A MIXED METHOD FOR AXISYMMETRIC DIV-CURL SYSTEMS

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ABSTRACT. We present a mixed method for a three-dimensional axisymmetric div-curl system reduced to a two-dimensional computational domain via cylindrical coordinates. We show that when the meridian axisymmetric Maxwell problem is approximated by a mixed method using the lowest order Nédélec elements (for the vector variable) and linear elements (for the Lagrange multiplier), one obtains optimal error estimates in certain weighted Sobolev norms. The main ingredient of the analysis is a sequence of projectors in the weighted norms satisfying some commutativity properties.

1. INTRODUCTION

We present a mixed method for solving three-dimensional axisymmetric div-curl systems. We are interested in the case when symmetry with respect to an axis permits a reduction of the computational domain from three to two dimensions via cylindrical coordinates. The dimension reduction results in computationally efficient methods for applications in magnetostatics. However, the necessary use of cylindrical coordinates complicates the mathematical analysis of numerical methods. In particular, the coordinate transformation results in differential operators with singularities on the axis, and function spaces weighted by the radial variable are necessary. This note gives a few techniques to overcome the difficulties caused by the degenerate weights. Specifically, we construct simple interpolation operators satisfying a commuting diagram property involving a sequence of weighted spaces. This allows the application of standard techniques in the theory of mixed methods to axisymmetric div-curl systems.

An efficient method for the dimension-reduced div-curl system was recently proposed in [9], using a negative-norm least-squares technique. This method was shown to have excellent conditioning independent of jumps in the coefficient, but it allowed only for a piecewise constant coefficient. As an alternative, in this paper we present a mixed method which allows for a piecewise smooth coefficient. Another advantage is that the implementation of the mixed method involves finite element spaces more commonly used than those of the negative-norm least-squares method. The mixed method simply uses standard two-dimensional Nédélec elements [17] and

continuous piecewise linear functions. Thus the use of cylindrical coordinates does not necessitate special finite element spaces.

As a model problem, we consider Maxwell's system for magnetostatics. Let $\Omega \subset \mathbb{R}^3$ be a convex, simply-connected, bounded domain in $\mathbb{R}^3$, with a connected boundary. We assume that Ω is axisymmetric, i.e., it is invariant under rotations about the z-axis. Moreover, let D be a polygonal domain representing the cross-section, i.e., Ω is the revolution of D about the z-axis. We assume that Ω intersects the z-axis on an interval of positive length. If Ω does not intersect the z-axis, then the analysis of the problem is simple and standard finite element methods can be applied. The magnetostatic system is

$$(1.1) \quad \begin{cases} \mathbf{curl}\,\mathbf{H} & = \mathbf{J} & \text{in } \Omega, \\ \mathrm{div}(\mu\mathbf{H}) & = 0 & \text{in } \Omega, \\ \mu\mathbf{H}\cdot\mathbf{n} & = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{H}$ is the magnetic field, $\mathbf{J}$ is the current density, μ is the magnetic permeability, and $\mathbf{n}$ is the unit outward normal on $\partial\Omega$. We assume that $\mathbf{J}$ is in $L^2(\Omega)^3$ and satisfies the compatibility condition $\mathrm{div}\,\mathbf{J} = 0$. The coefficient μ is assumed to be piecewise smooth and to satisfy the bounds $0 < \mu_0 \leq \mu(\mathbf{x}) \leq \mu_1$ for all $\mathbf{x} \in \Omega$, where μ_0 and μ_1 are positive constants.

A standard technique to solve the magnetostatic system is via vector potentials. Define $\mathbf{B} = \mu\mathbf{H}$, the magnetic intensity, and note that $\mathrm{div}\,\mathbf{B} = 0$. It is well known (see e.g. [1] or [10, Theorem I.3.6]) that our assumptions on Ω imply the existence of a vector potential $\mathbf{A}$ in $\mathbf{H}_0(\mathbf{curl}, \Omega)$ satisfying $\mathbf{B} = \mathbf{curl}\,\mathbf{A}$ and $\mathrm{div}\,\mathbf{A} = 0$. Substituting $\mathbf{H} = \mu^{-1}\mathbf{B}$ into (1.1), we obtain the div-curl system

$$(1.2) \quad \begin{cases} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\mathbf{A} & = \mathbf{J} & \text{in } \Omega, \\ \mathrm{div}\,\mathbf{A} & = 0 & \text{in } \Omega, \\ \mathbf{A}\times\mathbf{n} & = 0 & \text{on } \partial\Omega. \end{cases}$$

A scalar-valued function is said to be *invariant under rotation* if it is constant with respect to the angular variable in cylindrical coordinates, and a vector-valued function is said to be *axisymmetric* if its component functions in cylindrical coordinates are invariant under rotation. We assume that μ is invariant under rotation and $\mathbf{J}$ is axisymmetric. Therefore, we seek an axisymmetric solution $\mathbf{A}$ (cf. [5]).

It is well known that (1.2) decouples into two independent systems in the presence of axial symmetry [3]. To exhibit the decoupled system, let D be as before, namely $D = \{(r,z) : (r,\theta,z) \in \Omega\}$, where we have used cylindrical coordinates (r,θ,z). We assume that D has a polygonal boundary ∂D, which can be expressed as the disjoint union $\partial D = \Gamma_0 \cup \Gamma_1$, where $\Gamma_1 = \{(r,z) \in \partial D : r > 0\}$. Writing the system (1.2) in cylindrical coordinates and assuming that the three components in $\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z$ are θ-independent, we obtain the two decoupled systems

$$(1.3) \quad \begin{cases} \mathbf{curl}_{rz}\,\mu^{-1}\mathrm{curl}_{rz}(A_r, A_z) = (J_r, J_z) & \text{in } D, \\ \mathrm{div}_{rz}(A_r, A_z) \equiv \dfrac{1}{r}\dfrac{\partial}{\partial r}(rA_r) + \dfrac{\partial A_z}{\partial z} = 0 & \text{in } D, \\ (A_r, A_z)\cdot(-n_z, n_r) = 0 & \text{on } \Gamma_1, \end{cases}$$

and

$$(1.4) \quad \begin{cases} -\dfrac{\partial}{\partial r}\left(\mu^{-1}\dfrac{1}{r}\dfrac{\partial}{\partial r}(rA_\theta)\right) - \dfrac{\partial}{\partial z}\left(\mu^{-1}\dfrac{\partial A_\theta}{\partial z}\right) = J_\theta & \text{in } D, \\ A_\theta = 0 & \text{on } \Gamma_1. \end{cases}$$

Here, we have used row-vector notation, n_r and n_z are the r- and z-components of **n**, respectively, and the two curl operators are defined by

$$(1.5) \qquad \operatorname{curl}_{rz}(A_r, A_z) \equiv \partial_z A_r - \partial_r A_z \quad \text{and} \quad \mathbf{curl}_{rz}\phi \equiv \Big(-\partial_z\phi, \; \frac{1}{r}\,\partial_r(r\phi)\Big),$$

where we have abbreviated $\partial/\partial r$ to ∂_r, etc. The first system, namely (1.3), forms the so-called *meridian problem*, while (1.4) represents the *azimuthal problem*. Finite elements for such problems were considered in [8, 14], although no error analysis in the natural weighted spaces was given. The finite element suggested in [14] for the azimuthal problem was independently arrived at in [11], where a convergence analysis was also provided. Additionally, in [11], the convergence of a discretization of the azimuthal problem using a more standard element (namely, the bilinear finite element), as well as the uniform convergence of a V-cycle multigrid algorithm, was also proved. In this paper, we will analyze finite elements for the meridian problem (1.3). The main new ingredient here is an interpolation operator for a weighted $H(\mathrm{curl})$-space described next.

Due to the transformation to cylindrical coordinates, the subsequent analysis of problem (1.3) involves function spaces weighted by the radial variable r. We denote by $L_\alpha^2(D)$ the weighted Lebesgue space of measurable functions v on D bounded in the norm $\|v\|_{L_\alpha^2(D)} = (\int_D r^\alpha v^2 \, dr\, dz)^{1/2}$. Our notation for the natural norm on any Banach space X is simply $\|\cdot\|_X$, but for convenience in the case of the often used space $L_1^2(D)$ (i.e., when $\alpha = 1$), we abbreviate $\|\cdot\|_{L_1^2(D)}$ to $\|\cdot\|_r$. Additionally, we denote the inner product in $L_1^2(D)$ by

$$(u, v)_r = \int_D r\, uv \; dr\, dz$$

and abbreviate the norm $\|\cdot\|_{H_1^1(D)}$ to $\|\cdot\|_{1,r}$. The weighted Sobolev space $H_\alpha^k(D)$ for any $k \geq 1$ is the space of all functions in $L_\alpha^2(D)$ whose distributional derivatives of order at most k are in $L_\alpha^2(D)$. We shall need the following additional weighted spaces:

$$\mathbf{H}_r(\mathrm{curl}, D) = \{\mathbf{v} \in L_1^2(D)^2 : \operatorname{curl}_{rz}\mathbf{v} \in L_1^2(D)\},$$
$$H_{1,\diamond}^1(D) = \{\phi \in H_1^1(D) : \phi = 0 \text{ on } \Gamma_1\},$$
$$H_{-}^1(D) = \{\phi \in L_{-1}^2(D) : \mathbf{grad}_{rz}\phi \in L_1^2(D)^2\}.$$

Here $\mathbf{grad}_{rz}\phi \equiv (\partial_r\phi, \partial_z\phi)$ and the norm on $\mathbf{H}_r(\mathrm{curl}, D)$ is defined by

$$\|\mathbf{v}\|_{r,\mathrm{curl}} = \big(\|\mathbf{v}\|_r^2 + \|\operatorname{curl}_{rz}\mathbf{v}\|_r^2\big)^{1/2}.$$

It can be shown that the trace condition in the definition of $H_{1,\diamond}^1(D)$ is well defined (see [5] or Lemma A.1). These spaces have previously occurred in the literature, and we shall use their properties established in recent papers studying the axisymmetric Stokes problem [4] or the axisymmetric Maxwell system [3], as well as in earlier books [5, 13].

The remainder of the paper is outlined as follows. Section 2 is devoted to carefully defining the tangential trace operator on the weighted space $\mathbf{H}_r(\mathrm{curl}, D)$. In Section 3, we define and analyze a mixed formulation for (1.3) on the continuous level. The analysis proceeds by verifying the standard conditions in the theory of mixed methods [7]. In Sections 4 and 5, we consider discretization and approximation based on rectangular and triangular meshes, respectively. Section 6 contains an analysis of the resulting discrete mixed formulations, simultaneously treating

the cases of rectangular and triangular meshes. Numerical results are given in Section 7.

2. The tangential trace operator

The purpose of this section is to define the tangential trace of meridian vector fields in $\mathbf{H}_r(\mathrm{curl}, D)$. This is necessary to introduce a variational formulation incorporating the boundary condition of (1.3), namely $(A_r, A_z) \cdot \mathbf{t} = 0$ on Γ_1, where $\mathbf{t} = (-n_z, n_r)$ denotes the unit tangent vector to ∂D in the r-z plane, oriented counterclockwise. We will first show that smooth vector fields are dense in $\mathbf{H}_r(\mathrm{curl}, D)$. This will show that an integration by parts formula that holds for smooth functions also holds for $\mathbf{H}_r(\mathrm{curl}, D)$ functions. The continuity of a trace operator follows.

Before proving the density result, we first set some more notation. Let $\mathcal{D}(\mathbb{R}^2)$ denote the space of infinitely smooth real-valued functions with compact support in $\mathbb{R}^2$, and set $\mathcal{D}(\overline{D}) = \{\phi|_{\overline{D}} : \phi \in \mathcal{D}(\mathbb{R}^2)\}$. Also, let $\mathbb{R}^2_+ = \{(r, z) : r > 0\}$. Note that for any w in $L^2_1(\mathbb{R}^2_+)$ and ϕ in $\mathcal{D}(\mathbb{R}^2_+)^2$, the action of the distribution $\mathbf{curl}_{rz}w$ on $r\phi$, denoted by $\langle \mathbf{curl}_{rz}w, r\phi \rangle$, satisfies

$$(2.1) \qquad \langle \mathbf{curl}_{rz}w, r\phi \rangle = \int_{\mathbb{R}^2_+} r\, w\, \mathrm{curl}_{rz}\phi\ dr\, dz,$$

which also equals $\langle r\mathbf{curl}_{rz}w, \phi \rangle$. Here, products with r are to be considered in the sense of products of distributions with smooth functions.

Proposition 2.1. *The space of smooth vector fields $\mathcal{D}(\overline{D})^2$ is dense in $\mathbf{H}_r(\mathrm{curl}, D)$.*

Proof. Denoting the dual of a Hilbert space H by H', we show that any functional in $\mathbf{H}_r(\mathrm{curl}, D)'$ which vanishes on $\mathcal{D}(\overline{D})^2$ also vanishes on $\mathbf{H}_r(\mathrm{curl}, D)$. The proposition then follows from Banach space theory. (This technique is also used in [10, Theorem I.2.4].) Hence, we start by considering a linear functional l in $\mathbf{H}_r(\mathrm{curl}, D)'$ that vanishes on $\mathcal{D}(\overline{D})^2$. Since $\mathbf{H}_r(\mathrm{curl}, D)$ is a Hilbert space, the Riesz Representation Theorem yields some $\mathbf{u}$ in $\mathbf{H}_r(\mathrm{curl}, D)$ such that

$$(2.2) \qquad l(\mathbf{v}) = (\mathbf{u}, \mathbf{v})_{r,\mathrm{curl}} \qquad \text{for all } \mathbf{v} \text{ in } \mathbf{H}_r(\mathrm{curl}, D),$$

where $(\cdot, \cdot)_{r,\mathrm{curl}}$ is the inner product in $\mathbf{H}_r(\mathrm{curl}, D)$ with diagonal $\|\cdot\|^2_{r,\mathrm{curl}}$.

The remainder of this proof proceeds by observing the properties of the function $\mathbf{u}$ in (2.2). Let $Z : L^2_1(D) \mapsto L^2_1(\mathbb{R}^2_+)$ denote the operator that trivially extends functions by zero into $\mathbb{R}^2_+$. Also for vector functions, let $Z(u_r, u_z) = (Zu_r, Zu_z)$. Then

$$\int_{\mathbb{R}^2_+} r\big(Z\mathbf{u} \cdot \boldsymbol{\eta} + (Z\mathrm{curl}_{rz}\mathbf{u})(\mathrm{curl}_{rz}\boldsymbol{\eta})\big)\, dr\, dz = l(\boldsymbol{\eta}|_D) = 0 \qquad \text{for all } \boldsymbol{\eta} \text{ in } \mathcal{D}(\mathbb{R}^2_+)^2.$$

By (2.1), this implies that $r(Z\mathbf{u} + \mathbf{curl}_{rz}Z\mathrm{curl}_{rz}\mathbf{u}) = 0$ in the distributional sense. But since $rZ\mathbf{u}$ is in $L^2_{-1}(\mathbb{R}^2_+)^2$, we find that the same equality must hold in $L^2_{-1}(\mathbb{R}^2_+)^2$, or equivalently,

$$(2.3) \qquad Z\mathbf{u} = -\mathbf{curl}_{rz}(Z\mathrm{curl}_{rz}\mathbf{u}) \qquad \text{holds in } L^2_1(\mathbb{R}^2)^2.$$

Next, let $\check{\mathbf{v}}$ denote the revolution of $\mathbf{v}$, i.e., for any $\mathbf{v}$ in $L^2_1(D)$, define $\check{\mathbf{v}}$ a.e. in Ω by $\check{\mathbf{v}}(r, \theta, z) = v_r(r, z)\mathbf{e}_r + v_z(r, z)\mathbf{e}_z$. It is proved in [3] that for any $\mathbf{v}$ in $\mathbf{H}_r(\mathrm{curl}, D)$, the revolved function $\check{\mathbf{v}}$ is in $H(\mathbf{curl}, \Omega)$. Hence for the $\mathbf{u}$ in (2.2), the

function $\mathbf{curl}\,\check{\mathbf{u}}$ is in $L^2(\Omega)^3$. It is easy to see that if we revolve the functions on either side of (2.3) we obtain

$$(2.4) \qquad Z_\Omega \check{\mathbf{u}} = -\mathbf{curl}\,(Z_\Omega \mathbf{curl}\,\check{\mathbf{u}}) \qquad \text{in } L^2(\mathbb{R}^3)^3,$$

where $Z_\Omega : L^2(\Omega)^3 \mapsto L^2(\mathbb{R}^3)^3$ is the extension by zero. Thus, $Z_\Omega \mathbf{curl}\,\check{\mathbf{u}}$ is in $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$, and hence $\mathbf{curl}\,\check{\mathbf{u}}$ is in $\mathbf{H}_0(\mathbf{curl}, \Omega)$.

We need to show that $l(\mathbf{v}) = 0$ for all $\mathbf{v}$ in $\mathbf{H}_r(\mathrm{curl}, D)$. Since $\mathbf{curl}\,\check{\mathbf{u}}$ in $\mathbf{H}_0(\mathbf{curl}, \Omega)$ and $\check{\mathbf{v}}$ is in $H(\mathbf{curl}, \Omega)$, by an integration by parts formula (see e.g. [16, Theorem 3.31]),

$$2\pi\, l(\mathbf{v}) = 2\pi\,(\mathbf{u}, \mathbf{v})_{r,\mathrm{curl}} = (\check{\mathbf{u}}, \check{\mathbf{v}})_{L^2(\Omega)^3} + (\mathbf{curl}\,\check{\mathbf{u}}, \mathbf{curl}\,\check{\mathbf{v}})_{L^2(\Omega)^3}$$
$$= (\check{\mathbf{u}}, \check{\mathbf{v}})_{L^2(\Omega)^3} + (\mathbf{curl}\,\mathbf{curl}\,\check{\mathbf{u}}, \check{\mathbf{v}})_{L^2(\Omega)^3} = 0.$$

Thus l vanishes on $\mathbf{H}_r(\mathrm{curl}, D)$, and the proof is complete. $\qquad \square$

Next, we establish an integration by parts formula. Functions in $H^1_-(D)$ are well known to have zero trace on Γ_0 [3, 5, 11]. Since $H^1_-(D)$ is a subspace of $H^1_1(D)$, and since the latter is well known to have traces in $L^2_1(\Gamma_1)$ (cf. [11, Proposition 2.1]), we can define the trace operator $\gamma_t : \mathcal{D}(\overline{D})^2 \mapsto H^1_-(D)'$ by

$$(2.5) \qquad \langle \gamma_t(\mathbf{v}), \phi \rangle = \int_{\Gamma_1} r\,\mathbf{v} \cdot \mathbf{t}\,\phi\ ds \qquad \text{for all } \phi \in H^1_-(D).$$

Here, and elsewhere later, with slight abuse of notation we continue to use $\langle \cdot, \cdot \rangle$ for the duality pairing in different spaces. For example, in (2.5) it is the duality pairing in $H^1_-(D)$, while in (2.1) it is the duality pairing in $\mathcal{D}(\mathbb{R}^2_+)^2$. The space will be clear from context. The next result shows that the domain of γ_t can be expanded.

Proposition 2.2. *The trace operator extends to a continuous linear map from* $\mathbf{H}_r(\mathrm{curl}, D)$ *into* $H^1_-(D)'$. *Moreover, the integration by parts formula*

$$(2.6) \qquad \langle \gamma_t(\mathbf{v}), \phi \rangle = (\mathbf{v}, \mathbf{curl}_{rz}\phi)_r - (\mathrm{curl}_{rz}\mathbf{v}, \phi)_r$$

holds for all $\mathbf{v}$ *in* $\mathbf{H}_r(\mathrm{curl}, D)$ *and* ϕ *in* $H^1_-(D)$.

Proof. Equation (2.6) obviously holds for all smooth functions $\mathbf{v}$ in $\mathcal{D}(\overline{D})^2$ and ϕ in $\mathcal{D}(\overline{D})$. By [11, Lemma 3.1], the set $\{\phi \in \mathcal{D}(\overline{D}) : \phi$ vanishes in a neighborhood of $\Gamma_0\}$ is dense in $H^1_-(D)$. Hence, for any fixed $\mathbf{v}$ in $\mathcal{D}(\overline{D})^2$, the identity (2.6) holds for all ϕ in $H^1_-(D)$. Note that here we have used the easily verifiable fact that the map $\phi \mapsto \mathbf{curl}_{rz}\phi$ is continuous from $H^1_-(D)$ to $L^2_1(D)^2$. Finally, by the density asserted in Proposition 2.1, the identity (2.6) holds for all $\mathbf{v}$ in $\mathbf{H}_r(\mathrm{curl}, D)$. The continuity of the trace operator follows by standard arguments (see e.g. the proof of [10, Theorem I.2.5]). $\qquad \square$

In the next section, we will present a mixed formulation for (1.3). The boundary condition there becomes an essential boundary condition, so we will use the subspace

$$\mathbf{H}_{r,\diamond}(\mathrm{curl}, D) = \ker \gamma_t$$

in our variational formulation. Clearly, by Proposition 2.2, $\mathbf{H}_{r,\diamond}(\mathrm{curl}, D)$ is a closed subspace of $\mathbf{H}_r(\mathrm{curl}, D)$. Moreover, from (2.5) we see that smooth functions in $\ker \gamma_t$ have tangential components that vanish on Γ_1.

3. Mixed formulation

We can derive a mixed formulation of the meridian problem by multiplying the first two equations of (1.3) by test functions as usual and integrating by parts. The process is similar to the derivation of the standard mixed formulation [12, 16] for the three-dimensional magnetostatics equations. The appropriate integration by parts formula in weighted spaces for the curl and divergence operators are given in Proposition 2.2 and [9, Lemma 1], respectively. The resulting weak formulation seeks $\mathbf{u} \in \mathbf{H}_{r,\diamond}(\mathrm{curl}, D)$ and $p \in H^1_{1,\diamond}(D)$ satisfying

(3.1)
$$\begin{cases} (\mu^{-1}\mathrm{curl}_{rz}\mathbf{u}, \mathrm{curl}_{rz}\mathbf{v})_r + (\mathbf{v}, \mathbf{grad}_{rz}p)_r &= (\mathbf{f}, \mathbf{v})_r \quad \text{for all } \mathbf{v} \in \mathbf{H}_{r,\diamond}(\mathrm{curl}, D), \\ \qquad\qquad\qquad\qquad (\mathbf{u}, \mathbf{grad}_{rz}q)_r &= (g, q)_r \quad \text{for all } q \in H^1_{1,\diamond}(D), \end{cases}$$

where $\mathbf{f} = (J_r, J_z)$ and $g = -\,\mathrm{div}_{rz}(A_r, A_z)$. Although (A_r, A_z) is divergence-free in the model problem, we allow for g to be any function in $L^2_1(D)$. In the remainder of this section, we quickly verify the conditions of the Babuška-Brezzi theory [7] to conclude the unique solvability of (3.1).

Defining the linear operator

$$B : \mathbf{H}_{r,\diamond}(\mathrm{curl}, D) \to H^1_{1,\diamond}(D)' \quad \text{by} \quad B\mathbf{v}(q) = (\mathbf{v}, \mathbf{grad}_{rz}q)_r,$$

we denote $W^0 = \ker B$, i.e.

$$W^0 = \{\mathbf{v} \in \mathbf{H}_{r,\diamond}(\mathrm{curl}, D) : (\mathbf{v}, \mathbf{grad}_{rz}q)_r = 0 \text{ for all } q \in H^1_{1,\diamond}(D)\}.$$

Then we have the following theorem. In this theorem and in the remainder of the paper, the letter C will denote a generic constant whose value may vary in different occurrences.

Theorem 3.1. *The following holds for the axisymmetric mixed formulation* (3.1):

(1) Inf-sup condition: *There exists a constant $C > 0$ such that*

$$C\,\|p\|_{1,r} \leq \sup_{\mathbf{v} \in \mathbf{H}_{r,\diamond}(\mathrm{curl}, D)} \frac{(\mathbf{v}, \mathbf{grad}_{rz}p)_r}{\|\mathbf{v}\|_{r,\mathrm{curl}}}, \qquad \text{for all } p \in H^1_{1,\diamond}(D).$$

(2) Coercivity on $\ker B$: *There exists a constant $C > 0$ such that*

$$\|\mathbf{v}\|_{r,\mathrm{curl}} \leq C\mu_1^{1/2}\|\mu^{-1/2}\mathrm{curl}_{rz}\mathbf{v}\|_r, \qquad \text{for all } \mathbf{v} \in W^0.$$

(3) Unique solvability: *There is a unique $\mathbf{u}$ in $\mathbf{H}_{r,\diamond}(\mathrm{curl}, D)$ and a unique p in $H^1_{1,\diamond}(D)$ satisfying* (3.1).

Proof. (1) Denote by $\breve{H}^1_0(\Omega)$ the functions in $H^1_0(\Omega)$ invariant under rotation. The mapping $\breve{H}^1_0(\Omega) \to H^1_{1,\diamond}(D)$ defined by $v(r, z) = \breve{v}(r, \theta, z)$ is an isomorphism (see §II.4 of [5]). Applying this isomorphism and the Poincaré inequality on $H^1_0(\Omega)$ we get

(3.2) $$2\pi\|p\|^2_{1,r} \leq \|\breve{p}\|^2_{H^1(\Omega)} \leq C\|\mathbf{grad}\,\breve{p}\|^2_{L^2(\Omega)} = 2\pi C\|\mathbf{grad}_{rz}p\|^2_r,$$

where $\mathbf{grad}(\cdot)$ denotes the three-dimensional gradient operator. Note that for any p in $H^1_{1,\diamond}(D)$, the gradient $\mathbf{grad}\,\breve{p}$ is in $\mathbf{H}_0(\mathbf{curl}, \Omega)$ and hence $\mathbf{grad}_{rz}p$ is in $\mathbf{H}_{r,\diamond}(\mathrm{curl}, D)$ (cf. [3, Proposition 3.19]). Moreover, $\mathrm{curl}_{rz}(\mathbf{grad}_{rz}p) = 0$ implies

$$C\|p\|_{1,r} \leq \frac{(\mathbf{grad}_{rz}p, \mathbf{grad}_{rz}p)_r}{\|\mathbf{grad}_{rz}p\|_{r,\mathrm{curl}}} \leq \sup_{\mathbf{v} \in \mathbf{H}_{r,\diamond}(\mathrm{curl}, D)} \frac{(\mathbf{v}, \mathbf{grad}_{rz}p)_r}{\|\mathbf{v}\|_{r,\mathrm{curl}}}.$$

(2) Let $\mathbf{v} \in W^0$ and consider the revolved vector field $\breve{\mathbf{v}} = v_r \mathbf{e}_r + v_z \mathbf{e}_z$ defined in Ω. Clearly $\breve{\mathbf{v}} \times \mathbf{n} = 0$ on $\partial\Omega$. By [3, Proposition 3.19], $\breve{\mathbf{v}}$ is in $H(\mathbf{curl}, \Omega)$. Let us now also show that

$$(3.3) \qquad (\breve{\mathbf{v}}, \mathbf{grad}\, q)_{L^2(\Omega)} = 0, \qquad \text{for all } q \in H_0^1(\Omega).$$

The left hand side is

$$(\breve{\mathbf{v}}, \mathbf{grad}\, q)_{L^2(\Omega)} = \iint_D \int_0^{2\pi} r\, \breve{\mathbf{v}} \cdot \mathbf{grad}\, q \, d\theta\, dr\, dz$$

$$= \iint_D \int_0^{2\pi} r\, (v_r\, \partial_r q + v_z\, \partial_z q)\, d\theta\, dr\, dz$$

$$= 2\pi \iint_D r\, (v_r\, \partial_r \underset{\smile}{q} + v_z\, \partial_z \underset{\smile}{q})\, dr\, dz,$$

where

$$\underset{\smile}{q}(r, z) = \frac{1}{2\pi} \int_0^{2\pi} q(r, \theta, z)\, d\theta,$$

and the interchanges in the order of integration and differentiations are justified as usual. Since q is in $H_0^1(\Omega)$, by [5, Theorem II.3.1], the averaged function $\underset{\smile}{q}$ is in $H_{1,\diamond}^1(D)$. Consequently, the above calculation implies that

$$(\breve{\mathbf{v}}, \mathbf{grad}\, q)_{L^2(\Omega)} = 2\pi (\mathbf{v}, \mathbf{grad}_{rz} \underset{\smile}{q})_r = 0,$$

as $\mathbf{v}$ is given to be in W^0. By a standard Poincaré-Friedrichs estimate [1, 10] (sometimes also called Weber inequality [3, 18]) for functions satisfying (3.3), we have

$$\|\breve{\mathbf{v}}\|_{L^2(\Omega)} \leq C \|\mathbf{curl}\, \breve{\mathbf{v}}\|_{L^2(\Omega)}.$$

Since $\breve{\mathbf{v}}$ is axisymmetric, $\mathbf{curl}\, \breve{\mathbf{v}} = (\mathrm{curl}_{rz}\mathbf{v})\mathbf{e}_\theta$, so the above inequality implies a corresponding estimate on D, namely

$$\|\mathbf{v}\|_r \leq C \|\mathrm{curl}_{rz}\mathbf{v}\|_r \leq C\mu_1^{1/2} \|\mu^{-1/2}\mathrm{curl}_{rz}\mathbf{v}\|_r.$$

This is equivalent to the desired inequality.

(3) The final statement follows by applying the Babuška-Brezzi theory for mixed systems, as the previous two statements of the theorem verified the assumptions required to apply the theory. $\qquad \square$

4. Approximation on rectangular meshes

In this section, we introduce finite element spaces based on rectangular meshes, which approximate the spaces $\mathbf{H}_r(\mathrm{curl}, D)$ and $H_1^1(D)$. Triangular finite elements are given in the next section. Although triangular elements cover more general domains, we include the rectangular case, because its analysis readily reveals the main ideas, and they are simple to implement when applicable.

The rectangular finite element functions are simple bilinear functions for $H_1^1(D)$ and the two-dimensional edge-based Nédélec functions for $\mathbf{H}_r(\mathrm{curl}, D)$. To establish notation for our finite element spaces, first define the polynomial spaces

$$Q_{1,1} = \{p(r, z) = c_0 + c_1 r + c_2 z + c_3 rz : c_i \in \mathbb{R} \text{ for } 0 \leq i \leq 3\},$$
$$\mathcal{ND}_{1,1} = \{(a + bz,\ c + dr) : a, b, c, d \in \mathbb{R}\}.$$

Given a rectangular mesh $\mathcal{T}_h$ aligned with the coordinate axes, define the finite element spaces

$$V_h = \{u \in H_1^1(D) : u|_K \in Q_{1,1} \text{ for all } K \in \mathcal{T}_h\},$$
$$W_h = \{\mathbf{v} \in \mathbf{H}_r(\operatorname{curl}, D) : \mathbf{v}|_K \in \mathcal{ND}_{1,1} \text{ for all } K \in \mathcal{T}_h\},$$
$$S_h = \{u \in L_1^2(D) : u|_K \text{ is constant, for all } K \in \mathcal{T}_h\}.$$

Additionally, let

$$(4.1) \qquad \begin{aligned} V_{h,\diamond} &= \{v \in V_h : v = 0 \text{ on } \Gamma_1\}, \\ W_{h,\diamond} &= \{\mathbf{v} \in W_h : \mathbf{v} \cdot \mathbf{t} = 0 \text{ on } \Gamma_1\}. \end{aligned}$$

Our aim in this section is to construct projectors Π_h^V, Π_h^W and Π_h^S onto the above finite element spaces such that the following diagram commutes:

$$(4.2) \qquad \begin{array}{ccccc} H_1^1(D) & \xrightarrow{\ \mathbf{grad}_{rz}\ } & \mathbf{H}_r(\operatorname{curl}, D) & \xrightarrow{\ \operatorname{curl}_{rz}\ } & L_1^2(D) \\ \downarrow{\scriptstyle \Pi_h^V} & & \downarrow{\scriptstyle \Pi_h^W} & & \downarrow{\scriptstyle \Pi_h^S} \\ V_h & \xrightarrow{\ \mathbf{grad}_{rz}\ } & W_h & \xrightarrow{\ \operatorname{curl}_{rz}\ } & S_h. \end{array}$$

The above is somewhat of an abuse of notation, as the first two projectors are defined on sufficiently smooth functions in $H_1^1(D)$ and $\mathbf{H}_r(\operatorname{curl}, D)$, respectively. The importance of such a property in the analysis of mixed methods with the usual (un-weighted) Sobolev spaces is well known. As in the case of the standard unweighted Sobolev spaces, it is quite technical to construct projectors whose domain is the entire corresponding Sobolev space appearing in (4.2). Hence, the projectors we construct will require further regularity.

We first define the interpolation operator Π_h^V by its action element by element. On every element K whose boundary does not intersect Γ_0, $(\Pi_h^V v)|_K$ is the unique bilinear function whose value at each of the four vertices of K coincides with that of v. The remaining elements have vertices of the form $(0, z_0)$, (h_r, z_0), (h_r, z_1), and $(0, z_1)$, with $z_1 = z_0 + h_z$. On these elements, we define new degrees of freedom by

$$(4.3) \qquad \begin{aligned} m_1(v) &= v(h_r, z_0), & m_2(v) &= \int_0^{h_r} r\, \partial_r v(r, z_0)\, dr, \\ m_3(v) &= v(h_r, z_1), & m_4(v) &= \int_0^{h_r} r\, \partial_r v(r, z_1)\, dr. \end{aligned}$$

The interpolant $\Pi_h^V v$ on such an element is the unique function in $Q_{1,1}$ satisfying $m_i(\Pi_h^V v) = m_i(v)$ for $i = 1, \ldots, 4$.

Remark 4.1. The definition and analysis of the above interpolant are very similar to those of an interpolation operator defined in [11], where the weight $r^{1/2}$ was used in place of r in defining m_2 and m_4. We chose the r weight for convenience. A similar analysis could be provided for the $r^{1/2}$ weight. In [11], the reason for the introduction of integral degrees of freedom was (incorrectly) given to be the lack of continuity of the functional $v \mapsto v(0, z)$ in $H_1^2(D)$. However, this functional is indeed continuous in $H_1^2(D)$, as shown in [15, Theorem 4.7]. Relying on this continuity, an alternate approach to approximation theory in $H_1^2(D)$ is pursued in [15]. However, [15] does not give the other projectors and the commutativity properties in (4.2) needed for the analysis of mixed methods.

Next, we define the projection Π_h^W onto W_h. On rectangles K not intersecting Γ_0, Π_h^W is the standard Nédélec interpolant defined by the unweighted integrals of the tangential components along edges, i.e., the degrees of freedom are $\int_e (\Pi_h^W \mathbf{v}) \cdot \mathbf{t} \, ds$ for all edges e of K. On the remaining rectangles, the degrees of freedom are

$$
\begin{aligned}
(4.4) \quad & M_1(\mathbf{v}) = \int_{(0,z_0)}^{(h_r,z_0)} r \, \mathbf{v} \cdot \mathbf{t} \, ds, && M_2(\mathbf{v}) = \int_{(h_r,z_0)}^{(h_r,z_0+h_z)} \mathbf{v} \cdot \mathbf{t} \, ds, \\
& M_3(\mathbf{v}) = \int_{(h_r,z_0+h_z)}^{(0,z_0+h_z)} r \, \mathbf{v} \cdot \mathbf{t} \, ds, && M_4(\mathbf{v}) = \int_K r \, \mathrm{curl}_{rz} \mathbf{v} \, dr \, dz.
\end{aligned}
$$

Here $\int_{\mathbf{a}_1}^{\mathbf{a}_2}$ denotes the straight line integral from point $\mathbf{a}_1$ to point $\mathbf{a}_2$ in the r-z plane. For any function $\mathbf{v}$ for which the moments M_i exist, we define $\Pi_h^W \mathbf{v}|_K$ as a function in $\mathcal{N}\mathcal{D}_{1,1}$ satisfying $M_i(\Pi_h^W \mathbf{v}) = M_i(\mathbf{v})$ for $i = 1, \dots, 4$. That $\Pi_h^W \mathbf{v}$ is uniquely defined this way follows from the next lemma.

Finally, for the space S_h at the end of (4.2), we define $\Pi_h^S w$ for any w in $L_1^2(D)$ as the unique function in S_h such that on a mesh element K, its restriction $(\Pi_h^S w)|_K$ satisfies

$$
\begin{aligned}
(4.5) \quad & \int_K r \, (\Pi_h^S w) \, v \, dr \, dz = \int_K r \, w \, v \, dr \, dz \quad \text{if } \bar{K} \text{ intersects } \Gamma_0, \text{ and} \\
& \int_K (\Pi_h^S w) \, v \, dr \, dz = \int_K w \, v \, dr \, dz \quad \text{otherwise,}
\end{aligned}
$$

for all v in S_h. With these definitions, we have the following lemma verifying the commutative properties in (4.2).

Lemma 4.1. *The interpolation operators defined above have the following properties:*

(1) *The degrees of freedom in (4.3) are continuous linear functionals on $H_1^2(D)$ and are unisolvent. Hence Π_h^V is well defined and continuous on $H_1^2(D)$.*

(2) *The degrees of freedom in (4.4) are continuous linear functionals on $H_1^1(D)^2$ and are unisolvent. Hence Π_h^W is well defined and continuous on $H_1^1(D)^2$.*

(3) *The commutative properties*

$$
\mathrm{curl}_{rz} \Pi_h^W \mathbf{v} = \Pi_h^S \mathrm{curl}_{rz} \mathbf{v},
$$
$$
\Pi_h^W \mathbf{grad}_{rz} u = \mathbf{grad}_{rz} \Pi_h^V u,
$$

hold for any u and $\mathbf{v}$ for which all of the above quantities are bounded.

(4) *Π_h^V and Π_h^W preserve zero boundary conditions, i.e., if u is in $H_{1,\diamond}^1(D) \cap H_1^2(D)$, then Π_h^V is in $V_{h,\diamond}$, and if $\mathbf{v}$ is in $\mathbf{H}_{r,\diamond}(\mathrm{curl}, D) \cap H_1^1(D)^2$, then $\Pi_h^W \mathbf{v}$ is in $W_{h,\diamond}$.*

Proof. The continuity of m_1 and m_3 on $H_1^2(D)$ follows from the standard Sobolev inequality (because the points involved are away from the z-axis). For m_2 and m_4, the same continuity follows from [11, Proposition 2.1] or Lemma A.1. Similarly, M_i is continuous on $H_1^1(D)^2$ for $1 \le i \le 4$.

Next, let us prove the unisolvence of $\{M_i\}$. (We omit the proof of unisolvence of $\{m_i\}$, as it is simpler.) Suppose $\mathbf{v}_h = (a + bz, \ c + dr)$ on a rectangle K intersecting Γ_0, with $M_i(\mathbf{v}_h) = 0$ for $1 \le i \le 4$. We need to show that $\mathbf{v}_h$ vanishes. Since

$$
\mathrm{curl}_{rz} \mathbf{v}_h = b - d,
$$

$M_4(\mathbf{v}_h) = 0$ implies $b = d$. Consequently, $\mathbf{v}_h = \mathbf{grad}_{rz}\psi_h$ for some bilinear function ψ_h in $Q_{1,1}$. It follows from $M_1(\mathbf{v}_h) = 0$ and $M_3(\mathbf{v}_h) = 0$ that $\partial\psi_h/\partial r$ vanishes on the horizontal edges of K. Thus, $\psi_h(r, z) = c_0 + c_2 z$. Now

$$\int_{(h_r,z_0)}^{(h_r,z_0+h_z)} \mathbf{grad}_{rz}\psi_h \cdot \mathbf{t}\, ds = M_2(\mathbf{v}_h) = 0$$

implies $\psi_h(r, z) = c_0$ is constant, i.e., $\mathbf{v}_h = \mathbf{grad}_{rz}\psi_h = 0$.

The commutative properties follow, in a standard way, elements away from Γ_0. For the remaining elements, the identity $\operatorname{curl}_{rz}\Pi_h^W \mathbf{v} = \Pi_h^S \operatorname{curl}_{rz}\mathbf{v}$ follows immediately from the definition of M_4, since $\operatorname{curl}_{rz}\mathbf{w}$ is a scalar constant for any $\mathbf{w}$ in $\mathcal{ND}_{1,1}$. To verify the second identity $\Pi_h^W \mathbf{grad}_{rz}u = \mathbf{grad}_{rz}\Pi_h^V u$, it suffices to show that the degrees of freedom M_i agree for $\mathbf{grad}_{rz}u$ and $\mathbf{grad}_{rz}\Pi_h^V u$, because the gradient of any function in V_h is in W_h. By the definition (4.3) of Π_h^V,

$$M_1(\mathbf{grad}_{rz}(u - \Pi_h^V u)) = \int_0^{h_r} r\bigl(\partial_r u(r, z_0) - \partial_r \Pi_h^V u(r, z_0)\bigr)\, dr = 0,$$

$$\begin{aligned} M_2(\mathbf{grad}_{rz}(u - \Pi_h^V u)) &= \int_{z_0}^{z_0+h_z} \bigl(\partial_z u(h_r, z) - \partial_z \Pi_h^V u(h_r, z)\bigr)\, dz \\ &= u(h_r, z_0+h_z) - \Pi_h^V u(h_r, z_0+h_z) - u(h_r, z_0) + \Pi_h^V u(h_r, z_0) \\ &= 0, \end{aligned}$$

$$M_3(\mathbf{grad}_{rz}(u - \Pi_h^V u)) = \int_{h_r}^0 -r\bigl(\partial_r u(r, z_0 + h_z) - \partial_r \Pi_h^V u(r, z_0 + h_z)\bigr)\, dr = 0,$$

$$M_4(\mathbf{grad}_{rz}(u - \Pi_h^V u)) = \int_K r\operatorname{curl}_{rz}\mathbf{grad}_{rz}(u - \Pi_h^V u)\, dr\, dz = 0.$$

This proves the second commutative identity.

Finally, the last assertion on the boundary preservation properties is obvious by construction. $\qquad\square$

We conclude this section by proving approximation properties for the interpolation operators. From now on, whenever we use the generic constant C, its value will always be independent of the mesh size $h = \max_{K \in \mathcal{T}_h} \operatorname{diam}(K)$.

Lemma 4.2 (Approximation estimates). *There exists a constant C such that*

$$(4.6) \qquad \|\mathbf{grad}_{rz}(u - \Pi_h^V u)\|_{L_1^2(D)} \le Ch|u|_{H_1^2(D)},$$

$$(4.7) \qquad \|\mathbf{v} - \Pi_h^W \mathbf{v}\|_{L_1^2(D)^2} \le Ch|\mathbf{v}|_{H_1^1(D)^2},$$

$$(4.8) \qquad \|\operatorname{curl}_{rz}(\mathbf{v} - \Pi_h^W \mathbf{v})\|_{L_1^2(D)} \le Ch|\operatorname{curl}_{rz}\mathbf{v}|_{H_1^1(D)},$$

$$(4.9) \qquad \|w - \Pi_h^S w\|_{L_1^2(D)} \le Ch|w|_{H_1^1(D)},$$

for all $u \in H_1^2(D)$, $w \in H_1^1(D)$, and $\mathbf{v} \in H_1^1(D)^2$. For (4.8), we additionally require $\operatorname{curl}_{rz}\mathbf{v} \in H_1^1(D)$.

Proof. These estimates are established by proving analogous local estimates on each element, squaring them, and summing. For elements away from the z-axis, standard arguments prove the estimates. For the remaining elements K intersecting Γ_0, we will now show that all of the estimates follow from the fact that whenever ϕ in

$H_1^1(K)$ satisfies any one of the conditions

$$(4.10) \qquad \int_e r\,\phi(r,z)\,dr = 0, \quad \text{for an edge } e \text{ of } K \text{ having a vertex in } \Gamma_0, \text{ or}$$

$$(4.11) \qquad \int_e \phi(r,z)\,dr = 0, \quad \text{for the vertical edge } e \text{ of } K \text{ not intersecting } \Gamma_0, \text{ or}$$

(4.12)
$$\int_K r\,\phi(r,z)\,dr\,dz = 0,$$

then

$$(4.13) \qquad \|\phi\|_{L_1^2(K)} \le Ch|\phi|_{H_1^1(K)}.$$

The fact that (4.13) holds under conditions (4.10) and (4.11) is contained in [11, Lemma 2.1]. Its proof under condition (4.12) is very similar, so we omit it.

Consider the last operator Π_h^S first. Since $\phi = (w - \Pi_h^S w)|_K$ satisfies (4.12), the estimate $\|\phi\|_{L_1^2(K)} \le Ch|w|_{H_1^1(K)}$ follows from (4.13) and $|\Pi_h^S w|_{H_1^1(K)} = 0$. This proves (4.9). To prove estimate (4.7) for Π_h^W, let $(\phi_r, \phi_z) = \mathbf{v} - \Pi_h^W \mathbf{v}$ and observe that by construction, ϕ_r satisfies (4.10) and ϕ_z satisfies (4.11). Hence

$$\|\phi_r\|_{L_1^2(K)} \le Ch|\phi_r|_{H_1^1(K)}, \qquad \|\phi_z\|_{L_1^2(K)} \le Ch|\phi_z|_{H_1^1(K)}.$$

It is easy to verify by direct calculation that $|\Pi_h^W \mathbf{v}|_{H_1^1(K)^2} \le C|\mathbf{v}|_{H_1^1(K)^2}$. Using this to bound the right hand sides above, we get (4.7). Estimate (4.8) follows from Lemma 4.1, because

$$\begin{aligned}
\|\mathrm{curl}_{rz}(\mathbf{v} - \Pi_h^W \mathbf{v})\|_{L_1^2(D)} &= \|\mathrm{curl}_{rz}\mathbf{v} - \Pi_h^S \mathrm{curl}_{rz}\mathbf{v}\|_{L_1^2(D)} && \text{by Lemma 4.1(3),} \\
&\le Ch|\mathrm{curl}_{rz}\mathbf{v}|_{H_1^1(D)} && \text{by (4.9).}
\end{aligned}$$

Proof of the final estimate proceeds similarly using Lemma 4.1:

$$\begin{aligned}
\|\mathbf{grad}_{rz}(u - \Pi_h^V u)\|_{L_1^2(D)} &= \|\mathbf{grad}_{rz}u - \Pi_h^W \mathbf{grad}_{rz}u\|_{L_1^2(D)} && \text{by Lemma 4.1(3),} \\
&\le Ch|\mathbf{grad}_{rz}u|_{H_1^1(D)} && \text{by (4.7),}
\end{aligned}$$

from which (4.6) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5. Approximation on triangular meshes

We now consider triangular meshes, proving results analogous to those given in the previous section for rectangular meshes. In particular, we want to construct projectors onto triangular finite element spaces satisfying the commuting diagram property (4.2) and possessing optimal approximation properties.

With triangular elements, we are now able to mesh more general polygonal domains D. Let $\mathcal{T}_h$ be a mesh of D consisting of triangles, satisfying the usual finite element assumptions. For simplicity, we assume that $\mathcal{T}_h$ is quasiuniform. Define the polynomial spaces

$$\begin{aligned}
P_1 &= \{p(r,z) = c_0 + c_1 r + c_2 z : c_i \in \mathbb{R} \text{ for } 0 \le i \le 2\}, \\
\mathcal{ND}_1 &= \{(a - bz,\ c + br) : a, b, c \in \mathbb{R}\}.
\end{aligned}$$

Then for each $\mathbf{v}$ in $\mathcal{ND}_1(\tau)$, $\mathbf{v} \cdot \mathbf{t}$ is constant along each edge of τ. The global finite element spaces are

$$V_h = \{u \in H_1^1(D) : u|_\tau \in P_1 \text{ for all } \tau \in \mathcal{T}_h\},$$
$$W_h = \{\mathbf{v} \in \mathbf{H}_r(\text{curl}, D) : \mathbf{v}|_\tau \in \mathcal{ND}_1 \text{ for all } \tau \in \mathcal{T}_h\},$$
$$S_h = \{u \in L_1^2(D) : u|_\tau \in \mathbb{R} \text{ is constant, for all } \tau \in \mathcal{T}_h\}.$$

One obvious set of global degrees of freedom for V_h consists of the function values at all mesh vertices. However, we cannot consider the nodal interpolant corresponding to these degrees of freedom, as the values on the z-axis are undefined for functions in our weighted spaces. As in the rectangular case, we therefore define new degrees of freedom through weighted edge moments. In the triangular case, there is an additional complication due to multiple interior edges meeting at a vertex on the z-axis. Hence, we associate with each vertex $\mathbf{a}_i$ in Γ_0 one edge $e(\mathbf{a}_i)$ such that $\mathbf{a}_i$ is an endpoint of $e(\mathbf{a}_i)$ and $e(\mathbf{a}_i) \not\subseteq \Gamma_0$. If $\mathbf{a}_i$ is an endpoint of Γ_0, then we choose $e(\mathbf{a}_i)$ to be the mesh edge connected to $\mathbf{a}_i$ lying on Γ_1. Then, we set one global degree of freedom per mesh vertex by

(5.1)
$$\begin{cases} \sigma_i^V(u) = u(\mathbf{a}_i), & \text{if } \mathbf{a}_i \notin \Gamma_0, \\ \sigma_i^V(u) = \int_{e(\mathbf{a}_i)} r\, \mathbf{t} \cdot \mathbf{grad}_{rz} u \ ds, & \text{if } \mathbf{a}_i \in \Gamma_0. \end{cases}$$

Proposition 5.1. *The linear functionals in* (5.1) *are continuous on* $H_1^2(D)$ *and form unisolvent degrees of freedom for* V_h.

Proof. For $\mathbf{a}_i \notin \Gamma_0$, by a standard Sobolev inequality,

$$|\sigma_i^V(u)| = |u(\mathbf{a}_i)| \leq c_i \|u\|_{H_1^2(D)},$$

while for $\mathbf{a}_i \in \Gamma_0$, by Lemma A.1,

$$|\sigma_i^V(u)| = \left| \int_{e(\mathbf{a}_i)} r\, \mathbf{t} \cdot \mathbf{grad}_{rz} u \ ds \right| \leq c_i \|u\|_{H_1^2(D)},$$

for some constants c_i independent of u. Hence, the degrees of freedom are continuous linear functionals on $H_1^2(D)$. It only remains to show that these degrees of freedom are unisolvent. If $v_h \in V_h$ has $\sigma_i^V(v_h) = 0$ for all i, then obviously v_h vanishes at all vertices $\mathbf{a}_i \notin \Gamma_0$. It also vanishes on the remaining vertices $\mathbf{a}_i \in \Gamma_0$ because the zero integral degrees of freedom imply that $v_h(\mathbf{a}_i)$ must equal the value of v_h at the other endpoint of $e(\mathbf{a}_i)$. Hence $v_h \equiv 0$. $\square$

Next, we define the set of global degrees of freedom for the Nédélec space W_h as

(5.2)
$$\begin{cases} \sigma_i^W(\mathbf{v}) = \int_{e(\mathbf{a}_i)} r\, \mathbf{v} \cdot \mathbf{t} \ ds & \text{for all mesh vertices } \mathbf{a}_i \in \overline{\Gamma}_0, \\ \sigma_e^W(\mathbf{v}) = \int_e \mathbf{v} \cdot \mathbf{t} \ ds & \text{for all mesh edges } e \text{ with } \overline{e} \cap \overline{\Gamma}_0 = \emptyset, \\ \sigma_\tau^W(\mathbf{v}) = \int_\tau r\, \text{curl}_{rz} \mathbf{v} \ dr\, dz & \text{for all mesh triangles } \tau \text{ with } \overline{\tau} \cap \overline{\Gamma}_0 \neq \emptyset. \end{cases}$$

Note that these differ from the standard Nédélec degrees of freedom [17] in three respects: (i) We use weighted edge moments near Γ_0. (ii) Not all interior edges are used in defining the degrees of freedom. (iii) Elements near Γ_0 have an interior

degree of freedom even though we are only considering the lowest order case. Due to these differences, we must now verify that the new degrees of freedom control W_h.

Proposition 5.2. *The linear functionals in* (5.2) *are continuous on* $H_1^1(D)^2$ *and form unisolvent degrees of freedom for* W_h.

Proof. To prove unisolvence, suppose that all the degrees of freedom in (5.2) vanish for a function $\mathbf{v}_h$ in W_h. Then, clearly $\mathbf{v}_h$ vanishes on elements τ not intersecting Γ_0. It remains to show that $\mathbf{v}_h$ vanishes on the remaining elements, which we collect into a set S. The elements in S can be classified in two types: for $k = 1$ or 2, we say that a triangle τ in S is of type k if τ has exactly k vertices on Γ_0 (see Figure 1). For any $\tau \in S$, writing $\mathbf{v}_h|_\tau = (a - bz, c + br)$ and using the fact that the interior degree of freedom is zero, we have

$$-2b \int_\tau r \, dr \, dz = \int_\tau r \operatorname{curl}_{rz}(a - bz, c + br) \ dr \, dz = \int_\tau r \operatorname{curl}_{rz}\mathbf{v}_h \ dr \, dz = 0.$$

Hence $b = 0$ and $\mathbf{v}_h$ is constant on τ. Therefore, to prove that $\mathbf{v}_h$ vanishes on any τ in S, it suffices to show that the tangential component of $\mathbf{v}_h$ is zero on any two edges of τ.

Note that around any interior vertex of Γ_0, there are two triangles of type two separated by a number (possibly none) of type one triangles. The case of an end point is similar but only involves one triangle of type two.

Now let us prove that $\mathbf{v}_h$ vanishes on all type one triangles in S. Let τ be a type one triangle and let $\boldsymbol{a}$ be its vertex on $\overline{\Gamma}_0$ (including the endpoints of Γ_0). Then, $e(\boldsymbol{a})$ cannot be an edge shared by type two triangles only, i.e., there is at least one type one triangle, say τ', having $e(\boldsymbol{a})$ as an edge (see Figure 1). This triangle τ' need not coincide with τ. It has an edge not intersecting Γ_0 where the tangential component of $\mathbf{v}_h$ is given to be zero. Therefore, $\mathbf{v}_h \cdot \mathbf{t}$ is zero on two edges of τ', hence $\mathbf{v}_h$ vanishes on τ'. We can repeat this argument on any type one triangle adjacent to τ' (since $\mathbf{v}_h \cdot \mathbf{t} = 0$ on all edges of τ'). Therefore $\mathbf{v}_h$ vanishes on all type one triangles connected to $\boldsymbol{a}$, including τ.

Next, consider a type two triangle τ. It has two edges not on Γ_0. We claim that $\mathbf{v}_h \cdot \mathbf{t}$ vanishes on these two edges. Indeed, if e is such an edge, and e is shared by a type one triangle, then by the conclusion of the previous paragraph, $\mathbf{v}_h \cdot \mathbf{t} = 0$ on e. Also, if e is on Γ_1, then obviously $\mathbf{v}_h \cdot \mathbf{t} = 0$ on e. The only other possibility is that e is shared by another type two triangle τ'. But in this case, e coincides with $e(\boldsymbol{a})$ where $\boldsymbol{a}$ is the common vertex of τ and τ' on Γ_0. Obviously, $\mathbf{v}_h \cdot \mathbf{t} = 0$ on $e(\boldsymbol{a})$ since all degrees of freedom are given to be zero. Thus, $\mathbf{v}_h \cdot \mathbf{t}$ vanishes on two edges of each type two triangle, so $\mathbf{v}_h$ vanishes on type two triangles as well.

The continuity of the linear functionals on $H_1^1(D)^2$ follows easily from Lemma A.1 and standard results. $\square$

Define the projector Π_h^V onto V_h by

$$\sigma_i^V(\Pi_h^V v) = \sigma_i^V(v),$$

for all the degrees of freedom σ_i^V in (5.1). Similarly define the projector Π_h^W onto W_h using all the degrees of freedom in (5.2). By Propositions 5.1 and 5.2, these projectors are well defined. In particular, $\Pi_h^W \mathbf{u}$ for any $\mathbf{u}$ in $H_1^1(D)^2$ automatically satisfies the tangential continuity constraints of W_h along all mesh edges even

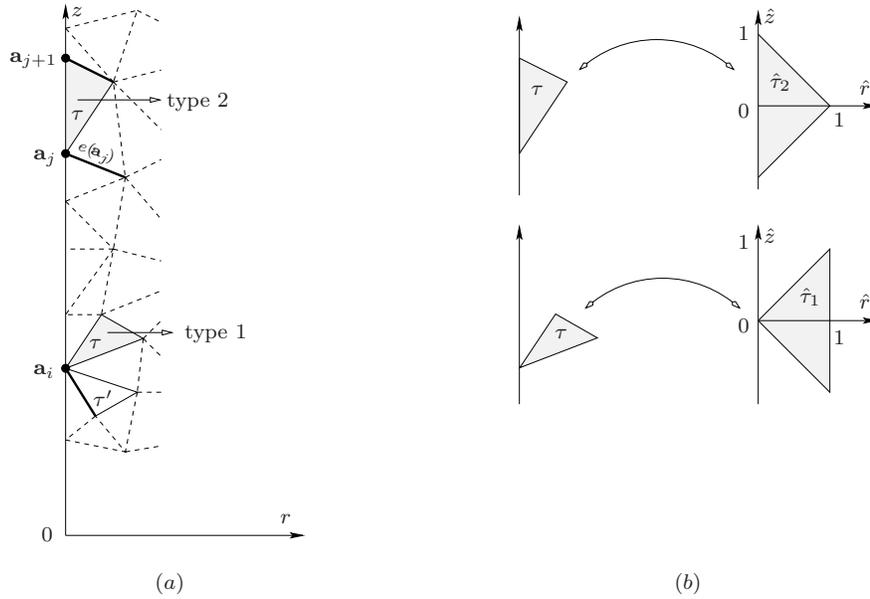(a) (b)

FIGURE 1. (a) Illustration of the notations used in some of the proofs in this section. The edges that define degrees of freedom connected to vertices $\mathbf{a}_i$, $\mathbf{a}_j$ and $\mathbf{a}_{j+1}$ are marked by thicker lines. (b) The two types of mappings and the two corresponding reference triangles.

though not all edges are degrees of freedom, i.e.,

$$\int_e r\mathbf{v} \cdot \mathbf{t}\, ds \text{ and } \int_e r(\Pi_h^W \mathbf{v}) \cdot \mathbf{t}\, ds$$

are not equal in general.

Additionally, we have the following local norm estimate.

Proposition 5.3. *There exists a constant C such that*

$$|\Pi_h^W \mathbf{v}|_{H_1^1(\tau)^2} \le C|\mathbf{v}|_{H_1^1(\tau)^2},$$

for all mesh triangles τ in $\mathcal{T}_h$ and all $\mathbf{v}$ in $H_1^1(\tau)^2$.

Proof. The result is standard in the case when τ does not intersect Γ_0. Writing $\Pi_h^W \mathbf{v}|_\tau = (a - bz, c + br)$ on the remaining triangles τ, we have

$$-2b \int_\tau r\, dr\, dz = \int_\tau r\, \mathrm{curl}_{rz}(a - bz, c + br)\, dr\, dz = \int_\tau r\, \mathrm{curl}_{rz}\Pi_h^W \mathbf{v}\, dr\, dz$$

$$= \sigma_\tau^W(\Pi_h^W \mathbf{v}) = \sigma_\tau^W(\mathbf{v}) = \int_\tau r\, \mathrm{curl}_{rz}\mathbf{v}\, dr\, dz$$

and hence

$$|\Pi_h^W \mathbf{v}|_{H_1^1(\tau)^2}^2 = \int_\tau r\, b^2\, dr\, dz = \frac{\left(\int_\tau r\, \mathrm{curl}_{rz}\mathbf{v}\, dr\, dz\right)^2}{4 \int_\tau r\, dr\, dz} \le C|\mathbf{v}|_{H_1^1(\tau)^2}^2. \qquad \square$$

We also need a projector Π_h^S onto S_h. Its definition is similar to the rectangular case (4.5); namely, on elements τ intersecting Γ_0, we define $(\Pi_h^S w)|_\tau$ to be the $L_1^2(\tau)$-orthogonal projection of $w|_\tau$ onto the space of constants on τ. On the remaining elements, $(\Pi_h^S w)|_\tau$ is the $L^2(\tau)$-orthogonal projection of w onto the space of constants on τ. Then, we have the following analogue of Lemma 4.1 in the rectangular case.

Lemma 5.1. *Items* (3) *and* (4) *of Lemma* 4.1 *hold verbatim in the triangular case with the above defined projectors.*

Proof. Since the commutative properties are standard for the interpolants based on the standard degrees of freedom, it suffices to verify that they hold on elements τ with nonempty $\overline{\tau} \cap \overline{\Gamma}_0$. On such elements τ, because of the last set of degrees of freedom in (5.2), the identity

$$(5.3) \qquad \mathrm{curl}_{rz} \Pi_h^W \mathbf{v} = \Pi_h^S \mathrm{curl}_{rz} \mathbf{v}$$

obviously holds. In order to prove the remaining commutative identity, it suffices to show that the degrees of freedom in (5.2) vanish when they are applied to $(\Pi_h^W \mathbf{grad}_{rz} p - \mathbf{grad}_{rz} \Pi_h^V p)|_\tau$. We need only check σ_i^W and σ_τ^W, as the case of σ_e^W is standard. Now, for any vertex $\mathbf{a}_i$ in Γ_0, the definition of Π_h^V gives

$$\int_{e(\mathbf{a}_i)} r\,\mathbf{grad}_{rz}\Pi_h^V p \cdot \mathbf{t}\,ds = \int_{e(\mathbf{a}_i)} r\,\mathbf{grad}_{rz} p \cdot \mathbf{t}\,ds = \int_{e(\mathbf{a}_i)} r\,\Pi_h^W\mathbf{grad}_{rz} p \cdot \mathbf{t}\,ds,$$

so $\sigma_i^W(\Pi_h^W\mathbf{grad}_{rz} p - \mathbf{grad}_{rz}\Pi_h^V p) = 0$. Finally, if τ is a triangle in $\mathcal{T}_h$ intersecting Γ_0 nontrivially, then

$$\int_\tau r\,\mathrm{curl}_{rz}\mathbf{grad}_{rz}\Pi_h^V p\,dr\,dz = 0 = \int_\tau r\,\mathrm{curl}_{rz}\mathbf{grad}_{rz} p\,dr\,dz$$

$$= \int_\tau r\,\mathrm{curl}_{rz}\Pi_h^W\mathbf{grad}_{rz} p\,dr\,dz,$$

so $\sigma_\tau^W(\Pi_h^W\mathbf{grad}_{rz} p - \mathbf{grad}_{rz}\Pi_h^V p) = 0$. Thus $\Pi_h^W\mathbf{grad}_{rz} p$ and $\mathbf{grad}_{rz}\Pi_h^V p$ agree on all the degrees of freedom in W_h. Hence the commutativity property

$$(5.4) \qquad \Pi_h^W\mathbf{grad}_{rz} p = \mathbf{grad}_{rz}\Pi_h^V p$$

follows.

Finally, the assertion of the lemma on the boundary conditions is obvious by the construction of the interpolants (where we use the fact that $e(\mathbf{a}_i)$ is contained in Γ_1 if $\mathbf{a}_i$ is an endpoint of $\overline{\Gamma}_0$). $\qquad \square$

In proving an approximation estimate for the interpolation operator Π_h^W, we shall use the next lemma to control the error on edges not having degrees of freedom. In the triangular case, local arguments of the type we used in the rectangular case are not applicable, as the degrees of freedom are defined more globally.

Lemma 5.2. *Let τ be a mesh triangle in $\mathcal{T}_h$ having edges e_1, e_2, e_3. Set*

$$I_j^r(\mathbf{u}) = \int_{e_j} r\,\mathbf{u} \cdot \mathbf{t}\,ds \quad and \quad I_j^1(\mathbf{u}) = \int_{e_j} \mathbf{u} \cdot \mathbf{t}\,ds.$$

(1) *If the edge e_3 is contained in Γ_0, then*

$$\|\mathbf{u}\|_{L_1^2(\tau)^2}^2 \le C\left(h^2|\mathbf{u}|_{H_1^1(\tau)^2}^2 + h^{-1}|I_1^r(\mathbf{u})|^2 + h^{-1}|I_2^r(\mathbf{u})|^2\right)$$

for all $\mathbf{u}$ in $H_1^1(\tau)^2$.

(2) *If e_3 does not intersect Γ_0, but e_1 and e_2 have a common endpoint on Γ_0, then*

$$\|\mathbf{u}\|^2_{L^2_1(\tau)^2} + h^{-1}|I^r_1(\mathbf{u})|^2 \leq C\left(h^2|\mathbf{u}|^2_{H^1_1(\tau)^2} + h^{-1}|I^r_2(\mathbf{u})|^2 + h|I^1_3(\mathbf{u})|^2\right)$$

for all $\mathbf{u}$ in $H^1_1(\tau)^2$.

Proof. The triangle τ is either a type one or type two triangle (using the terminology in the proof of Proposition 5.2). This proof uses scaling arguments (as in [4]) employing maps from two reference triangles, one for each type of τ. The reference triangles are $\hat{\tau}_1$ and $\hat{\tau}_2$, defined to be the triangles in the $\hat{r}$-$\hat{z}$ plane with vertices $\{(0,0), (1,1), (1,-1)\}$ and $\{(0,-1), (1,0), (0,1)\}$, respectively (see Figure 1).

To prove the first estimate, let $\mathbf{F}$ be an affine homeomorphism that maps $\hat{\tau}_2$ onto τ such that $\mathbf{F}$ maps the edge of $\hat{\tau}_2$ on the z-axis onto e_3. Map $\mathbf{u}(r,z)$ covariantly to define the function $\hat{\mathbf{u}}(\hat{r},\hat{z}) = J(\mathbf{F})^t\mathbf{u}(r,z)$ on $\hat{\tau}_2$, where $J(\mathbf{F})$ is the Fréchet derivative of $\mathbf{F}$. If λ_3 is the barycentric coordinate function of τ that vanishes on $e_3 \subseteq \Gamma_0$, then $r = h_3\lambda_3$, where h_3 is the distance from e_3 to the vertex of τ opposite of e_3. Using this, it is easy to show that

$$(5.5) \qquad\qquad \|\mathbf{u}\|^2_{L^2_1(\tau)^2} \leq C\,h\|\hat{\mathbf{u}}\|^2_{L^2_1(\hat{\tau}_2)^2},$$

$$(5.6) \qquad\qquad |\hat{\mathbf{u}}|^2_{H^1_1(\hat{\tau}_2)^2} \leq C\,h\,|\mathbf{u}|^2_{H^1_1(\tau)^2},$$

$$(5.7) \qquad \left|\int_{\hat{e}_i} \hat{r}\,\hat{\mathbf{u}}\cdot\hat{\mathbf{t}}_i\,d\hat{s}\right|^2 \leq Ch^{-2}\left|\int_{e_i} r\,\mathbf{u}\cdot\mathbf{t}\,ds\right|^2, \qquad \text{for } i = 1 \text{ and } 2,$$

where $\hat{e}_i = \mathbf{F}^{-1}(e_i)$ and $\hat{\mathbf{t}}_i$ denotes the unit tangent on $\hat{e}_i$. Therefore, once we show that

$$(5.8) \quad C\|\hat{\mathbf{u}}\|^2_{L^2_1(\hat{\tau}_2)^2} \leq |\hat{\mathbf{u}}|^2_{H^1_1(\hat{\tau}_2)^2} + |\hat{I}^r_1(\hat{\mathbf{u}})|^2 + |\hat{I}^r_2(\hat{\mathbf{u}})|^2, \qquad \text{for all } \hat{\mathbf{u}} \in H^1_1(\hat{\tau}_2)^2,$$

the first inequality of the lemma follows using (5.5)–(5.7). Here, $\hat{I}^r_j(\hat{\mathbf{u}}) = \int_{\hat{e}_j} \hat{r}\,\hat{\mathbf{u}}\cdot\hat{\mathbf{t}}\,d\hat{s}$.

To establish (5.8), suppose on the contrary that there exists a sequence $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ in $H^1_1(\hat{\tau}_2)^2$ such that

$$(5.9) \qquad\qquad \frac{1}{n}\|\mathbf{v}_n\|^2_{L^2_1(\hat{\tau}_2)^2} > |\mathbf{v}_n|^2_{H^1_1(\hat{\tau}_2)^2} + |\hat{I}^r_1(\mathbf{v}_n)|^2 + |\hat{I}^r_2(\mathbf{v}_n)|^2.$$

Normalizing, we may assume without loss of generality that $\|\mathbf{v}_n\|_{L^2_1(\hat{\tau}_2)} = 1$ for all $n \in \mathbb{N}$. Thus (5.9) implies that $\lim_{n\to\infty}|\mathbf{v}_n|^2_{H^1_1(\hat{\tau}_2)^2} = 0$. In particular, $\{\mathbf{v}_n\}$ is a uniformly bounded sequence in $H^1_1(\hat{\tau}_2)^2$. Consequently, the compact imbedding of $H^1_1(\hat{\tau}_2)^2$ in $L^2_1(\hat{\tau}_2)^2$ [15, Lemma 4.2] yields a subsequence $\{\mathbf{v}_{j_n}\}_{n\in\mathbb{N}}$ converging in $L^2_1(\hat{\tau}_2)^2$. This convergence in $L^2_1(\hat{\tau}_2)^2$ and $\lim_{n\to\infty}|\mathbf{v}_{j_n}|_{H^1_1(\hat{\tau}_2)^2} = 0$ together imply the existence of a $\mathbf{v} \in H^1_1(\hat{\tau}_2)^2$ such that

$$\lim_{n\to\infty} \mathbf{v}_{j_n} = \mathbf{v} \qquad \text{in } H^1_1(\hat{\tau}_2)^2.$$

Moreover, since $|\mathbf{v}|_{H^1_1(\hat{\tau}_2)^2} = 0$, we find that $\mathbf{v}$ is constant. By Lemma A.1, $\hat{I}^r_1(\cdot)$ and $\hat{I}^r_2(\cdot)$ are continuous linear functionals on $H^1_1(\hat{\tau}_2)^2$, so we also have that

$$\hat{I}^r_j(\mathbf{v}) = (\mathbf{v}\cdot\hat{\mathbf{t}}_j)\int_{\hat{e}_j} r\,d\hat{s} = 0 \qquad \text{for } i = 1, 2.$$

Since the tangent vectors $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$ are linearly independent, $\mathbf{v}\cdot\hat{\mathbf{t}}_1 = \mathbf{v}\cdot\hat{\mathbf{t}}_2 = 0$ implies that $\mathbf{v}$ vanishes. But this contradicts $\|\mathbf{v}\|^2_{L^2_1(\hat{\tau}_2)^2} = \lim_{n\to\infty}\|\mathbf{v}_n\|^2_{L^2_1(\hat{\tau}_2)^2} = 1$,

so we conclude that (5.8) holds on the reference element $\hat{\tau}_2$. Mapping to a target element τ we complete the proof of the first inequality of the lemma.

The proof of the second inequality of the lemma is similar. For this case, since τ is a type one triangle, we must use a mapping from the other reference triangle $\hat{\tau}_1$. Also, now r is given by a different expression, namely $r = r_1 \lambda_1 + r_2 \lambda_2$, where r_i denotes the r-coordinate of the vertex of τ opposite of e_i and λ_i is the barycentric coordinate function vanishing on e_i. But inequalities like (5.5)–(5.7) continue to hold, because r_1/h and r_2/h are bounded above and below by fixed constants (independent of mesh size, but depending on element angles and on the angle that Γ_1 makes with Γ_0). The argument on the reference element $\hat{\tau}_1$ by contradiction is analogous to that above. $\qquad\square$

Lemma 5.3 (Approximation estimates). *The projectors defined in the triangular case also satisfy the estimates of Lemma 4.2, i.e., there exists a constant C such that*

$$(5.10) \qquad \|\mathbf{grad}_{rz}(u - \Pi_h^V u)\|_{L_1^2(D)} \leq Ch|u|_{H_1^2(D)},$$

$$(5.11) \qquad \|\mathbf{v} - \Pi_h^W \mathbf{v}\|_{L_1^2(D)^2} \leq Ch|\mathbf{v}|_{H_1^1(D)^2},$$

$$(5.12) \qquad \|\mathrm{curl}_{rz}(\mathbf{v} - \Pi_h^W \mathbf{v})\|_{L_1^2(D)} \leq Ch|\mathrm{curl}_{rz}\mathbf{v}|_{H_1^1(D)},$$

$$(5.13) \qquad \|w - \Pi_h^S w\|_{L_1^2(D)} \leq Ch|w|_{H_1^1(D)},$$

for all $u \in H_1^2(D)$, $w \in H_1^1(D)$, and $\mathbf{v} \in H_1^1(D)^2$. For (5.12), we additionally require $\mathrm{curl}_{rz}\mathbf{v} \in H_1^1(D)$.

Proof. First, (5.13) follows from a result already established in the literature [4, Lemma 5].

As in the proof of Lemma 4.2, (5.12) follows from the commutativity property of Lemma 5.1 and (5.13).

The proof of (5.11) is more involved. Recalling the notations in the proof of Proposition 5.2 (see Figure 1), we first consider a type one triangle τ with a vertex $\boldsymbol{a}$ on Γ_0. Then there is a sequence of type one triangles $T_0, T_1, \ldots, T_n$, all connected to $\boldsymbol{a}$, with $T_0 = \tau$, and T_n having $e(\boldsymbol{a})$ as an edge. Let $\ell_j = \partial T_{j-1} \cap \partial T_j$ denote the corresponding sequence of edges connected to $\boldsymbol{a}$. Then, by Lemma 5.2(2) and Proposition 5.3,

$$\|\mathbf{v} - \Pi_h^W \mathbf{v}\|_{L_1^2(\tau)^2}^2 \leq C\left(h^2 |\mathbf{v}|_{H_1^1(T_0)^2}^2 + h^{-1}\left|\int_{\ell_1} r\,(\mathbf{v} - \Pi_h^W \mathbf{v}) \cdot \mathbf{t}\,ds\right|^2 \right),$$

where we have used the fact that σ_e^W-type degrees of freedom in (5.2) coincide when applied to $\mathbf{v}$ and $\Pi_h^W \mathbf{v}$. We use Lemma 5.2(2) again, but this time to bound the integral over ℓ_1 appearing above using a seminorm over the next triangle T_1. Combining with further applications of Lemma 5.2(2) and Proposition 5.3, we have

$$\|\mathbf{v} - \Pi_h^W \mathbf{v}\|_{L_1^2(\tau)^2}^2 \leq C\left(h^2 |\mathbf{v}|_{H_1^1(T_0 \cup T_1)^2}^2 + h^{-1}\left|\int_{\ell_2} r\,(\mathbf{v} - \Pi_h^W \mathbf{v}) \cdot \mathbf{t}\,ds\right|^2 \right)$$

$$\leq C\left(h^2 |\mathbf{v}|_{H_1^1(T_0 \cup T_1 \cup \cdots \cup T_n)^2}^2 + h^{-1}\left|\int_{e(\boldsymbol{a})} r\,(\mathbf{v} - \Pi_h^W \mathbf{v}) \cdot \mathbf{t}\,ds\right|^2 \right).$$

The last integral is zero because of the first type of degrees of freedom in (5.2). Thus, if $D_{\boldsymbol{a}}$ denotes the union of all triangles connected to $\boldsymbol{a}$, we have

$$(5.14) \qquad \|\mathbf{v} - \Pi_h^W \mathbf{v}\|_{L_1^2(D_{\boldsymbol{a}})^2}^2 \leq Ch^2 |\mathbf{v}|_{H_1^1(D_{\boldsymbol{a}})^2}^2,$$

where we have tacitly used the fact that the number n above is bounded by a fixed constant due to the shape regularity of the mesh.

Next consider a type two triangle τ with edges e_1, e_2, e_3 such that $e_3 \subseteq \Gamma_0$. Applying Lemma 5.2(1) and Proposition 5.3,

$$\|\mathbf{v} - \Pi_h^W \mathbf{v}\|_{L_1^2(\tau)^2}^2 \leq C \left(h^2 \, |\mathbf{v}|_{H_1^1(\tau)^2}^2 + h^{-1} \left| \int_{e_1 \cup e_2} r \, (\mathbf{v} - \Pi_h^W \mathbf{v}) \cdot \mathbf{t} \, ds \right|^2 \right).$$

The portions of the last integral over e_i (for $i = 1$ or 2) vanish if e_i is a degree of freedom. Otherwise, they are bounded as above and we conclude that for any type two triangle τ,

$$(5.15) \qquad \|\mathbf{v} - \Pi_h^W \mathbf{v}\|_{L_1^2(\tau)^2}^2 \leq Ch^2 \, |\mathbf{v}|_{H_1^1(D_{\mathbf{x}_1} \cup D_{\mathbf{x}_2})^2}^2$$

where $\mathbf{x}_i$ is the vertex of τ opposite of e_i.

For triangles τ not intersecting Γ_0, the estimate

$$(5.16) \qquad \|\mathbf{v} - \Pi_h^W \mathbf{v}\|_{L_1^2(\tau)^2}^2 \leq Ch^2 |\mathbf{v} - \Pi_h^W \mathbf{v}|_{H_1^1(\tau)^2}^2 \leq Ch^2 |\mathbf{v}|_{H_1^1(\tau)^2}^2$$

follows by standard arguments. Hence by summing over all triangles, and using (5.14), (5.15), or (5.16) as appropriate, we obtain

$$\|\mathbf{v} - \Pi_h^W \mathbf{v}\|_{L_1^2(D)^2} \leq Ch |\mathbf{v}|_{H_1^1(D)^2}.$$

This proves (5.11).

The estimate (5.10) is proved as in the rectangular case using Lemma 5.1 and commutativity. This completes the proof of all the lemmas.   □

6. THE DISCRETE MIXED METHOD

With the tools constructed in the previous sections, we are now able to analyze a discrete mixed method suggested by the variational formulation in Section 3. The mixed Galerkin method develops approximate solutions in the discrete subspaces considered in the previous sections by requiring them to satisfy the equations of the variational formulation (3.1) in the discrete subspaces. To limit the technicalities in the analysis, we assume throughout this section that the revolution of D, namely Ω, is convex.

The mixed Galerkin approximation is the function pair $(\mathbf{u}_h, p_h) \in W_{h,\diamond} \times V_{h,\diamond}$ satisfying

$$(6.1)$$
$$\begin{cases} (\mu^{-1} \mathrm{curl}_{rz} \mathbf{u}_h, \mathrm{curl}_{rz} \mathbf{v}_h)_r + (\mathbf{v}_h, \mathbf{grad}_{rz} p_h)_r &= (\mathbf{f}, \mathbf{v}_h)_r & \text{for all } \mathbf{v}_h \in W_{h,\diamond}, \\ (\mathbf{u}_h, \mathbf{grad}_{rz} q_h)_r &= (g, q_h)_r & \text{for all } q_h \in V_{h,\diamond}. \end{cases}$$

It follows from the next theorem that such a pair is uniquely determined by the above equations. In (6.1), the finite element spaces can be either those defined in Section 4 (for rectangular meshes) or those in Section 5 (for triangular meshes), i.e., $V_{h,\diamond}$ and $W_{h,\diamond}$ are defined by (4.1), where V_h and W_h can be either as defined in Section 4 or as in Section 5. The analysis of this section treats both cases

simultaneously, following a standard approach (cf. [16, 6, 7]). Define $W_{h,\diamond}^0$ as the subspace of $W_{h,\diamond}$ consisting of discretely divergence-free functions, i.e.,

$$W_{h,\diamond}^0 = \{\boldsymbol{\phi}_h \in W_{h,\diamond} : (\boldsymbol{\phi}_h, \mathbf{grad}_{rz} p_h)_r = 0 \text{ for all } p_h \in V_{h,\diamond}\}.$$

Then we have the following theorem (cf. Theorem 3.1).

Theorem 6.1. *The following holds for the axisymmetric mixed method* (6.1)*:*

(1) Discrete inf-sup condition: *For all p_h in $V_{h,\diamond}$,*

$$C\,\|p_h\|_{1,r} \ \leq\ \sup_{\mathbf{v}_h \in W_{h,\diamond}} \frac{(\mathbf{v}_h, \mathbf{grad}_{rz} p_h)_r}{\|\mathbf{v}_h\|_{r,\mathrm{curl}}}.$$

(2) Coercivity on the discrete kernel: *For all $\mathbf{v}_h$ in $W_{h,\diamond}^0$,*

$$\|\mathbf{v}_h\|_{r,\mathrm{curl}} \leq C\mu_1\|\mu^{-1/2}\mathrm{curl}_{rz}\mathbf{v}_h\|_r.$$

(3) Unique solvability: *There is a unique $\mathbf{u}_h$ in $W_{h,\diamond}$ and a unique p_h in $V_{h,\diamond}$ satisfying* (6.1)*.*

(4) Optimal error estimates: *If $(\mathbf{u}, p)$ in $\mathbf{H}_{r,\diamond}(\mathrm{curl}, D) \times H_{1,\diamond}^1(D)$ is the exact solution satisfying* (3.1) *and $(\mathbf{u}_h, p_h)$ in $W_{h,\diamond} \times V_{h,\diamond}$ is the discrete solution satisfying* (6.1)*, then*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}_r(\mathrm{curl}, D)} + \|p - p_h\|_{H_1^1(D)} \leq Ch\big(|\mathbf{u}|_{H_1^1(D)} + |\mathrm{curl}_{rz}\mathbf{u}|_{H_1^1(D)} + |p|_{H_1^2(D)}\big).$$

Here, we have implicitly assumed that $\mathbf{u}$ and p are smooth enough so that the above norms make sense.

Proof. (1) The proof of the inf-sup condition is similar to the proof of the exact inf-sup condition in Theorem 3.1(1). The only change is that we use the discrete version of $\mathbf{grad}_{rz} H_{1,\diamond}^1(D) \subseteq \mathbf{H}_{r,\diamond}(\mathrm{curl}, D)$, namely that for all p_h in $V_{h,\diamond}$, the gradient $\mathbf{grad}_{rz} p_h$ is in $W_{h,\diamond}$.

(2) The proof of this inequality is an adaptation of the proof of Proposition 5.1(III) of [10]. Let $p \in H_{1,\diamond}^1(D)$ be the unique solution (cf. [11]) to

$$(\mathbf{grad}_{rz} p, \mathbf{grad}_{rz} q)_r = (\mathbf{v}_h, \mathbf{grad}_{rz} q)_r \quad \text{for all } q \in H_{1,\diamond}^1(D),$$

and put

$$\mathbf{w} := \mathbf{v}_h - \mathbf{grad}_{rz} p.$$

Define the revolved vector field $\breve{\mathbf{w}} = w_r \mathbf{e}_r + w_z \mathbf{e}_z$ on Ω. By the same argument given in the proof of Theorem 3.1(2), $\breve{\mathbf{w}}$ is in $H_0(\mathbf{curl}, \Omega)$ and satisfies

$$(\breve{\mathbf{w}}, \mathbf{grad}\, q)_{L^2(\Omega)} = 0 \qquad \text{for all } q \in H_0^1(\Omega),$$

i.e., $\breve{\mathbf{w}}$ is divergence free. Now, a well known imbedding of $H_0(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$ into $H^1(\Omega)^3$ for convex Ω [10, Proposition I.3.1] yields

$$(6.2) \qquad \|\breve{\mathbf{w}}\|_{H^1(\Omega)^3} \leq C\|\mathbf{curl}\,\breve{\mathbf{w}}\|_{L^2(\Omega)^3} = C\|\mathbf{curl}\,\breve{\mathbf{v}}_h\|_{L^2(\Omega)^3},$$

where, in the last step, we used $\mathbf{curl}\,\breve{\mathbf{w}} = \mathbf{curl}\,\breve{\mathbf{v}}_h$. By [3, Proposition 3.17], $\mathbf{w} = (w_r, w_z)$ is in $H_-^1(D) \times H_1^1(D)$ (and is bounded by an accompanying norm estimate). Hence,

$$(6.3) \qquad \|\mathbf{w}\|_{H_1^1(D)^2} \leq \|\mathbf{w}\|_{H_-^1(D) \times H_1^1(D)} \leq C\|\breve{\mathbf{w}}\|_{H^1(\Omega)^3} \leq C\|\mathbf{curl}\,\breve{\mathbf{v}}_h\|_{L^2(\Omega)^3}.$$

To prove the required inequality for $\mathbf{v}_h$, we use the above estimates for $\mathbf{w}$. Since $\mathbf{w}$ is in $H_1^1(D)^2$, the projection $\Pi_h^W \mathbf{w}$ is well defined by Lemma 4.1(2) (rectangular case) or Proposition 5.2 (triangular case). Obviously, $\mathbf{grad}_{rz} p = \mathbf{v}_h - \mathbf{w}$ implies

that on each element τ, $(\mathbf{grad}_{rz}p)|_{\tau}$ is in $H_1^1(\tau)^2$, so it possesses enough regularity for the application of Π_h^W. Furthermore, since $\Pi_h^W \mathbf{v}_h = \mathbf{v}_h$,

$$\mathbf{v}_h = \Pi_h^W \mathbf{w} + \Pi_h^W \mathbf{grad}_{rz}p$$
$$= \Pi_h^W \mathbf{w} + \mathbf{grad}_{rz}\Pi_h^V p,$$

using the commutativity properties of Lemma 4.1 (rectangular case) or Lemma 5.1 (triangular case). Now since $\mathbf{v}_h$ is $L_1^2(D)$-orthogonal to $\mathbf{grad}_{rz}V_{h,\diamond}$, the above equation implies

$$(\mathbf{v}_h, \mathbf{v}_h)_r = (\Pi_h^W \mathbf{w}, \mathbf{v}_h)_r,$$

which, by the Cauchy-Schwarz inequality, gives

$$\|\mathbf{v}_h\|_{L_1^2(D)^2} \leq \|\Pi_h^W \mathbf{w}\|_{L_1^2(D)^2}.$$

Combining this with the approximation properties of Lemma 4.2 or Lemma 5.3,

$$\|\mathbf{v}_h\|_{L_1^2(D)^2} \leq \|\Pi_h^W \mathbf{w}\|_{L_1^2(D)^2} \leq \|\Pi_h^W \mathbf{w} - \mathbf{w}\|_{L_1^2(D)^2} + \|\mathbf{w}\|_{L_1^2(D)^2}$$
$$\leq Ch|\mathbf{w}|_{H_1^1(D)} + \|\mathbf{w}\|_{L_1^2(D)^2}$$
$$\leq C\|\mathrm{curl}_{rz}\mathbf{v}_h\|_{L_1^2(D)},$$

where in the last step we have used (6.3). This proves the coercivity on the discrete kernel.

(3) The unique solvability is a well known consequence of (1) and (2) (cf. [7, 10]).

(4) The error estimate now follows from the Babuška-Brezzi theory [7, 10]. Indeed, (1) and (2) imply quasioptimality, i.e.,

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}_r(\mathrm{curl},D)} + \|p - p_h\|_{H_1^1(D)}$$
$$\leq C\left(\inf_{\mathbf{v}_h \in W_{h,\diamond}} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}_r(\mathrm{curl},D)} + \inf_{q_h \in V_{h,\diamond}} \|p - q_h\|_{H_1^1(D)}\right)$$
$$\leq C\left(\|\mathbf{u} - \Pi_h^W \mathbf{u}\|_{\mathbf{H}_r(\mathrm{curl},D)} + \|p - \Pi_h^V p\|_{H_1^1(D)}\right).$$

Here we used the previously noted fact that the projectors Π_h^W and Π_h^V preserve zero boundary conditions (see Lemma 4.1(4) or Lemma 5.1), so that $\Pi_h^W \mathbf{u}$ and $\Pi_h^V p$ are indeed in $W_{h,\diamond}$ and $V_{h,\diamond}$, respectively. Estimating the right hand side above by Lemma 4.2 (rectangular case) or Lemma 5.3 (triangular case) and applying a Poincaré inequality completes the proof of the theorem. $\qquad\square$

7. NUMERICAL RESULTS

In this section, we verify that the finite element convergence rates predicted by our analysis are achieved in a numerical example. We also make suggestions on how to iteratively solve the discrete matrix system. As in the previous section, the finite element spaces $V_{h,\diamond}$ and $W_{h,\diamond}$ can be defined on rectangular or triangular meshes. Define the operators $A_h : W_{h,\diamond} \to (W_{h,\diamond})'$ and $B_h : W_{h,\diamond} \to (V_{h,\diamond})'$ by

$$A_h\mathbf{v}_h(\mathbf{w}_h) = (\mu^{-1}\mathrm{curl}_{rz}\mathbf{v}_h, \mathrm{curl}_{rz}\mathbf{w}_h)_r \quad \text{for all } \mathbf{v}_h, \mathbf{w}_h \in W_{h,\diamond},$$
$$B_h\mathbf{v}_h(q_h) = (\mathbf{v}_h, \mathbf{grad}_{rz}q_h)_r \quad\quad\quad \text{for all } \mathbf{v}_h \in W_{h,\diamond}, q_h \in V_{h,\diamond}.$$

Let A_h and B_h denote the matrix representations of the operators A_h and B_h, respectively, in the standard bases for $V_{h,\diamond}$ and $W_{h,\diamond}$. Then (6.1) is rewritten as the linear system

$$(7.1) \qquad\qquad\qquad\qquad \mathsf{C}_h\mathsf{x}_h = \mathsf{b}_h,$$

TABLE 7.1. Constant coefficient ($\mu \equiv 1$) on the unit square D.

level	L_1^2 error of (A_r, A_z)	Order	PCG	Points	Elements	Unknowns
1	0.0851831		8	49	72	240
2	0.0515349	0.725	8	169	288	1032
3	0.0255215	1.014	8	625	1152	4128
4	0.0126821	1.009	8	2401	4608	16368
5	0.0063197	1.004	8	9409	18432	65040
6	0.0031543	1.003	8	37249	73728	259152
7	0.0015757	1.001	8	148225	294912	1034448

where

$$\mathsf{C}_h = \left[\begin{array}{cc} \mathsf{A}_h & \mathsf{B}_h^t \\ \mathsf{B}_h & 0 \end{array}\right], \qquad \mathsf{x}_h = \left[\begin{array}{c} \mathsf{u}_h \\ \mathsf{p}_h \end{array}\right], \qquad \text{and } \mathsf{b}_h = \left[\begin{array}{c} \mathsf{f}_h \\ \mathsf{g}_h \end{array}\right].$$

Here, u_h and p_h denote the vectors of coefficients in the expansions of $\mathbf{u}_h$ and p_h in the standard bases, respectively, and the right hand side vector b_h is computed from the right hand side of (6.1) as usual. Note that the matrix C_h is symmetric but indefinite.

For practical problems with a large number of degrees of freedom, since the matrix system (7.1) is sparse, we solve (7.1) using a preconditioned iterative method. To precondition the linear system (7.1), we employ multigrid methods on the spaces $V_{h,\diamond}$ and $W_{h,\diamond}$ and forms coming from the $\mathbf{H}_r(\mathrm{curl}, D)$ and $H_1^1(D)$ inner-products, respectively. The uniform convergence of a V-cycle multigrid with standard Gauss-Seidel smoothing was proved in [11]. For the Nédélec subspace $W_{h,\diamond}$ of $\mathbf{H}_r(\mathrm{curl}, D)$, we use the V-cycle multigrid with Arnold-Falk-Winther smoothing [2]. Let M_V and M_W denote the matrices representing the multigrid V-cycles acting on the matrices with entries given by the inner products of basis functions in $V_{h,\diamond}$ and $W_{h,\diamond}$, respectively. Instead of the indefinite system (7.1), we solve the symmetric positive definite system

(7.2) $$\left(\mathsf{A}_h + \mathsf{B}_h^t \mathsf{M}_V \mathsf{B}_h\right) \mathsf{u}_h = \mathsf{f}_h + \mathsf{B}_h^t \mathsf{M}_V \mathsf{g}_h$$

by the preconditioned conjugate gradient method (PCG), with M_W as a preconditioner. The system (7.2) is obtained from (7.1) by replacing the zero block with $-\mathsf{M}_V^{-1}$, since $\mathsf{p}_h = 0$. In all the numerical experiments reported below, the termination condition for the PCG iterations is the reduction of the energy norm of the residual by a factor of 10^{-12}.

In Table 7.1, we list the $L_1^2(D)^2$-norm of the error in approximating (A_r, A_z) on a sequence of uniformly refined, unstructured meshes of the unit square. These results are for the constant coefficient $\mu \equiv 1$ and data

$$\mathbf{f} = \left(\pi^2 \sin \pi z, \frac{\pi}{r}(\cos \pi z - \cos \pi r) + \pi^2 \sin \pi r\right),$$
$$g = \frac{\sin \pi z}{r}.$$

In this case, the exact solution is the bounded and smooth vector field $(A_r, A_z) = (\sin \pi z, \sin \pi r)$. The coarsest mesh (level 1) has a mesh size of approximately $1/6$. Each refinement of a mesh is performed by connecting the midpoints of edges of each mesh triangle, so the finest mesh (level 7) is roughly of size $1/384$. The third

TABLE 7.2. PCG iteration counts for discontinuous coefficients (7.3) and (7.4) on the unit square D.

level	PCG for (7.3)	PCG for (7.4)	Points	Elements	Unknowns
1	15	23	45	64	246
2	16	23	153	256	954
3	17	23	561	1024	3714
4	17	23	2145	4096	14610
5	17	23	8385	16384	57906
6	17	23	33153	65536	230514
7	17	26	131841	262144	919794

column lists the observed order of convergence (the entry in the jth row of this column is equal to $\log_2(\varepsilon_{j-1}/\varepsilon_j)$, where ε_j is the $L_1^2(D)^2$-norm of the error at level j). We observe the theoretically predicted first-order convergence rate.

The effectiveness of the iterative solution technique by multigrid preconditioning is also evident from Table 7.1 – the fourth column of the table lists the number of PCG iterations taken to satisfy the termination condition. Observe that only 8 iterations are needed in all cases. Table 7.2 lists the PCG iteration counts for the following two discontinuous coefficients:

$$(7.3) \qquad \mu = \begin{cases} (1 + \sin r)/2, & \text{if } z > 1/2, \\ 1, & \text{otherwise,} \end{cases}$$

$$(7.4) \qquad \mu = \begin{cases} 10^4, & \text{if } z > 1/2, \\ 1, & \text{otherwise.} \end{cases}$$

The finest mesh size (level 7) is $1/256$. For both of these discontinuous coefficients, we observe that the preconditioned iterative solution technique works well.

APPENDIX A. A TRACE RESULT

In this appendix we show a trace result for $H_1^1(\tau)$ for a triangle τ connected to Γ_0. As seen from Figure 1, τ can be either of type one or type two. If h_τ and ρ_τ are the diameters of τ and the largest circle inscribed in τ, respectively, then the ratio ρ_τ/h_τ measures the shape regularity of the triangle. We expect the trace inequality to have a constant C_{trc} dependent on ρ_τ/h_τ. This dependence can be translated into a dependence on the interior angles of τ. Our purpose here is to show that the trace constant also depends on the angle τ makes with the z-axis and that the weights in the trace inequality are different for type one and type two triangles. Of the two nonzero angles that the edges of τ make with the z-axis, let θ be the one with the smaller sine.

Lemma A.1. *Suppose e is an edge of τ that intersects $\overline{\Gamma}_0$ at one and only one point. Then there is a constant C_{trc} depending on the shape regularity constant ρ_τ/h_τ, but otherwise independent of τ, such that for all v in $H_1^1(\tau)$, we have the following:*

(1) *If τ is a type one triangle, then*

$$\frac{1}{|e|}\|v\|_{L_2^2(e)}^2 \leq C_{\text{trc}}\frac{h_\tau}{\sin\theta}\left(\frac{1}{h_\tau^2}\|v\|_{L_1^2(\tau)}^2 + |v|_{H_1^1(\tau)}^2\right).$$

(2) *If τ is a type two triangle, then*

$$\frac{1}{|e|}\|v\|_{L_1^2(e)}^2 \le C_{\text{trc}} \frac{1}{\sin\theta} \left(\frac{1}{h_\tau^2}\|v\|_{L_1^2(\tau)}^2 + |v|_{H_1^1(\tau)}^2 \right).$$

Proof. It suffices to prove the stated inequalities for v in $\mathcal{D}(\overline{\tau})$ because of the density of $\mathcal{D}(\overline{\tau})$ in $H_1^1(\tau)$ [13]. First, consider the type one case. Let the type one reference triangle $\hat{\tau}_1$ (see Figure 1) be mapped onto τ by an affine map $\mathbf{F}$. Without loss of generality, assume that $e = \mathbf{F}(\hat{e})$, where $\hat{e}$ is the edge of $\hat{\tau}_1$ in the first quadrant of the $\hat{r}$-$\hat{z}$ plane. If $\hat{v} = v \circ \mathbf{F}$, then the following estimates are easy to verify:

$$(A.1) \qquad \frac{1}{|e|}\|v\|_{L_2^2(e)}^2 \le \hat{c}\,\overline{r}_\tau^2 \|\hat{v}\|_{L_2^2(\hat{e})}^2,$$

$$(A.2) \qquad \|\hat{v}\|_{L_1^2(\hat{\tau}_1)}^2 \le \frac{\hat{c}}{h_\tau^2 \underline{r}_\tau}\|v\|_{L_1^2(\tau)}^2,$$

$$(A.3) \qquad |\hat{v}|_{H_1^1(\hat{\tau}_1)}^2 \le \frac{\hat{c}}{\underline{r}_\tau}|v|_{H_1^1(\tau)}^2,$$

where $\overline{r}_\tau$ and $\underline{r}_\tau$ denote the maximum and minimum of the r-coordinate values of the two vertices of τ not on $\overline{\Gamma}_0$, respectively, and $\hat{c}$ depends only on the shape regularity constant. Note that $\overline{r}_\tau/\underline{r}_\tau \le C_{\text{trc}}/\sin\theta$. We will now show that

$$(A.4) \qquad \frac{1}{\sqrt{2}}\|\hat{v}\|_{L_2^2(\hat{e})}^2 \le \|\hat{v}\|_{L_1^2(\hat{\tau}_1)}^2 + 2\|\partial_z\hat{v}\|_{L_3^2(\hat{\tau}_1)}^2.$$

Once this is established, the required inequality follows from (A.1)–(A.3).

To prove (A.4), we consider a smooth function v on $\hat{\tau}_1$ (dropping the accent from $\hat{r}$, $\hat{z}$, and $\hat{v}$ for convenience) and use the fundamental theorem of calculus to write

$$v(r,r) - v(r,z) = \int_z^r \partial_z v(r,\zeta)\,d\zeta,$$

so that

$$|v(r,r)|^2 \le 2|v(r,z)|^2 + 2\left|\int_z^r \partial_z v(r,\zeta)\,d\zeta\right|^2.$$

Integrating over z from $-r$ to r,

$$r|v(r,r)|^2 \le \int_{-r}^r |v(r,z)|^2\,dz + \int_{-r}^r \left|\int_z^r \partial_z v(r,\zeta)\,d\zeta\right|^2 dz$$

$$\le \int_{-r}^r |v(r,z)|^2\,dz + 2r^2 \int_{-r}^r |\partial_z v(r,\zeta)|^2\,d\zeta.$$

Multiplying by r and integrating over r from 0 to 1 yields (A.4). This completes the proof for the type one case.

The proof of the type two case proceeds similarly, but using the reference triangle $\hat{\tau}_2$ instead (see Figure 1). Again, we denote by $\hat{e}$ the edge of $\hat{\tau}_2$ in the first quadrant of the $\hat{r}$-$\hat{z}$ plane. As in the previous case,

$$v(r,1-r) - v(r,z) = \int_z^{1-r} \partial_z v(r,\zeta)\,d\zeta$$

implies

$$|v(r,1-r)|^2 \le 2|v(r,z)|^2 + 4\int_z^{1-r} |\partial_z v(r,\zeta)|^2\,d\zeta.$$

Integrating over $-1 + r < z < 1 - r$, we have

$$(1-r)\,|v(r,1-r)|^2 \leq \int_{-1+r}^{1-r} |v(r,z)|^2\ dz + 4\int_{-1+r}^{1-r} |\partial_z v(r,\zeta)|^2\ d\zeta.$$

Unlike the previous case, we now observe the presence of the factor $(1-r)$ above. Upon further integration over $0 < r < 1$, this will result in an unwanted degenerate weight. Therefore, we integrate only over $0 < r < 1/2$, obtaining

$$\frac{1}{2}\int_0^{1/2} r\,|v(r,1-r)|^2\ dr \leq \int_0^{1/2} r(1-r)\,|v(r,1-r)|^2\ dr$$

$$\text{(A.5)} \qquad\qquad\qquad \leq \|v\|_{L_1^2(\hat\tau_2)}^2 + 4\|\partial_z v\|_{L_1^2(\hat\tau_2)}^2.$$

For the remaining piece of $\hat e$ where $1/2 < r < 1$, we integrate along the line segment $\{(s, s - 2r + 1) : r - 1/2 < s < r\}$ of length $\sqrt{2}/2$, perpendicular to $\hat e$ at $(r, 1-r)$. This gives

$$|v(r,1-r)|^2 \leq 2|v(s, s - 2r + 1)|^2 + \left| \int_s^r (1,1) \cdot \mathbf{grad}_{rz} v(\zeta, \zeta - 2r + 1)\, d\zeta \right|^2.$$

Integrating over $r - 1/2 < s < r$ and then over $1/2 < r < 1$, we have

$$\text{(A.6)} \qquad \frac{1}{2}\int_{1/2}^1 r\,|v(r,1-r)|^2\ dr \leq 2\|v\|_{L_1^2(\hat\tau_2)}^2 + \frac{1}{2}|v|_{H_1^1(\hat\tau_2)}^2.$$

Combining (A.5) and (A.6) yields

$$\frac{1}{\sqrt{2}}\|v\|_{L_1^2(\hat e)}^2 \leq 6\|v\|_{L_1^2(\hat\tau_2)}^2 + 9|v|_{H_1^1(\hat\tau_2)}^2.$$

The scaling argument to complete the proof is similar to the type one case. The only difference is that the norm on $L_1^2(e)$ gives one less factor of $\overline{r}_\tau$ than the norm on $L_2^2(e)$, resulting in one less factor of h_τ. $\qquad\qquad\square$

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