi-uniform triangulation satisfying the minimum angle condition. By Lemma 2.1 we see that there exist positive constants  $K_1$  and  $K_2$  such that

$$(3.2) \quad K_1 h^{2/p-1} \leq \|\phi_{v,i}\|_{p,\Omega} \leq K_2 h^{2/p-1} \quad \text{and} \quad K_1 h^{2/p-1} \leq \|\tilde{\phi}_{v,i}\|_{p,\Omega} \leq K_2 h^{2/p-1},$$

where  $h$  is the mesh size of  $\mathcal{T}$ .

**Lemma 3.1.** *For  $1 \leq p \leq \infty$ ,  $\{h^{1-2/p}\phi_{v,i} : v \in V, i = 0, 1, 2\}$  forms a stable basis for  $\tilde{S}_2^1(\mathcal{T}) \cap L_p(\Omega)$ , i.e., any  $f \in \tilde{S}_2^1(\mathcal{T}) \cap L_p(\Omega)$  can be represented as  $f = \sum_{v \in V} \sum_{i=0}^2 a_{v,i} \phi_{v,i}$  for some  $a_{v,i} \in \mathbb{R}$  ( $v \in V$  and  $i = 0, 1, 2$ ), and*

$$A h^{2/p-1} \left( \sum_{v \in V} \sum_{i=0}^2 |a_{v,i}|^p \right)^{1/p} \leq \|f\|_{p,\Omega} \leq B h^{2/p-1} \left( \sum_{v \in V} \sum_{i=0}^2 |a_{v,i}|^p \right)^{1/p},$$

where  $A$  and  $B$  are positive constants independent of  $h$  and  $f$ .

*Proof.* Suppose that  $(a_{v,i})_{v \in V, i=0,1,2}$  is a sequence of real numbers with the property that  $(\sum_{v \in V} \sum_{i=0}^2 |a_{v,i}|^p)^{1/p} < \infty$ . Since each  $\phi_{v,i}$  ( $i = 0, 1, 2$ ) is supported on  $\text{St}_{\mathcal{T}}(v)$  and satisfies the inequalities in (3.2), by Lemma 3.2 of [18] we see that there exists a positive constant  $B$  such that

$$\left\| \sum_{v \in V} \sum_{i=0}^2 a_{v,i} \phi_{v,i} \right\|_{p,\Omega} \leq B h^{2/p-1} \left( \sum_{v \in V} \sum_{i=0}^2 |a_{v,i}|^p \right)^{1/p}.$$

On the other hand, suppose  $f \in \tilde{S}_2^1(\mathcal{T}) \cap L_p(\Omega)$ . Then  $f = \sum_{v \in V} \sum_{i=0}^2 a_{v,i} \phi_{v,i}$  for some  $a_{v,i} \in \mathbb{R}$  ( $v \in V$  and  $i = 0, 1, 2$ ). By (3.1) we have  $a_{v,i} = \langle f, \tilde{\phi}_{v,i} \rangle$  for  $v \in V$  and  $i = 0, 1, 2$ . By Lemma 3.1 of [18], there exists a positive constant  $C$  such that

$$\left( \sum_{v \in V} \sum_{i=0}^2 |a_{v,i}|^p \right)^{1/p} \leq C h^{1-2/p} \|f\|_{p,\Omega}.$$

This shows that  $\{h^{1-2/p}\phi_{v,i} : v \in V, i = 0, 1, 2\}$  forms a stable basis for  $\tilde{S}_2^1(\mathcal{T}) \cap L_p(\Omega)$ . □

By Lemma 2.2, there exists a constant  $C > 0$  such that  $|\phi_{v,i}|_{B_{p,p}^{\mu}} \leq C h^{2/p-1-\mu}$  for  $0 < \mu < 2 + 1/p$ . With the help of Lemma 3.3 of [18], we deduce from Lemma 3.1 that  $\tilde{S}_2^1(\mathcal{T}) \cap L_p(\Omega) \subset B_{p,p}^{\mu}(\Omega)$  for  $0 < \mu < 2 + 1/p$ .

Let  $Q$  be the linear operator given by

$$Qf := \sum_{v \in V} \sum_{i=0}^2 \langle f, \tilde{\phi}_{v,i} \rangle \phi_{v,i},$$

where  $f$  is a locally integrable function on  $\Omega$ . In light of (3.1) we see that  $Q$  is a projection onto  $\tilde{S}_2^1(\mathcal{T})$ . In particular,  $Qq = q$  for  $q \in \Pi_2|_\Omega$ . Hence, the following Jackson type inequality is valid. See [17] for details of the proof.

**Lemma 3.2.** *If  $f \in L_p(\Omega)$  for  $1 \leq p < \infty$  or  $f \in C(\Omega)$  for  $p = \infty$ , then*

$$\|f - Qf\|_{p,\Omega} \leq C\omega_3(f, h)_p,$$

where  $C$  is a constant independent of  $h$  and  $f$ .

#### 4. REFINABLE SPACES OF $C^1$ QUADRATIC SPLINES

In this section we discuss multiresolution analysis of Besov spaces by using a sequence of refinable spaces of  $C^1$  quadratic splines. Then we establish norm equivalence for Besov spaces on the basis of multilevel decomposition.

Let  $\Omega$  be a polygonal domain in  $\mathbb{R}^2$ , and let  $\mathcal{T}$  be a quasi-uniform triangulation of  $\Omega$ . We assume that the mesh size of  $\mathcal{T}$  is 1.

Let  $\mathcal{T}_0 := \mathcal{T}$  and  $\mathcal{T}_k := \delta_4(\mathcal{T}_{k-1})$  for  $k = 1, 2, \dots$ . Clearly, the mesh size of  $\mathcal{T}_k$  is  $2^{-k}$ . Also,  $\mathcal{T}_{k+1}^*$  is a subdivision of  $\mathcal{T}_k^*$ . Suppose  $f \in \tilde{S}_2^1(\mathcal{T}_k)$  and  $e$  is an edge of  $\mathcal{T}_{k+1}$ . Then the normal derivative of  $f$  on  $e$  is a linear function. Hence, the normal derivative of  $f$  at the middle point of  $e$  is equal to the average of the normal derivatives of  $f$  at its two end points. This shows that  $\tilde{S}_2^1(\mathcal{T}_k) \subset \tilde{S}_2^1(\mathcal{T}_{k+1})$  for  $k \in \mathbb{N}_0$ .

Suppose  $1 \leq p \leq \infty$ . Fix  $p$  for the time being. In what follows,  $\|\cdot\|_{p,\Omega}$  will be abbreviated as  $\|\cdot\|_p$ . Let

$$(4.1) \quad F_k := \tilde{S}_2^1(\mathcal{T}_k) \cap L_p(\Omega), \quad k = 0, 1, 2, \dots$$

For  $0 < \mu < 2 + 1/p$  we have

$$F_0 \subset F_1 \subset F_2 \subset \dots \subset B_{p,p}^\mu(\Omega).$$

It will be shown later that  $\bigcup_{k=0}^\infty F_k$  is dense in  $B_{p,p}^\mu(\Omega)$  for  $1 \leq p < \infty$ .

For each  $k \in \mathbb{N}_0$ , let  $V_k$  denote the set of vertices of  $\mathcal{T}_k$ , and let  $E_k$  denote the set of edges of  $\mathcal{T}_k$ . For  $v \in V_k$  and  $i = 0, 1, 2$ , let  $\varphi_{v,i,k}$  be the unique spline in  $\tilde{S}_2^1(\mathcal{T}_k)$  such that

$$(4.2) \quad \begin{bmatrix} \varphi_{v,0,k}(v) & D_1\varphi_{v,0,k}(v) & D_2\varphi_{v,0,k}(v) \\ \varphi_{v,1,k}(v) & D_1\varphi_{v,1,k}(v) & D_2\varphi_{v,1,k}(v) \\ \varphi_{v,2,k}(v) & D_1\varphi_{v,2,k}(v) & D_2\varphi_{v,2,k}(v) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$(4.3) \quad \varphi_{v,i,k}(w) = 0, \quad D_1\varphi_{v,i,k}(w) = 0, \quad D_2\varphi_{v,i,k}(w) = 0, \quad i = 0, 1, 2, \quad w \in V_k \setminus \{v\}.$$

For  $k \in \mathbb{N}_0$ ,  $v \in V_k$ , and  $i = 0, 1, 2$ , let

$$\phi_{v,i,k} := \varphi_{v,i,k} / \|\varphi_{v,i,k}\|_2.$$

**Lemma 4.1.** *Suppose  $1 \leq p \leq \infty$  and  $0 < \mu < 2+1/p$ . Let  $f_k \in F_k$ ,  $k = 0, 1, \dots, n$ . Then there exists a positive constant  $C$  independent of  $n$  such that*

$$\left\| \sum_{k=0}^n f_k \right\|_{B_{p,p}^\mu(\Omega)} \leq C \left( \sum_{k=0}^n [2^{k\mu} \|f_k\|_p]^p \right)^{1/p}.$$

*Proof.* Each  $f_k$  may be represented as  $f_k = \sum_{v \in V_k} \sum_{i=0}^2 c_{v,i,k} \phi_{v,i,k}$ , where  $c_{v,i,k}$  are real coefficients. By Lemma 2.1, there exist positive constants  $K_1$  and  $K_2$  such that

$$K_1(2^{-k})^{2/p-1} \leq \|\phi_{v,i,k}\|_p \leq K_2(2^{-k})^{2/p-1}.$$

Moreover, by Lemma 2.2, there exists a positive constant  $K$  such that

$$|\phi_{v,i,k}|_{B_{p,p}^\mu(\Omega)} \leq K(2^{-k})^{-\mu}(2^{-k})^{2/p-1}.$$

By Theorem 5.1 of [18],  $(\phi_{v,i,k})_{k \in \mathbb{N}_0, v \in V_k, i=0,1,2}$  is a Bessel sequence in  $B_{p,p}^\mu(\Omega)$  with respect to the semi-norm. More precisely, there exists a constant  $C_1 > 0$  such that

$$\left| \sum_{k=0}^n \sum_{v \in V_k} \sum_{i=0}^2 c_{v,i,k} \phi_{v,i,k} \right|_{B_{p,p}^\mu(\Omega)} \leq C_1 \left( \sum_{k=0}^n \sum_{v \in V_k} \sum_{i=0}^2 |2^{k\mu}(2^{-k})^{2/p-1} c_{v,i,k}|^p \right)^{1/p}.$$

On the other hand, Lemma 3.1 tells us that

$$C_2(2^{-k})^{2/p-1} \left( \sum_{v \in V_k} \sum_{i=0}^2 |c_{v,i,k}|^p \right)^{1/p} \leq \|f_k\|_p,$$

where  $C_2$  is a constant independent of  $k$ . Consequently, there exists a positive constant  $C$  independent of  $n$  such that

$$\left| \sum_{k=0}^n f_k \right|_{B_{p,p}^\mu(\Omega)} \leq C \left( \sum_{k=0}^n [2^{k\mu} \|f_k\|_p]^p \right)^{1/p}.$$

Furthermore,

$$\begin{aligned} \left\| \sum_{k=0}^n f_k \right\|_p &\leq \sum_{k=0}^n \|f_k\|_p = \sum_{k=0}^n (2^{-k\mu}) (2^{k\mu} \|f_k\|_p) \\ &\leq \left( \sum_{k=0}^n (2^{-k\mu})^q \right)^{1/q} \left( \sum_{k=0}^n [2^{k\mu} \|f_k\|_p]^p \right)^{1/p}, \end{aligned}$$

where  $1/q + 1/p = 1$  and Hölder’s inequality has been used to derive the last inequality. Combining the above two estimates together, we establish the desired inequality.  $\square$

It was shown in §3 that there exist functions  $\tilde{\phi}_{v,i,k} \in S_2(\mathcal{T}_k^*) \cap L_p(\Omega)$  such that

$$\langle \tilde{\phi}_{v,i,k}, \phi_{w,j,k} \rangle = \begin{cases} 1 & \text{if } v = w \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Each  $\tilde{\phi}_{v,i,k}$  is supported on  $\text{St}_{\mathcal{T}_k}(v)$ . Moreover, there exist two positive constants  $K_1$  and  $K_2$  such that

$$K_1(2^{-k})^{2/p-1} \leq \|\tilde{\phi}_{v,i,k}\|_p \leq K_2(2^{-k})^{2/p-1}.$$

Let  $Q_k$  be the linear operator given by

$$(4.4) \quad Q_k f := \sum_{v \in V_k} \sum_{i=0}^2 \langle f, \tilde{\phi}_{v,i,k} \rangle \phi_{v,i,k},$$

where  $f$  is a locally integrable function on  $\Omega$ . Then  $Q_k$  is a linear projection onto  $F_k$ . If  $f \in L_p(\Omega)$  for  $1 \leq p < \infty$  or  $f \in C(\Omega)$  for  $p = \infty$ , then it follows from Lemma 3.2 that

$$\|f - Q_k f\|_p \leq C \omega_3(f, 2^{-k})_p,$$

where  $C$  is a constant independent of  $k$  and  $f$ .

**Lemma 4.2.** *Suppose  $f \in B_{p,p}^\mu(\Omega)$ , where  $1 \leq p < \infty$  and  $0 < \mu < 2 + 1/p$ . Then*

$$(4.5) \quad \lim_{n \rightarrow \infty} \|Q_n f - f\|_{B_{p,p}^\mu(\Omega)} = 0.$$

Consequently,  $\bigcup_{n=0}^\infty F_n$  is dense in  $B_{p,p}^\mu(\Omega)$ .

*Proof.* There exists a positive constant  $C_1$  such that

$$\|Q_{k+1} f - Q_k f\|_p \leq \|Q_{k+1} f - f\|_p + \|f - Q_k f\|_p \leq C_1 \omega_3(f, 2^{-k})_p \quad \forall k \in \mathbb{N}_0.$$

It follows that

$$(4.6) \quad \left( \sum_{k=0}^n [2^{k\mu} \|Q_{k+1} f - Q_k f\|_p]^p \right)^{1/p} \leq C_1 \left( \sum_{k=0}^n [2^{k\mu} \omega_3(f, 2^{-k})_p]^p \right)^{1/p} \leq C_1 \|f\|_{B_{p,p}^\mu(\Omega)}.$$

In particular, if  $f \in B_{p,p}^\mu(\Omega)$  for  $1 \leq p < \infty$ , then  $\sum_{k=0}^\infty [2^{k\mu} \|Q_{k+1} f - Q_k f\|_p]^p$  is a convergent series. Suppose  $0 \leq m < n < \infty$ . By Lemma 4.1 we have

$$(4.7) \quad \|Q_n f - Q_m f\|_{B_{p,p}^\mu(\Omega)} \leq C_2 \left( \sum_{k=m}^{n-1} [2^{k\mu} \|Q_{k+1} f - Q_k f\|_p]^p \right)^{1/p},$$

where  $C_2$  is a positive constant independent of  $n$ . Hence,

$$\lim_{m,n \rightarrow \infty} \|Q_n f - Q_m f\|_{B_{p,p}^\mu(\Omega)} = 0.$$

In other words,  $(Q_n f)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $B_{p,p}^\mu(\Omega)$ . Consequently, there exists some  $g \in B_{p,p}^\mu(\Omega)$  such that

$$\lim_{n \rightarrow \infty} \|Q_n f - g\|_{B_{p,p}^\mu(\Omega)} = 0;$$

but  $\lim_{n \rightarrow \infty} \|Q_n f - f\|_p = 0$ . Therefore,  $g = f$ . This proves (4.5). It follows that  $\bigcup_{n=0}^\infty F_n$  is dense in  $B_{p,p}^\mu(\Omega)$ .  $\square$

In [11], DeVore, Jawerth, and Popov investigated norm equivalence based on wavelet decomposition. In [28] Oswald further considered norm equivalence for finite element spaces. The following result on norm equivalence is pertinent to our purpose.

**Lemma 4.3.** *Suppose  $1 \leq p < \infty$  and  $0 < \mu < 2 + 1/p$ . Then the norm  $\|f\|_{B_{p,p}^\mu(\Omega)}$  is equivalent to*

$$\|Q_0 f\|_p + \left( \sum_{k=0}^\infty [2^{k\mu} \|Q_{k+1} f - Q_k f\|_p]^p \right)^{1/p}.$$

*Proof.* Suppose  $f \in B_{p,p}^\mu(\Omega)$ . By (4.5) and (4.7) there exists a positive constant  $C_1$  independent of  $f$  such that

$$(4.8) \quad \begin{aligned} \|f - Q_0f\|_{B_{p,p}^\mu(\Omega)} &= \lim_{n \rightarrow \infty} \|Q_n f - Q_0f\|_{B_{p,p}^\mu(\Omega)} \\ &\leq C_1 \left( \sum_{k=0}^{\infty} [2^{k\mu} \|Q_{k+1}f - Q_kf\|_p]^p \right)^{1/p}. \end{aligned}$$

This in connection with (4.6) gives

$$\|f - Q_0f\|_{B_{p,p}^\mu(\Omega)} \leq C_2 \|f\|_{B_{p,p}^\mu(\Omega)},$$

where  $C_2$  is a positive constant independent of  $f$ . Moreover,

$$\|Q_0f\|_p \leq \|f\|_p + \|Q_0f - f\|_p.$$

Therefore, there exists a positive constant  $C_3$  such that

$$\|Q_0f\|_p + \left( \sum_{k=0}^{\infty} [2^{k\mu} \|Q_{k+1}f - Q_kf\|_p]^p \right)^{1/p} \leq C_3 \|f\|_{B_{p,p}^\mu(\Omega)} \quad \forall f \in B_{p,p}^\mu(\Omega).$$

On the other hand,

$$\|f\|_{B_{p,p}^\mu(\Omega)} \leq \|Q_0f\|_{B_{p,p}^\mu(\Omega)} + \|f - Q_0f\|_{B_{p,p}^\mu(\Omega)}.$$

By Lemma 4.1, there exists a positive constant  $C_4$  such that

$$\|Q_0f\|_{B_{p,p}^\mu(\Omega)} \leq C_4 \|Q_0f\|_p.$$

This together with (4.8) tells us that there exists a constant  $C > 0$  such that

$$\|f\|_{B_{p,p}^\mu(\Omega)} \leq C \left[ \|Q_0f\|_p + \left( \sum_{k=0}^{\infty} [2^{k\mu} \|Q_{k+1}f - Q_kf\|_p]^p \right)^{1/p} \right].$$

The proof of the lemma is complete. □

### 5. STABLE WAVELET BASES

The purpose of this section is to develop a general theory for wavelet bases in Besov spaces. Let us begin with a useful inequality related to Banach spaces.

Let  $X$  be a Banach space with norm  $\|\cdot\|$ . Let  $F_0, F_1, \dots, F_n$  be closed subspaces of  $X$  such that

$$F_0 \subseteq F_1 \subseteq \dots \subseteq F_n.$$

For  $j = 0, 1, \dots, n$ , let  $R_j$  be a linear projection from  $F_n$  onto  $F_j$ . Set  $R_{-1} := 0$ .

**Lemma 5.1.** *Suppose there exist some  $\nu > 0$  and a constant  $K > 0$  such that for  $k \geq j$ ,*

$$(5.1) \quad \|R_j f\| \leq K 2^{\nu(k-j)} \|f\| \quad \forall f \in F_k.$$

*If  $f = \sum_{k=0}^n f_k$  with  $f_k \in F_k$ ,  $k = 0, 1, \dots, n$ , then for  $\mu > \nu$  and  $1 \leq p \leq \infty$ ,*

$$(5.2) \quad \left( \sum_{j=0}^n [2^{\mu j} \|(R_j - R_{j-1})f\|]^p \right)^{1/p} \leq C \left( \sum_{k=0}^n [2^{\mu k} \|f_k\|]^p \right)^{1/p},$$

*where  $C$  is a constant depending only on  $K$ ,  $\mu$  and  $\nu$ .*

*Proof.* It follows from (5.1) that

$$\|(R_j - R_{j-1})f\| \leq \|R_j f\| + \|R_{j-1} f\| \leq K(1 + 2^\nu)2^{\nu(k-j)}\|f\| \quad \forall f \in F_k.$$

For  $k < j$  we have  $(R_j - R_{j-1})f_k = 0$ . Hence,

$$(R_j - R_{j-1})f = \sum_{k=j}^n (R_j - R_{j-1})f_k.$$

Consequently,

$$2^{\mu j} \|(R_j - R_{j-1})f\| \leq K(1 + 2^\nu) \sum_{k=j}^n 2^{-(k-j)(\mu-\nu)} [2^{k\mu} \|f_k\|].$$

Note that

$$\sum_{k=j}^n 2^{-(k-j)(\mu-\nu)} \leq \frac{1}{1 - 2^{-(\mu-\nu)}}.$$

Therefore, (5.2) is valid with  $C := K(1 + 2^\nu)/(1 - 2^{-(\mu-\nu)})$ . See Lemma 4.1 of [18].  $\square$

Let  $\Omega$  be a polygonal domain in  $\mathbb{R}^2$ , and let  $\mathcal{T}$  be a quasi-uniform triangulation of  $\Omega$  satisfying the minimum angle condition. We assume that the mesh size of  $\mathcal{T}$  is 1. For  $k = 0, 1, 2, \dots$ , let  $F_k$  be the space as defined in (4.1). We have

$$F_0 \subset F_1 \subset F_2 \subset \dots \subset B_{p,p}^\mu(\Omega),$$

where  $1 \leq p < \infty$  and  $0 < \mu < 2 + 1/p$ . Moreover,  $\bigcup_{k=0}^\infty F_k$  is dense in  $B_{p,p}^\mu(\Omega)$ .

Suppose  $P_k$  is a linear projection from  $F_k$  onto  $F_{k-1}$  ( $k = 1, 2, \dots$ ) and  $P_0 = 0$ . The operator norm  $\|P_k\|$  is induced from the  $L_p$  norm on  $F_k$ . Let  $G_k := \ker P_k$ ,  $k = 0, 1, 2, \dots$ . Then  $F_0 = G_0$  and  $F_k = F_{k-1} + G_k$ ,  $k = 1, 2, \dots$

**Lemma 5.2.** *If there exists some  $\nu < \mu$  and a positive constant  $K$  such that*

$$\|P_{k+1} \cdots P_l\| \leq K2^{\nu(l-k)}$$

*for all integers  $k$  and  $l$  with  $0 \leq k < l$ , then there exists a positive constant  $C$  independent of  $n$  such that*

$$(5.3) \quad \left( \sum_{k=0}^n [2^{k\mu} \|g_k\|_p]^p \right)^{1/p} \leq C \left\| \sum_{k=0}^n g_k \right\|_{B_{p,p}^\mu(\Omega)},$$

*provided  $g_k \in G_k$  for  $k = 0, 1, \dots, n$ .*

*Proof.* Let  $n \in \mathbb{N}$  be fixed. For  $k = 0, 1, \dots, n - 1$ , define  $R_k := P_{k+1} \cdots P_n$ , and let  $R_n$  be the identity mapping on  $F_n$ . For  $f \in F_n$ , set  $f_0 := Q_0 f$  and  $f_k := Q_k f - Q_{k-1} f$  for  $k = 1, \dots, n$ . Then  $f = \sum_{k=0}^n f_k$  with  $f_k \in F_k$  for each  $k$ . By Lemma 4.3, there exists a positive constant  $C_1$  independent of  $n$  such that

$$\left( \sum_{k=0}^n [2^{\mu k} \|f_k\|]^p \right)^{1/p} \leq C_1 \|f\|_{B_{p,p}^\mu(\Omega)}.$$

This in connection with (5.2) gives

$$\left( \sum_{k=0}^n [2^{\mu k} \|(R_k - R_{k-1})f\|]^p \right)^{1/p} \leq C \|f\|_{B_{p,p}^\mu(\Omega)} \quad \forall f \in F_n,$$



where  $R_{-1} := 0$ , and  $C$  is a constant independent of  $n$ . Now suppose  $f = \sum_{k=0}^n g_k$ , where  $g_k \in G_k$ . Then  $(R_k - R_{k-1})f = g_k$ ,  $k = 0, 1, \dots, n$ . Consequently, (5.3) follows from the above inequality at once.  $\square$

Suppose  $\psi_{jk} \in G_k$  for  $j \in J_k$ ,  $k = 0, 1, \dots$ . Each  $\psi_{jk}$  is supported in  $\text{St}_{\mathcal{T}_k}^N(v_{jk})$  for some vertex  $v_{jk}$  of  $\mathcal{T}_k$ , where  $N \in \mathbb{N}$  is independent of  $j$  and  $k$ . The functions  $\psi_{jk}$  are so normalized that  $\|\psi_{jk}\|_p = 2^{-k\mu}$ . Suppose  $\{2^{k\mu}\psi_{jk} : j \in J_k\}$  forms a stable basis for the space  $G_k$  equipped with the  $L_p$  norm. In other words, there exist two positive constants  $C_1$  and  $C_2$  such that the inequalities

$$(5.4) \quad C_1 2^{-k\mu} \left[ \sum_{j \in J_k} |c_{jk}|^p \right]^{1/p} \leq \left\| \sum_{j \in J_k} c_{jk} \psi_{jk} \right\|_p \leq C_2 2^{-k\mu} \left[ \sum_{j \in J_k} |c_{jk}|^p \right]^{1/p}$$

hold true for all sequences  $(c_{jk})_{j \in J_k}$  with  $\sum_{j \in J_k} |c_{jk}|^p < \infty$ .

**Theorem 5.3.** *Suppose  $P_k$  is a linear projection from  $F_k$  onto  $F_{k-1}$  ( $k = 1, 2, \dots$ ) and  $P_0 = 0$ . If there exists some  $\nu < \mu$  and a positive constant  $K$  such that*

$$\|P_{k+1} \cdots P_l\| \leq K 2^{\nu(l-k)}$$

for all integers  $k$  and  $l$  with  $0 \leq k < l$ , then there exist two positive constants  $A$  and  $B$  such that

$$(5.5) \quad A \left( \sum_{k=0}^{\infty} \sum_{j \in J_k} |c_{jk}|^p \right)^{1/p} \leq \left\| \sum_{k=0}^{\infty} \sum_{j \in J_k} c_{jk} \psi_{jk} \right\|_{B_{p,p}^{\mu}(\Omega)} \leq B \left( \sum_{k=0}^{\infty} \sum_{j \in J_k} |c_{jk}|^p \right)^{1/p}.$$

*Proof.* Since  $\|\psi_{jk}\|_p = 2^{-k\mu}$ , there exists a constant  $K$  such that  $\|\psi_{jk}\|_{B_{p,p}^{\mu}(\Omega)} \leq K$ , by Lemma 2.2. The second inequality in (5.5) can be easily derived from Theorem 5.1 of [18].

For  $k = 0, 1, \dots$ , let  $g_k := \sum_{j \in J_k} c_{jk} \psi_{jk}$ . By Lemma 5.2, there exists a positive constant  $C$  independent of  $n$  such that

$$\left( \sum_{k=0}^n [2^{k\mu} \|g_k\|_p]^p \right)^{1/p} \leq C \left\| \sum_{k=0}^n \sum_{j \in J_k} c_{jk} \psi_{jk} \right\|_{B_{p,p}^{\mu}(\Omega)}.$$

This in connection with (5.4) verifies the first inequality in (5.5).  $\square$

### 6. CONSTRUCTION OF WAVELETS

We are in a position to construct  $C^1$  wavelet bases on general triangulations. In particular, for the three-direction mesh, the wavelets will be given explicitly.

Let  $E_k$  be the set of edges of  $\mathcal{T}_k$ . For  $f, g \in F_k$  and  $e = [v_1, v_2] \in E_k$ , define

$$(6.1) \quad \langle f, g \rangle_e := 2^{-2k} [f(v_1)g(v_1) + D_e f(v_1)D_e g(v_1) + f(v_2)g(v_2) + D_e f(v_2)D_e g(v_2)].$$

Accordingly,  $\|f\|_e^2 = \langle f, f \rangle_e$ . Furthermore, define

$$(6.2) \quad \langle f, g \rangle_{\mathcal{T}_k} := \sum_{e \in E_k} \langle f, g \rangle_e.$$

Consequently,

$$\|f\|_{\mathcal{T}_k}^2 = \sum_{e \in E_k} \|f\|_e^2.$$

It is easily seen from Lemma 3.1 that the norm  $\|f\|_{\mathcal{T}_k}$  is equivalent to the norm  $\|f\|_{2,\Omega}$ .

For  $k \in \mathbb{N}$ , let  $P_k$  be the orthogonal projection from  $F_k$  to  $F_{k-1}$  with respect to the inner product given in (6.2). Thus, for  $f \in F_k$ ,  $P_k f \in F_{k-1}$  satisfies the following condition:

$$\langle f - P_k f, g \rangle_{\mathcal{T}_k} = 0 \quad \forall g \in F_{k-1}.$$

Consequently,

$$\|P_k f\|_{\mathcal{T}_k} \leq \|f\|_{\mathcal{T}_k} \quad \forall f \in F_k.$$

**Lemma 6.1.** *There exists a positive constant  $\lambda_0 < 4$  such that*

$$(6.3) \quad \|P_k f\|_{\mathcal{T}_{k-1}} \leq \lambda_0 \|f\|_{\mathcal{T}_k} \quad \forall f \in F_k.$$

*Proof.* We have

$$\frac{\|P_k f\|_{\mathcal{T}_{k-1}}}{\|f\|_{\mathcal{T}_k}} \leq \frac{\|P_k f\|_{\mathcal{T}_{k-1}}}{\|P_k f\|_{\mathcal{T}_k}}.$$

Thus, it suffices to estimate  $\|g\|_{\mathcal{T}_{k-1}}/\|g\|_{\mathcal{T}_k}$  for  $g \in F_{k-1}$ .

Recall that

$$\|g\|_{\mathcal{T}_{k-1}}^2 = \sum_{e \in E_{k-1}} \|g\|_e^2.$$

Suppose  $e = [v_1, v_2] \in E_{k-1}$ . Let  $v_0 := (v_1 + v_2)/2$  be the middle point of  $e$ . Then  $e = e_1 \cup e_2$ , where  $e_1 := [v_1, v_0]$  and  $e_2 := [v_0, v_2]$  are two edges of  $\mathcal{T}_k$ . Let us estimate

$$\frac{\|g\|_{e_1}^2 + \|g\|_{e_2}^2}{\|g\|_e^2}.$$

Note that  $D_{e_1} = D_{e_2}$ . Let

$$a_i := g(v_i) \quad \text{and} \quad b_i = D_{e_1} g(v_i), \quad i = 0, 1, 2.$$

For  $0 \leq t \leq 1$ , let  $q(t) := g((1-t)v_1 + tv_2)$ . Then  $q$  is a quadratic polynomial on  $[0, 1]$ . Suppose  $q(t) = \alpha t^2 + \beta t + \gamma$ , where  $\alpha, \beta, \gamma \in \mathbb{R}$ . Clearly,

$$2q(0) + q'(0) = 2\gamma + \beta = 2q(1) - q'(1).$$

But  $q(0) = g(v_1) = a_1$  and  $q'(0) = D_{e_1} g(v_1) = b_1$ . Similarly,  $q(1) = g(v_2) = a_0$  and  $q'(1) = D_{e_1} g(v_2) = b_0$ . Consequently,

$$2a_1 + b_1 = 2a_0 - b_0.$$

Similarly,

$$2a_0 + b_0 = 2a_2 - b_2.$$

It follows that

$$a_0^2 + b_0^2 \geq a_0^2 + \frac{1}{4} b_0^2 = \frac{1}{2} [(a_0 - b_0/2)^2 + (a_0 + b_0/2)^2] = \frac{1}{2} [(a_1 + b_1/2)^2 + (a_2 - b_2/2)^2].$$

Therefore, we obtain

$$\begin{aligned} \|g\|_{e_1}^2 + \|g\|_{e_2}^2 &= 2^{-2k}(a_1^2 + b_1^2 + 2a_0^2 + 2b_0^2 + a_2^2 + b_2^2) \\ &\geq 2^{-2k}[a_1^2 + b_1^2 + (a_1 + b_1/2)^2 + (a_2 - b_2/2)^2 + a_2^2 + b_2^2]. \end{aligned}$$

Let  $c_1 := D_e g(v_1)$  and  $c_2 = D_e g(v_2)$ . Then  $b_1 = c_1/2$  and  $b_2 = c_2/2$ . We have

$$a_1^2 + b_1^2 + (a_1 + b_1/2)^2 = a_1 + c_1^2/4 + (a_1 + c_1/4)^2 = 2a_1^2 + \frac{1}{2} a_1 c_1 + \frac{5}{16} c_1^2 \geq \rho_0(a_1^2 + c_1^2),$$

where  $\rho_0$  is the minimum of the two eigenvalues of the matrix

$$\begin{bmatrix} 2 & 1/4 \\ 1/4 & 5/16 \end{bmatrix}.$$

A simple computation yields

$$\rho_0 = \frac{37 - \sqrt{793}}{32}.$$

The same argument tells us that

$$a_2^2 + b_2^2 + (a_2 - b_2/2)^2 = 2a_2^2 - \frac{1}{2}a_2c_2 + \frac{5}{16}c_2^2 \geq \rho_0(a_2^2 + c_2^2).$$

Therefore,

$$\|g\|_{e_1}^2 + \|g\|_{e_2}^2 \geq 2^{-2k} \rho_0(a_1^2 + c_1^2 + a_2^2 + c_2^2) = \frac{\rho_0}{4} [2^{-2(k-1)}(a_1^2 + c_1^2 + a_2^2 + c_2^2)] = \frac{\rho_0}{4} \|g\|_e^2.$$

Consequently,

$$\|g\|_{\mathcal{T}_{k-1}} \leq \lambda_0 \|g\|_{\mathcal{T}_k} \quad \forall g \in F_{k-1},$$

where

$$\lambda_0 := \sqrt{\frac{4}{\rho_0}} = \frac{\sqrt{13} + \sqrt{61}}{3} < 4.$$

We conclude that (6.3) is valid for this  $\lambda_0$ . □

For  $k \in \mathbb{N}_0$ ,  $v \in V_k$ , and  $i = 0, 1, 2$ , let  $\varphi_{v,i,k}$  be the unique spline in  $\tilde{S}_2^1(\mathcal{T}_k)$  satisfying (4.2) and (4.3). We have  $\|\varphi_{v,0,k}\|_2 \sim 2^{-k}$  and  $\|\varphi_{v,i,k}\|_2 \sim 2^{-2k}$  for  $i = 1, 2$ .

If  $v, w \in V_k$  and  $v \neq w$ , then  $\langle \varphi_{v,i,k}, \varphi_{w,j,k} \rangle_{\mathcal{T}_k} = 0$ ,  $i, j = 0, 1, 2$ . If  $v \in V_{k-1}$  and  $w \in V_k$ , then  $\langle \varphi_{v,i,k-1}, \varphi_{w,j,k} \rangle_{\mathcal{T}_k} = 0$  unless  $w = v$  or  $w$  is the middle point of an edge in  $\mathcal{T}_{k-1}$  incident to  $v$ . Moreover, since the values and the gradients of  $\varphi_{v,j,k-1}$  and  $\varphi_{v,j,k}$  at  $v$  are equal, we have

$$\langle \varphi_{v,i,k}, \varphi_{v,j,k-1} \rangle_{\mathcal{T}_k} = \langle \varphi_{v,i,k}, \varphi_{v,j,k} \rangle_{\mathcal{T}_k}, \quad i, j = 0, 1, 2.$$

Let us consider the matrix

$$\left( \langle \varphi_{v,i,k}, \varphi_{v,j,k} \rangle_{\mathcal{T}_k} \right)_{i,j=0,1,2}.$$

We have  $\langle \varphi_{v,0,k}, \varphi_{v,0,k} \rangle_{\mathcal{T}_k} = 2^{-2k}$  and  $\langle \varphi_{v,0,k}, \varphi_{v,j,k} \rangle_{\mathcal{T}_k} = 0$  for  $j = 1, 2$ . Suppose  $e_1, \dots, e_N$  ( $N \geq 2$ ) are all the edges in  $\mathcal{T}_k$  incident to  $v$  and  $e_m = (r_m, s_m)$  for  $m = 1, \dots, N$ . Then  $D_{e_m} = r_m D_1 + s_m D_2$ . Consequently,

$$\begin{bmatrix} \langle \varphi_{v,1,k}, \varphi_{v,1,k} \rangle_{\mathcal{T}_k} & \langle \varphi_{v,1,k}, \varphi_{v,2,k} \rangle_{\mathcal{T}_k} \\ \langle \varphi_{v,2,k}, \varphi_{v,1,k} \rangle_{\mathcal{T}_k} & \langle \varphi_{v,2,k}, \varphi_{v,2,k} \rangle_{\mathcal{T}_k} \end{bmatrix} = 2^{-2k} \begin{bmatrix} \sum_{m=1}^N r_m^2 & \sum_{m=1}^N r_m s_m \\ \sum_{m=1}^N r_m s_m & \sum_{m=1}^N s_m^2 \end{bmatrix}.$$

Note that

$$\left| \begin{array}{cc} \sum_{m=1}^N r_m^2 & \sum_{m=1}^N r_m s_m \\ \sum_{m=1}^N r_m s_m & \sum_{m=1}^N s_m^2 \end{array} \right| = \sum_{1 \leq m < n \leq N} (r_m s_n - r_n s_m)^2.$$

Hence, the matrix  $\left( \langle \varphi_{v,i,k}, \varphi_{v,j,k} \rangle_{\mathcal{T}_k} \right)_{i,j=0,1,2}$  is invertible. Moreover, we have

$$(r_1 s_2 - r_2 s_1)^2 = (r_1^2 + s_1^2)(r_2^2 + s_2^2) \sin^2 \theta,$$

where  $\theta$  is the angle between the edges  $e_1 = (r_1, s_1)$  and  $e_2 = (r_2, s_2)$ . Therefore, the inverse of the matrix

$$\begin{bmatrix} \sum_{m=1}^N r_m^2 & \sum_{m=1}^N r_m s_m \\ \sum_{m=1}^N r_m s_m & \sum_{m=1}^N s_m^2 \end{bmatrix}$$

is bounded by  $2^{-2k}K$ , where  $K$  is a constant depending only on the smallest angle of the triangulation  $\mathcal{T}$ .

Let

$$\phi_{v,i,k} := \varphi_{v,i,k} / \|\varphi_{v,i,k}\|_2.$$

It follows that

$$\langle \phi_{v,i,k}, \phi_{v,j,k-1} \rangle_{\mathcal{T}_k} = \frac{\langle \varphi_{v,i,k}, \varphi_{v,j,k-1} \rangle_{\mathcal{T}_k}}{\|\varphi_{v,i,k}\|_2 \|\varphi_{v,j,k-1}\|_2} = \frac{\langle \varphi_{v,i,k}, \varphi_{v,j,k} \rangle_{\mathcal{T}_k}}{\|\varphi_{v,i,k}\|_2 \|\varphi_{v,j,k-1}\|_2}.$$

Consequently, the inverse of the matrix  $(\langle \phi_{v,i,k}, \phi_{v,j,k-1} \rangle_{\mathcal{T}_k})_{i,j=1,2}$  is bounded by a constant depending only on the smallest angle of  $\mathcal{T}$ . Therefore, for each  $v \in V_{k-1}$ , the inverse of the matrix

$$\left( \langle \phi_{v,i,k}, \phi_{v,j,k-1} \rangle_{\mathcal{T}_k} \right)_{i,j=0,1,2}$$

is bounded by a constant depending only on the smallest angle of  $\mathcal{T}$ .

Suppose  $e = [v_0, v_1] \in E_{k-1}$ . Let  $v_e$  be the middle point of  $e$ . Consider the functions  $\psi_{v_e,i,k}$  ( $i = 0, 1, 2$ ) given by

$$(6.4) \quad \begin{bmatrix} \psi_{v_e,0,k} \\ \psi_{v_e,1,k} \\ \psi_{v_e,2,k} \end{bmatrix} = \begin{bmatrix} \phi_{v_e,0,k} \\ \phi_{v_e,1,k} \\ \phi_{v_e,2,k} \end{bmatrix} - B_0 \begin{bmatrix} \phi_{v_0,0,k} \\ \phi_{v_0,1,k} \\ \phi_{v_0,2,k} \end{bmatrix} - B_1 \begin{bmatrix} \phi_{v_1,0,k} \\ \phi_{v_1,1,k} \\ \phi_{v_1,2,k} \end{bmatrix},$$

where  $B_0$  and  $B_1$  are the  $3 \times 3$  matrices such that  $\langle \psi_{v_e,i,k}, \phi_{w,j,k-1} \rangle_{\mathcal{T}_k} = 0$  for all  $w \in V_{k-1}$  and  $i, j = 0, 1, 2$ . In light of the preceding discussion we see that this happens if

$$(6.5) \quad B_0 \left( \langle \phi_{v_0,i,k}, \phi_{v_0,j,k-1} \rangle_{\mathcal{T}_k} \right)_{i,j=0,1,2} = \left( \langle \phi_{v_e,i,k}, \phi_{v_0,j,k-1} \rangle_{\mathcal{T}_k} \right)_{i,j=0,1,2}$$

and

$$(6.6) \quad B_1 \left( \langle \phi_{v_1,i,k}, \phi_{v_1,j,k-1} \rangle_{\mathcal{T}_k} \right)_{i,j=0,1,2} = \left( \langle \phi_{v_e,i,k}, \phi_{v_1,j,k-1} \rangle_{\mathcal{T}_k} \right)_{i,j=0,1,2}.$$

The matrices on the right-hand side of (6.5) and (6.6) have upper bounds independent of  $k$  and the choice of the edge  $e$ ; but the inverses of the matrices  $(\langle \phi_{v_0,i,k}, \phi_{v_0,j,k-1} \rangle_{\mathcal{T}_k})_{i,j=0,1,2}$  and  $(\langle \phi_{v_1,i,k}, \phi_{v_1,j,k-1} \rangle_{\mathcal{T}_k})_{i,j=0,1,2}$  are bounded by a constant depending only on the smallest angle of  $\mathcal{T}$ . Therefore, both  $B_0$  and  $B_1$  have upper bounds independent of  $k$  and the choice of the edge  $e$ .

**Theorem 6.2.** *For  $i = 0, 1, 2$ ,  $k \in \mathbb{N}$  and  $e \in E_{k-1}$ , let  $\psi_{v_e,i,k}$  be the function given in (6.4) with  $B_0$  and  $B_1$  determined by (6.5) and (6.6). Then*

$$(6.7) \quad \{ \phi_{v,i,0} : v \in V_0, i = 0, 1, 2 \} \cup \bigcup_{k=1}^{\infty} \{ 2^{-k\mu} \psi_{v_e,i,k} : e \in E_{k-1}, i = 0, 1, 2 \}$$

is a Riesz basis in  $H^\mu(\Omega)$  for  $\mu_0 < \mu < 5/2$ , where

$$\mu_0 := \log_2 \left( \frac{\sqrt{13} + \sqrt{61}}{3} \right) \approx 1.927.$$

*Proof.* Suppose  $0 \leq k < l$ . By Lemma 6.1 we have

$$\|P_{k+1} \cdots P_l f\|_{\mathcal{T}_k} \leq 2^{\mu_0(l-k)} \|f\|_{\mathcal{T}_l} \quad \forall f \in F_l.$$

Hence, there exists a positive constant  $K$  such that

$$\|P_{k+1} \cdots P_l f\|_2 \leq 2^{\mu_0(l-k)} K \|f\|_2 \quad \forall f \in F_l.$$

Let  $G_k := \ker P_k$ ,  $k = 0, 1, 2, \dots$ . Then  $F_0 = G_0$  and  $F_k = F_{k-1} + G_k$ ,  $k = 1, 2, \dots$ . Clearly,  $\psi_{v_e, i, k} \in G_k$ .

For  $k \in \mathbb{N}_0$ ,  $\{\phi_{v, i, k} : v \in V_k, i = 0, 1, 2\}$  forms a Riesz basis for  $F_k$  equipped with the  $L_2$  norm and the Riesz bounds are independent of  $k$ . Indeed, by Lemma 3.1, there exist two positive constants  $A_1$  and  $A_2$  independent of  $k$  such that

$$A_1 \left( \sum_{v \in V_k} \sum_{i=0}^2 |a_{v, i, k}|^2 \right)^{1/2} \leq \left\| \sum_{v \in V_k} \sum_{i=0}^2 a_{v, i, k} \phi_{v, i, k} \right\|_2 \leq A_2 \left( \sum_{v \in V_k} \sum_{i=0}^2 |a_{v, i, k}|^2 \right)^{1/2}$$

for all real coefficients  $a_{v, i, k}$ . By the construction of the wavelets  $\psi_{v_e, i, k}$  in (6.4), we see that  $\{\psi_{v_e, i, k} : e \in E_{k-1}, i = 0, 1, 2\}$  is a Riesz sequence in  $G_k$ . More precisely, there exist two positive constants  $C_1$  and  $C_2$  independent of  $k$  such that

$$\begin{aligned} C_1 \left( \sum_{e \in E_{k-1}} \sum_{i=0}^2 |c_{v_e, i, k}|^2 \right)^{1/2} &\leq \left\| \sum_{e \in E_{k-1}} \sum_{i=0}^2 c_{v_e, i, k} \psi_{v_e, i, k} \right\|_2 \\ &\leq C_2 \left( \sum_{e \in E_{k-1}} \sum_{i=0}^2 |c_{v_e, i, k}|^2 \right)^{1/2} \end{aligned}$$

for all real coefficients  $c_{v_e, i, k}$ . Suppose  $f \in G_k$ . Then  $f$  can be represented as

$$f = \sum_{e \in E_{k-1}} \sum_{i=0}^2 c_{v_e, i, k} \psi_{v_e, i, k} + \sum_{v \in V_{k-1}} \sum_{i=0}^2 d_{v, i, k} \phi_{v, i, k},$$

where  $c_{v_e, i, k}$  and  $d_{v, i, k}$  are real coefficients. Since  $\psi_{v_e, i, k} \in G_k$  for  $e \in E_{k-1}$  and  $i = 0, 1, 2$ , we have

$$\left\langle \sum_{v \in V_{k-1}} \sum_{i=0}^2 d_{v, i, k} \phi_{v, i, k}, \phi_{w, j, k-1} \right\rangle_{\mathcal{T}_k} = 0, \quad w \in V_{k-1}, j = 0, 1, 2.$$

For  $v, w \in V_{k-1}$  and  $v \neq w$ , we have  $\langle \phi_{v, i, k}, \phi_{w, j, k-1} \rangle_{\mathcal{T}_k} = 0$ ,  $i, j = 0, 1, 2$ . Hence, it follows that

$$\sum_{i=0}^2 d_{w, i, k} \langle \phi_{w, i, k}, \phi_{w, j, k-1} \rangle_{\mathcal{T}_k} = 0, \quad j = 0, 1, 2;$$

but the matrix  $(\langle \phi_{w, i, k}, \phi_{w, j, k-1} \rangle_{\mathcal{T}_k})_{i, j=0,1,2}$  is invertible. Therefore,  $d_{w, i, k} = 0$  for all  $w \in V_{k-1}$  and  $i = 0, 1, 2$ . This shows that the set  $\{\psi_{v_e, i, k} : e \in E_{k-1}, i = 0, 1, 2\}$  is a Riesz basis in  $G_k$  with the corresponding Riesz bounds independent of  $k$ . By Theorem 5.3, we conclude that the set given in (6.7) indeed forms a Riesz basis in  $H^\mu(\Omega)$  for  $\mu_0 < \mu < 5/2$ .  $\square$

*Remark 1.* The range of  $\mu$  in Theorem 6.2 can be improved by choosing a different inner product in (6.1). For  $f, g \in F_k$  and  $e = [v_1, v_2] \in E_k$ , define

$$\begin{aligned} \langle f, g \rangle_e &:= 2^{-2k} (f(v_1)g(v_1) + [f(v_1) + D_e f(v_1)/4][g(v_1) + D_e g(v_1)/4] \\ &\quad + [f(v_2) - D_e f(v_2)/4][g(v_2) - D_e g(v_2)/4] + f(v_2)g(v_2)). \end{aligned}$$

Furthermore, define  $\langle f, g \rangle_{\mathcal{T}_k} := \sum_{e \in E_k} \langle f, g \rangle_e$ . Let us estimate  $\|g\|_{\mathcal{T}_{k-1}} / \|g\|_{\mathcal{T}_k}$  for  $g \in F_{k-1}$ . We have

$$\|g\|_{\mathcal{T}_{k-1}}^2 = \sum_{e \in E_{k-1}} \|g\|_e^2.$$

Suppose  $e = [v_1, v_2] \in E_{k-1}$ . Let  $v_0 := (v_1 + v_2)/2$  be the middle point of  $e$ . Then  $e = e_1 \cup e_2$ , where  $e_1 := [v_1, v_0]$  and  $e_2 := [v_0, v_2]$  are two edges of  $\mathcal{T}_k$ . Let

$$a_1 := g(v_1), \quad b_1 := g(v_1) + D_e g(v_1)/4, \quad \text{and} \quad a_2 := g(v_2), \quad b_2 := g(v_2) - D_e g(v_2)/4.$$

A simple computation gives

$$\begin{aligned} \|g\|_{e_1}^2 + \|g\|_{e_2}^2 &= 2^{-2k} \left[ a_1^2 + \left(\frac{a_1 + b_1}{2}\right)^2 + \left(\frac{3b_1 + b_2}{4}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 \right. \\ &\quad \left. + \left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + 3b_2}{4}\right)^2 + \left(\frac{a_2 + b_2}{2}\right)^2 + a_2^2 \right] \\ &= 2^{-2k} [a_1, b_1, a_2, b_2] A [a_1, b_1, a_2, b_2]^T, \end{aligned}$$

where  $A$  is the  $4 \times 4$  matrix

$$\frac{1}{8} \begin{bmatrix} 10 & 2 & 0 & 0 \\ 2 & 11 & 7 & 0 \\ 0 & 7 & 11 & 2 \\ 0 & 0 & 2 & 10 \end{bmatrix}.$$

The smallest eigenvalue of  $A$  is  $\lambda = (7 - \sqrt{13})/8$ . Hence,

$$\|g\|_{e_1}^2 + \|g\|_{e_2}^2 \geq \frac{7 - \sqrt{13}}{8} 2^{-2k} (a_1^2 + b_1^2 + b_2^2 + a_2^2) = \frac{7 - \sqrt{13}}{32} \|g\|_e^2.$$

It follows that

$$\|g\|_{\mathcal{T}_{k-1}} \leq \sqrt{\frac{32}{7 - \sqrt{13}}} \|g\|_{\mathcal{T}_k} = \frac{2(1 + \sqrt{13})}{3} \|g\|_{\mathcal{T}_k} \quad \forall g \in F_{k-1}.$$

Following the approach of Theorem 6.2 we can construct a similar wavelet basis in  $H^\mu(\Omega)$  for  $\mu_1 < \mu < 5/2$ , where

$$\mu_1 := \log_2 \frac{2(1 + \sqrt{13})}{3} \approx 1.618.$$

*Remark 2.* The wavelet basis constructed in Theorem 6.2 can be modified so that they satisfy the homogeneous boundary condition

$$f|_{\partial\Omega} = 0 \quad \text{and} \quad \frac{\partial f}{\partial n} \Big|_{\partial\Omega} = 0,$$

where  $\frac{\partial f}{\partial n}$  denotes the normal derivative of  $f$ .

Let  $\Omega$  be a bounded Lipschitz-graph domain in  $\mathbb{R}^2$ . For  $1 \leq p < \infty$  and  $0 < \mu < 2 + 1/p$ , we use  $\overset{\circ}{B}_{p,p}^\mu(\Omega)$  to denote the closure of  $C_c^1(\Omega) \cap B_{p,p}^\mu(\Omega)$  in  $B_{p,p}^\mu(\Omega)$ , where  $C_c^1(\Omega)$  denotes the linear space of all continuously differentiable functions with support contained in  $\Omega$ . In particular,  $H_0^\mu(\Omega) = \overset{\circ}{B}_{2,2}^\mu(\Omega)$ . Suppose in addition that  $\Omega$  is a polygonal domain and  $\mathcal{T}$  is a triangulation of  $\Omega$ . For  $k \in \mathbb{N}_0$ , let  $\overset{\circ}{V}_k$  and  $\overset{\circ}{E}_k$  be the set of interior vertices and interior edges of  $\mathcal{T}_k$ , respectively. Let  $\overset{\circ}{F}_k$  be the space of those functions in  $\overset{\circ}{S}_2^1(\mathcal{T}_k)$  for which  $f(v) = D_1 f(v) = D_2 f(v) = 0$  for all  $v \in \overset{\circ}{V}_k \setminus \overset{\circ}{V}_k$ . Thus, for  $f \in \overset{\circ}{F}_k$ , we have  $f|_{\partial\Omega} = 0$  and  $\frac{\partial f}{\partial n} \Big|_{\partial\Omega} = 0$ . Moreover,

$$\overset{\circ}{F}_0 \subset \overset{\circ}{F}_1 \subset \overset{\circ}{F}_2 \subset \dots \subset \overset{\circ}{B}_{p,p}^\mu(\Omega).$$

Let  $Q_k$  be the projection operator given in (4.4). For  $g \in C_c^1(\Omega) \cap B_{p,p}^\mu(\Omega)$ , we have  $Q_n g \in \overset{\circ}{F}_n$  for sufficiently large  $n$ . By Lemma 4.2,

$$\lim_{n \rightarrow \infty} \|Q_n f - f\|_{B_{p,p}^\mu(\Omega)} = 0.$$

Consequently,  $\bigcup_{n=0}^\infty \overset{\circ}{F}_n$  is dense in  $\overset{\circ}{B}_{p,p}^\mu(\Omega)$ . Let  $P_k$  be the orthogonal projection from  $\overset{\circ}{F}_k$  to  $\overset{\circ}{F}_{k-1}$  with respect to the inner product given in (6.2). Then there exists a positive constant  $\lambda_0 < 4$  such that

$$\|P_k f\|_{\mathcal{T}_{k-1}} \leq \lambda_0 \|f\|_{\mathcal{T}_k} \quad \forall f \in \overset{\circ}{F}_k.$$

Without loss of any generality, we may assume that at least one of the end points of every interior edge of  $\mathcal{T}$  is an interior vertex. Indeed, for any triangulation of  $\mathcal{T}$ ,  $\mathcal{T}_1 := \delta_4(\mathcal{T})$  must satisfy the above condition. Thus, if necessary, we may replace  $\mathcal{T}$  by  $\mathcal{T}_1$  in our consideration. Suppose  $e = [v_0, v_1]$  is an interior edge of  $\mathcal{T}_{k-1}$ . Let  $v_e$  be the middle point of  $e$ . We choose  $\psi_{v_e, i, k}$  ( $i = 0, 1, 2$ ) as in (6.4), but we set  $B_0 = 0$  if  $v_0$  is a boundary vertex. In light of the same argument as in Theorem 6.2, we may conclude that

$$\{\phi_{v, i, 0} : v \in \overset{\circ}{V}_0, i = 0, 1, 2\} \cup \bigcup_{k=1}^\infty \{2^{-k\mu} \psi_{v_e, i, k} : e \in \overset{\circ}{E}_{k-1}, i = 0, 1, 2\}$$

is a Riesz basis in  $H_0^\mu(\Omega)$  for  $\mu_0 < \mu < 5/2$ , where  $\mu_0 = \log_2 \lambda_0 < 2$ . □

Finally, let us consider the triangulation  $\mathcal{T}$  of  $\mathbb{R}^2$  given by

$$\mathcal{T} = \bigcup_{\gamma \in \mathbb{Z}^2} ((\tau_1 + \gamma) \cup (\tau_2 + \gamma)),$$

where  $\tau_1$  is the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 1)$ , and  $\tau_2$  is the triangle with vertices  $(0, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ . This triangulation is generated from three families of lines:  $x_1 = j$ ,  $x_2 = j$ , and  $x_1 - x_2 = j$ ,  $j \in \mathbb{Z}$ . Thus, such a mesh is often called the three-direction mesh.

Let  $\phi_0, \phi_1, \phi_2$  be the elements in  $\tilde{S}_2^1(\mathcal{T})$  satisfying the following conditions:

$$\begin{bmatrix} \phi_0(0) & D_1 \phi_0(0) & D_2 \phi_0(0) \\ \phi_1(0) & D_1 \phi_1(0) & D_2 \phi_1(0) \\ \phi_2(0) & D_1 \phi_2(0) & D_2 \phi_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\phi_i(\beta) = D_1 \phi_i(\beta) = D_2 \phi_i(\beta) = 0 \quad \forall \beta \in \mathbb{Z}^2 \setminus \{0\} \text{ and } i = 0, 1, 2.$$

For  $k \in \mathbb{N}_0$ ,  $i = 0, 1, 2$  and  $\beta \in \mathbb{Z}^2$ , let

$$\phi_{\beta/2^k, i, k}(x) := 2^k \phi_i(2^k x - \beta), \quad x \in \mathbb{R}^2.$$

Let  $\mathcal{E}_k$  denote the collection of all horizontal and vertical edges of  $\mathcal{T}_k$ . For  $f, g \in \tilde{S}_2^1(\mathcal{T}_k)$  and  $e \in \mathcal{E}_k$ , let  $\langle f, g \rangle_e$  be the inner product as given in (6.1). Moreover, define

$$(6.8) \quad \langle f, g \rangle_{\mathcal{T}_k} := \sum_{e \in \mathcal{E}_k} \langle f, g \rangle_e \quad \text{and} \quad \|f\|_{\mathcal{T}_k} := \sqrt{\langle f, f \rangle_{\mathcal{T}_k}}.$$

Note that the edges of  $\mathcal{T}_k$  in the direction of diagonals are not included in  $\mathcal{E}_k$ . But the norms  $\|f\|_2$  and  $\|f\|_{\mathcal{T}_k}$  are still equivalent. Moreover, Lemma 6.1 remains valid

for the inner product and norm given in (6.8). It is easily seen that

$$\begin{aligned} \langle f, g \rangle_{\mathcal{T}_k} &= 2^{-2k} \sum_{\beta \in \mathbb{Z}^2} [4f(2^{-k}\beta)g(2^{-k}\beta) + 2^{-2k+1}D_1f(2^{-k}\beta)D_1g(2^{-k}\beta) \\ &\quad + 2^{-2k+1}D_2f(2^{-k}\beta)D_2g(2^{-k}\beta)]. \end{aligned}$$

In particular,

$$\langle f, g \rangle_{\mathcal{T}_1} = \sum_{\beta \in \mathbb{Z}^2} \left[ f(\beta/2)g(\beta/2) + \frac{1}{8}D_1f(\beta/2)D_1g(\beta/2) + \frac{1}{8}D_2f(\beta/2)D_2g(\beta/2) \right].$$

Let  $V := \{v_1, v_2, v_3\}$ , where  $v_1 = (1/2, 0)$ ,  $v_2 = (0, 1/2)$ , and  $v_3 = (1/2, 1/2)$ . As was done before, the wavelets  $\psi_{v,i}$  ( $v \in V$ ,  $i = 0, 1, 2$ ) are constructed as follows:

$$\begin{aligned} \begin{bmatrix} \psi_{v_1,0} \\ \psi_{v_1,1} \\ \psi_{v_1,2} \end{bmatrix} &= \begin{bmatrix} \phi_{v_1,0,1} \\ \phi_{v_1,1,1} \\ \phi_{v_1,2,1} \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & 2 & 0 \\ -2 & -2 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} \phi_{(0,0),0,1} \\ \phi_{(0,0),1,1} \\ \phi_{(0,0),2,1} \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} \phi_{(1,0),0,1} \\ \phi_{(1,0),1,1} \\ \phi_{(1,0),2,1} \end{bmatrix}, \\ \begin{bmatrix} \psi_{v_2,0} \\ \psi_{v_2,1} \\ \psi_{v_2,2} \end{bmatrix} &= \begin{bmatrix} \phi_{v_2,0,1} \\ \phi_{v_2,1,1} \\ \phi_{v_2,2,1} \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 2 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} \phi_{(0,0),0,1} \\ \phi_{(0,0),1,1} \\ \phi_{(0,0),2,1} \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & 0 & -2 \\ -1 & 2 & 2 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} \phi_{(0,1),0,1} \\ \phi_{(0,1),1,1} \\ \phi_{(0,1),2,1} \end{bmatrix}, \end{aligned}$$

and

$$\begin{bmatrix} \psi_{v_3,0} \\ \psi_{v_3,1} \\ \psi_{v_3,2} \end{bmatrix} = \begin{bmatrix} \phi_{v_3,0,1} \\ \phi_{v_3,1,1} \\ \phi_{v_3,2,1} \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & 2 & 2 \\ -1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} \phi_{(0,0),0,1} \\ \phi_{(0,0),1,1} \\ \phi_{(0,0),2,1} \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & -2 & -2 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} \phi_{(1,1),0,1} \\ \phi_{(1,1),1,1} \\ \phi_{(1,1),2,1} \end{bmatrix}.$$

The following result is a consequence of Theorem 6.2.

**Theorem 6.3.** *For  $v \in V$  and  $i = 0, 1, 2$ , let  $\psi_{v,i}$  be the wavelets constructed above. Then*

$$\{\phi_i(\cdot - \beta) : i = 0, 1, 2, \beta \in \mathbb{Z}^2\} \cup \bigcup_{k=1}^{\infty} \{2^{-k\mu} \psi_{v,i}(2^k \cdot - \beta) : v \in V, i = 0, 1, 2, \beta \in \mathbb{Z}^2\}$$

is a Riesz basis in  $H^\mu(\mathbb{R}^2)$  for  $\mu_0 < \mu < 5/2$ .

Let  $\mathcal{T}$  be a general triangulation of a polygonal domain  $\Omega$  in  $\mathbb{R}^2$ . For  $k \in \mathbb{N}_0$ , the mesh formed by the triangles of  $\mathcal{T}_k$  contained in a fixed triangle  $\tau$  of  $\mathcal{T}$  can be viewed as a three-direction mesh. Let  $e$  be an edge that is not a part of any edge of  $\mathcal{T}$ . Then the wavelets  $\psi_{v_e,i,k}$  ( $i = 0, 1, 2$ ) in Theorem 6.2 can be obtained by applying a suitable affine transform to the wavelets constructed just before Theorem 6.3.

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DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA,  
EDMONTON, CANADA T6G 2G1

*E-mail address:* rjia@ualberta.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

*E-mail address:* songtao@uchicago.edu