

## A COUNTEREXAMPLE CONCERNING THE $L_2$ -PROJECTOR ONTO LINEAR SPLINE SPACES

PETER OSWALD

ABSTRACT. For the  $L_2$ -orthogonal projection  $P_V$  onto spaces of linear splines over simplicial partitions in polyhedral domains in  $\mathbb{R}^d$ ,  $d > 1$ , we show that in contrast to the one-dimensional case, where  $\|P_V\|_{L_\infty \rightarrow L_\infty} \leq 3$  independently of the nature of the partition, in higher dimensions the  $L_\infty$ -norm of  $P_V$  cannot be bounded uniformly with respect to the partition. This fact is folklore among specialists in finite element methods and approximation theory but seemingly has never been formally proved.

### 1. INTRODUCTION

Variational methods based on piecewise polynomial approximations are a workhorse in numerical methods for PDEs and data analysis. In particular, least-squares methods lead to the study of the  $L_2$ -orthogonal projection operator  $P_V : L_2(\Omega) \rightarrow V$  onto a given spline space  $V$  defined on a domain  $\Omega \in \mathbb{R}^d$ . A question of considerable interest is the uniform boundedness of the  $L_\infty$ -norm

$$\|P_V\|_{L_\infty \rightarrow L_\infty} := \max_{f \in L_\infty(\Omega): \|f\|_{L_\infty} = 1} \|P_V f\|_{L_\infty}$$

of  $P_V$  with respect to families of spline spaces  $V$ . If  $\Omega$  is a bounded interval in  $\mathbb{R}^1$ , then the question has been intensively studied for the family of spaces of smooth splines of fixed degree  $r$  over arbitrary partitions, where Shadrin [7] has recently established that

$$(1) \quad \|P_V\|_{L_\infty \rightarrow L_\infty} \leq C(r) < \infty, \quad r \geq 1,$$

for any partition. This result was known for a long time for small values of  $r$ , e.g., Ciesielski [3] proved that for linear splines one can take  $C(1) = 3$ , while de Boor [2] solved the case  $r \leq 4$ . The estimate (1) plays an important role in numerical analysis and for the investigation of orthonormal spline systems such as the Franklin system in  $L_p$ -based scales of function spaces,  $1 \leq p \leq \infty$ .

In higher dimensions, the study of the  $L_\infty$ -norm of  $P_V$  arose mostly in the context of obtaining  $L_p$  error estimates in the finite element Galerkin method [5, 6], where sufficient conditions on the underlying partition and nodal basis  $\{\phi_i\}$  of a finite element space  $V$  are formulated under which the norms  $\|P_V\|_{L_\infty \rightarrow L_\infty}$  are bounded by a certain finite constant. Interestingly enough, these results suggest that such conditions on partitions or finite element type are essential for obtaining uniform bounds but formal proof of their *necessity* was not given.

---

Received by the editor December 20, 2006.

2000 *Mathematics Subject Classification*. Primary 65N30, 41A15.

©2007 American Mathematical Society  
Reverts to public domain 28 years from publication

Similarly, in the theory of multivariate splines final results on the uniform boundedness of  $\|P_V\|_{L_\infty \rightarrow L_\infty}$  could not be localized. Recently, Ciesielski [4] asked about the extension of his result for linear splines [3] to the higher-dimensional case, and the unanimous opinion of the audience was that in higher dimensions a similar result cannot hold. However, other than a vague reference to unpublished work by A. A. Privalov, no concrete proof could be found.

It is the aim of this paper to provide an elementary example of triangulations  $\mathcal{T}_J$  of a square  $\Omega \subset \mathbb{R}^2$  into  $O(J)$  triangles for which

$$(2) \quad \|P_{V(\mathcal{T}_J)}\|_{L_\infty \rightarrow L_\infty} \geq J, \quad J \geq 1,$$

where  $V(\mathcal{T}_J)$  is the space of linear  $C^0$  splines (or finite element functions) on  $\mathcal{T}_J$ ; see Theorem 1 below. As  $J \rightarrow \infty$ , the triangulations  $\mathcal{T}_J$  will not satisfy the minimum angle condition, which is natural since for these types of triangulations a uniform bound can easily be established. However, they satisfy the maximum angle condition, [1], and do not possess vertices of high valence. The example can easily be extended to  $d \geq 3$ , and implies that  $C(1) = \infty$  for all  $d > 1$ .

## 2. NOTATION AND RESULT

We concentrate on  $d = 2$ , the case  $d \geq 3$  is mentioned in Section 3. Let  $\Omega \subset \mathbb{R}^2$  be a bounded polygonal domain equipped with a finite triangulation  $\mathcal{T}$  into non-degenerate closed triangles  $\Delta$  satisfying the usual regularity condition that two different triangles may intersect at a common vertex respectively, edge only. The set of vertices of  $\mathcal{T}$  is denoted by  $\mathcal{V}_\mathcal{T}$ . Let  $|E|$  denote the Lebesgue measure of a measurable set  $E \subset \mathbb{R}^2$ . By  $0 < \underline{\alpha}_\mathcal{T} \leq \bar{\alpha}_\mathcal{T} < \pi$  we denote the minimal and maximal interior angle of all triangles in  $\mathcal{T}$ .

Let  $V(\mathcal{T})$  denote the linear space of all continuous functions  $g$  whose restriction to any of the triangles  $\Delta \in \mathcal{T}$  is a linear polynomial. Any  $g \in V(\mathcal{T})$  has a unique representation of the form w.r.t. the interpolating nodal basis  $\Phi_\mathcal{T} := \{\phi_P, P \in \mathcal{V}_\mathcal{T}\}$ :

$$(3) \quad g = \sum_{P \in \mathcal{V}_\mathcal{T}} g(P) \phi_P,$$

where the Courant hat functions  $\phi_P \in V(\mathcal{T})$  are characterized by the conditions  $\phi_P(P) = 1$ , and  $\phi_P(Q) = 0$ , where  $Q \neq P$  is any of the remaining vertices of  $\mathcal{T}$ . Thus,  $\dim V(\mathcal{T}) = \#\mathcal{V}_\mathcal{T}$ , and

$$\text{supp } \phi_P = \Omega_P := \bigcup_{\Delta \in \mathcal{T}: P \in \Delta} \Delta.$$

The set  $\Omega_P$  corresponds to the 1-ring neighborhood of  $P$  in  $\mathcal{T}$ , and we denote by  $\mathcal{V}_P = \{Q \in \Omega_P \cap \mathcal{V} : Q \neq P\}$  the set of all neighboring vertices to  $P$ .

The  $L_2$ -orthogonal projection of a function  $f \in L_2(\Omega)$  onto  $V(\mathcal{T})$  is given by the unique  $g := P_{V(\mathcal{T})}f \in V(\mathcal{T})$  such that

$$(f - g, \phi_P) = 0 \quad \forall P \in \mathcal{V}_\mathcal{T}.$$

Here and in the sequel,  $(\cdot, \cdot)$  stands for the  $L_2(\Omega)$  inner product. Using (3) with unknown nodal values  $x_P = g(P)$  as ansatz, this is equivalent to the linear system

$$(4) \quad \sum_{Q \in \mathcal{V}} (\phi_Q, \phi_P) x_Q = (f, \phi_P) \quad \forall P \in \mathcal{V}.$$

A simple calculation shows that

$$(\phi_P, \phi_P) = \frac{|\Omega_P|}{6}.$$

Similarly, if  $Q \in \mathcal{V}_P$  is a neighbor of  $P$ , then

$$(\phi_Q, \phi_P) = \sum_{\Delta \in \mathcal{T}: P, Q \in \Delta} \frac{|\Delta|}{12},$$

in all other cases we have  $(\phi_Q, \phi_P) = 0$ . This shows, in particular, that

$$(\phi_P, \phi_P) = \sum_{Q \neq P} (\phi_Q, \phi_P) = (1, \phi_P)/2.$$

For example, if we normalize (4) by  $(\phi_P, \phi_P)$ , then (4) turns into a linear system

$$(5) \quad Ax = b, \quad x := (x_Q : Q \in \mathcal{V}_T)^T, \quad b := (b_P = \frac{(f, \phi_P)}{(\phi_P, \phi_P)} : P \in \mathcal{V}_T)^T,$$

where  $\|b\|_\infty \leq 2\|f\|_{L_\infty}$ , and  $A := (a_{PQ})$  satisfies

$$(6) \quad a_{PQ} = \begin{cases} 1, & Q = P, \\ \frac{(\phi_Q, \phi_P)}{(\phi_P, \phi_P)} > 0, & Q \in \mathcal{V}_P, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(7) \quad 1 = \sum_{Q \neq P} a_{PQ} = \sum_{Q \in \mathcal{V}_P} a_{PQ}, \quad \forall P \in \mathcal{V}_T.$$

Thus, the matrix  $A$  is only weakly diagonally dominant, and not strictly diagonally dominant as in the one-dimensional case. Otherwise, we could estimate  $\|A^{-1}\|_\infty$  in a trivial way, and use the inequality

$$(8) \quad \|P_{V(\mathcal{T})}\|_{L_\infty \rightarrow L_\infty} \leq 2\|A^{-1}\|_\infty = 2 \max_{\|Ay\|_\infty \leq 1} \|y\|_\infty,$$

which follows from the above, in conjunction with the obvious equality  $\|P_{V(\mathcal{T})}f\|_{L_\infty} = \|x\|_\infty$ . Let us mention without proof that (8) implies the following partial result.

**Proposition 1.** *If for any two neighboring vertices  $P \neq Q$  from  $\mathcal{V}_T$  we have  $a_{PQ} \geq c_0 > 0$ , then*

$$\|P_{V(\mathcal{T})}\|_{L_\infty \rightarrow L_\infty} \leq (1 + 2c_0)c_0^{-2}.$$

For triangulations satisfying the minimum angle condition uniformly, i.e.,  $\underline{\alpha}_T \geq \underline{\alpha}_0 > 0$ , this result is applicable with  $c_0$  determined solely by  $\alpha_0$ , and thus covers the bounds considered in the finite element literature [5, 6]. The main result of this paper is the following:

**Theorem 1.** *For any  $J \geq 1$ , there is a triangulation  $\mathcal{T}_J$  of a square into  $8J + 4$  triangles such that the norm of the  $L_2$ -projector  $P_{V(\mathcal{T}_J)}$  satisfies*

$$\|P_{V(\mathcal{T}_J)}\|_{L_\infty \rightarrow L_\infty} \geq 2J.$$

Thus, for spatial dimension  $d = 2$  we have

$$C(1) := \sup_{\mathcal{T}} \|P_{V(\mathcal{T})}\|_{L_\infty \rightarrow L_\infty} = \infty.$$

We conjecture that in terms of the number of triangles our result is asymptotically sharp for bounded polygonal domains in  $\mathbb{R}^2$ , i.e.,

$$(9) \quad \sup_{\mathcal{T}: \#\mathcal{T} \leq N} \|P_V(\mathcal{T})\|_{L_\infty \rightarrow L_\infty} \asymp N, \quad N \rightarrow \infty.$$

That  $\Omega$  is a square is not crucial. The examples given below can easily be modified to show  $C(1) = \infty$  for simplicial partitions in higher dimensions as well.

### 3. PROOF OF THEOREM 1

We will use the following notation. Let  $S_a = [-a, a]^2$  be the square of side-length  $2a$ ,  $a > 0$ , with center at the origin. Set  $\Omega := S_1$ , with vertices denoted (in clockwise direction) by  $P_{0,i}$ ,  $i = 1, \dots, 4$ , and fix the parameter  $t \in (0, 1)$ . The triangulation  $\mathcal{T}_J$ ,  $J \geq 1$ , of  $\Omega$  is obtained by inserting the squares  $S_t, S_{t^2}, \dots, S_{t^J}$ , whose vertices will be denoted similarly by  $P_{j,i}$ ,  $i = 1, \dots, 4$ ,  $j = 1, \dots, J$ , placing an additional vertex  $P_{J+1}$  at the origin, connecting  $P_{J+1}$  by straight lines with the 4 vertices  $P_{0,i}$  of  $S_1$ , and finally subdividing each of the remaining trapezoidal regions  $P_{j-1,i-1}P_{j,i-1}P_{j,i}P_{j-1,i}$  into two triangles by connecting  $P_{j-1,i-1}$  with  $P_{j,i}$  in a consistent way. The outer rings of the resulting triangulation are shown in Figure 1a.

Let the function  $f \in L_\infty(\Omega)$  with  $\|f\|_{L_\infty} = 1$  be defined as follows:

$$f(x) = (-1)^j, \quad x \in \Omega_j := \begin{cases} S_{t^{j-1}} \setminus S_{t^j}, & j = 1, \dots, J, \\ S_{t^J}, & j = J + 1. \end{cases}$$

We use the same notation as above, and consider the linear system (5) corresponding to the  $L_2$ -orthogonal projection  $g = P_V(\mathcal{T}_J)f$  of this  $f$  onto  $V(\mathcal{T}_J)$ . Because of uniqueness of orthogonal projections and the rotational symmetry of  $\mathcal{T}_J$  and  $f$ , the entries of the vector  $x$  that represent the nodal values of  $g$  corresponding to the vertices  $P_{j,i}$ ,  $i = 1, \dots, 4$ , are equal, and will be denoted by  $x_j$ ,  $j = 0, \dots, J$  (the value at the origin is denoted by  $x_{J+1}$ ). Moreover, the 4 equations in (5) corresponding to the 4 vertices of a square  $S_{t^j}$ ,  $j = 0, \dots, J$ , can be replaced by one, thus turning the original system  $Ax = b$  of dimension  $4J + 5$  into a reduced tridiagonal system  $\tilde{A}\tilde{x} = \tilde{b}$  of dimension  $J + 2$  for the vector  $\tilde{x} = (x_0, \dots, x_{J+1})^T$ . Since the equations in system (5) are invariant under affine transformations, and

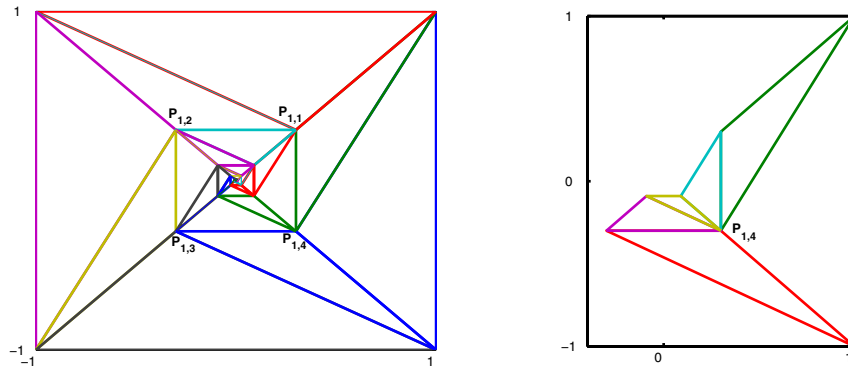


FIGURE 1. Triangulation  $\mathcal{T}_J$  for  $t = 0.3$  (left), and typical  $\Omega_{P_{j,i}}$  (right)

because of the definition of  $\mathcal{T}_J$  via squares  $S_{tj}$  shrinking at a fixed geometric rate, it is easy to see that, with the exception of the first and last two, all equations in  $\tilde{A}\tilde{x} = \tilde{b}$  have the same form:

$$(10) \quad \alpha x_{j-1} + \beta x_j + \gamma x_{j+1} = (-1)^j \delta, \quad j = 1, \dots, J - 1.$$

The coefficients can be found from the triangular neighborhood  $\Omega_{P_{j,i}}$  of any of the  $P_{j,i}$ ,  $j = 1, \dots, J - 1$  (see Figure 1 for an illustration), and the definitions leading to (5). We do not need their exact expressions, just their limit behavior as  $t \rightarrow 0$ , i.e., we will be looking for the entries of  $\hat{A} := \lim_{t \rightarrow 0} \tilde{A}$  and  $\hat{b} := \lim_{t \rightarrow 0} \tilde{b}$ . Indeed, since for  $t \rightarrow 0$  the whole  $\Omega_{P_{j,i}}$  is essentially covered by the single triangle with vertices  $P_{j,i}$ ,  $P_{j-1,i-1}$ ,  $P_{j-1,i}$ , and since  $f(x) = (-1)^j$  on the latter, we have

$$\alpha = 1 + O(t), \quad \beta = 1 + O(t), \quad \gamma = O(t^2), \quad \delta = 2 + O(t).$$

From (10), we obtain in the limit  $t \rightarrow 0$  the equations

$$\hat{x}_{j-1} + \hat{x}_j = 2(-1)^j, \quad j = 1, \dots, J - 1,$$

where  $\hat{x}_j = \lim_{t \rightarrow \infty} x_j$  (the existence of these limits follows from the invertibility of the limit matrix  $\hat{A}$ , see below). Similar considerations for the first and the last two equations of  $\tilde{A}\tilde{x} = \tilde{b}$  yield the remaining three equations of  $\hat{A}\hat{x} = \hat{b}$  as follows:

$$\frac{3}{2}\hat{x}_0 + \frac{1}{2}\hat{x}_1 = -2, \quad \hat{x}_{j-1} + \hat{x}_j = 2(-1)^j, \quad j = J, J + 1.$$

The resulting matrix  $\hat{A}$  is obviously invertible. After finding  $\hat{x}_0 = \hat{x}_1 = -1$  from the first two equations, forward substitution gives  $\hat{x}_j = (2j - 1)(-1)^j$ ,  $j = 2, \dots, J + 1$ . This implies that for any  $\epsilon > 0$  one can find a sufficiently small  $t > 0$  such that

$$\|\tilde{x}\|_\infty \geq \|\hat{x}\|_\infty - \epsilon = 2J + 1 - \epsilon.$$

This proves Theorem 1. Note that the above reasoning does not work for type-I triangulations of a square obtained from a non-uniform rectangular tensor-product partition.

We conclude with the straightforward extension of the above example to arbitrary  $d > 2$ . Let  $e^m$  denote the  $m$ -th unit coordinate vector in  $\mathbb{R}^d$ ,  $m = 1, \dots, d$ . As  $\Omega$  we take the convex polyhedral domain with vertices

$$P_{0,1} = e^1 + e^2, \quad P_{0,2} = e^1 - e^2, \quad P_{0,3} = -e^1 - e^2, \quad P_{0,4} = -e^1 + e^2,$$

and  $P'_m = e^m$ ,  $m = 3, \dots, d$ . For  $d = 3$ , this domain is a pyramid with square base in the  $xy$ -plane, and tip on the  $z$ -axis.

A suitable simplicial partition of  $\Omega$  is obtained as follows. The base square with vertices  $P_{0,i}$ ,  $i = 1, \dots, 4$ , is triangulated into  $\mathcal{T}_J$  which depends on the parameters  $0 < t < 1$  and  $J$  as described for  $d = 2$ . The resulting triangulation (now embedded into  $\mathbb{R}^d$ ) and its vertices are again denoted by  $\mathcal{T}_J$  resp. by  $P_{j,i}$ ,  $i = 1, \dots, 4$ ,  $j = 0, \dots, J$ , and  $P_{J+1}$ . Then each simplex in the associated simplicial partition  $\mathcal{P}_J$  of  $\Omega$  is generated by the  $d - 2$  vertices  $P'_m$ ,  $m = 3, \dots, d$ , and the three vertices of a triangle in  $\mathcal{T}_J$ . The latter is called base triangle of the associated simplex. Obviously, the  $d$ -dimensional volume of each simplex is proportional to the 2-dimensional area of its base triangle, with proportionality constant  $2/d!$ . To obtain a suitable function  $f \in L_\infty(\Omega)$  with  $\|f\|_{L_\infty} = 1$  we prescribe values  $\pm 1$  on the simplices by inheritance from the values on the base triangles of the above 2-dimensional  $f$ .

From the symmetry properties of  $f$  and  $\mathcal{P}_J$ , it is obvious that the  $L_2$ -orthogonal projection  $P_{V(\mathcal{P}_J)}f$  of  $f$  onto the linear spline space  $V(\mathcal{P}_J)$  is characterized by its

value  $x_{J+1}$  at the origin  $P_{J+1}$ , by values  $x_j$  taken at the vertices  $P_{j,i}$ ,  $i = 1, \dots, 4$ , where  $j = 0, \dots, J$ , and a common value  $x'$  taken at the remaining vertices  $P'_m$ ,  $m = 3, \dots, d$ . To estimate these values which, in complete analogy to the 2-dimensional case, are represented by the solution vector  $\tilde{x}$  of a certain linear system  $\tilde{A}\tilde{x} = \tilde{b}$  (now of dimension  $J + 3$ ), we need the limit version  $\hat{A}\hat{x} = \hat{b}$  of this linear system for  $t \rightarrow 0$ . We spare the reader the elementary calculations, and state it without proof:

$$\begin{aligned} \hat{x}_j + \hat{x}_{j-1} + \frac{d-2}{2}\hat{x}' &= (-1)^j \frac{d+2}{2}, & j = 1, \dots, J+1, \\ \frac{3}{2}\hat{x}_0 + \frac{1}{2}\hat{x}_1 + \frac{d-2}{2}\hat{x}' &= -\frac{d+2}{2}, \\ \hat{x}_0 + \frac{1}{2}\hat{x}_1 + \left(1 + \frac{d-3}{2}\right)\hat{x}' &= -\frac{d+2}{2}. \end{aligned}$$

From this system one easily concludes that  $\|\hat{x}\|_\infty \geq cJ$ , and consequently  $\|P_{V(\mathcal{P}_J)}f\|_{L_\infty} \geq cJ$  for a small enough  $t > 0$  which implies the desired result. The lower bound  $cJ$  obtained does not seem to accurately reflect the possible growth of the projector norms in  $L_\infty$  as a function of the number of simplices, for  $d \geq 3$  we would rather expect an exponential rate.

#### REFERENCES

- [1] I. Babuska, A. K. Aziz, On the angle condition in the finite element method, *SIAM J. Numer. Anal.* **13** (1976), 214–226. MR0455462 (56:13700)
- [2] C. de Boor, On a max-norm bound for the least-squares spline approximant, in *Approximation and Function Spaces* (Gdansk, 1979), Z. Ciesielski (ed.), pp. 163–175, North-Holland, Amsterdam, 1981. MR649424 (84k:41012)
- [3] Z. Ciesielski, Properties of the orthonormal Franklin system, *Studia Math.* **23** (1963), 141–157. MR0157182 (28:419)
- [4] Z. Ciesielski, Private communication, *Int. Conf. Approximation Theory and Probability*, Bedlowo, 2004.
- [5] J. Desloux, On finite element matrices, *SIAM J. Numer. Anal.* **9**, 2 (1972), 260–265. MR0309292 (46:8402)
- [6] J. Douglas, Jr., T. Dupont, L. Wahlbin, The stability in  $L^q$  of the  $L^2$ -projection into finite element function spaces, *Numer. Math.* **23** (1975), 193–197. MR0383789 (52:4669)
- [7] A. Yu. Shadrin, The  $L_\infty$ -norm of the  $L_2$ -spline projector is bounded independently of the knot sequence: a proof of de Boor's conjecture, *Acta Math.* **187** (2001), 59–137. MR1864631 (2002j:41007)

SCHOOL OF ENGINEERING AND SCIENCE, JACOBS UNIVERSITY, D-28759 BREMEN, GERMANY  
*E-mail address:* poswald@jacobs-university.de