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THE ZEROS OF DEDEKIND ZETA FUNCTIONS AND CLASS NUMBERS OF CM-FIELDS

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ABSTRACT. Let F'/F be a finite normal extension of number fields with Galois group $Gal(F'/F)$. Let χ be an irreducible character of $Gal(F'/F)$ of degree greater than one and $L(s, \chi)$ the associated Artin L-function. Assuming the truth of Artin's conjecture, we have explicitly determined a zero-free region about 1 for $L(s, \chi)$. As an application we show that, for a CM-field K of degree $2n$ with solvable normal closure over $\mathbb{Q}$, if $n \geq 370$ as well as $n \notin \{384, 400, 416, 448, 512\}$, then the relative class number of K is greater than one.

1. INTRODUCTION

For a number field M we let D_M, h_M, and κ_M be the absolute value of the discriminant, the class number of M, and the residue of $\zeta_M(s)$, the Dedekind zeta function of M at $s = 1$, respectively. Let K be a CM-field of degree $2n$ and let k be its maximal totally real subfield. Then we have $\zeta_K(s) = \zeta_k(s)L(s, \chi)$ and $D_K = D_k^2 f$, where f is the norm of the relative discriminant. Let $h_K^- = h_K/h_k$ be the relative class number of K.

In [M2], Kumar Murty proved that there are only finitely many CM-fields K with class number one if the normal closure N of k has a solvable Galois group. In this paper we determine explicitly a zero-free region about 1 for $\zeta_K(s)$, assuming the truth of Artin's conjecture. We apply this result to get an explicit lower bound for the relative class number of CM-fields.

Before stating our result we define the following functions as in [M2]. For an integer n, let

$$e(n) = \max_{p^\alpha || n} \alpha$$

and

$$\delta(n) = (e(n) + 1)^2 3^{1/3} 12^{e(n)-1}.$$

Our result is as follows.

Theorem 1. (1) *Let F'/F be a normal extension of number fields with Galois group $Gal(F'/F)$. Let χ be an irreducible character of $Gal(F'/F)$ of degree larger than*

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one, $\mathfrak{f}_\chi$ its Artin conductor, and $L(s,\chi)$ the associated Artin L-function. Assume the truth of Artin's conjecture for F'/F. Then $L(\sigma,\chi)$ has no real zeros in the range

$$1-\frac{0.023932}{\chi(1)^3\log A_\chi}\leq\sigma\leq 1 \tag{1.1}$$

where $A_\chi=D_F^{\chi(1)}\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{f}_\chi)$.

(2) *Let L/E be an extension of number fields of degree n whose Galois closure is solvable. Suppose that $\zeta_L(s)$ has a real zero σ in the range*

$$1-\frac{0.011966}{n^{e(n)}\delta(n)\log d_L}\leq\sigma\leq 1.$$

Then, there is a field B with $E\subseteq B\subseteq L$ and $[B:E]\leq 2$ such that $\zeta_B(\sigma)=0$.

(3) *Let K be a CM-field of degree $2n$ with solvable normal closure over $\mathbb{Q}$. Then $\zeta_K(\sigma)$ has no real zeros in the range*

$$1-\frac{0.000443}{n(2n)^{e(2n)}\delta(n)}D_K^{-\frac{1}{2n}}\leq\sigma\leq 1. \tag{1.2}$$

In particular, if $n\geq 774$, then $\zeta_K(\sigma)$ has no real zeros in the range

$$1-\frac{D_K^{-\frac{1}{2n}}}{1.78244\times 10^{40}\times(1.03412)^n}\leq\sigma\leq 1. \tag{1.3}$$

The proof is given in Section 2. We apply these estimates to get lower bounds for h_K^-.

Theorem 2. *Let K be a CM-field of degree $2n$ with solvable normal closure over $\mathbb{Q}$. If $n\geq 774$, then*

$$h_K^-\geq\frac{(1.12806)^n}{3.04616\times 10^{40}}\geq 1.05043>1. \tag{1.4}$$

If, in addition, $401\leq n\leq 773$ and $n\notin\{405,416,432,448,480,512\}$, then we have $h_K^->1$.

This theorem is proved in Section 3. In Section 4 we will give a better lower bound for h_K^- than (1.4).

2. Proof of Theorem 1

We use Kumar Murty's approach.

Proof of point (1). For any character χ (i.e., not necessarily irreducible) of $Gal(F'/F)$, we let V_χ be the underlying space of χ. For each infinite place ν of F, denote by σ_ν the conjugacy class of Frobenius elements at primes of F' dividing ν and denote by $\chi_\nu^\pm(1)$ the dimension of the subspace on which σ_ν acts by ± 1. Following Murty ([M1] and [MM]) we set

$$a_\chi=\sum\chi_\nu^+(1), b_\chi=\sum\chi_\nu^-(1),\text{ and } c_\chi=\chi(1)r_2(F),$$

where the sums range over the real infinite places ν of F and $r_2(F)$ is the number of complex places of F. Set

$$\mathfrak{G}(s,\chi)=(\pi^{-s/2}\Gamma(s/2))^{a_\chi}(\pi^{-(s+1)/2}\Gamma((s+1)/2))^{b_\chi}((2\pi)^{-s}\Gamma(s))^{c_\chi}$$

and

$$\Lambda(s,\chi)=A_\chi^{s/2}\mathfrak{G}(s,\chi)L(s,\chi).$$

Let $\delta(\chi) = (\chi, 1)$ be the multiplicity of the trivial representation in χ. We assume Artin's conjecture. Then, $(s(s-1))^{\delta(\chi)}\Lambda(s,\chi)$ is entire and, using its infinite product, we obtain that for $\mathrm{Re}(s) > 1$, we have

$$-\mathrm{Re}\frac{L'}{L}(s,\chi) \leq -\sum_{\rho}\mathrm{Re}(\frac{1}{s-\rho}) + \frac{1}{2}\log A_\chi + \mathrm{Re}\frac{\mathfrak{G}'}{\mathfrak{G}}(s,\chi) + \delta(\chi)\mathrm{Re}(\frac{1}{s} + \frac{1}{s-1}),$$

where the sum over ρ is over any subset of zeros of $\Lambda(s,\chi)$ with $0 < \mathrm{Re}\,\rho < 1$. (See [M1, p. 291] and [Od1, p. 391].) For real σ with $1 < \sigma < 3/2$, $\mathrm{Re}\,\mathfrak{G}'/\mathfrak{G}(\sigma,\chi) < 0$ because all of the three terms $\Gamma'/\Gamma(\sigma/2) - \log\pi$, $\Gamma'/\Gamma((\sigma+1)/2) - \log\pi$, and $\Gamma'/\Gamma(\sigma) - \log(2\pi)$ are negative. Hence, for $1 < \sigma < 3/2$,

$$-\mathrm{Re}\frac{L'}{L}(\sigma,\chi) \leq -\sum_{\rho}\mathrm{Re}(\frac{1}{\sigma-\rho}) + \frac{1}{2}\log A_\chi + \delta(\chi)(\frac{1}{\sigma} + \frac{1}{\sigma-1}). \tag{2.1}$$

(See [M1, (2.2)].) Now we are ready to prove (1.1). From now on we assume that χ is an irreducible character of $Gal(F'/F)$ of degree larger than one. Let ϕ be an irreducible constituent of $\chi\otimes\bar{\chi}$ which is not the identity character or χ. Let $\rho = \beta$ be a real zero of $L(s,\chi)$ with $0 < \beta < 1$. Following the proof of [M1, Proposition 3.7] and [HR, p. 297], we have that for $\sigma > 1$,

$$\begin{aligned}0 \leq & -2\left(\frac{L'}{L}(\sigma,\chi) + \frac{L'}{L}(\sigma,\bar{\chi})\right) - \frac{\zeta_F'}{\zeta_F}(\sigma) \\ & -\frac{L'}{L}(\sigma,\chi\otimes\bar{\chi}) - \frac{L'}{L}(\sigma,\phi\otimes\bar{\phi}) - \left(\frac{L'}{L}(\sigma,\phi) + \frac{L'}{L}(\sigma,\bar{\phi})\right) - \frac{I'}{I}(\sigma),\end{aligned}$$

where $I(s)$ is some entire function that is real for real s. Since $\mathfrak{f}_{\chi\otimes\bar{\chi}}$ divides $\mathfrak{f}_\chi^{\bar{\chi}(1)}\mathfrak{f}_{\bar{\chi}}^{\chi(1)} = \mathfrak{f}_\chi^{2\chi(1)}$ (see [Ma, p. 80] and [Od1, Lemma1]), we have $\log A_{\chi\otimes\bar{\chi}} \leq 2\chi(1)\log A_\chi$. Similarly, $\mathfrak{f}_{\phi\otimes\bar{\phi}} | \mathfrak{f}_\phi^{2\phi(1)}$ and $\mathfrak{f}_\phi | \mathfrak{f}_{\chi\otimes\bar{\chi}} | \mathfrak{f}_\chi^{2\chi(1)}$, so $\log A_\phi \leq 2\chi(1)\log A_\chi$ and $\log A_{\phi\otimes\bar{\phi}} \leq 4\chi(1)^3\log A_\chi$.

Using (2.1), we find that for $1 < \sigma < 3/2$,

$$\begin{aligned}-\frac{L'}{L}(\sigma,\chi) - \frac{L'}{L}(\sigma,\bar{\chi}) &\leq -\frac{2}{\sigma-\beta} + \log A_\chi, \\ -\frac{\zeta_F'}{\zeta_F}(\sigma) &\leq \frac{1}{\sigma} + \frac{1}{\sigma-1} + \frac{1}{2}\log A_\chi, \\ -\frac{L'}{L}(\sigma,\chi\otimes\bar{\chi}) &\leq \chi(1)\log A_\chi + \frac{1}{\sigma} + \frac{1}{\sigma-1}, \\ -\frac{L'}{L}(\sigma,\phi\otimes\bar{\phi}) &\leq 2\chi(1)^3\log A_\chi + \frac{1}{\sigma} + \frac{1}{\sigma-1}, \\ -\frac{L'}{L}(\sigma,\phi) - \frac{L'}{L}(\sigma,\bar{\phi}) &\leq 2\chi(1)\log A_\chi,\end{aligned}$$

and $\log A_\chi \leq \frac{1}{2}\chi(1)\log A_\chi \leq \frac{1}{4}\chi(1)^2\log A_\chi \leq \frac{1}{8}\chi(1)^3\log A_\chi$. Hence,

$$0 \leq -\frac{4}{\sigma-\beta} + \frac{3}{\sigma-1} + 3\chi(1)^3\log A_\chi.$$

Now choosing

$$\sigma = 1 + \frac{\alpha}{\chi(1)^3\log A_\chi}$$

for a sufficiently small $\alpha > 0$ gives the zero-free region

$$1-\beta \geq \frac{-3\alpha^2+\alpha}{(3\alpha+3)\chi(1)^3 \log A_\chi}.$$

For $x > 0$, the function $h(x) = (-3x^2+x)/(3x+3)$ reaches its maximum $(7-4\sqrt{3})/2 = 0.023932\cdots$ at $x = 2/\sqrt{3}-1 = 0.1547\cdots$. Taking $\alpha = 0.1547$ yields the desired zero-free region (1.1).

The point (2) follows immediately from [M2, Theorem 2.1] with $c_1 = 0.023932$. □

Proof of point (3). The first statement follows from [M2, Proposition 3.2] and the proof of [M2, Theorem 3.2]. For the second statement we consider the function

$$f(x) = 2\log_2 x + \log_2 x(\log_2 24 + \log_2 x) + 2\log_2(\log_2 x + 1)$$

for $x > 0$. Then

$$f'(x) = \frac{2+\log_2 24}{x\log 2} + \frac{2\log_2 x}{x\log 2} + \frac{2}{x\log 2(\log_2 x+1)},$$

which is decreasing for $x \geq e$. So, $f(x) \leq f(x_0) + (x-x_0)f'(x_0)$ for $x \geq x_0 \geq e$.

For $n \geq n_0 \geq 3$,

$$\begin{aligned} n(2n)^{e(2n)}\delta(n) &\leq n(2n)^{\log_2 n+1}(\log_2 n+1)^2 3^{1/3} 12^{\log_2 n-1} \\ &= (3^{1/3}/6)n^2(24n)^{\log_2 n}(\log_2 n+1)^2 \\ &= (3^{1/3}/6)2^{f(n)} \\ &\leq (3^{1/3}/6)2^{f(n_0)-n_0 f'(n_0)}(2^{f'(n_0)})^n. \end{aligned}$$

We evaluate $f(774) = 162.0883\cdots$ and $f'(774) = 0.048399\cdots$, so

$$n(2n)^{e(2n)}\delta(n) \leq 7.8962 \times 10^{36} \times (1.03412)^n \qquad \text{for } n \geq 774. \tag{2.2}$$

Substituting (2.2) into (1.2) yields (1.3). This completes the proof of Theorem 1. The reason why we take $n_0 = 774$ will become apparent in Section 3 below. □

3. Proof of Theorem 2

Our proof of Theorem 2 is similar to [B]. From the analytic class number formula we have

$$h_K^- = \frac{Q_K\omega_K}{(2\pi)^n}\sqrt{\frac{D_K}{D_k}}\frac{\kappa_K}{\kappa_k} \geq \frac{2D_K^{1/4}}{(2\pi)^n}\frac{\kappa_K}{\kappa_k}, \tag{3.1}$$

where $Q_K \in \{1,2\}$ is the Hasse unit index of K and ω_K denotes the number of roots of unity in K ([W]). Using Weil's explicit formula we get a lower bound for D_K (see (3.2) below) and an upper bound for κ_k (see (3.3) below). To get a lower bound for κ_K we combine Louboutin's result in [Lou] and our estimate of the zero-free region about 1 for $\zeta_K(s)$ obtained in Section 2. Gathering together those three bounds we get an explicit lower bound for h_K^-. (See (3.5) below.)

For a real-valued function F satisfying the conditions in [B, Proposition 3] we set

$$I_n(F) = \frac{4}{n}\int_0^\infty F(x)\cosh(x/2)dx + \int_0^\infty \frac{(1-F(x))e^{x/2}}{\sinh(x)}dx.$$

We use Weil's explicit formula with

$$F(x)=\begin{cases} 9\Big(\dfrac{\sin(x/b)-(x/b)\cos(x/b)}{(x/b)^3}\Big)^2/\cosh(x/2) & \text{for } x\neq 0,\\ 1 & \text{for } x=0,\end{cases}$$

where b is the constant which is chosen so as to minimize the value $I_n(F)$ for a given n. Then we obtain a good lower bound

$$\frac{1}{n}\log D_k \geq \log(8\pi e^{\gamma})+\frac{\pi}{2}-I_n(F), \tag{3.2}$$

where γ is an Euler constant. (For Weil's explicit formula, see [La, Ch. XVII]. For lower bounds on discriminants, see [Poi1] and [Poi2], [Od2], and [B, Subsection 3.1].)

Remark. Integrating by parts five times and using $\int_0^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$ we obtain the following formula, which is useful to evaluate $I_n(F)$:

$$\int_0^{\infty}\Big(\frac{\sin x - x\cos x}{x^3}\Big)^2 dx=\frac{\pi}{15}.$$

In [B, Theorem 8] it was shown that for any totally real number field k of degree $n\geq m$ we have

$$\kappa_k \leq E_{\sigma}\Big(\frac{D_k}{C_5(m)^n}\Big)^{c_4(m)}, \tag{3.3}$$

where $E_{\sigma}=1$ if $\zeta_k(s)$ has no real zero in the range $1/2<\beta<1$, $E_{\sigma}=(1-\beta)/(\sigma_m-\beta)$ otherwise, σ_m is a constant depending only on m, and $c_4(m)$ and $C_5(m)$ are explicitly computable constants. The possible values for $c_4(m)$ and $C_5(m)$ for some m with $400\leq m\leq 774$ are given in Table 1. Note that σ_m is not too small (i.e., $\sigma_m>1.001$ for $400\leq m\leq 774$).

TABLE 1

m	σ_m	$c_4(m)$	$C_5(m)$	m	σ_m	$c_4(m)$	$C_5(m)$
400	1.003274	0.339768	50.76955	448	1.003011	0.338791	51.37075
401	1.003268	0.339746	50.78313	456	1.002973	0.338645	51.46186
405	1.003244	0.339657	50.83698	464	1.002935	0.338504	51.55068
408	1.003226	0.339592	50.87684	472	1.002898	0.338367	51.63728
412	1.003203	0.339505	50.92936	480	1.002863	0.338234	51.72177
416	1.003180	0.339421	50.98112	496	1.002795	0.337980	51.88471
420	1.003158	0.339338	51.03217	512	1.002731	0.337739	52.04011
424	1.003136	0.339256	51.08253	576	1.002507	0.336891	52.59673
432	1.003093	0.339096	51.18119	640	1.002323	0.336192	53.06872
440	1.003052	0.338941	51.27722	774	1.002026	0.335057	53.86114

For a lower bound of κ_K we use Theorem 1 in [Lou] (see also Theorem 16 in [B]): for any totally imaginary number field K of degree ≥ 10 and root discriminant $\rho_K\geq 2\pi^2$, we have unconditionally

$$\kappa_K\geq\begin{cases}\dfrac{1}{ce^{1/(2c)}\log D_K} & \text{if } \zeta_K(1-1/(c\log D_K))\leq 0,\\ \dfrac{1-\beta}{2e^{1/(2c)}} & \text{if } \zeta_K(\beta)\leq 0 \text{ with } 1-1/(c\log D_K)\leq\beta<1,\end{cases} \tag{3.4}$$

where $c=(2+\sqrt{3})/4$.

Set $I = [1 - 1/c\log D_K, 1)$. We consider the three following cases:
(i) $\zeta_k(s)$ has a zero $\beta \in I$.
(ii) $\zeta_k(s)$ has no zero in I and $\zeta_K(s)$ has a simple zero β in I.
(iii) $\zeta_k(s)$ has no zero in I and $\zeta_K(s)$ has no simple zero in I.

Case (i): If $\zeta_k(s)$ has a zero $\beta \in I$, then $\zeta_K(\beta) = 0$. Note that $(1-\beta)/E_\sigma = \sigma_m - \beta > \sigma_m - 1$. From (3.1), (3.3), and (3.4), it follows that

$$h_K^- \geq \frac{\sigma_m - 1}{e^{1/(2c)}}\Big(\frac{C_5(m)^{c_4(m)}}{2\pi}\Big)^n D_K^{\frac{1}{4}-\frac{1}{2}c_4(m)}.$$

Case (ii): Since $\zeta_K(s) = \zeta_k(s)L(s,\chi)$, we have $L(\beta,\chi) = 0$. Note that $E_\sigma = 1$ here. Combining (3.1), (3.3), and (3.4), we have

$$h_K^- \geq \frac{1-\beta}{e^{1/(2c)}}\Big(\frac{C_5(m)^{c_4(m)}}{2\pi}\Big)^n D_K^{\frac{1}{4}-\frac{1}{2}c_4(m)}.$$

Using the lower bound for $1-\beta$ in (1.2) we have

$$h_K^- \geq \frac{0.000443}{e^{1/(2c)}n(2n)^{e(2n)}\delta(n)}\Big(\frac{C_5(m)^{c_4(m)}}{2\pi}\Big)^n D_K^{\frac{1}{4}-\frac{1}{2}c_4(m)-\frac{1}{2n}}. \tag{3.5}$$

Case (iii): If $\zeta_k(s)$ has no zero in I and $\zeta_K(s)$ has no simple zero in I, then either $\zeta_K(s)$ has no zero at all in I or $\zeta_K(s)$ has a double zero in I. This is because $\zeta_K(s)$ has at most two zeros with multiplicity in I by [LLO, Lemma 15]. Then $\zeta_K(1 - 1/(c\log D_K)) \leq 0$ and

$$h_K^- \geq \frac{2}{ce^{1/(2c)}}\Big(\frac{C_5(m)^{c_4(m)}}{2\pi}\Big)^n \frac{D_K^{\frac{1}{4}-\frac{1}{2}c_4(m)}}{\log D_K}$$

by (3.1), (3.3) and (3.4).

Now, we compare the three following terms:

$$\sigma_m - 1,\ \frac{0.000433 D_K^{-1/(2n)}}{n(2n)^{e(2n)}\delta(n)},\ \frac{2}{c\log D_K}.$$

The second term is the smallest one among the three terms; hence (3.5) holds.

Using (3.2) with $b = 6.6467$ we verify that

$$D_K^{\frac{1}{2n}} \geq D_k^{\frac{1}{n}} \geq 55.1658 \tag{3.6}$$

for any CM-field of degree $2n \geq 2 \cdot 774$. Substituting (2.2), (3.6), and the values $c_4(774)$ and $C_5(774)$ in Table 1 into (3.5) yields (1.4).

In a similar way we have explicitly computed the lower bounds for h_K^- for all $n \leq 773$ and have verified that $h_K^- > 1$ for $401 \leq n \leq 773$ except for $n = 405$, 416, 432, 448, 480, and 512. Our computational results are summarized in Table 2.

TABLE 2

n	b	$D_k^{1/n} \geq$	$h_K^- \geq$	n	b	$D_k^{1/n} \geq$	$h_K^- \geq$
400	5.2873	52.2941	0.0000278	460	5.5516	52.9951	2.6×10^8
401	5.2919	52.3071	1.4×10^{11}	464	5.5684	53.0368	1.5425
402	5.2966	52.3201	2.1×10^8	472	5.6017	53.1184	109302
403	5.3012	52.3330	2.1×10^{11}	480	5.6345	53.1979	0.001496
404	5.3058	52.3459	14272	486	5.6589	53.2562	4.0920
405	5.3103	52.3587	0.05238	488	5.6670	53.2754	1835091
406	5.3149	52.3714	4.4×10^8	496	5.6991	53.3509	406.6
407	5.3195	52.3841	4.2×10^{11}	500	5.7151	53.3880	1.5×10^{10}
408	5.3241	52.3968	1.64046	512	5.7623	53.4964	6.0×10^{-18}
410	5.3332	52.4220	8.9×10^8	520	5.7934	53.5664	5.4×10^8
412	5.3422	52.4470	57211	540	5.8698	53.7345	2.1×10^{13}
416	5.3603	52.4964	2.8×10^{-8}	544	5.8848	53.7669	104.2
420	5.3782	52.5451	230434	576	6.0025	54.0135	1.6017
424	5.3961	52.5930	25.61	600	6.0880	54.1847	1.0×10^{15}
430	5.4226	52.6637	3.0×10^{10}	625	6.1745	54.3520	5.2×10^{15}
432	5.4314	52.6869	0.006293	640	6.2253	54.4474	6.9917
440	5.4663	52.7781	407.8	720	6.4833	54.9019	1.5×10^{20}
448	5.5008	52.8667	4.3×10^{-10}	773	6.6437	55.1612	1.8×10^{41}
450	5.5093	52.8885	4.0×10^{10}	774	6.6467	55.1658	5.5×10^{36}
456	5.5348	52.9529	6618				

This completes our proof of Theorem 2.

4. AN IMPROVEMENT ON THEOREMS 1 AND 2

The second statement of Theorem 1 point (3) and Theorem 2 can be refined as follows.

Theorem 3. *Let K be as above.*

(1) *If $n \geq 726$, then $\zeta_K(\sigma)$ has no real zeros in the range*

$$1 - \frac{D_K^{-\frac{1}{2n}}}{4.06231 \times 10^{39} \times (1.0362)^n} \leq \sigma \leq 1.$$

(2) *If $n \geq 726$, then*

$$h_K^- \geq \frac{(1.14136)^n}{3.8137 \times 10^{41}} \geq 1.2808 > 1.$$

If, in addition, $370 \leq n \leq 725$ and $n \notin \{384, 400, 416, 448, 512\}$, then we have $h_K^- > 1$.

The proof of point (1) of Theorem 3 is similar to that of (1.3). To prove point (2) of Theorem 3 we proceed as in [LK]. We have used Weil's explicit formula twice: i.e., to get lower bounds for D_K and to get upper bounds for κ_k. In addition we take care of prime ideals of small norms when dealing with this explicit formula of Weil. This allows us to improve especially upper bounds for κ_k. Analogously to the proof of Theorem 2 we can prove Theorem 3 point (2). We do not give the details of our proof, which are somewhat computational. We have explicitly computed the lower bounds for h_K^- for all $n \leq 725$ and give our computational results in Table 3 and Table 4.

TABLE 3

Ranges of n	n's for which $h_K^- > 1$
$n \leq 250$	221, 223, 227, 229, 231, 233, 235, 237, 239, 241, 247, 249
$251 \leq n \leq 270$	251, 253, 255, 257, 258, 259, 262, 263, 265, 266, 267, 269
$271 \leq n \leq 285$	271, 273, 274, 277, 278, 279, 281, 282, 283, 285
$286 \leq n \leq 300$	286, 287, 289, 290, 291, 293, 294, 295, 298, 299
$301 \leq n \leq 330$	all n's except for 304, 308, 312, 320, 324, 328
$331 \leq n \leq 370$	all n's except for 336, 344, 352, 360, 368
$371 \leq n \leq 725$	all n's except for 384, 400, 416, 448, 512

TABLE 4

n	$h_K^- \geq$	n	$h_K^- \geq$	n	$h_K^- \geq$
370	3.7×10^8	400	0.0160	480	1.3725
371	3.2×10^{11}	410	5.3×10^{11}	500	1.5×10^{13}
372	27570	416	1.7×10^{-5}	512	6.4×10^{-15}
373	4.8×10^{11}	420	1.4×10^8	520	6.1×10^{11}
374	7.7×10^8	430	2.1×10^{13}	540	2.6×10^{16}
375	2202	440	296533	600	1.6×10^{18}
380	114506	448	3.2×10^{-7}	625	9.3×10^{18}
384	4.8×10^{-16}	450	2.9×10^{13}	640	13578
390	1.4×10^{10}	460	2.1×10^{11}	720	4.0×10^{23}

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