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UNIFORM ERROR ESTIMATES IN THE FINITE ELEMENT METHOD FOR A SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEM

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Abstract. Consider the problem $-\epsilon^2\Delta u+u=f$ with homogeneous Neumann boundary condition in a bounded smooth domain in $\mathbb{R}^N$. The whole range $0<\epsilon\le 1$ is treated. The Galerkin finite element method is used on a globally quasi-uniform mesh of size h; the mesh is fixed and independent of ϵ.

A precise analysis of how the error at each point depends on h and ϵ is presented. As an application, first order error estimates in h, which are uniform with respect to ϵ, are given.

1. Introduction

Consider the following problem: find a function $u(x,\varepsilon)$ that satisfies the following partial differential equation with homogeneous Neumann boundary conditions:

$$\begin{aligned} -\varepsilon^2\Delta u+u&=f(x,\varepsilon) \qquad \text{in } \Omega,\\ \frac{\partial u}{\partial n}&=0 \qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N\ge 2$. Here ε is a parameter, $0<\varepsilon\le 1$, and $f(x,\varepsilon)$ is a uniformly bounded function in $L_2(\Omega)$.

In this paper we consider the whole range $0<\varepsilon\le 1$. In contrast to many other investigations (cf. below), the mesh is not allowed to vary with ε. We assume that the mesh is globally quasi-uniform, not necessarily regular, of size h. When ε is of order one, the problem is uniformly elliptic, the solution u is "well behaved", and the precise theory of A.H. Schatz [7] explains in detail how the error behaves (cf. below in this introduction). On the other hand, when ε approaches zero, the problem becomes singularly perturbed, and the solution may develop boundary layers. These boundary layers are somewhat less pronounced in our case of Neumann boundary conditions than in the case of Dirichlet boundary conditions. Hence, in our investigation with Neumann conditions, we can establish first order convergence in h, uniformly in ε, with a mesh independent of ε.

To achieve first order convergence in the Dirichlet case, or, to achieve higher order convergence than first in the Neumann case, will require remeshing according to each ε. In practice, this is rather undesirable if one wants to solve a number of problems (1.1) with varying ε.

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A great amount of research has been done on numerical methods for singularly perturbed reaction-diffusion problems. Most of the work has been focused on the problems either in one space dimension or on very special domains in the plane. For instance, in a recent paper [2], the authors considered the problem with Dirichlet boundary conditions on a unit square and proved second order convergence in h uniformly in ε for the standard central finite difference method with mesh refinement depending on ε.

Results for general domains in $\mathbb{R}^N$, $N \geq 2$, are rare, especially in the maximum norm. Two such results we would like to mention are [1] and [5], where the problem was considered on a general smooth plane domain with Dirichlet boundary conditions. In those papers, with special meshes depending on ε, the authors obtained a second order estimate in the maximum norm over the whole domain, including the boundary layer, uniformly in ε. Furthermore, as in [2], the degrees of freedom of the used spaces are bounded by Ch^{-2} uniformly in ε.

The aim of this paper is somewhat different. We consider the standard Galerkin finite element method on a globally quasi-uniform mesh of size h. The mesh is independent of ε. The Galerkin finite element solution $u_h \in S_h^r$ satisfies

$$\varepsilon^2(\nabla u_h, \nabla \chi) + (u_h, \chi) \quad = \quad (f, \chi), \text{ for all } \chi \in S_h^r, \tag{1.2}$$

where (v, w) denotes the $L_2(\Omega)$ inner product $\int_\Omega v(x)w(x)dx$. The precise definition of S_h^r is given in Chapter 2. For now, we may think of S_h^r as a set of continuous piecewise polynomials of total degree $r-1$ on globally quasi-uniform partitions of Ω.

Instead of deriving an "ε-specific" method that guarantees a certain order of convergence uniformly in ε, we give a precise analysis of how the error between the real solution u and the Galerkin solution u_h at each point depends on h and ε. Then as an application of our main result, we show that the error is of first order in h, uniformly in ε.

Before we describe the main result, let us review pointwise error estimates in two extreme cases, $\varepsilon = 0$ and $\varepsilon = 1$.

When $\varepsilon = 0$, problem (1.2) degenerates formally into the zero order equation

$$(u_h, \chi) = (f, \chi),$$

i.e. u_h is the L_2 projection onto S_h^r. Pointwise behavior of L_2 projections are well analyzed (cf. Chapter 7 in [13]), and it can be shown that the error satisfies

$$|(u - u_h)(x)| \leq C \min_{\chi \in S_h^r} \|e^{-c\frac{|x-y|}{h}}(u - \chi)(y)\|_{L_\infty(\Omega)}, \tag{1.3}$$

for some positive constants c and C independent of u, u_h, x, and h.

When $\varepsilon = 1$, the equation (1.2) is uniformly elliptic and sharp pointwise error estimates were obtained by A.H. Schatz in [7]. To describe his main result we need to introduce some notation. Fix $x \in \Omega$ and consider the weight

$$\sigma(y) = \sigma_{h,x}(y) = \frac{h}{h + |x - y|}, \text{ for } y \in \mathbb{R}^N. \tag{1.4}$$

Notice that $\sigma(y) = O(1)$ if $|x - y| = O(h)$ and $\sigma(y) = O(h)$ if $|x - y| = O(1)$.

For $1 \leq p \leq \infty$, a real number s, and a fixed x, we define the weighted norms over domains Ω by

$$\|u\|_{L_p(\Omega),\sigma,s} = \|\sigma_{h,x}^s(y)u(y)\|_{L_p(\Omega)}. \tag{1.5}$$

The main result of [7] says that, for any $0 \le s \le r-2$,

$$|(u-u_h)(x)| \le C\ell_h \min_{\chi\in S_h^r} \|u-\chi\|_{L_\infty(\Omega),\sigma,s}\,, \tag{1.6}$$

where the constant C is independent of u, u_h, h, and x, and the logarithmic term $\ell_h = |\log h|$ is necessary only when $s = r-2$.

The main result in this paper can be thought of as an interpolation between these two extreme cases and may roughly be stated as follows: Let $0 < \varepsilon \le 1$. Then, for any fixed $x \in \Omega$ and $0 \le s \le r-2$,

$$|(u-u_h)(x)| \le C\ell_h \min_{\chi\in S_h^r} \left\| e^{-c\frac{|x-y|}{\varepsilon+h}}(u-\chi)(y)\right\|_{L_\infty(\Omega),\sigma,s}, \tag{1.7}$$

where C and c are independent of u, u_h, h, ε, and x, and the logarithmic term $\ell_h = |\log h|$ is necessary only when $s = r-2$ and $\varepsilon \gg h$.

From (1.7) it is easy to see that if $\varepsilon = O(h)$, then u_h behaves essentially like the L_2 projection, and if $\varepsilon = O(1)$, we get the A.H. Schatz's weighted result (1.6).

The estimate (1.7) is useful for analyzing singularly perturbed problems, i.e. when ε is small. We now give some applications.

For the rest of the introduction we assume that ε is small, for example $\varepsilon = O(h^\alpha)$, for some $\alpha > 0$.

Let B_d denote a ball of radius d centered at x. From (1.7), taking into consideration only the exponential weight, we have

$$|(u-u_h)(x)| \le C\ell_h \min_{\chi\in S_h^r} \|u-\chi\|_{L_\infty(B_d)} + C\ell_h e^{-\frac{cd}{\varepsilon+h}}\|u\|_{L_\infty(\Omega\backslash B_d)}. \tag{1.8}$$

If $u \in W_\infty^r(B_d)$, $u \in L_\infty(\Omega\backslash B_d)$, and $d > \kappa(\varepsilon+h)|\log h|$, for κ sufficiently large, then $|(u-u_h)(x)| \le C\ell_h h^r$. Thus we can conclude that Galerkin solution u_h approximates u to the optimal order on subdomains where the solution u is sufficiently smooth.

On the other hand, in the boundary layer we have to be careful since the derivatives of u may depend on ε. In Corollary 2.3 we show, assuming $f \in W_\infty^1(\Omega)$, that, for any $x \in \Omega$, there exists a positive constant C independent of ε and h, such that

$$|(u-u_h)(x)| \le C|\log h|^3 \min\{h^2/\varepsilon, h\}\|f\|_{W_\infty^1(\Omega)}. \tag{1.9}$$

Therefore, we may conclude that the Galerkin approximation for the Neumann problem is of almost first order uniformly in ε in the global maximum norm, provided $\|f\|_{W_\infty^1}$ is uniformly bounded in ε.

One way to increase the order of convergence in the boundary layer is by using matched asymptotic expansion (cf. [4]). For example, let $x' \in \partial\Omega$ denote the point where the normal from x meets the $\partial\Omega$. Set

$$u_\varepsilon(x) = f(x) + \frac{\partial f}{\partial n}(x')e^{-\frac{|x-x'|}{\varepsilon}}, \tag{1.10}$$

where f is evaluated at $\varepsilon = 0$. The first term on the right is called the "regular inner expansion" and the second term is the "boundary layer correction". It is not hard to show that in the boundary layer $\|u-u_\varepsilon\|_{L_\infty} \le C\varepsilon^2$. Thus in the boundary layer, switching from the Galerkin approximation u_h to the matched expansion u_ε when $\varepsilon < O(h^{2/3})$, gives a "method" of uniform order almost 4/3 in the global maximum norm. Of course if more terms in the matched asymptotic expansion are available we can increase the order, but in general they are much harder to compute.

Remark 1. Using the same techniques we can prove a similar result for the above problem with Dirichlet boundary conditions on convex bounded domains in $\mathbb{R}^N$ for piecewise linear finite element spaces.

In the case of Dirichlet boundary conditions, the boundary layer is more pronounced, and under the same basic assumptions using similar techniques we can only show

$$|(u-u_h)(x)| \leq C\ell_h \min\{h^2/\varepsilon^2, 1\}\|f\|_{L_\infty}.$$

The matched asymptotic expansion in the Dirichlet case is

$$u_\varepsilon(x) = f(x) - f(x')e^{-\frac{|x-x'|}{\varepsilon}}, \tag{1.11}$$

and on the boundary layer we have $\|u-u_\varepsilon\|_{L_\infty} \leq C\varepsilon$. Thus switching from the Galerkin solution u_h to the matched expansion u_ε in the boundary layer when $\varepsilon < O(h^{2/3})$ gives a method of uniform order only 2/3 in the global maximum norm.

This work is based on the paper [11] by A.H. Schatz and L.B. Wahlbin, in which the authors showed a somewhat similar result restricted to the piecewise linear case $r = 2$ and space dimension $N = 2$. This paper sharpens the above result and removes the restrictions on the dimension and the order of the finite element spaces in the case when $a \equiv 1$.

The proof of our main result (1.7) is based on a Green's function estimate for the continuous problem, which is obtained from a Green's function estimate for the parabolic problems [3], and local energy estimates for the approximate Green's function. An essential analytical tool for the derivation of (1.7) is a "kick-back" argument, which was developed by A.H. Schatz and L.B. Wahlbin and was used in a number of papers, for example [8], [9], [10].

Outline of the paper. Section 2 contains the assumptions on the finite element spaces, the statement of the main result, and Corollary 2.3 with a proof. Sections 3-4 are preliminary and contain global and local energy estimates, which are used in the proof of the main result. In Sections 5-6 we prove the main result. Finally, in the Appendix we prove Lemma 2.2, the pointwise estimate of the Green's function for the continuous problem.

2. Preliminaries and statement of the main result

With $0 < h < 1/2$ a parameter, let $\tau_j^h, j = 1, ..., J_h$, be disjoint open sets, elements, which form a partition of Ω and fit the boundary exactly, i.e. $\overline{\Omega} = \bigcup_{j=1}^{J_h} \overline{\tau}_j^h$. For each such partition, let $S_h^r = S_h^r(\Omega) \subset W_\infty^1(\Omega)$ be a finite-dimensional space. We will use $W_p^l(D)$, with $1 \leq p \leq \infty$, $l = 0, 1, ...$, and a set D to denote the standard Sobolev spaces with $\|\cdot\|_{W_p^l(D)}$ and $|\cdot|_{W_p^l(D)}$ their norms and semi-norms respectively. When needed, we will also use the piecewise norms

$$\|u\|_{W_p^l(D)}^{(h)} = \left(\sum_{\tau_j^h \cap D \neq \emptyset} \|u\|_{W_p^l(\tau_j^h \cap D)}^p\right)^{1/p}. \tag{2.1}$$

Similarly, we have the weighted piecewise norms

$$\|u\|_{W_p^l(D),\sigma,s}^{(h)} = \sum_{0 \leq |\alpha| \leq l} \|\sigma^s D_x^\alpha u\|_{L_p(D)}^{(h)}. \tag{2.2}$$

Next, we will state some standard assumptions about finite element spaces. Assume there exist positive constants δ, k, $\underline{k}$, $\overline{k}$, C_1, C_2, C_3, C_4, and an integer $r \geq 2$, all independent of h, such that the assumptions 2.1 through 2.4 below hold.

The first assumption expresses the global quasi-uniformity of the partition of Ω and a trace inequality at the boundary of each element.

2.1. **Quasi-uniformity and trace.** (i) Each τ_j^h contains a ball of radius $\underline{k}h$ and is contained in a ball of radius $\overline{k}h$.

(ii) For $0 < h < \frac{1}{2}$ and $j = 1, 2, ..., J_h$,

$$\int_{\partial\tau_j^h} |\nabla v| dS_j \leq C_1 \left(h^{-1} |v|_{W_1^1(\tau_j^h)} + |v|_{W_1^2(\tau_j^h)} \right), \ \forall v \in W_1^2(\tau_j^h).$$

The second assumption is a standard inverse property. For $D \subset \Omega$, $S_h^r(D)$ will denote the restriction of S_h^r to D.

2.2. **Inverse property.** Let $\chi \in S_h^r(D)$, where D is any union of closures of elements. Then for $0 \leq k \leq l \leq 2$, $1 \leq q \leq p \leq \infty$,

$$\|\chi\|^{(h)}_{W_p^l(D)} \leq C_2 h^{-(l-k)-N\left(\frac{1}{q}-\frac{1}{p}\right)} \|\chi\|^{(h)}_{W_q^k(D)}.$$

Our third assumption is about local approximation properties of the finite element spaces. For D a subset of Ω we let $D_d = \{x \in \Omega : dist(x, D) \leq d\}$.

2.3. **Local approximation.** Let $d \geq kh$. There exists a linear operator $I_h : W_1^1(\Omega) \to S_h^r(\Omega)$ such that for any D the following holds:

$$\|v - I_h v\|^{(h)}_{W_p^s(D)} \leq C_3 h^{l-s} \|v\|_{W_p^l(D_d)}, \text{ for } \ 0 \leq s \leq l \leq r, \ 1 \leq p \leq \infty.$$

2.4. **Superapproximation.** If the function to be approximated is of a certain special form, we have an assumption known as superapproximation.

Let $d \geq kh$ and $\omega \in C_0^\delta(\bar{D}_{2d})$; then for any $\psi \in S_h^r(D_{3d})$ there exists $\eta \in S_h^r(D_{3d})$, vanishing outside of D_{3d} such that

$$\|\omega\psi - \eta\|_{W_2^l(D_{3d})} \leq C_4 h \|\omega\|_{W_\infty^\delta(D_{2d})} \|\psi\|_{W_2^l(D_{3d})}, \quad l = 0, 1.$$

Furthermore, if $\omega \equiv 1$ on D_d, then $\eta = \psi$ on D, and the last factor may be replaced by $\|\psi\|_{W_2^l(D_{3d}\setminus D)}$.

We can now state our main result, which expresses how the error at a point depends on the continuous solution.

Theorem 2.1. *Suppose that assumptions* 2.1 *through* 2.4 *hold and* u *and* $u_h \in S_h^r$ *satisfy* (1.1) *and* (1.2) *respectively. Let* $x \in \Omega$, $0 < \varepsilon \leq 1$, *and let* s *satisfy* $0 \leq s \leq r-2$, *for* $r \geq 2$. *Furthermore assume* $1 - \varepsilon c_2 > 0$, *where* c_2 *is the smallest real number such that the estimate in Lemma* 7.1 *holds. Then there exist constants* C *and* c *independent of* x, u, u_h, ε, *and* h *such that*

$$|(u - u_h)(x)| \leq C \ell_h \min_{\chi \in S_h^r} \left\| e^{-c\frac{|x-y|}{\varepsilon + h}} (u - \chi)(y) \right\|_{L_\infty(\Omega), \sigma, s},$$

where $\ell_h = 1$, *if* $s < r-2$ *or* $\varepsilon = O(h)$ *and* $\ell_h = |\log h|$, *if* $s = r-2$ *and* $\varepsilon \gg h$.

Remark 2. If $\varepsilon = O(h)$, then the exponential weight is the dominating one and we have

$$|(u - u_h)(x)| \leq C \min_{\chi \in S_h^r} \left\| e^{-c\frac{|x-y|}{h}} (u - \chi)(y) \right\|_{L_\infty(\Omega)},$$

i.e. u_h behaves like the L_2 projection.

The major tool in obtaining the main result is the following estimate for the Green's function of the continuous problem (1.1).

Lemma 2.2. *The solution of* (1.1) *may be represented in terms of the Green's function* $K^\varepsilon(x,y)$, *for* $x, y \in \Omega$, *as*

$$u(x) = \int_\Omega K^\varepsilon(x,y) f(y) dy.$$

Assume that the boundary $\partial\Omega$ *is sufficiently smooth and* $1 - \varepsilon c_2 > 0$, *where* c_2 *is the smallest real number such that the estimate in Lemma* 7.1 *holds. Then for any multi-integer* m, *there exist constants* C *and* $c_0 > 0$ *such that for the Green's function* $K^\varepsilon(x,y)$, $x, y \in \Omega$, *we have*

$$|D_x^m K^\varepsilon(x,y)| \le \frac{Ce^{-c_0\frac{|x-y|}{\varepsilon}}}{\varepsilon^{N+|m|}} \times \begin{cases} 1, & \text{if } N+|m| = 1, \\ 1 + |\log \frac{|x-y|}{\varepsilon}|, & \text{if } N+|m| = 2, \\ \left(\frac{|x-y|}{\varepsilon}\right)^{2-N-|m|}, & \text{if } N+|m| \ge 3. \end{cases}$$

The proof of this result is given in the Appendix. It is based on [3].

Remark 3. If $\varepsilon = O(1)$, then the above estimate reduces to the well known estimate for the Green's function for the uniformly elliptic problem (cf. Krasovski [6]).

Corollary 2.3. *Under the assumptions of Theorem* 2.1 *and assuming* $S_h^r \subset C(\overline{\Omega})$ *and* $f \in W_\infty^r(\Omega)$, *we have for any* $1 \le s \le r$

$$|(u-u_h)(x)| \le C\ell_h \cdot \min \begin{cases} \frac{h^2|\log h| \log\left(\frac{1}{\varepsilon}\right)}{\varepsilon}\|f\|_{W_\infty^1(\Omega)}, \\ \varepsilon|\log h|^2 \log\left(\frac{1}{\varepsilon}\right)\|f\|_{W_\infty^1(\Omega)} + h^s\|f\|_{W_\infty^s(\Omega)}. \end{cases} \tag{2.3}$$

Proof. Since $S_h^r \subset C(\overline{\Omega})$, the standard interpolant satisfies (cf. [12] Section 4)

$$\|u - I_h u\|_{L_\infty(\Omega)} \le C|\log h| h^2 \|\Delta u\|_{L_\infty(\Omega)}.$$

From Theorem 2.1 we have

$$|(u-u_h)(x)| \le C\ell_h \min_{\chi \in S_h^r} \|u-\chi\|_{L_\infty(\Omega)} \le C\ell_h |\log h| h^2 \|\Delta u\|_{L_\infty(\Omega)}. \tag{2.4}$$

The top part of estimate (2.3) will follow from (2.4) and the following lemma.

Lemma 2.4. *There exists a constant* C *independent of* ε *such that*

$$\|\Delta u\|_{L_\infty(\Omega)} \le \frac{C}{\varepsilon} \log\left(\frac{1}{\varepsilon}\right)\|f\|_{W_\infty^1(\Omega)}. \tag{2.5}$$

Proof. Since the case $\varepsilon > 1/2$ is easy, we assume $\varepsilon \le 1/2$. Assuming that u and f are sufficiently smooth, we have

$$\|\Delta u\|_{L_\infty(\Omega)} = \frac{1}{\varepsilon^2}\|u - f\|_{L_\infty(\Omega)}.$$

For $x \in \Omega$,

$$u(x) - f(x) = \int_\Omega K^\varepsilon(x,y) f(y) dy - f(x) = \int_\Omega K^\varepsilon(x,y)\big(f(y) - f(x)\big) dy,$$

where we used that $\int_\Omega K^\varepsilon(x,y)dy = 1$ for any x since the function $v \equiv 1$ solves

$$\begin{aligned} -\varepsilon^2 \Delta u + u &= 1 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.6}$$

Thus,

$$u(x) - f(x) = \int_{\Omega\setminus B_d} K^\varepsilon(x,y)\big(f(y)-f(x)\big)dy + \int_{B_d\cap\Omega} K^\varepsilon(x,y)\big(f(y)-f(x)\big)dy$$
$$= J_1 + J_2,$$

where B_d denotes a ball centered at x of radius d. Choose $d = \kappa\varepsilon\log\big(\frac{1}{\varepsilon}\big)$, with κ sufficiently large. Using the estimates of Lemma 2.2 in the case $N \geq 3$, we have

$$|J_1| \leq C\|f\|_{L_\infty(\Omega)}\frac{1}{\varepsilon^N}e^{-c_0\kappa\log\left(\frac{1}{\varepsilon}\right)} \leq C\varepsilon\|f\|_{L_\infty(\Omega)},$$

provided $c_0\kappa \geq N+1$.

By the Mean Value Theorem we can bound J_2 by

$$|J_2| \leq C\kappa\log\big(\frac{1}{\varepsilon}\big)\varepsilon\|f\|_{W^1_\infty(B_d)}\int_\Omega |K^\varepsilon(x,y)|dy.$$

It remains to show that $\int_\Omega |K^\varepsilon(x,y)|dy \leq C$. Using Lemma 2.2 with $N \geq 3$,

$$\int_\Omega |K^\varepsilon(x,y)|dy \leq C\int_\Omega e^{-c_0\frac{|x-y|}{\varepsilon}}\frac{dy}{\varepsilon^2|x-y|^{N-2}}.$$

Switching to polar coordinates, $|x-y| = \rho$, $dy = C\rho^{N-1}d\rho$, we have

$$\int_\Omega |K^\varepsilon(x,y)|dy \leq C\int_0^R e^{-c_0\frac{\rho}{\varepsilon}}\frac{\rho}{\varepsilon}\frac{d\rho}{\varepsilon} \leq C. \tag{2.7}$$

Thus we have the first estimate of the corollary in the case $N \geq 3$. The case $N = 2$ is very similar. □

To show the other part of estimate (2.3), we notice that

$$u - u_h = \varepsilon^2\Delta u + f - \varepsilon^2\Delta_h u_h - P_h f, \tag{2.8}$$

where $P_h : L_2(\Omega) \to S_h^r$ is the L_2 projection defined by

$$(P_h v, \chi) = (v, \chi), \text{ for } \chi \in S_h^r,$$

and $\Delta_h : S_h^r \to S_h^r$ is the discrete Laplacian defined by

$$-(\Delta_h v, \chi) = (\nabla v, \nabla\chi), \text{ for } \chi \in S_h^r.$$

Using the triangle inequality we have

$$\|u - u_h\|_{L_\infty(\Omega)} \leq \varepsilon^2\|\Delta u - \Delta_h u_h\|_{L_\infty(\Omega)} + \|f - P_h f\|_{L_\infty(\Omega)}. \tag{2.9}$$

Using the approximation properties of the L_2 projection we can bound the second term as

$$\|f - P_h f\|_{L_\infty(\Omega)} \leq Ch^s\|f\|_{W^s_\infty(\Omega)}, \text{ for any } 0 \leq s \leq r. \tag{2.10}$$

For the first term on the right hand side in (2.9) by the triangle inequality, we have

$$\|\Delta u - \Delta_h u\|_{L_\infty(\Omega)} \leq \|\Delta u - \Delta_h R_h u\|_{L_\infty(\Omega)} + \|\Delta_h R_h u - \Delta_h u_h\|_{L_\infty(\Omega)}, \tag{2.11}$$

where $R_h : H^1(\Omega) \to S_h^r$ is the Ritz projection defined by

$$(\nabla R_h v, \nabla\chi) = (\nabla v, \nabla\chi), \text{ for } \chi \in S_h^r.$$

Using the operator identity $\Delta_h R_h = P_h \Delta$, the stability of the L_2 projection in L_∞ norm, and (2.5), we can bound the first term on the right hand side of (2.11) as

$$\begin{aligned}\|\Delta u - \Delta_h R_h u\|_{L_\infty(\Omega)} &= \|\Delta u - P_h \Delta u\|_{L_\infty(\Omega)} \\ &\le C\|\Delta u\|_{L_\infty(\Omega)} \le \frac{C}{\varepsilon}\log\big(\frac{1}{\varepsilon}\big)\|f\|_{W^1_\infty(\Omega)}.\end{aligned} \tag{2.12}$$

Applying the inverse inequality and the triangle inequality on the second term on the right hand side of (2.11), we have

$$\|\Delta_h R_h u - \Delta_h u_h\|_{L_\infty(\Omega)} \le Ch^{-2}\left(\|R_h u - u\|_{L_\infty(\Omega)} + \|u - u_h\|_{L_\infty(\Omega)}\right). \tag{2.13}$$

By (2.4), the estimate $\|R_h u - u\|_{L_\infty(\Omega)} \le Ch^2|\log h|^2\|\Delta u\|_{L_\infty(\Omega)}$ (cf. Lemma 4.1 in [12]), and (2.5), we finally obtain

$$\|R_h u - u\|_{L_\infty(\Omega)} + \|u - u_h\|_{L_\infty(\Omega)} \le \frac{Ch^2|\log h|^2 \log\left(\frac{1}{\varepsilon}\right)}{\varepsilon}\|f\|_{W^1_\infty(\Omega)}. \tag{2.14}$$

Combining estimates (2.9), (2.10), (2.11), (2.12), (2.13), and (2.14) we have the corollary. □

In the next sections we will collect some results which we will use later.

3. Global energy estimates

For $v \in H^1(\Omega)$, define $P_h^\varepsilon v \in S_h^r$ by

$$A_\varepsilon(v - P_h^\varepsilon v, \chi) = 0, \text{ for any } \chi \in S_h^r, \tag{3.1}$$

where

$$A_\varepsilon(w, \chi) := \varepsilon^2(\nabla w, \nabla \chi) + (w, \chi). \tag{3.2}$$

Lemma 3.1. *There exists a constant C independent of $0 < \varepsilon \le 1$ and $0 < h < 1/2$ such that*

$$\|\nabla(v - P_h^\varepsilon v)\|_{L_2(\Omega)} \le \begin{cases} C\ \|v\|_{H^1(\Omega)}, \\ Ch\|v\|_{H^2(\Omega)}, \end{cases}$$

and

$$\|v - P_h^\varepsilon v\|_{L_2(\Omega)} \le \begin{cases} Ch\ \|v\|_{H^1(\Omega)}, \\ Ch^2\|v\|_{H^2(\Omega)}. \end{cases}$$

The proof of this result, which is valid for $N \ge 2$, is in [11], Lemma 4.1.

4. Local energy estimates

In the results below we assume that $d \ge \underline{k}h$ for some positive constant $\underline{k}$.

Lemma 4.1. *Let $0 < \varepsilon \le 1$ and $0 < h \le 1/2$ be parameters, and $v_h \in S_h^r(D_d)$ satisfy*

$$A_\varepsilon(v_h, \chi) = 0, \text{ for any } \chi \in S_h^r(D_d).$$

There exist positive constants c_1 and C independent of ε and h, such that

$$\|v_h\|_D + d\|\nabla v_h\|_D \le Ce^{-\frac{c_1 d}{\varepsilon + h}}\|v_h\|_{D_d}.$$

Lemma 4.2. *Let $0 < \varepsilon \leq 1$ and $0 < h \leq 1/2$ be parameters, and $v_h \in S_h^r(D_d)$ satisfy*

$$A_\varepsilon(v_h, \chi) = 0, \text{ for any } \chi \in S_h^r(D_d).$$

There exist positive constants c_1 and C independent of ε and h, such that

$$\|v - v_h\|_{H^1(D)} \leq C\left(\|\nabla(v-\chi)\|_{D_d} + d^{-1}\|v-\chi\|_{D_d}\right) + Cd^{-1}e^{-\frac{c_1 d}{\varepsilon+h}}\|v - v_h\|_{D_d}.$$

The proofs of these two results are in [11], Lemma 5.1. and Lemma 5.2. respectively. Although the main result in that paper was done in the plane domains, the proofs of these lemmas are valid in any number of dimensions.

Lemma 4.3. *Let $0 < \varepsilon \leq 1$ and $0 < h \leq 1/2$ be parameters, and $v_h \in S_h^r(D_d)$ satisfy*

$$A_\varepsilon(v - v_h, \chi) = 0, \text{ for any } \chi \in S_h^r(D_d).$$

There exist positive constants c_1 and C independent of ε and h, such that

$$\|v - v_h\|_D \leq Ch\left(\|\nabla(v-\chi)\|_{D_{2d}} + d^{-1}\|v-\chi\|_{D_{2d}}\right) + Ce^{-\frac{c_1 d}{\varepsilon+h}}\|v - v_h\|_{D_{2d}}.$$

Proof. Let $\omega \in C_0^\infty(D_d)$ be a cut-off function with the following properties:

$$\omega \equiv 1 \text{ on } D_d \text{ and } \|\omega\|_{l,D_{2d}} \leq Cd^{-l},\ l = 0,1.$$

Define $\tilde{v} = \omega v$ and $\tilde{v}_h = P_h^\varepsilon \tilde{v}$. Then we have

$$\|v - v_h\|_D \leq \|\tilde{v} - \tilde{v}_h\|_D + \|\tilde{v}_h - v_h\|_D. \tag{4.1}$$

Since $A^\varepsilon(\tilde{v}_h - v_h, \chi) = 0$, for $\chi \in S_h^r(D_d)$, by Lemma 4.1 we have

$$\begin{aligned}\|\tilde{v}_h - v_h\|_D &\leq Ce^{-\frac{c_1 d}{\varepsilon+h}}\|\tilde{v}_h - v_h\|_{D_d} \\ &\leq Ce^{-\frac{c_1 d}{\varepsilon+h}}\left(\|\tilde{v} - \tilde{v}_h\|_{D_d} + \|v - v_h\|_{D_d}\right).\end{aligned} \tag{4.2}$$

Thus we only need to estimate $\|\tilde{v} - \tilde{v}_h\|$. Using global energy estimates Lemma 3.1

$$\|\tilde{v} - \tilde{v}_h\| \leq Ch\|\tilde{v}\|_1 \leq Ch\left(\|\nabla v\|_{D_{2d}} + d^{-1}\|v\|_{D_{2d}}\right). \tag{4.3}$$

Combining estimates (4.1), (4.2), (4.3), and writing $v - v_h = (v - \chi) - (v_h - \chi)$ for $\chi \in S_h^r$, we complete the proof. □

5. Proof of the main result: Part 1

Let $x \in \overline{\tau}_0$. For any $\chi \in S_h^r$ using the triangle inequality and assumptions 2.2 and 2.3 we have

$$\begin{aligned}|(u-u_h)(x)| &\leq |(u-\chi)(x)| + Ch^{-N/2}\|\chi - u_h\|_{L_2(\tau_0)} \\ &\leq |(u-\chi)(x)| + Ch^{-N/2}\left(\|u-\chi\|_{L_2(\tau_0)} + \|u-u_h\|_{L_2(\tau_0)}\right) \\ &\leq C\|u-\chi\|_{L_\infty(\tau_0)} + Ch^{-N/2}\|u-u_h\|_{L_2(\tau_0)}.\end{aligned} \tag{5.1}$$

Define a function

$$\eta(y) = \begin{cases} h^{-N/2}(u-u_h)(y)/\|u-u_h\|_{L_2(\tau_0)}, & \text{for } y \in \tau_0, \\ 0, & \text{otherwise.}\end{cases} \tag{5.2}$$

It is easy to see that $\|\eta\|_{L_2(\Omega)} \leq Ch^{-N/2}$ and $\|\eta\|_{L_1(\Omega)} \leq C$.

Define a function g^ε to satisfy

$$A_\varepsilon(v, g^\varepsilon) = (\eta, v), \quad \text{for } v \in W_2^1(\Omega), \tag{5.3}$$

and define $g_h^\varepsilon \in S_h^r$ to be a unique solution of

$$A_\varepsilon(\chi, g^\varepsilon - g_h^\varepsilon) = 0, \quad \text{for all } \chi \in S_h^r. \tag{5.4}$$

First we will show the global a priori estimates.

Lemma 5.1. *There exists a constant C independent of $0 < \varepsilon \leq 1$ such that*

$$\begin{aligned} \|g^\varepsilon\|_{L_2(\Omega)} &\leq C\|\eta\|_{L_2(\Omega)} = Ch^{-N/2}, \\ \|g^\varepsilon\|_{H^1(\Omega)} &\leq C\varepsilon^{-1}\|\eta\|_{L_2(\Omega)} = C\varepsilon^{-1}h^{-N/2}, \\ \|g^\varepsilon\|_{H^2(\Omega)} &\leq C\varepsilon^{-2}\|\eta\|_{L_2(\Omega)} = C\varepsilon^{-2}h^{-N/2}. \end{aligned}$$

Proof. From (5.3) we have

$$\varepsilon^2\|\nabla g^\varepsilon\|^2_{L_2(\Omega)} + \|g^\varepsilon\|^2_{L_2(\Omega)} = A_\varepsilon(g^\varepsilon, g^\varepsilon) = (\eta, g^\varepsilon) \leq \|\eta\|_{L_2(\Omega)}\|g^\varepsilon\|_{L_2(\Omega)}.$$

Thus $\|g^\varepsilon\|_{L_2(\Omega)} \leq \|\eta\|_{L_2(\Omega)}$ and $\|\nabla g^\varepsilon\|_{L_2(\Omega)} \leq \varepsilon^{-1}\|\eta\|_{L_2(\Omega)}$, which proves the first two estimates.

To prove the last estimate we notice that

$$\|g^\varepsilon\|_{H^2(\Omega)} \leq C\| - \Delta g^\varepsilon + g^\varepsilon\|_{L_2(\Omega)},$$

hence

$$\begin{aligned} \|g^\varepsilon\|_{H^2(\Omega)} &\leq C\varepsilon^{-2}\| - \varepsilon^2\Delta g^\varepsilon + g^\varepsilon\|_{L_2(\Omega)} + C(1 + \varepsilon^{-2})\|g^\varepsilon\|_{L_2(\Omega)} \\ &\leq C\varepsilon^{-2}\|\eta\|_{L_2(\Omega)} = C\varepsilon^{-2}h^{-N/2}, \end{aligned}$$

which completes the proof of the lemma. □

Thus we have

$$\begin{aligned} h^{-N/2}&\|u - u_h\|_{L_2(\tau_0)} = (u - u_h, \eta) = A_\varepsilon(u - u_h, g^\varepsilon) = A_\varepsilon(u - u_h, g^\varepsilon - g_h^\varepsilon) \\ &= A_\varepsilon(u - \chi, g^\varepsilon - g_h^\varepsilon) \\ &= -\varepsilon^2 \sum_i \left(\int_{\tau_i^h} (u - \chi)\Delta(g^\varepsilon - g_h^\varepsilon) + \oint_{\partial\tau_i^h} (u - \chi)\nabla(g^\varepsilon - g_h^\varepsilon) \cdot n \right) \\ &\quad + (u - \chi, g^\varepsilon - g_h^\varepsilon). \end{aligned} \tag{5.5}$$

Letting $F^\varepsilon \equiv g^\varepsilon - g_h^\varepsilon$ and using Trace Inequality 2.1 we have,

$$\begin{aligned} h^{-N/2}\|u - u_h\|_{L_2(\tau_0)} \leq C\|e^{-c\frac{|x-y|}{\varepsilon+h}}(u - \chi)\|_{L_\infty(\Omega),\sigma,s} \Big(&\varepsilon^2\|e^{c\frac{|x-y|}{\varepsilon+h}} D^2F^\varepsilon\|^{(h)}_{L_1(\Omega),\sigma,-s} \\ &+ \varepsilon^2h^{-1}\|e^{c\frac{|x-y|}{\varepsilon+h}} \nabla F^\varepsilon\|_{L_1(\Omega),\sigma,-s} + \|e^{c\frac{|x-y|}{\varepsilon+h}} F^\varepsilon\|_{L_1(\Omega),\sigma,-s} \Big). \end{aligned} \tag{5.6}$$

By the triangle inequality

$$\begin{aligned} \|e^{c\frac{|x-y|}{\varepsilon+h}} D^2F^\varepsilon\|^{(h)}_{L_1(\Omega),\sigma,-s} &\leq \|e^{c\frac{|x-y|}{\varepsilon+h}} D^2(g^\varepsilon - \chi)\|^{(h)}_{L_1(\Omega),\sigma,-s} \\ &\quad + \|e^{c\frac{|x-y|}{\varepsilon+h}} D^2(g_h^\varepsilon - \chi)\|^{(h)}_{L_1(\Omega),\sigma,-s}, \text{ for any } \chi \in S_h^r. \end{aligned}$$

Let $y_\tau \in \tau$ be the center of the circumscribed sphere over an element τ. Using the triangle inequality $|x - y| \leq |x - y_\tau| + |y_\tau - y|$, assumption 2.1, and inverse inequality 2.2 in the case $D=\tau$, we have

$$\|e^{c\frac{|x-y|}{\varepsilon+h}}D^2(g_h^\varepsilon-\chi)\|^{(h)}_{L_1(\Omega),\sigma,-s}=\sum_\tau\int_\tau\left|e^{c\frac{|x-y|}{\varepsilon+h}}D^2(g_h^\varepsilon-\chi)\left(\frac{h+|x-y|}{h}\right)^s\right|$$
$$\le\sum_\tau e^{c\frac{|x-y_\tau|+\bar{k}h}{\varepsilon+h}}\left(\frac{h+|x-y_\tau|+\bar{k}h}{h}\right)^s\int_\tau\left|D^2(g_h^\varepsilon-\chi)\right|$$
$$\le Ch^{-1}\sum_\tau e^{c\frac{|x-y_\tau|+\bar{k}h}{\varepsilon+h}}\left(\frac{h+|x-y_\tau|+\bar{k}h}{h}\right)^s\left(\int_\tau\left|\nabla(g^\varepsilon-\chi)\right|+\left|\nabla(g^\varepsilon-g_h^\varepsilon)\right|\right).$$

Using the triangle inequality $-|x-y|\le|y_\tau-y|-|x-y_\tau|$, we have

$$\sum_\tau e^{c\frac{|x-y_\tau|+\bar{k}h}{\varepsilon+h}}\left(\frac{h+|x-y_\tau|+\bar{k}h}{h}\right)^s\left(\int_\tau\left|\nabla(g^\varepsilon-\chi)\right|+\left|\nabla(g^\varepsilon-g_h^\varepsilon)\right|\right)$$
$$\le\sum_\tau e^{c\frac{2\bar{k}h}{\varepsilon+h}}(1+2\bar{k})^s\int_\tau\left|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla(g^\varepsilon-\chi)\left(\frac{h+|x-y|}{h}\right)^s\right|$$
$$+\sum_\tau e^{c\frac{2\bar{k}h}{\varepsilon+h}}(1+2\bar{k})^s\int_\tau\left|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla(g^\varepsilon-g_h^\varepsilon)\left(\frac{h+|x-y|}{h}\right)^s\right|$$
$$\le e^{2c\bar{k}}(1+2\bar{k})^s\left(\|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla(g^\varepsilon-\chi)\|_{L_1(\Omega),\sigma,-s}+\|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla F^\varepsilon\|_{L_1(\Omega),\sigma,-s}\right).$$

Thus, we have shown

$$\begin{aligned}\|e^{c\frac{|x-y|}{\varepsilon+h}}D^2F^\varepsilon\|^{(h)}_{L_1(\Omega),\sigma,-s}&\le Ch^{-1}\|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla F^\varepsilon\|_{L_1(\Omega),\sigma,-s}\\&+Ch^{-1}\|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla(g^\varepsilon-\chi)\|_{L_1(\Omega),\sigma,-s}+\|e^{c\frac{|x-y|}{\varepsilon+h}}D^2(g^\varepsilon-\chi)\|^{(h)}_{L_1(\Omega),\sigma,-s}.\end{aligned}\tag{5.7}$$

Putting it all together, we have

$$\begin{aligned}&|(u-u_h)(x)|\le C\|e^{-c\frac{|x-y|}{\varepsilon+h}}(u-\chi)\|_{L_\infty(\Omega),\sigma,s}\\&\quad\times\Bigg(1+\varepsilon^2h^{-1}\|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla F^\varepsilon\|_{L_1(\Omega),\sigma,-s}+\|e^{c\frac{|x-y|}{\varepsilon+h}}F^\varepsilon\|_{L_1(\Omega),\sigma,-s}\\&\qquad+\varepsilon^2h^{-1}\|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla(g^\varepsilon-\chi)\|_{L_1(\Omega),\sigma,-s}+\varepsilon^2\|e^{c\frac{|x-y|}{\varepsilon+h}}D^2(g^\varepsilon-\chi)\|^{(h)}_{L_1(\Omega),\sigma,-s}\Bigg).\end{aligned}\tag{5.8}$$

Thus in order to prove the theorem we need to show that

$$\begin{aligned}I_1=\varepsilon^2h^{-1}&\|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla(g^\varepsilon-\chi)\|_{L_1(\Omega),\sigma,-s}\\&+\varepsilon^2\|e^{c\frac{|x-y|}{\varepsilon+h}}D^2(g^\varepsilon-\chi)\|^{(h)}_{L_1(\Omega),\sigma,-s}\le C\ell_h\end{aligned}\tag{5.9}$$

and

$$I_2=\varepsilon^2h^{-1}\|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla F^\varepsilon\|_{L_1(\Omega),\sigma,-s}+\|e^{c\frac{|x-y|}{\varepsilon+h}}F^\varepsilon\|_{L_1(\Omega),\sigma,-s}\le C\ell_h.$$

6. Proof of the main results: Part 2

To prove (5.9), we shall decompose Ω into “annuli”. For j an integer, let $d_j=2^{-j}$ and $\Omega_j=\{y\in\Omega:d_j\le|y-x|\le 2d_j\}$. Then, with J_0 fixed such that $|y-x|\le 2d_{J_0}=2^{1-J_0}$ in Ω, and any $J_*>J_0$,

$$\Omega=\left(\bigcup_{j=J_0}^{J_*}\Omega_j\right)\cup\Omega_*,\ \text{where }\Omega_*=\{y\in\Omega:|y-x|\le d_{J_*}\}.$$

We shall refer to Ω_* as the "innermost" set. Ultimately, we shall choose $J_* = J_*(h)$ such that $d_{J_*} \approx C_* h$ for small h, where C_* is a sufficiently large number to be chosen later. Note that then $J_* \approx C|\log h|$. Constants C and c will, as usual, change freely but will be independent of C_*. We shall write $\sum_{*,j}$ when the innermost set is included and $\sum_j$ when it is not. We also define $\Omega_j' = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$, $\Omega_j'' = (\Omega_j')'$, and so on.

Proposition 6.1. *There exist constants C and c such that $I_1 \le CC_*^{N/2+s} e^{c\frac{C_* h}{\varepsilon+h}} + C\ell_h$, where I_1 is defined in* (5.9).

Proof. First we shall bound the second term in I_1 on Ω_*. Since on Ω_* the weights $e^{c\frac{|y-x|}{\varepsilon+h}} \le e^{c\frac{C_* h}{\varepsilon+h}}$ and $\sigma \le C_*$, it is sufficient to estimate $\|g^\varepsilon - I_h g^\varepsilon\|_{W_1^2(\Omega_*)}$. Using the Cauchy-Schwarz inequality, the local approximation assumption 2.3, a priori estimate of Lemma 5.1 and (5.2), we have

$$\|g^\varepsilon - I_h g^\varepsilon\|_{W_1^2(\Omega_*)} \le C(C_* h)^{N/2}\|g^\varepsilon\|_{W_2^2(\Omega)} \le C(C_* h)^{N/2}\varepsilon^{-2}\|\eta\|_{L_2(\Omega)} \le CC_*^{N/2}\varepsilon^{-2}.$$

To estimate I_1 on $\Omega \backslash \Omega_*'$ we use the representation $g^\varepsilon(x) = \int_\Omega K^\varepsilon(x,y)\eta(y)dy$. The Green's function $K^\varepsilon(x,y)$ is singular only for $x = y$. Hence if $x \notin supp(\eta)$, the representation $D^\alpha g^\varepsilon(x) = \int_\Omega D_x^\alpha K^\varepsilon(x,y)\eta(y)dy$ is valid for multi-index α.

Using local approximation 2.3 and Lemma 2.2, for any $|\alpha| = r$ and $c < c_0$ we have

$$\begin{aligned}
&\|e^{c\frac{|y-x|}{\varepsilon+h}} D^2(g^\varepsilon - I_h g^\varepsilon)\|^{(h)}_{L^1(\Omega\backslash\Omega_*'),\sigma,-s} \le \sum_{j=J_0}^{J_*-1}(d_j/h)^s e^{\frac{cd_j}{h+\varepsilon}}\|g^\varepsilon - I_h g^\varepsilon\|^{(h)}_{W_1^2(\Omega_j)} \\
&\le C\sum_{j=J_0}^{J_*-1}(d_j/h)^s e^{\frac{cd_j}{h+\varepsilon}} h^{r-2}\|g^\varepsilon\|_{W_1^r(\Omega_j)} \\
&\le C\varepsilon^{-2}\sum_{j=J_0}^{J_*-1}(d_j/h)^s e^{\frac{cd_j}{h+\varepsilon}} h^{r-2} d_j^N e^{-c_0\frac{d_j}{\varepsilon}} d_j^{2-N-r}\|\eta\|_{L_1(\Omega)} \\
&\le C\varepsilon^{-2}\sum_{j=J_0}^{J_*-1}\left(\frac{h}{d_j}\right)^{r-2-s} e^{-\tilde{c}\frac{d_j}{\varepsilon}} \le \begin{cases} C\varepsilon^{-2}, & \text{if } r-2 > s \text{ or } \varepsilon = O(h), \\ C\varepsilon^{-2}|\log h|, & \text{if } r-2 = s \text{ and } \varepsilon \gg h. \end{cases}
\end{aligned}$$

The proof is very similar for the other term in I_1. □

To conclude the proof of Theorem 2.1, it remains to prove the following result.

Proposition 6.2. *There exist constants c, C, and C_*, with the latter large enough, such that $I_2 \le CC_*^{N/2+s} e^{c\frac{C_* h}{h+\varepsilon}} + C\ell_h$, where I_2 is defined in* (5.9).

Proof. In this proof, almost all norms occurring in the estimates will be L_2 based. We shall write $\|v\|_D$ for L_2-norms over a set D and $\|v\|_{k,D}$ when up to k spatial derivatives are included.

Using Cauchy-Schwarz inequality

$$I_2 \le \sum_{*,j}(d_j/h)^s(2d_j)^{N/2} e^{\frac{cd_j}{h+\varepsilon}}\left(\varepsilon^2 h^{-1}\|\nabla F^\varepsilon\|_{\Omega_j} + \|F^\varepsilon\|_{\Omega_j}\right).$$

The part of I_2 over Ω_*, which we will call I_2^*, can be bounded by

$$\begin{aligned}
I_2^* &\le CC_*^{N/2+s} h^{N/2} e^{c\frac{C_* h}{h+\varepsilon}}\left(\|F^\varepsilon\|_\Omega + \varepsilon^2 h^{-1}\|\nabla F^\varepsilon\|_\Omega\right) \\
&\le CC_*^{N/2+s} h^{N/2} e^{c\frac{C_* h}{h+\varepsilon}}\left(\|g^\varepsilon\|_\Omega + \|g_h^\varepsilon\|_\Omega + \varepsilon^2\|g^\varepsilon\|_{2,\Omega}\right)
\end{aligned} \tag{6.1}$$

by using the global estimate from Lemma 3.1. Using a priori estimates in Lemma 5.1 and the fact that $\|g_h^\varepsilon\|_\Omega \le \|\eta\|_\Omega$, we get

$$I_2^* \le CC_*^{N/2+s} h^{N/2} e^{c\frac{C_* h}{h+\varepsilon}} \|\eta\|_\Omega \le CC_*^{N/2+s} e^{c\frac{C_* h}{h+\varepsilon}}. \tag{6.2}$$

The remaining terms are bounded by $Cd_j^{N/2}(d_j/h)^s e^{\frac{cd_j}{h+\varepsilon}} M_j$, where

$$M_j = \|F^\varepsilon\|_{\Omega_j} + \varepsilon^2 h^{-1} \|\nabla F^\varepsilon\|_{\Omega_j}. \tag{6.3}$$

Thus so far we have

$$I_2 \le CC_*^{N/2+s} e^{c\frac{C_* h}{h+\varepsilon}} + CM, \text{ where } M = \sum_j d_j^{N/2}(d_j/h)^s e^{\frac{cd_j}{h+\varepsilon}} M_j. \tag{6.4}$$

To treat the terms involved in M_j, we shall consider two cases, $\varepsilon \le h$ and $\varepsilon > h$.

6.1. **Case 1:** $\varepsilon \le h$**.**

$$M \le \sum_j d_j^{N/2} (d_j/h)^s e^{\frac{cd_j}{\varepsilon+h}} \big(\|g^\varepsilon\|_{\Omega_j} + \|g_h^\varepsilon\|_{\Omega_j} + \varepsilon^2 h^{-1} \|\nabla g^\varepsilon\|_{\Omega_j} + \varepsilon^2 h^{-1} \|\nabla g_h^\varepsilon\|_{\Omega_j}\big).$$

Using the Green's function representation and Lemma 2.2 for $N \ge 3$, we have

$$|g^\varepsilon(x)| \le \int_\Omega |K^\varepsilon(x,y)| \cdot |\eta(y)| dy \le C\varepsilon^{-2} d_j^{2-N} e^{-c_0\frac{d_j}{\varepsilon}} \|\eta\|_{L_1(\Omega)}.$$

Hence,

$$\|g^\varepsilon\|_{\Omega_j} \le Cd_j^{N/2} \varepsilon^{-2} d_j^{2-N} e^{-c_0\frac{d_j}{\varepsilon}} \|\eta\|_{L_1} \le Cd_j^{2-N/2} \varepsilon^{-2} e^{-c_0\frac{d_j}{\varepsilon}}.$$

Using the fact that $\varepsilon \le h$,

$$\sum_j d_j^{N/2}(d_j/h)^s e^{\frac{cd_j}{\varepsilon+h}} \|g^\varepsilon\|_{\Omega_j} \le C \sum_j e^{-\tilde{c}\frac{d_j}{\varepsilon}} (d_j/\varepsilon)^{2+s} \le C. \tag{6.5}$$

Very similarly

$$\|\nabla g^\varepsilon\|_{\Omega_j} \le Cd_j^{1-N/2} \varepsilon^{-2} e^{-c_0\frac{d_j}{\varepsilon}},$$

and using the fact that $\varepsilon \le h$,

$$\sum_j d_j^{N/2}(d_j/h)^s e^{\frac{cd_j}{\varepsilon+h}} \varepsilon^2 h^{-1} \|\nabla g^\varepsilon\|_{\Omega_j} \le C \sum_j e^{-\tilde{c}\frac{d_j}{\varepsilon}} (d_j/\varepsilon)^{1+s} \le C. \tag{6.6}$$

The case $N = 2$ is similar, and we leave it to the reader.

Applying Lemma 4.1 to $\|g_h^\varepsilon\|_{\Omega_j}$ and $\|\nabla g_h^\varepsilon\|_{\Omega_j}$, we get

$$\|g_h^\varepsilon\|_{\Omega_j} + d_j \|\nabla g_h^\varepsilon\|_{\Omega_j} \le Ce^{-c_1\frac{d_j}{h}} \|g_h^\varepsilon\|_{\Omega_j'} \le Ce^{-c_1\frac{d_j}{h}} \|\eta\|_\Omega \le Ce^{-c_1\frac{d_j}{h}} h^{-N/2}.$$

Thus again using the fact that $\varepsilon \le h$,

$$\begin{aligned} &\sum_j d_j^{N/2}(d_j/h)^s e^{\frac{cd_j}{h}} (\|g_h^\varepsilon\|_{\Omega_j} + \varepsilon^2 h^{-1} \|\nabla g_h^\varepsilon\|_{\Omega_j}) \\ &\le C \sum_j e^{-\tilde{c}\frac{d_j}{h}} (d_j/h)^{N/2+s} + C \sum_j e^{-\tilde{c}\frac{d_j}{h}} (d_j/h)^{N/2+1+s} \le C. \end{aligned} \tag{6.7}$$

Combining estimates (6.5), (6.6), and (6.7) we complete the proof in the case when $\varepsilon \le h$.

6.2. **Case 2:** $\varepsilon > h$**.** To treat the terms involved in M_j in (6.4), we shall use the local energy-based estimates from Section 4.

By Lemma 4.2, we have

$$\|\nabla F^\varepsilon\|_{\Omega_j} \le C\left(\|\nabla(g^\varepsilon - \chi)\|_{\Omega_j'} + d_j^{-1}\|g^\varepsilon - \chi\|_{\Omega_j'}\right) + Cd_j^{-1}e^{-\frac{c_1 d_j}{\varepsilon+h}}\|F^\varepsilon\|_{\Omega_j'}, \tag{6.8}$$

for any $\chi \in S_h^r$. Taking $\chi = I_h g^\varepsilon$, using Green's function representation and the fact that $h < d_j$, we can estimate the first two terms in (6.8) as

$$\begin{aligned}&\|\nabla(g^\varepsilon - I_h g^\varepsilon)\|_{\Omega_j'} + d_j^{-1}\|g^\varepsilon - I_h g^\varepsilon\|_{\Omega_j'}\\ &\qquad \le Ch^{r-1}\|g^\varepsilon\|_{r,\Omega_j''} \le Ch^{r-1}\varepsilon^{-2}e^{-c_0\frac{d_j}{\varepsilon}}d_j^{2-N/2-r}.\end{aligned} \tag{6.9}$$

Hence the contribution to M is bounded by

$$\begin{aligned}&C\sum_j d_j^{N/2}(d_j/h)^s e^{\frac{cd_j}{h+\varepsilon}}h^{r-2}e^{-c_0\frac{d_j}{\varepsilon}}d_j^{2-N/2-r}\\ &\qquad \le C\sum_j (h/d_j)^{r-2-s}e^{-\tilde{c}\frac{d_j}{\varepsilon}} \le C\ell_h.\end{aligned} \tag{6.10}$$

We now apply Lemma 4.3 to the other term in M_j, namely $\|F^\varepsilon\|_{\Omega_j}$:

$$\|F^\varepsilon\|_{\Omega_j} \le Ch\left(\|\nabla(g^\varepsilon - \chi)\|_{\Omega_j'} + d_j^{-1}\|g^\varepsilon - \chi\|_{\Omega_j'}\right) + Ce^{-\frac{c_1 d_j}{\varepsilon+h}}\|F^\varepsilon\|_{\Omega_j'}. \tag{6.11}$$

Using estimates (6.9) and the fact that $\varepsilon > h$, we see that the contribution to M is bounded by

$$\begin{aligned}&C\sum_j d_j^{N/2}(d_j/h)^s e^{\frac{cd_j}{h+\varepsilon}}h^{r}\varepsilon^{-2}e^{-c_0\frac{d_j}{\varepsilon}}d_j^{2-N/2-r}\\ &\qquad \le C\sum_j (h/d_j)^{r-2-s}e^{-\tilde{c}\frac{d_j}{\varepsilon}} \le C\ell_h.\end{aligned} \tag{6.12}$$

Thus we have

$$M \le C\ell_h + C\sum_j d_j^{N/2}(d_j/h)^s\left(1 + \varepsilon^2 h^{-1}d_j^{-1}\right)e^{-\tilde{c}\frac{d_j}{\varepsilon}}\|F^\varepsilon\|_{\Omega_j'}. \tag{6.13}$$

In the following lemma we will estimate $\|F^\varepsilon\|_{\Omega_j'}$ by a duality argument.

Lemma 6.3. *The following estimate holds:*

$$\begin{aligned}\|F^\varepsilon\|_{\Omega_j'} &\le Ch\|\nabla F^\varepsilon\|_{\Omega_j'''} + Ch^2\varepsilon^{-2}\|F^\varepsilon\|_{\Omega_j'''}\\ &\quad + Ch^r e^{-c_0\frac{d_j}{\varepsilon}}d_j^{2-N/2-r}\varepsilon^{-2}\left(\varepsilon^2 h^{-1}\|\nabla F^\varepsilon\|_{L_1(\Omega)} + \|F^\varepsilon\|_{L_1(\Omega)}\right).\end{aligned}$$

Proof. Using $(v,w)_D$ for the L_2 inner product over a set D, we have

$$\|F^\varepsilon\|_{\Omega_j'} = sup\{(F^\varepsilon, v)_\Omega : \text{supp } v \subset \Omega_j',\ \|v\|_{\Omega_j'} = 1\}. \tag{6.14}$$

For each such fixed v, let w solve the dual problem $-\varepsilon^2\Delta w+w=v$ in Ω. Integrating by parts, we obtain for any $\chi\in S_h^r$,

$$
\begin{aligned}
(F^\varepsilon,v)_\Omega &= \varepsilon^2(\nabla F^\varepsilon,\nabla w)_\Omega+(F^\varepsilon,w)_\Omega\\
&= \varepsilon^2(\nabla F^\varepsilon,\nabla(w-\chi))_\Omega+(F^\varepsilon,w-\chi)_\Omega\\
&= \varepsilon^2(\nabla F^\varepsilon,\nabla(w-\chi))_{\Omega_j'''}+(F^\varepsilon,w-\chi)_{\Omega_j'''}\\
&\quad+\varepsilon^2(\nabla F^\varepsilon,\nabla(w-\chi))_{\Omega\setminus\Omega_j'''}+(F^\varepsilon,w-\chi)_{\Omega\setminus\Omega_j'''}\\
&\le \varepsilon^2\|\nabla F^\varepsilon\|_{\Omega_j'''}\|\nabla(w-\chi)\|_\Omega+\|F^\varepsilon\|_{\Omega_j'''}\|w-\chi\|_\Omega\\
&\quad+\varepsilon^2\|\nabla F^\varepsilon\|_{L_1(\Omega)}\|\nabla(w-\chi)\|_{L_\infty(\Omega\setminus\Omega_j''')}\\
&\quad+\|F^\varepsilon\|_{L_1(\Omega)}\|w-\chi\|_{L_\infty(\Omega\setminus\Omega_j''')}.
\end{aligned}
\tag{6.15}
$$

Take $\chi=I_hw$. Using the approximation and the global stability, we obtain

$$
\|w-\chi\|_\Omega+h\|\nabla(w-\chi)\|_\Omega\le Ch^2\|w\|_{H^2(\Omega)}\le C\frac{h^2}{\varepsilon^2}\|v\|_\Omega=C\frac{h^2}{\varepsilon^2},
\tag{6.16}
$$

and

$$
\begin{aligned}
&\|w-\chi\|_{L_\infty(\Omega\setminus\Omega_j''')}+h\|\nabla(w-\chi)\|_{L_\infty(\Omega\setminus\Omega_j''')}\le Ch^r\|w\|_{W_\infty^r(\Omega\setminus\Omega_j'')}\\
&\qquad\le C\frac{h^r}{\varepsilon^2}e^{-c_0\frac{d_j}{\varepsilon}}d_j^{2-N/2-r}.
\end{aligned}
\tag{6.17}
$$

In the last estimate we used the Green's function representation, Lemma 2.2, and Cauchy-Schwarz inequality, i.e.

$$
\begin{aligned}
|D^rw(x)|\le C\int_{\Omega_j'}|D_x^rK(x,y)v(y)|dy &\le C\varepsilon^{-2}e^{-c_0\frac{d_j}{\varepsilon}}d_j^{2-N-r}\|v\|_{L_1(\Omega_j')}\\
&\le C\varepsilon^{-2}e^{-c_0\frac{d_j}{\varepsilon}}d_j^{2-N/2-r}.
\end{aligned}
$$

Combining estimates (6.15), (6.16) , (6.17), and taking the supremum over v, we have the lemma. □

Now we are ready to conclude the proof of Proposition 6.2. By the lemma above and (6.13), we have

$$
\begin{aligned}
M&\le C\ell_h+C\sum_j d_j^{N/2}(d_j/h)^s\left(\varepsilon^{-2}h^2+hd_j^{-1}\right)\left(\varepsilon^2h^{-1}\|\nabla F^\varepsilon\|_{\Omega_j'''}+\|F^\varepsilon\|_{\Omega_j'''}\right)\\
&\quad+C\left(\varepsilon^2h^{-1}\|\nabla F^\varepsilon\|_{L_1(\Omega)}+\|F^\varepsilon\|_{L_1(\Omega)}\right)\sum_j(d_j/h)^s\left(\varepsilon^{-2}h^2+hd_j^{-1}\right)h^{r-2}e^{-c\frac{d_j}{\varepsilon}}d_j^{2-r}.
\end{aligned}
\tag{6.18}
$$

In the first sum on the right hand side we can replace $\varepsilon^2h^{-1}\|\nabla F^\varepsilon\|_{\Omega_j'''}+\|F^\varepsilon\|_{\Omega_j'''}$ by $\varepsilon^2h^{-1}\|\nabla F^\varepsilon\|_{\Omega_j}+\|F^\varepsilon\|_{\Omega_j}$. This multiplies the sum at most by seven. The overshooting contribution near the innermost Ω_* is estimated as before by $CC_*^{N/2+s}e^{c\frac{C_*h}{h+\varepsilon}}$.

Using that $\sigma^{-s} \geq 1$ and $e^{c\frac{|x-y|}{\varepsilon+h}} \geq 1$, and the inequality $e^{-c\frac{d_j}{\varepsilon}} \leq C(\frac{\varepsilon}{d_j})^p$ for any $p > 0$, from (6.18) we obtain

$$
\begin{aligned}
M &\leq CC_*^{N/2+s} e^{c\frac{C_* h}{h+\varepsilon}} \\
&\quad + C\ell_h + C\sum_j d_j^{N/2}(d_j/h)^s h d_j^{-1} e^{\frac{c d_j}{\varepsilon+h}} \left(\varepsilon^2 h^{-1}\|\nabla F^\varepsilon\|_{\Omega_j} + \|F^\varepsilon\|_{\Omega_j}\right) \\
&\quad + C\left(\|e^{c\frac{|x-y|}{\varepsilon+h}}\nabla F^\varepsilon\|_{L_1(\Omega),\sigma,-s} + \varepsilon^2 h^{-1}\|e^{c\frac{|x-y|}{\varepsilon+h}} F^\varepsilon\|_{L_1(\Omega),\sigma,-s}\right)\sum_j (h/d_j)^{r-1-s}.
\end{aligned}
$$

Recalling the definitions of I_2, M_j, and M, (5.9), (6.3), and (6.4) respectively, and using that $h/d_j \leq C_*^{-1}$, we have

$$M \leq CC_*^{N/2+s} e^{c\frac{C_* h}{h+\varepsilon}} + C\ell_h + CC_*^{-1}M + I_2 C\sum_j (h/d_j)^{r-1-s}. \tag{6.19}$$

By choosing C_* large enough, from (6.19) we can conclude that

$$M \leq CC_*^{N/2+s} e^{c\frac{C_* h}{h+\varepsilon}} + C\ell_h + I_2 C\sum_j (h/d_j)^{r-1-s}. \tag{6.20}$$

Inserting it into (6.4), we have

$$I_2 \leq CC_*^{N/2+s} e^{c\frac{C_* h}{h+\varepsilon}} + C\ell_h + I_2 C\sum_j (h/d_j)^{r-1-s}.$$

Since $r - 1 - s > 1$, choosing C_* once again large enough, we can conclude that

$$I_2 \leq CC_*^{N/2+s} e^{c\frac{C_* h}{h+\varepsilon}} + C\ell_h.$$

Thus the proof of Proposition 6.2 is complete. □

7. Appendix. Proof of Lemma 2.2

Proof. To show the estimates for $K^\varepsilon(x,y)$, we use the Green's function $G(x,y;t)$ for the parabolic problem

$$
\begin{aligned}
G_t(x,y;t) - \Delta G(x,y;t) &= 0 \quad \text{in } \Omega,\ t > 0, \\
\frac{\partial G(x,y;t)}{\partial n} &= 0 \quad \text{on } \partial\Omega, \\
G(x,y;0) &= \delta_x(y).
\end{aligned} \tag{7.1}
$$

Since u satisfies

$$
\begin{aligned}
-\Delta u + \frac{u}{\varepsilon^2} &= \frac{f}{\varepsilon^2}, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } \partial\Omega,
\end{aligned}
$$

by the Theorem 4 in [3], we have the following representation:

$$u(x) = \int_\Omega \left[\int_0^\infty e^{-\frac{z}{\varepsilon^2}} G(x,y;z)dz\right] \frac{f(y)}{\varepsilon^2} dy, \tag{7.2}$$

where G is the Green's function for the parabolic problem. With a change of variables $t = z/\varepsilon^2$, we obtain

$$\varepsilon^{-2}\int_0^\infty e^{-\frac{z}{\varepsilon^2}} G(x,y;z)dz = \int_0^\infty e^{-t} G(x,y;\varepsilon^2 t)dt.$$

Define

$$K^\varepsilon(x,y) = \int_0^\infty e^{-t} G(x,y;\varepsilon^2 t)dt.$$

Thus we have the following representation:

$$u(x) = \int_\Omega K^\varepsilon(x,y) f(y) dy. \tag{7.3}$$

Since the coefficients in the parabolic equation (7.1) are time independent, we have the following estimate for the parabolic Green's function.

Lemma 7.1. *Assume that $\partial\Omega$ in problem* (7.1) *is sufficiently smooth. Then for any multi-index m there exist constants c_2, c_3, C such that for $0 < t < \infty$,*

$$|D_x^m G(x,y;t)| \le C t^{-\frac{N+|m|}{2}} e^{c_2 t - c_3 \frac{|x-y|^2}{t}}.$$

The proof of this result can be found in [3], Theorem 3 in particular.

Using Lemma 7.1, we have

$$|D_x^m K^\varepsilon(x,y)| \le C \int_0^\infty \frac{e^{-t(1-\varepsilon^2 c_2) - c_3 \frac{|x-y|^2}{\varepsilon^2 t}}}{(\varepsilon^2 t)^{\frac{N+|m|}{2}}} dt. \tag{7.4}$$

To estimate this integral we use the following lemma.

Lemma 7.2. *There exist constants C and c_0 independent of d such that*

$$\int_0^\infty \frac{e^{-c_4 t - c_3 \frac{d^2}{t}}}{t^{M/2}} dt \le C e^{-c_0 d} \begin{cases} 1, & \text{if } M = 1, \\ 1 + |\log d|, & \text{if } M = 2, \\ d^{2-M}, & \text{if } M > 2. \end{cases}$$

Proof. The proof is adapted from [3]. First we split the integral into two parts.

$$\int_0^\infty \frac{e^{-c_4 t - c_3 \frac{d^2}{t}}}{t^{M/2}} dt = \int_0^1 \frac{e^{-c_4 t - c_3 \frac{d^2}{t}}}{t^{M/2}} dt + \int_1^\infty \frac{e^{-c_4 t - c_3 \frac{d^2}{t}}}{t^{M/2}} dt = I_1 + I_2.$$

In order to estimate I_1, we consider two cases, $d \le 1$ and $d > 1$:

Case 1: $d \le 1$,

$$I_1 \le \int_0^1 \frac{e^{-c_3 \frac{d^2}{t}}}{t^{M/2}} dt.$$

For $M > 2$, by making a change of variables $z = \frac{d}{\sqrt{t}}$, we have

$$I_1 \le \frac{2}{d^{M-2}} \int_d^\infty e^{-c_3 z^2} z^{M-3} dz \le \frac{C}{d^{M-2}}.$$

For $M = 2$ by letting $z = c_3 d^2$ and making a change of variables $w = \frac{z}{t}$, we have

$$I_1 \le \int_0^1 \frac{e^{-\frac{z}{t}}}{t} dt = \int_z^\infty \frac{e^{-w}}{w} dw \le \left| \int_z^1 \frac{dw}{w} \right| + \int_1^\infty e^{-w} dw = |\log z| + e^{-1}.$$

Finally for $M = 1$,

$$I_1 \le \int_0^1 \frac{e^{-c_3 \frac{d^2}{t}}}{\sqrt{t}} dt \le \int_0^1 \frac{1}{\sqrt{t}} dt = 2.$$

Case 2: $d > 1$,

$$I_1 = \int_0^1 \frac{e^{-c_4 t - c_3 \frac{d^2}{t}}}{t^{M/2}} dt = \int_0^1 \frac{e^{-c_4 t - c_3 \frac{d^2}{2t}} e^{-c_3 \frac{d^2}{2t}}}{t^{M/2}} dt. \tag{7.5}$$

The function $-c_4 t - \frac{c_3 d^2}{2t}$ has a maximum at $t = d\sqrt{\frac{c_4}{2c_3}}$ equal to $-d\sqrt{2c_4c_3}$, and the function $e^{-c_3\frac{d^2}{2t}}t^{-M/2}$ has a maximum at $t = \frac{2c_3d^2}{M}$ equal to $e^{-M/2}\left(\frac{M}{2c_3d^2}\right)^{M/2}$. Thus,

$$I_1 \leq e^{-d\sqrt{2c_4c_3}} \int_0^1 \frac{e^{-c_3\frac{d^2}{2t}}}{t^{M/2}}dt \leq Ce^{-d\sqrt{2c_4c_3}}. \tag{7.6}$$

Now we estimate I_2 for any $d > 0$. We have

$$I_2 = \int_1^\infty \frac{e^{-c_4t - c_3\frac{d^2}{t}}}{t^{M/2}}dt = \int_1^\infty \frac{e^{-\frac{c_4t}{2} - c_3\frac{d^2}{t}}e^{-\frac{c_4t}{2}}}{t^{M/2}}dt. \tag{7.7}$$

Again using that $-\frac{c_4t}{2} - \frac{c_3d^2}{t}$ has a maximum at $t = d\sqrt{\frac{2c_4}{c_3}}$ equal to $-2d\sqrt{2c_4c_3}$, we have

$$I_2 \leq e^{-2d\sqrt{2c_4c_3}} \int_1^\infty \frac{e^{-\frac{c_4t}{2}}}{t^{M/2}}dt \leq Ce^{-2d\sqrt{2c_4c_3}}, \tag{7.8}$$

and the proof of Lemma 7.2 is complete. □

Provided that $1 - \varepsilon^2 c_2 > 0$, we apply the previous lemma with $d = \frac{|x-y|}{\varepsilon}$ and $c_4 = 1 - \varepsilon^2 c_2$, to conclude the proof of Lemma 2.2. □

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