THE ZEROS OF DEDEKIND ZETA FUNCTIONS AND CLASS NUMBERS OF CM-FIELDS

GEON-NO LEE AND SOUN-HI KWON

Abstract. Let $F'/F$ be a finite normal extension of number fields with Galois group $\text{Gal}(F'/F)$. Let $\chi$ be an irreducible character of $\text{Gal}(F'/F)$ of degree greater than one and $L(s, \chi)$ the associated Artin $L$-function. Assuming the truth of Artin’s conjecture, we have explicitly determined a zero-free region about 1 for $L(s, \chi)$. As an application we show that, for a CM-field $K$ of degree $2n$ with solvable normal closure over $\mathbb{Q}$, if $n \geq 370$ as well as $n \notin \{384, 400, 416, 448, 512\}$, then the relative class number of $K$ is greater than one.

1. Introduction

For a number field $M$ we let $D_M$, $h_M$, and $\kappa_M$ be the absolute value of the discriminant, the class number of $M$, and the residue of $\zeta_M(s)$, the Dedekind zeta function of $M$ at $s = 1$, respectively. Let $K$ be a CM-field of degree $2n$ and let $k$ be its maximal totally real subfield. Then we have $\zeta_K(s) = \zeta_k(s)L(s, \chi)$ and $D_K = D_k^2f$, where $f$ is the norm of the relative discriminant. Let $h_K = h_K/h_k$ be the relative class number of $K$.

In [M2], Kumar Murty proved that there are only finitely many CM-fields $K$ with class number one if the normal closure $N$ of $k$ has a solvable Galois group. In this paper we determine explicitly a zero-free region about 1 for $\zeta_K(s)$, assuming the truth of Artin’s conjecture. We apply this result to get an explicit lower bound for the relative class number of CM-fields.

Before stating our result we define the following functions as in [M2]. For an integer $n$, let

$$e(n) = \max_{p^a||n} \alpha$$

and

$$\delta(n) = (e(n) + 1)^23^{1/3}12^{e(n)-1}.$$

Our result is as follows.

Theorem 1. (1) Let $F'/F$ be a normal extension of number fields with Galois group $\text{Gal}(F'/F)$. Let $\chi$ be an irreducible character of $\text{Gal}(F'/F)$ of degree larger than...
one, $f_\chi$ its Artin conductor, and $L(s, \chi)$ the associated Artin $L$-function. Assume the truth of Artin’s conjecture for $F'/F$. Then $L(\sigma, \chi)$ has no real zeros in the range

$$1 - \frac{0.023932}{\chi(1)^3 \log A_\chi} \leq \sigma \leq 1$$

where $A_\chi = D_\chi^{(1)} N_{F'/F}(f_\chi)$.

(2) Let $L/E$ be an extension of number fields of degree $n$ whose Galois closure is solvable. Suppose that $\zeta_L(s)$ has a real zero $\sigma$ in the range

$$1 - \frac{0.011966}{n^{(1)}} \leq \sigma \leq 1.$$ 

Then, there is a field $B$ with $E \subseteq B \subseteq L$ and $[B : E] \leq 2$ such that $\zeta_B(\sigma) = 0$.

(3) Let $K$ be a CM-field of degree $2n$ with solvable normal closure over $\mathbb{Q}$. Then $\zeta_K(\sigma)$ has no real zeros in the range

$$1 - \frac{0.000443}{n^{(2n)} \beta(n)} D_K^{-1/n} \leq \sigma \leq 1.$$ 

In particular, if $n \geq 774$, then $\zeta_K(\sigma)$ has no real zeros in the range

$$1 - \frac{D_K^{-1/n}}{1.78244 \times 10^{40} \times (1.03412)^n} \leq \sigma \leq 1.$$

The proof is given in Section 2. We apply these estimates to get lower bounds for $h_K$.

**Theorem 2.** Let $K$ be a CM-field of degree $2n$ with solvable normal closure over $\mathbb{Q}$. If $n \geq 774$, then

$$h_K^- \geq \frac{(1.12806)^n}{3.04616 \times 10^{40}} \geq 1.05043 > 1.$$

If, in addition, $401 \leq n \leq 773$ and $n \notin \{405, 416, 432, 448, 480, 512\}$, then we have $h_K^- > 1$.

This theorem is proved in Section 3. In Section 4 we will give a better lower bound for $h_K^-$ than (1.4).

2. PROOF OF THEOREM 1

We use Kumar Murty’s approach.

**Proof of point (1).** For any character $\chi$ (i.e., not necessarily irreducible) of $\text{Gal}(F'/F)$, we let $V_\chi$ be the underlying space of $\chi$. For each infinite place $\nu$ of $F$, denote by $\sigma_\nu$ the conjugacy class of Frobenius elements at primes of $F$ dividing $\nu$ and denote by $\chi_\nu^\pm(1)$ the dimension of the subspace on which $\sigma_\nu$ acts by $\pm 1$. Following Murty ([1] and [2]) we set

$$a_\chi = \sum \chi_\nu^+(1), b_\chi = \sum \chi_\nu^-(1), \text{ and } c_\chi = \chi(1)r_2(F),$$

where the sums range over the real infinite places $\nu$ of $F$ and $r_2(F)$ is the number of complex places of $F$. Set

$$\Theta(s, \chi) = (\pi^{-s/2}\Gamma(s/2))^{a_\chi}(\pi^{-(s+1)/2}\Gamma((s+1)/2))^{b_\chi}((2\pi)^{-s}\Gamma(s))^{c_\chi}$$

and

$$\Lambda(s, \chi) = A_\chi^{s/2}\Theta(s, \chi)L(s, \chi).$$
Let \( \delta(\chi) = (\chi, 1) \) be the multiplicity of the trivial representation in \( \chi \). We assume Artin’s conjecture. Then, \( (s(s-1))^{\delta(\chi)} A(s, \chi) \) is entire and, using its infinite product, we obtain that for \( \Re(s) > 1 \), we have

\[
-\Re \frac{L'}{L}(s, \chi) \leq -\sum_{\rho} \Re \left( \frac{1}{s - \rho} \right) + \frac{1}{2} \log A_{\chi} + \Re \frac{\vartheta'}{\vartheta}(s, \chi) + \delta(\chi) \Re \left( \frac{1}{s} + \frac{1}{s-1} \right),
\]

where the sum over \( \rho \) is over any subset of zeros of \( A(s, \chi) \) with \( 0 < \Re \rho < 1 \). (See [M1, p. 291] and [Od1, p. 391].) For real \( \sigma \) with \( 1 < \sigma < 3/2 \), \( \Re \log(2^{-1}) \) is entire and, using its infinite product, we obtain that for \( \Re(s, \chi) \leq 0 \), \( \Re \left( \frac{1}{s} \right) \) is some entire function that is real for real \( \sigma \), since \( f(\chi, \sigma, \varphi) \) is the multiplicity of the trivial representation in \( \chi \), which is not the identity character or \( \chi \). Let \( \rho = \beta \) be a real zero of \( L(s, \chi) \) with \( 0 < \beta < 1 \). Following the proof of [M1 Proposition 3.7] and [HR p. 297], we have that for \( \sigma > 1 \),

\[
0 \leq -2 \left( \frac{L'}{L}(\sigma, \chi) + \frac{L'}{L}(\sigma, \bar{\chi}) \right) - \frac{\zeta'(\sigma)}{\zeta(\sigma)}
- \frac{L'}{L}(\sigma, \chi) - \frac{L'}{L}(\sigma, \phi \otimes \bar{\chi}) - \left( \frac{L'}{L}(\sigma, \phi) + \frac{L'}{L}(\sigma, \bar{\phi}) \right) - \frac{L'}{L}(\sigma),
\]

where \( I(s) \) is some entire function that is real for real \( s \). Since \( f_{\chi \otimes \bar{\chi}}^{(1)}(1) = f_{\chi}^{(1)}(1) \) (see [Ma p. 80] and [Od1 Lemma1]), we have \( \log A_{\chi \otimes \bar{\chi}} \leq 2\chi(1) \log A_{\chi} \). Similarly, \( f_{\phi \otimes \bar{\chi}}^{(2)}(1) \) and \( f_{\phi}|_{\chi \otimes \bar{\chi}}^{(2)}(1) \), so \( \log A_{\phi \otimes \bar{\chi}} \leq 2\chi(1) \log A_{\chi} \) and \( \log A_{\phi \otimes \bar{\phi}} \leq 4\chi(1)^3 \log A_{\chi} \).

Using (2.1), we find that for \( 1 < \sigma < 3/2 \),

\[
-\frac{L'}{L}(\sigma, \chi) - \frac{L'}{L}(\sigma, \bar{\chi}) \leq -\frac{2}{\sigma - \beta} + \log A_{\chi},
- \frac{\zeta'(\sigma)}{\zeta(\sigma)} \leq \frac{1}{\sigma} + \frac{1}{\sigma - 1} + \frac{1}{2} \log A_{\chi},
- \frac{L'}{L}(\sigma, \chi) \leq \chi(1) \log A_{\chi} + \frac{1}{\sigma} + \frac{1}{\sigma - 1},
- \frac{L'}{L}(\sigma, \phi \otimes \bar{\chi}) \leq 2\chi(1)^3 \log A_{\chi} + \frac{1}{\sigma} + \frac{1}{\sigma - 1},
- \frac{L'}{L}(\sigma, \phi) - \frac{L'}{L}(\sigma, \bar{\phi}) \leq 2\chi(1) \log A_{\chi},
\]

and \( \log A_{\chi} \leq \frac{1}{5}\chi(1) \log A_{\chi} \leq \frac{1}{4}\chi(1)^2 \log A_{\chi} \leq \frac{1}{6}\chi(1)^3 \log A_{\chi} \). Hence,

\[
0 \leq -\frac{4}{\sigma - \beta} + \frac{3}{\sigma - 1} + 3\chi(1)^3 \log A_{\chi}.
\]

Now choosing

\[
\sigma = 1 + \frac{\alpha}{\chi(1)^3 \log A_{\chi}}
\]
for a sufficiently small $\alpha > 0$ gives the zero-free region

$$1 - \beta \geq \frac{-3\alpha^2 + \alpha}{(3\alpha + 3)(1)^3 \log A_n}.$$ 

For $x > 0$, the function $h(x) = (-3x^2 + x)/(3x + 3)$ reaches its maximum $(7 - 4\sqrt{3})/2 = 0.023932\cdots$ at $x = 2/\sqrt{3} - 1 = 0.1547\cdots$. Taking $\alpha = 0.1547$ yields the desired zero-free region (1.1).

The point (2) follows immediately from [M2 Theorem 2] with $c_1 = 0.023932$.

Proof of point (3). The first statement follows from [M2 Proposition 3.2] and the proof of [M2 Theorem 3.2]. For the second statement we consider the function

$$f(x) = 2 \log x + \log x(\log x 24 + \log x) + 2 \log(\log x + 1)$$

for $x > 0$. Then

$$f'(x) = \frac{2 + \log x 24}{x \log 2} + \frac{2 \log x}{x \log 2} + \frac{2}{x \log(\log x + 1)},$$

which is decreasing for $x \geq e$. So, $f(x) \leq f(x_0) + (x - x_0)f'(x_0)$ for $x \geq x_0 \geq e$.

For $n \geq n_0 \geq 3$,

$$n(2n)^c(2n)^\delta(n) \leq n(2n)^{\log n + 1}(\log n + 1)^{3/2} 12 \log n + 1^{-}\log n - 1$$

$$= (3^{1/3}/6)n^2(24n)\log n(\log n + 1)^2$$

$$= (3^{1/3}/6)2^{f(n)}$$

$$\leq (3^{1/3}/6)2^{f(n_0) - n_0f'(n_0)}(2f'(n_0)).$$

We evaluate $f(774) = 162.0883\cdots$ and $f'(774) = 0.048399\cdots$, so

$$n(2n)^{c(2n)} \delta(n) \leq 7.8962 \times 10^{36} \times (1.03412)^n$$

for $n \geq 774$.

Substituting (2.2) into (1.2) yields (1.3). This completes the proof of Theorem 1. The reason why we take $n_0 = 774$ will become apparent in Section 3 below.

3. Proof of Theorem 2

Our proof of Theorem 2 is similar to [B]. From the analytic class number formula we have

$$h^{-1}_K = \frac{Q_K \omega_K}{(2\pi)^n} \sqrt{\frac{D_K \kappa_K}{D_k \kappa_k}} \geq \frac{2D_K^{1/4} \kappa_K}{(2\pi)^n \kappa_k},$$

where $Q_K \in \{1, 2\}$ is the Hasse unit index of $K$ and $\omega_K$ denotes the number of roots of unity in $K$ ([W]). Using Weil’s explicit formula we get a lower bound for $D_K$ (see (3.2) below) and an upper bound for $\kappa_k$ (see (3.3) below). To get a lower bound for $\kappa_k$ we combine Louboutin’s result in [L0] and our estimate of the zero-free region about 1 for $\zeta_K(s)$ obtained in Section 2. Gathering together those three bounds we get an explicit lower bound for $h_K^{-1}$. (See [B] below.)

For a real-valued function $F$ satisfying the conditions in [B Proposition 3] we set

$$I_n(F) = \frac{4}{n} \int_0^\infty F(x) \cosh(x/2)dx + \int_0^\infty \frac{(1 - F(x))e^{x/2}}{\sinh(x)}dx.$$
We use Weil’s explicit formula with
\[ F(x) = \begin{cases} 
9 \left( \frac{\sin(x/b) - (x/b) \cos(x/b)}{(x/b)^3} \right)^2 / \cosh(x/2) & \text{for } x \neq 0, \\
1 & \text{for } x = 0,
\end{cases} \]

where \( b \) is the constant which is chosen so as to minimize the value \( I_n(F) \) for a given \( n \). Then we obtain a good lower bound
\[
(3.2) \quad \frac{1}{n} \log D_k \geq \log(8 \pi e) + \frac{\pi}{2} - I_n(F),
\]

where \( \gamma \) is an Euler constant. (For Weil’s explicit formula, see \[La\] Ch. XVII. For lower bounds on discriminants, see \[Po1\] and \[Po2\], \[Od2\], and \[B\] Subsection 3.1.)

**Remark.** Integrating by parts five times and using \( \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \) we obtain the following formula, which is useful to evaluate \( I_n(F) \):
\[
\int_0^\infty \left( \frac{\sin x - x \cos x}{x^3} \right)^2 dx = \frac{\pi}{15}.
\]

In \[B\] Theorem 8 it was shown that for any totally real number field \( k \) of degree \( n \geq m \) we have
\[
\kappa_n \leq E_\sigma \left( \frac{D_k}{C_5(m)^n} \right)^{c_4(m)},
\]

where \( E_\sigma = 1 \) if \( \zeta_\sigma(s) \) has no real zero in the range \( 1/2 < \beta < 1 \), \( E_\sigma = (1 - \beta)/(\sigma_m - \beta) \) otherwise, \( \sigma_m \) is a constant depending only on \( m \), and \( c_4(m) \) and \( C_5(m) \) are explicitly computable constants. The possible values for \( c_4(m) \) and \( C_5(m) \) for some \( m \) with \( 400 \leq m \leq 774 \) are given in Table 1. Note that \( \sigma_m \) is not too small (i.e., \( \sigma_m > 1.001 \) for \( 400 \leq m \leq 774 \)).

<table>
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<th>( c_4(m) )</th>
<th>( C_5(m) )</th>
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For a lower bound of \( \kappa_K \) we use Theorem 1 in \[Lou\] (see also Theorem 16 in \[B\]): for any totally imaginary number field \( K \) of degree \( \geq 10 \) and root discriminant \( \rho_K \geq 2\pi^2 \), we have unconditionally
\[
(3.4) \quad \kappa_K \geq \begin{cases} 
\frac{1}{c \log D_K} & \text{if } \zeta_K(1 - 1/(c \log D_K)) \leq 0, \\
\frac{1 - \beta}{2e^{1/(2\beta)}} & \text{if } \zeta_K(\beta) \leq 0 \text{ with } 1 - 1/(c \log D_K) \leq \beta < 1,
\end{cases}
\]

where \( c = (2 + \sqrt{3})/4 \).
Set $I = [1 - 1/c\log D_K, 1)$. We consider the three following cases:

(i) $\zeta_K(s)$ has a zero $\beta \in I$.

(ii) $\zeta_K(s)$ has no zero in $I$ and $\zeta_K(s)$ has a simple zero $\beta$ in $I$.

(iii) $\zeta_K(s)$ has no zero in $I$ and $\zeta_K(s)$ has no simple zero in $I$.

Case (i): If $\zeta_K(s)$ has a zero $\beta \in I$, then $\zeta_K(\beta) = 0$. Note that $(1 - \beta)/E_\sigma = \sigma_m - \beta > \sigma_m - 1$. From (3.1), (3.3), and (3.4), it follows that

$$h_K^- \geq \frac{\sigma_m - 1}{e^{1/(2c)}} \left( \frac{C_5(m)c_4(m)}{2\pi} \right)^n D_K^{\frac{1}{2} - \frac{1}{4}c_4(m)}.$$

Case (ii): Since $\zeta_K(s) = \zeta_K(s)L(s,\chi)$, we have $L(\beta, \chi) = 0$. Note that $E_\sigma = 1$ here. Combining (3.1), (3.3), and (3.4), we have

$$h_K^- \geq \frac{1 - \beta}{e^{1/(2c)}} \left( \frac{C_5(m)c_4(m)}{2\pi} \right)^n D_K^{\frac{1}{2} - \frac{1}{4}c_4(m)}.$$  

Using the lower bound for $1 - \beta$ in (1.2) we have

$$h_K^- \geq \frac{0.000443}{e^{1/(2c)}} \left( \frac{C_5(m)c_4(m)}{2\pi} \right)^n D_K^{\frac{1}{2} - \frac{1}{4}c_4(m)}.$$  

Case (iii): If $\zeta_K(s)$ has no zero in $I$ and $\zeta_K(s)$ has no simple zero in $I$, then either $\zeta_K(s)$ has no zero at all in $I$ or $\zeta_K(s)$ has a double zero in $I$. This is because $\zeta_K(s)$ has at most two zeros with multiplicity in $I$ by [LLO, Lemma 15]. Then $\zeta_K(1 - 1/(c\log D_K)) \leq 0$ and

$$h_K^- \geq \frac{2}{cc^1/(2c)} \left( \frac{C_5(m)c_4(m)}{2\pi} \right)^n \frac{D_K^{\frac{1}{2} - \frac{1}{4}c_4(m)}}{\log D_K}$$

by (3.1), (3.3) and (3.4).

Now, we compare the three following terms:

$$\sigma_m - 1, \frac{0.000433D_K^{-1/(2n)}}{n(2n)^c(2n)^\delta(n)} \frac{2}{c\log D_K}.$$

The second term is the smallest one among the three terms; hence (3.5) holds.

Using (3.2) with $b = 6.6467$ we verify that

$$D_K^{\frac{1}{10}} \geq D_k^{\frac{1}{10}} \geq 55.1658$$

for any CM-field of degree $2n \geq 2 \cdot 774$. Substituting (2.2), (3.6), and the values $c_4(774)$ and $C_5(774)$ in Table 1 into (3.5) yields (1.4).

In a similar way we have explicitly computed the lower bounds for $h_K^-$ for all $n \leq 773$ and have verified that $h_K^- > 1$ for $401 \leq n \leq 773$ except for $n = 405, 416, 432, 448, 480$, and $512$. Our computational results are summarized in Table 2.
This completes our proof of Theorem 2.

4. AN IMPROVEMENT ON THEOREMS 1 AND 2

The second statement of Theorem 1 point (3) and Theorem 2 can be refined as follows.

**Theorem 3.** Let $K$ be as above.

1. If $n \geq 726$, then $\zeta_K(\sigma)$ has no real zeros in the range

   $$1 - \frac{D_K^{1/n}}{4.06231 \times 10^{39} \times (1.0362)^n} \leq \sigma \leq 1.$$

2. If $n \geq 726$, then

   $$h_K^n \geq \frac{(1.14136)^n}{3.8137 \times 10^{41}} \geq 1.2808 > 1.$$

   If, in addition, $370 \leq n \leq 725$ and $n \notin \{384, 400, 416, 448, 512\}$, then we have $h_K^n > 1$.

The proof of point (1) of Theorem 3 is similar to that of [1,3]. To prove point (2) of Theorem 3 we proceed as in [3]. We have used Weil's explicit formula twice: i.e., to get lower bounds for $D_K$ and to get upper bounds for $\kappa_k$. In addition we take care of prime ideals of small norms when dealing with this explicit formula of Weil. This allows us to improve especially upper bounds for $\kappa_k$. Analogously to the proof of Theorem 2 we can prove Theorem 3 point (2). We do not give the details of our proof, which are somewhat computational. We have explicitly computed the lower bounds for $h_K$ for all $n \leq 725$ and give our computational results in Table 3 and Table 4.

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**Table 2**

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<td>5.5093</td>
<td>52.8885 4.0 × 10^{10}</td>
</tr>
<tr>
<td>456</td>
<td>5.5348</td>
<td>52.9529 6618</td>
</tr>
</tbody>
</table>

---

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Table 3

<table>
<thead>
<tr>
<th>Ranges of n</th>
<th>n’s for which $h_K &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \leq 250$</td>
<td>221, 223, 227, 229, 231, 233, 235, 237, 239, 241, 247, 249</td>
</tr>
<tr>
<td>$251 \leq n \leq 270$</td>
<td>251, 253, 255, 257, 258, 259, 262, 263, 265, 266, 267, 269</td>
</tr>
<tr>
<td>$271 \leq n \leq 285$</td>
<td>271, 273, 274, 277, 278, 279, 281, 282, 283, 285</td>
</tr>
<tr>
<td>$286 \leq n \leq 300$</td>
<td>286, 287, 289, 290, 291, 293, 294, 295, 298, 299</td>
</tr>
<tr>
<td>$301 \leq n \leq 330$</td>
<td>all n’s except for 304, 308, 312, 320, 324, 328</td>
</tr>
<tr>
<td>$331 \leq n \leq 370$</td>
<td>all n’s except for 336, 344, 352, 360, 368</td>
</tr>
<tr>
<td>$371 \leq n \leq 725$</td>
<td>all n’s except for 384, 400, 416, 448, 512</td>
</tr>
</tbody>
</table>

Table 4

<table>
<thead>
<tr>
<th>n</th>
<th>$h_K \geq$</th>
<th>n</th>
<th>$h_K \geq$</th>
<th>n</th>
<th>$h_K \geq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>370</td>
<td>$3.7 \times 10^8$</td>
<td>400</td>
<td>0.0160</td>
<td>480</td>
<td>1.3725</td>
</tr>
<tr>
<td>371</td>
<td>$3.2 \times 10^{11}$</td>
<td>410</td>
<td>$5.3 \times 10^{11}$</td>
<td>500</td>
<td>$1.5 \times 10^{13}$</td>
</tr>
<tr>
<td>372</td>
<td>27570</td>
<td>416</td>
<td>$1.7 \times 10^{-5}$</td>
<td>512</td>
<td>$6.4 \times 10^{-15}$</td>
</tr>
<tr>
<td>373</td>
<td>$4.8 \times 10^{11}$</td>
<td>420</td>
<td>$1.4 \times 10^8$</td>
<td>520</td>
<td>$6.1 \times 10^{11}$</td>
</tr>
<tr>
<td>374</td>
<td>$7.7 \times 10^{8}$</td>
<td>430</td>
<td>$2.1 \times 10^{13}$</td>
<td>540</td>
<td>$2.6 \times 10^{16}$</td>
</tr>
<tr>
<td>375</td>
<td>2202</td>
<td>440</td>
<td>296533</td>
<td>600</td>
<td>$1.6 \times 10^{-8}$</td>
</tr>
<tr>
<td>380</td>
<td>114506</td>
<td>448</td>
<td>$3.2 \times 10^{-7}$</td>
<td>625</td>
<td>$9.3 \times 10^{-8}$</td>
</tr>
<tr>
<td>384</td>
<td>$4.8 \times 10^{-16}$</td>
<td>450</td>
<td>$2.9 \times 10^{13}$</td>
<td>640</td>
<td>13578</td>
</tr>
<tr>
<td>390</td>
<td>$1.4 \times 10^{10}$</td>
<td>460</td>
<td>$2.1 \times 10^{11}$</td>
<td>720</td>
<td>$4.0 \times 10^{23}$</td>
</tr>
</tbody>
</table>

Acknowledgments

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References


Department of Mathematics, Korea University, 136-701, Seoul, Korea  
E-mail address: thisknow@korea.ac.kr

Department of Mathematics Education, Korea University, 136-701, Seoul, Korea  
E-mail address: sounhikwon@korea.ac.kr